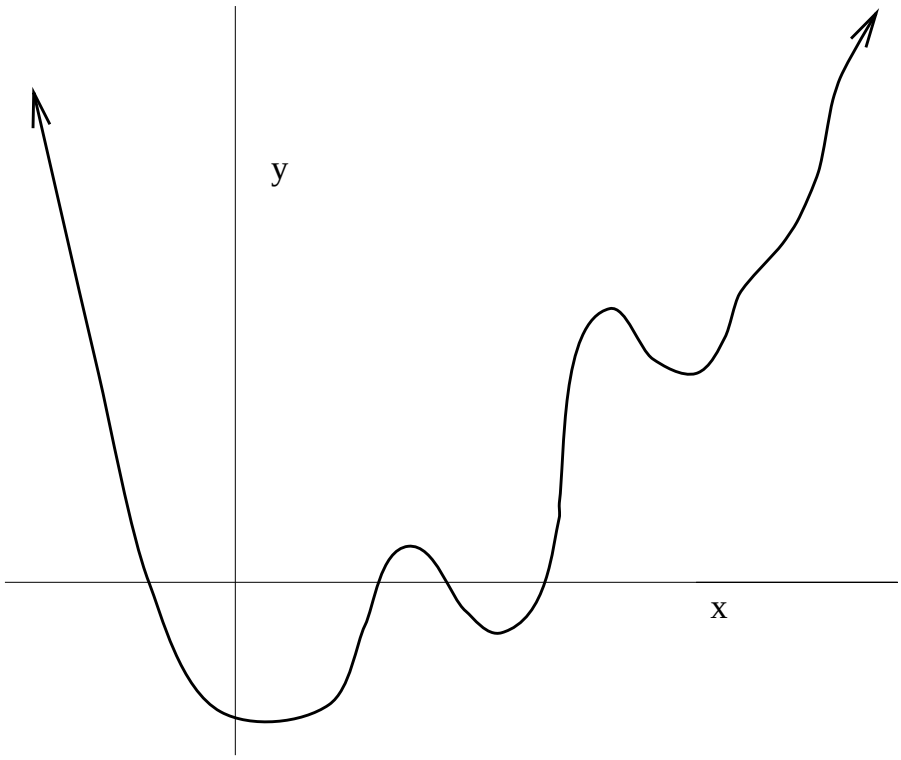
Problems, Week 11

1. Suppose that the picture below is the graph of a polynomial. What is the lowest degree that this polynomial could have?



2. Let $s(x) = \dfrac{\ln x}{x}$, $x > 0$. Sketch the graph of s completely. Be sure to indicate what happens as x approaches 0^+ and as x approaches ∞, where s is positive and negative, where s is increasing and decreasing, the concavity, and any maxima, minima, and inflection points.

3. Let $t(x) = x \ln x$, $x > 0$. Sketch the graph of t completely. Be sure to indicate what happens as x approaches 0^+ (apply L'Hosp carefully: make sure that the hypotheses are verified!) and as x approaches ∞, where t is positive and negative, where t is increasing and decreasing, the concavity, and any maxima, minima, and inflection points.

4. a) Compute $(1.5)^{1/(1.5)}$, $2^{1/2}$, $(2.5)^{1/(2.5)}$, $3^{1/3}$, and $(3.5)^{1/(3.5)}$.

b) Now consider the function $V(x) = x^{1/x}$, whose domain is $(0, \infty)$. Compute $V'(x)$ and find any critical points of V. Either compute V'' or study V' more closely to conclude information about the nature of the critical point(s?) of V.

c) Compare your results from a) and b).

5. Suppose $f(x)$, $f'(x)$, and $f''(x)$ are defined for all x.

a) If $f'(a) = 0$ for some a, does it necessarily follow that f has a local max or a local min at a? (If your answer is "Yes," explain why. If your answer is "No," give two examples in which f does not have a local max or min at a.)

b) If $f''(a) = 0$ for some a, does it necessarily follow that f has an inflection point at a? (If your answer is "Yes," explain why. If your answer is "No," give two examples in which f does not have an inflection point at a.)

6. Suppose you know all of the following:

Information about g

The domain of g is all real numbers except for -1, and $\lim_{x \to -1^-} g(x) = 4$ and $\lim_{x \to -1^+} g(x) = +\infty$ and $\lim_{x \to +\infty} g(x) = 3$ and $\lim_{x \to -\infty} g(x) = 5$.

Information about g'

The domain of g' is all real numbers except for -1, and g' is 0 at -2 (and $g(-2) = 7$) and g' is 0 at 1 (and $g(1) = 1$) and g' is *not* 0 at any other point. Also, $\lim_{x \to -1^-} g'(x) = -1$.

Information about g''

The domain of g'' is all real numbers except for -1, and g'' is 0 at -4 (and $g(-4) = 6$) and g'' is 0 at 2 (and $g(2) = 2$) and g'' is *not* 0 at any other point.

Sketch a graph of $y = g(x)$ as precisely as you can given the above information. You should be able to answer such questions as: in what intervals is g {in/de}creasing? In what intervals is g concave {up/down}? Does the graph of g have any {horizontal/vertical} asymptotes (and if so, what are they)? Is g ever 0?

7. a) Suppose that a function has a *positive* derivative in some interval. Explain why such a function is one-to-one.

b) Explain why $F(x) = x^7 + 34x^5 + 7x^3 + 87x + 23 + \cos(8 \sin x)$ is one-to-one on all of the real numbers.

8. a) Consider the function $H(x) = \dfrac{e^{100x}}{1 + e^{100x}}$. Graph this function, and find the notable features of this graph, including any local extrema and points of inflection.

b) a) Consider the function $K(x) = \dfrac{x^{100}}{1 + x^{100}}$. Graph this function, and find the notable features of this graph, including any local extrema and points of inflection.