# MULTIPLICITY-FREE SPACES AND SCHUR-WEYL-HOWE DUALITY

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*Dedicated to the memory of Irving E. Segal, who introduced me to the beauties and mysteries of representation theory.*

## Introduction

The unifying theme of these lectures is the *duality* between the irreducible representations occuring in a linear group action and irreducible representations of the commuting algebra relative to this action. This notion of duality in representation theory was introduced by Schur a century ago, and it has developed into an important tool with many applications. In keeping with the tutorial aspect, I have tried to tell the story starting from the beginning. Complete proofs of all the major results stated are given in the volume [22] (at several points I refer to the lectures of Benson-Ratcliff in [22] for details). Of course, this limits the scope of the lectures to the more classical parts of the theory: Schur-Weyl-Brauer duality for finite-dimensional representations, and Howe duality between finite-dimensional and infinite-dimensional highest-weight representations. Substantial parts of these lectures are based on joint work with Nolan Wallach and I would like to acknowledge his contributions to my understanding of representation theory. I would also like to thank Eng-Chye Tan and Chen-Bo Zhu for inviting me to give these lectures and for their wonderful hospitality.

## Lecture 1. Representations and Duality

1.1. **Representations of Algebraic Groups.** Assume that $G \subset \mathrm{GL}(n,\mathbb{C})$ is an *algebraic group* (defined by a set of polynomial equations in the matrix entry functions). We denote by $\mathrm{Aff}(G)$ the commutative algebra of *regular functions* on $G$ (the restrictions to $G$ of polynomials in the matrix entry functions $x_{ij}$ and $\det^{-1}$).

Let $(\rho, L)$ be a representation of $G$ on a complex vector space $L$. If $L$ is finite-dimensional, then we say that $\rho$ is *regular* (rational) if the representative functions $g \mapsto \mathrm{tr}(\rho(g)E)$, for $E \in \mathrm{End}(L)$, are regular. Every regular function on $G$ arises as such a representative function. When $L$ is infinite dimensional, we say that $\rho$ is *locally regular* if for all $x \in L$ there is finite-dimensional $G$-invariant subspace $M$ containing $x$ so that $(\rho|_M,\, M)$ is a regular representation.

The most fundamental tool in representation theory is **Schur's Lemma:** *If $E$ and $F$ are irreducible, finite-dimensional representations of a group $G$, then*

$$\dim \mathrm{Hom}_G(E,F) = \begin{cases} 1 & \text{if } E \cong F \\ 0 & \text{if } E \ncong F \end{cases}$$

($\mathrm{Hom}_G(E, F)$ denotes the space of linear transformations $T : E \to F$ that intertwine the $G$ actions on the two spaces). To prove Schur's Lemma, observe that the null space and range of $T$ are $G$-invariant subspaces, so $T$ must be either zero or bijective, with the first case holding if $E \not\cong F$. When $E \cong F$ and $S, T$ are two nonzero intertwining maps, take $\lambda$ to be an eigenvalue of $S^{-1}T$. Since $S^{-1}T - \lambda I$ commutes with the action of $G$ on $E$ and has a nonzero null space, it must be zero.

1.2. **Examples.**

**1.** Let $(\pi, V)$ be any regular (finite-dimensional) representation of $G$. We denote by $\mathcal{P}(V)$ the algebra of complex-valued polynomial functions on $V$. Define a representation of $G$ on $\mathcal{P}(V)$ by

$$\rho(g)f(v) = f(\pi(g)^{-1}v) \quad \text{for } f \in \mathcal{P}(V) \text{ and } g \in G.$$

Since the $G$ action is linear, it commutes with the $\mathbb{C}^\times$ action on $V$ by scalar multiplication, and we have the direct-sum decomposition into finite-dimensional $G$-invariant subspaces

$$\mathcal{P}(V) = \bigoplus_{k \geq 0} \mathcal{P}^k(V),$$

where $\mathcal{P}^k(V)$ is the space of homogeneous polynomials of degree $k$. The action of $G$ on each of these spaces is regular, so the representation $\rho$ is locally regular. Furthermore, the $G$ action preserves the multiplication on $\mathcal{P}(V)$.

