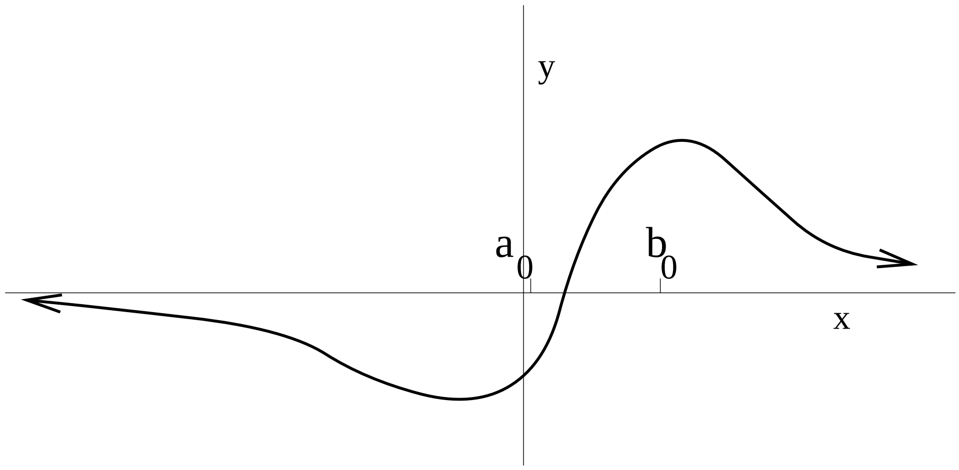
Problems, Week 7

1. Consider the function $y = f(x)$ whose graph is sketched below.



a) If Newton's method is applied repeatedly with initial value $x = a_0$, a sequence of numbers $\{a_n\}$ is obtained (here a_{n+1} is the result of applying Newton's method to a_n). Display graphically a_1, a_2, and a_3. What do you think will happen to a_n as $n \to \infty$?

b) If Newton's method is applied repeatedly with initial value $x = b_0$, a sequence of numbers $\{b_n\}$ is obtained (here b_{n+1} is the result of applying Newton's method to b_n). Display graphically b_1, b_2, and b_3. What do you think will happen to b_n as $n \to \infty$?

2. Graph the function $F(x) = \dfrac{x}{1+x^2}$. Suppose x_0 is the initial guess for Newton's method, and the sequence of numbers obtained using Newton's method is called $\{x_n\}$.

a) Draw pictures to convince yourself of the following facts:

i) If x_0 is "close" to 0 then the sequence $x_n \to 0$ as $n \to \infty$.

ii) If x_0 is "large" and positive then the sequence $x_n \to +\infty$ as $n \to \infty$.

iii) If x_0 is "large" and negative then the sequence $x_n \to -\infty$ as $n \to \infty$.

b) Draw a picture of the number line. Color each point, x_0, of the number line according to the following rule: x_0 is `RED` if Newton's method eventually converges to 0 starting at x_0, and x_0 is `GREEN` if Newton's method eventually goes to $+\infty$ starting at x_0, and x_0 is `BLUE` if Newton's method eventually goes to $-\infty$ starting at x_0.

c) Explain in complete English sentences why the preceding part of the problem is quite difficult.

Note This is a one-dimensional version of an iterative scheme featured on a number of popular posters and t-shirts: words like "fractal" and "chaos" are frequently associated with such images.

3. Suppose

$$5x^3y - 3xy^2 + y^3 = 6 .$$

Note that $(1,2)$ is a point on this curve. There's a nearby point on the curve whose coordinates are $(1.07, u)$. What is u? There's a nearby point on the curve whose coordinates are $(.98, v)$. What is v?

4. Choose an appropriate starting guess and then use three iterations of Newton's method to find the smallest positive solution to

$$\frac{1}{1+x^2} = \tan x$$

How many positive solutions to this equation are there? Why? What would you guess would be true asymptotically about the consecutive positive solutions to this equation? (Pictures will help answer this question.)

5. The numbers R_1, R_2, R_3, and R satisfy the following equation:

$$\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} = \frac{1}{R}$$

(Physics and electrical engineering people may recognize this as a formula for the total resistance, R, of a circuit composed of three resistances R_1, R_2, and R_3 connected in parallel. Others may merely regard it as another fantasy of calculus.)

a) If $R_1 = 1$ and $R_2 = 2$ and $R_3 = 3$, compute R exactly.

b) If both R_1 and R_3 are held constant, and R_2 is increased by .05, what is the approximate change in R?

c) If both R_1 and R_2 are held constant, and R_3 is increased by .05, what is the approximate change in R?

6. a) Sketch a graph of $F(x) = e^x$, find its derivative, give an equation for the line tangent to the curve at $x = 2$, and sketch on the same graph the line tangent to the curve when $x = 2$. Note any horizontal or vertical asymptotes on the graph.

The exponential function: the world's friendliest function.

b) Sketch a graph of $G(x) = e^{-\frac{1}{x}}$, find its derivative, give an equation for the line tangent to the curve at $x = 2$, and sketch on the same graph the line tangent to the curve when $x = 2$. Note any horizontal or vertical asymptotes on the graph.

The exponential function: the world's weirdest function.

7. Consider the function

$$H(x) = \frac{e^{Ax}}{1+e^{Ax}}$$

where A is a constant.

a) Make a rough sketch of $y = H(x)$ for x in $[-1,1]$ if $A = 10^{10}$.

b) Make a rough sketch of $y = H(x)$ for x in $[-1,1]$ if $A = 10^{-10}$.

c) Make a rough sketch of $y = H(x)$ for x in $[-1,1]$ if $A = -10^{10}$.

d) Make a rough sketch of $y = H(x)$ for x in $[-1,1]$ if $A = -10^{-10}$.