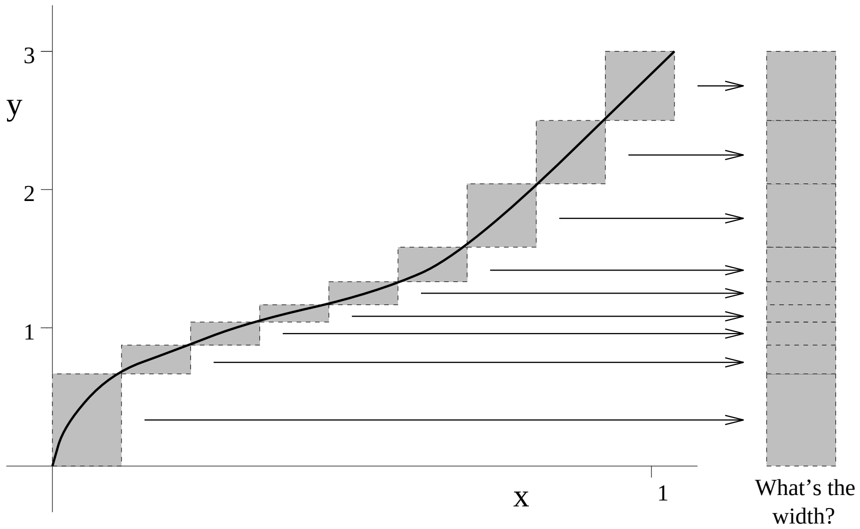
Problems, Week 4

1. Suppose $f(x) = \sqrt{3x + 6x^4}$.

a) Prove that f is increasing on the interval $[0, 1]$.

b) Write down a finite sum which will be within 10^{-10} of the true value of $\int_0^1 f(x)\,dx$. You are *not* asked to actually compute the sum, just describe it in any convenient fashion. Hint: your reasoning and your explanation may be guided by the picture below:



Note that the horizontal and vertical axes have different scales.
The rectangles represent the difference between right & left rectangles, trapping the area.

2. Consider the definite integral $\int_0^1 \cos(x^2)\,dx$.

a) Compute the results of approximating this integral with both the Trapezoidal Rule and Simpson's Rule with $n = 4$ and $n = 8$ and $n = 16$ subdivisions.

b) A twenty-digit approximation of the true value of this integral is

$$.90452423790027208145$$

Compute the differences between each of the approximations obtained in a) and the true value.

c) Multiply the Trapezoidal Rule errors by n^2 and the Simpson's Rule errors by n^4 in each case.

d) Make a table exhibiting all the numbers you have computed. Use the information in the table to answer the following questions:

> What happens as n increases to the error? What can you guess about the precise *rate* of error decrease in each case?

3. Can numerical methods *lie*? Create a function f on an interval $[a, b]$ so that both the Trapezoidal Rule and Simpson's Rule with, say, $n = 100$ will give 0 as their estimation of $\int_a^b f(x)\,dx$ but where the true value of the integral is greater than 100. Try to explain what goes "wrong" in your example using information about the error estimates in the text.

4. Many numerical methods for definite integrals arise by looking at what happens to simple polynomials.

a) Consider the equation

$$\int_0^1 f(x)\,dx = \frac{f(0) + f(1)}{2}$$

Verify that if $f(x) = A + Bx$ then this equation is correct. If $f(x) = x^2$ then this equation is not correct.

This equation leads to the Trapezoidal Rule.

b) Consider the equation

$$\int_0^1 f(x)\,dx = f(\tfrac{1}{2})$$

Verify that if $f(x) = A + Bx$ then this equation is correct. If $f(x) = x^2$ then this equation is not correct.

This equation leads to the Midpoint Rule.

c) Consider the equation

$$\int_0^1 f(x)\,dx = \tfrac{1}{3}f(0) + \tfrac{2}{3}f(\tfrac{1}{3})$$

Verify that if $f(x) = A + Bx + Cx^2$ then this equation is correct. If $f(x) = x^3$ then this equation is not correct.

This equation leads to Radau Quadrature.

Moral: Two function evaluations can get linear accuracy. One function evaluation can get linear accuracy. But two well-chosen function evaluations can get quadratic accuracy. If function evaluations are difficult to get (expensive or hard to measure), then ...

5. Many numerical methods for definite integrals arise by looking at what happens to simple polynomials.

a) Consider the equation

$$\int_0^1 f(x)\,dx = \frac{f(0) + 4f(\frac{1}{2}) + f(1)}{6}$$

Verify that if $f(x) = A + Bx + Cx^2 + Dx^3$ then this equation is correct. If $f(x) = x^4$ then this equation is not correct.

This equation leads to Simpson's Rule.

e) Consider the equation

$$\int_0^1 f(x)\,dx = \tfrac{1}{2} f\left(\tfrac{1}{2} - \tfrac{1}{2\sqrt{3}}\right) + \tfrac{1}{2} f\left(\tfrac{1}{2} + \tfrac{1}{2\sqrt{3}}\right)$$

Verify that if $f(x) = A + Bx + Cx^2 + Dx^3$ then this equation is correct. If $f(x) = x^4$ then this equation is not correct.

This equation leads to Gaussian Quadrature.

Moral: Three function evaluations can get cubic accuracy. The same accuracy can be gotten from only two function evaluations, if they are well-chosen. If function evaluations are difficult to get (expensive or hard to measure), then ...