Problems, Week 13

1. I looked up the sum of the first n 9^{th} powers. That is, if you define

$$S_9(n) = 1^9 + 2^9 + 3^9 + \cdots + n^9$$

(in summation notation, $S_9(n) = \sum_{k=1}^{n} k^9$) then an explicit formula for $S_9(n)$ is:

$$S_9(n) = \frac{1}{10}(n+1)^{10} - \frac{1}{2}(n+1)^9 + \frac{3}{4}(n+1)^8 - \frac{7}{10}(n+1)^6 + \frac{1}{2}(n+1)^4 - \frac{3}{20}(n+1)^2$$

(It isn't even clear that this formula gives an integer!) **Assume that this formula is true.**

a) Check the formula for S_9 by evaluating it for $n = 3$, and seeing that the answer is the same as $1^9 + 2^9 + 3^9$. (*Please* use a calculator – I did!)

b) Find some definite integral and some approximating sum which knowledge of this formula will allow you to evaluate exactly when $n \to \infty$. Write out the approximating sums, and evaluate the limit of these sums.

2. Let $f(x)$ be defined on $0 \leq x \leq 1$ by the following rule:

$$f(x) \text{ is the first digit in the decimal expansion for } x.$$

For example, $f(1/2) = 5$ and $f(0.719) = 7$.

a) Sketch the graph of $y = f(x)$. Be careful to use an appropriate scale for x and for y.

b) Use the geometric definition of the definite integral to find $\int_0^1 f(x)\,dx$.

c) Now let the function $g(x)$ be defined by

$$g(x) \text{ is the second digit in the decimal expansion for } x.$$

(For example, $g(0.437) = 3$.) Compute $\int_0^1 g(x)\,dx$.

3. Water is flowing through mighty Mill Brook (a local stream located only a short distance from where this was written). Two engineering students (Alpha and Beta) are assigned the task of estimating the water flow through the brook between 8 AM and noon. They are provided with instruments which will allow them to tell what the water flow is at any time in **gallons per minute**.

a) Lazy student Alpha makes observations of the water flow every hour on the hour from 8 AM to noon. Here are Alpha's observations:

Time	Flow, in Gals/Min	
8:00	5.6	
9:00	8.9	(Thunderstorm!)
10:00	12.4	
11:00	7.2	(Lunch break.)
12:00	.....	

What is Alpha's approximation for the total water drained through Mill Brook during those 4 hours?

b) Student Beta, more energetic, checks the water flow every *half* hour. Here's Beta's data:

Time	Flow, in Gals/Min	
8:00	5.6	
8:30	7.3	
9:00	8.9	
9:30	8.3	(Thunderstorm!)
10:00	12.4	
10:00	11.2	
11:00	7.2	
11:00	6.1	(Lunch break.)
12:00	.....	

What is Beta's estimate of the total water flow through Mill Brook during the period 8 AM to noon?

4. An alien spaceship is found. A major component of the spaceship is a thin metal bar, 120 meters long*. It is very **very** heavy. A metallurgist† takes small samplings of the bar at 20 meter intervals, and finds the density (in kilograms per centimeter). Here is her

* Metric aliens

† Spelled right, because I looked it up.

data:

Meter mark	Density	
0	30.4	
20	...	Data lost. (Stolen by aliens?)
40	46.5	
60	65.8	
80	29.2	
100	52.1	
120	...	Lunch break.

Estimate the total weight of the alien object.