**2.** With $(\pi, V)$ as above, we can take the full tensor algebra

$$\mathcal{T}(V) = \bigoplus_{k \geq 0} V^{\otimes k}$$

with $G$ action $\rho(g)(v_1 \otimes \cdots \otimes v_k) = \pi(g)v_1 \otimes \cdots \otimes \pi(g)v_k$. Since $G$ leaves invariant each subspace $V^{\otimes k}$, the representation $\rho$ is locally regular. As in the previous example, the $G$ action preserves the (noncommutative) multiplication on $\mathcal{T}(V)$.

**3.** Let $X \subset \mathbb{C}^m$ be an affine algebraic set (the zero set of a family of polynomials) and suppose that there is a regular $G$ action on $X$

$$G \times X \longrightarrow X, \quad (g, x) \mapsto g \cdot x.$$

Set $L = \mathrm{Aff}(X)$ (the restriction to $X$ of the polynomial functions on $\mathbb{C}^m$). Let $G$ act on $L$ by $\rho(g)f(x) = f(g^{-1} \cdot x)$. We can prove that this representation is locally regular as follows.

Given $f \in \mathrm{Aff}(X)$, set $V_f = \mathrm{Span}\{\rho(g)f : g \in G\}$. The function $(g, x) \mapsto f(g^{-1} \cdot x)$ on $G \times X$ is regular, and $\mathrm{Aff}(G \times X) = \mathrm{Aff}(G) \otimes \mathrm{Aff}(X)$, there are regular functions $\phi_i$ on $G$ and $\psi_i$ on $X$ so that

$$f(g^{-1} \cdot x) = \sum_{k=1}^{p} \phi_k(g)\psi_k(x).$$

In particular, $V_f \subset \mathrm{Span}\{\psi_k\}$ is finite-dimensional, so we can choose $g_1, \ldots, g_r$ in $G$ such that the functions $f_i = \rho(g_i)f$ give a basis for $V_f$. Now choose points $x_1, \ldots, x_k$ in $X$ so that the evaluation functionals $\delta_{x_i}$ are a basis for $V_f^*$. Since

$$\langle \rho(g)f_i, \delta_{x_j} \rangle = \rho(gg_i)f(x_j) = \sum_{k=1}^{r} \phi_k(gg_i)\psi(x_j),$$

we see that the representation of $G$ on $V_f$ is regular. Thus $(\rho, L)$ is locally regular.

1.3. **Reductive Groups and Isotypic Decompositions.** A complex algebraic group $G$ is called *reductive* if every finite dimensional regular representation decomposes as a direct sum of irreducible representations (this property is equivalent to every $G$-invariant subspace of a regular representation having a $G$-invariant complementary subspace). The *classical groups* are reductive:

- the general linear group $\mathrm{GL}(n, \mathbb{C})$ of invertible $n \times n$ complex matrices
- the special linear group $\mathrm{SL}(n, \mathbb{C})$ of $n \times n$ complex matrices of determinant one
- the orthogonal group $\mathrm{O}(\mathbb{C}^n, \omega)$ of $n \times n$ matrices preserving a non-degenerate symmetric bilinear form $\omega(x, y) = x^t By$ on $\mathbb{C}^n$, where $B$ is a symmetric invertible $n \times n$ matrix (defining equation $g^t Bg = B$)
- the special orthogonal group $\mathrm{SO}(\mathbb{C}^n, \omega)$ of orthogonal matrices of determinant one

- the symplectic group $\mathrm{Sp}(\mathbb{C}^{2n}, \omega)$ of $2n \times 2n$ matrices preserving a non-degenerate skew-symmetric bilinear form $\omega(x, y) = x^t J y$ on $\mathbb{C}^{2n}$, where $J$ is a skew-symmetric invertible $2n \times 2n$ matrix (defining equation $g^t J g = J$)

Finite groups are shown to be reductive by the method of averaging over the group. The proof that classical groups are reductive can be carried out analytically by integrating over a compact real form (Weyl's *unitary trick* – see [16, Theorem 2.4.7]), or algebraically by using a *Casimir operator* (see [16, Theorem 2.4.5]). Direct products of reductive groups are reductive, and the quotient of a reductive group by a closed normal subgroup is reductive (this is obvious). An algebraic group is reductive if and only if its identity component is reductive.

