Problems, Week 5

1. Sketch the curves $y = e^{-2x}$ and $y = e^{-3x}$ carefully. Is the area enclosed between them in the first quadrant finite?

2. Consider the following four integrals:

a) $\displaystyle\int_0^\infty \frac{x}{1+x^4}\,dx$ b) $\displaystyle\int_0^\infty \frac{x^2}{1+x^4}\,dx$

c) $\displaystyle\int_0^\infty \frac{x^3}{1+x^4}\,dx$ d) $\displaystyle\int_0^\infty \frac{x^4}{1+x^4}\,dx$

Which of these integrals converge? (Hint: compare to the "correct" pure powers of x.) Compute the exact value of at least one of the convergent integrals.

3. a) The following improper integral converges: $\displaystyle\int_0^\infty xe^{-x^2}\,dx$. What is its value?

b) The value of $\displaystyle\int_0^\infty e^{-x^2}\,dx$ (another convergent improper integral) is $\dfrac{\sqrt{\pi}}{2}$ (this amazing fact easiest to explain with some of the tools in third semester calculus). Improper integrals involving polynomials and e^{-x^2} often arise in statistics (and hence in statistical analyses of experiments). Use integration by parts to get a formula relating $\displaystyle\int_0^\infty x^n e^{-x^2}\,dx$ and $\displaystyle\int_0^\infty x^{n-2} e^{-x^2}\,dx$, where n is a positive integer bigger than 2. (The parts you take will be slightly tricky.)

c) Now find the values of

i) $\displaystyle\int_0^\infty x^2 e^{-x^2}\,dx$ ii) $\displaystyle\int_0^\infty x^3 e^{-x^2}\,dx$ iii) $\displaystyle\int_0^\infty x^4 e^{-x^2}\,dx$

You will need the reduction formula and the two initial values given in a) and b).

4. The integral $\displaystyle\int_1^\infty \frac{e^{-5x}}{1+x^2}\,dx$ certainly converges. One simple way to try to approximate its value is to write it as the sum of two integrals:

$$\int_1^\infty \frac{e^{-5x}}{1+x^2}\,dx = \int_1^A \frac{e^{-5x}}{1+x^2}\,dx + \int_A^\infty \frac{e^{-5x}}{1+x^2}\,dx$$

where A is some large number. Then any of our numerical techniques could be used to approximate the first (proper!) definite integral on the right-hand side of the equation. We could just neglect the second integral if we knew that it was small.

a) Compare the second integral on the right to an integral over the same interval of a simpler function to try to find an A so that the improper integral to be dropped will have a value smaller than 10^{-5}.

b) Use a numerical technique to try to evaluate the first integral with an error less than $\pm 10^{-5}$. *You will definitely need a programmable calculator or a computer for this part!*

c) Compute $\displaystyle\int_1^\infty \frac{e^{-5x}}{1+x^2}\,dx$ with an error less than $\pm 10^{-4}$.

Note: there is **no** unique or best possible answer to this question, either for the "simpler" function or for the A.

5. a) If m and n are constants, compute $\displaystyle\int_0^{2\pi} \cos mx\ \cos nx\ dx$.

b) Suppose $f(x) = A\cos x + B\cos 2x + C\cos 3x$, and that you also know

$$\int_0^{2\pi} f(x)\cos x\ dx = 5 \qquad \int_0^{2\pi} f(x)\cos 2x\ dx = 6 \qquad \int_0^{2\pi} f(x)\cos 3x\ dx = 7\ .$$

Find A and B and C.

Note: this is a computational beginning to Fourier series, a way of analyzing periodic phenomena.

6. If a function F is continuous on an interval $[a,b)$ and if $\lim_{x\to b^-} F(x) = +\infty$ then we'll say "F explodes at b."

a) At what point do the functions $\dfrac{x^2+1}{x-1}$ and $\dfrac{\cos x}{x-2}$ explode?

b) Write all solutions to the differential equation $y' = y^2$ subject to the initial condition $y(0) = y_0$. Can you find a solution that explodes at 10? And another one that explodes at 5? Can you find a solution that explodes at x_0 when $x_0 > 0$? How does the initial condition at $x = 0$ connect with a specified explosion at x_0? Draw a picture of an exploding solution.

7. A cylindrical metal tank, of height 10 feet, has circular cross section of radius 2 feet. There is a small hole in the bottom of the tank, through which water leaks at a rate proportional to the depth of the water in the tank. The tank is filled at noon and at 2:00 the depth of water in the tank is recorded to be 3 feet.

(a) Find the volume of water in the tank t hours after noon.

(b) Suppose that the water leaking from this tank is caught in a second tank in the shape of a right circular cone, whose top has radius 2 feet and whose height is 10 feet. At what rate is the depth of water in this tank increasing an instant before the tank begins to overflow?