Assume $G$ is reductive, and let $\widehat{G}$ be the equivalence classes of irreducible finite-dimensional regular representations of $G$. For each $\lambda \in \widehat{G}$ fix a representation $(\pi^\lambda), F^\lambda)$ in the class $\lambda$. Let $\lambda^*$ be the equivalence class of the *contragredient* representation on the dual space $(F^\lambda)^*$.

Given a locally regular representation $(\rho, L)$ of $G$, set

$$L_{(\lambda)} = \sum V \qquad (\text{sum of all } V \subset L \text{ such that } \rho|_V \cong F^\lambda).$$

Call $L_{(\lambda)}$ the $\lambda$*–isotypic component* of $L$. Define $\mathrm{Spec}(\rho) = \{\lambda \in \widehat{G} \,:\, L_{(\lambda)} \neq 0\}$ (the $G$-*spectrum* of $(\rho, L)$).

**Proposition 1.1.** $L = \bigoplus_{\lambda \in \mathrm{Spec}(L)} L_{(\lambda)}$ *(algebraic direct sum).*

**Corollary 1.2.** *There is a linear projection $x \mapsto x^\natural$ from $L$ onto the space $L^G$ of $G$-fixed vectors.*

We now turn to the $G$-module structure of the isotypic components of a representation $L$. Denote by $\mathrm{Hom}_G(F^\lambda, L)$ the vector space of all linear maps $T : F^\lambda \to L$ that intertwine the $G$ actions on these spaces. This is the space of *covariants of type* $\lambda$.

**Theorem 1.3.** *If $(\rho, L)$ is a locally regular representation of a complex reductive algebraic group $G$, then*

$$L \cong \bigoplus_{\lambda \in \mathrm{Spec}(\rho)} E^\lambda \otimes F^\lambda,$$

*where $E^\lambda = \mathrm{Hom}_G(F^\lambda, L)$ and $G$ acts by $1 \otimes \rho$ on each summand. In particular, the multiplicity of $\lambda$ in $\rho$ is the dimension of the space of covariants of type $\lambda$.*

1.4. **Multiplicities and Duality.** One says that $L$ is *multiplicity-free* as a $G$ module if $\dim E^\lambda = 1$ for all $\lambda \in \mathrm{Spec}(\rho)$. In this case $L$ is uniquely determined as a $G$-module by its spectrum. For a detailed analysis of such representations when $L = \mathcal{P}(X)$ and $X$ is a vector space or affine variety with regular $G$ action see the lectures by Benson–Ratcliff in this volume.

In these lectures we will study representations $(\rho, L)$ that are *not* multiplicity free. We want to determine

- The spectrum $\mathrm{Spec}(\rho) \subset \widehat{G}$
- The multiplicities $m_\lambda = \dim E^\lambda$
- Explicit models for the multiplicity spaces $E^\lambda$.

Let $\mathrm{End}_G(L)$ be the algebra of linear transformations on $L$ that commute with the $G$ action. There is a natural representation of this algebra on each multiplicity space $E^\lambda$. Indeed, if $A \in \mathrm{End}_G(L)$ and $T \in E^\lambda$, then the linear map $A \circ T : F^\lambda \to L$ also commutes with the $G$ action on $L$, and hence is an element of $E^\lambda$. Following the ideas of I. Schur, H. Weyl and R. Howe, the unifying theme in our approach will be

**Hidden Symmetry:** Study the spaces $E^\lambda$ as modules for *good* subalgebras of $\mathrm{End}_G(L)$.

The term *hidden symmetry* comes from applications of representation theory to quantum mechanics in cases where the *geometric symmetries* such as rotation invariance do not suffice to explain the multiplicities in the energy spectrum. In some cases, one can find a larger symmetry group containing $G$ and extend the representation of $G$ to a representation of this larger group on $L$ that is multiplicity free. In other cases the hidden symmetries are given by a Lie algebra of differential operators commuting with the $G$ action (see [31]).

When $L$ is infinite-dimension (for example, when $L = \mathrm{Aff}(X)$ with $X$ an affine $G$ variety), then $\mathrm{End}(L)$ is too big to deal with purely algebraically. In the context of *unitary representations* on a Hilbert space, one uses the von Neumann algebra of *bounded* operators that commute with $G$. In our algebraic setting we shall assume that $L$ is of countable dimension and that we have a subalgebra $\mathcal{R} \subset \mathrm{End}(L)$ that satisfies

(i) $\mathcal{R}$ acts irreducibly on $L$
(ii) $\mathcal{R}$ is invariant under $G$, relative to the action $\mathrm{Ad}(g)T = \rho(g)T\rho(g)^{-1}$, and the representation Ad of $G$ on $\mathcal{R}$ is locally regular

In case $\dim L < \infty$ we take $\mathcal{R} = \mathrm{End}(L) \cong L \otimes L^*$ and these conditions are always satisfied. When $L = \mathcal{P}(X)$ with $X$ a smooth affine $G$ variety, we take $\mathcal{R} = \mathbb{D}(X)$, the *algebraic* differential operators on $X$ (see Agricola [1]). In particular, if $X$ is a vector space with linear $G$ action, then $\mathbb{D}(X)$ is the *Weyl algebra* $\mathcal{PD}(X)$ of differential operators with polynomial coefficients, which we will examine in detail in Lecture 8.

Fix $\mathcal{R}$ satisfying the conditions (i) and (ii) and let

$$\mathcal{R}^G = \{T \in \mathcal{R} : \mathrm{Ad}(g)T = T \quad \text{for all } g \in G\}$$

(the commutant of $\rho(G)$ in $\mathcal{R}$).

**Theorem 1.4.** *Each multiplicity space $E^\lambda$ is an irreducible $\mathcal{R}^G$ module. Furthermore, if $\lambda, \mu \in \mathrm{Spec}(\rho)$ and $E^\lambda \cong E^\mu$ as an $\mathcal{R}^G$ module, then $\lambda = \mu$.*

In the next lecture we will prove this theorem.[a] At this point we derive some consequences. The following corollary plays a fundamental role in our approach to Howe duality.

**Corollary 1.5.** *Let $\sigma$ be the representation of $\mathcal{R}^G$ on $L$. Then $(\sigma, L)$ is a semisimple $\mathcal{R}^G$ module, and each irreducible submodule $E^\lambda$ occurs with finite multiplicity* $\dim F^\lambda$.

When $L$ is finite-dimensional then $\mathcal{R} = \mathrm{End}(L)$, and from the inequivalence of the representations $E^\lambda$ together with Schur's lemma we obtain the classical *Double Commutant Theorem:*

**Corollary 1.6.** *If* $\dim L < \infty$ *and* $\mathcal{B} = \mathrm{End}_G(L)$, *then* $\mathrm{Span}\{\rho(G)\}$ *consists of all linear transformations on $L$ that commute with $\mathcal{B}$.*

**Corollary 1.7** (Duality Correspondence)**.** *Let* $\mathrm{Spec}(\sigma)$ *denote the set of equivalence classes of the irreducible representations of the algebra $\mathcal{R}^G$ that occur in $L$. Then the map $F^\lambda \to E^\lambda$ sets up a bijection between* $\mathrm{Spec}(\rho)$ *and* $\mathrm{Spec}(\sigma)$.

## Lecture 2. Proof of Duality Theorem and Examples

### 2.1. **Density Lemmas.**

**Lemma 2.1** (Dixmier-Schur)**.** *Let $L$ be a vector space over $\mathbb{C}$ of countable dimension. Let $\mathcal{R} \subset \mathrm{End}(L)$ be a subalgebra that acts irreducibly on $L$. Suppose $A \in \mathrm{End}(L)$ commutes with $\mathcal{R}$. Then $A = \lambda I$ for some $\lambda \in \mathbb{C}$.*

Assume now that $L$ has countable dimension as a complex vector space and that $\mathcal{R} \subset \mathrm{End}(L)$ is a subalgebra that acts irreducibly on $L$.

**Lemma 2.2** (Jacobson)**.** *Let $X$ be any finite-dimensional subspace of $L$. Then every $f \in \mathrm{Hom}(X, L)$ is of the form $r|_X$ for some $r \in \mathcal{R}$.*

**Corollary 2.3** (Burnside)**.** *If* $\dim L < \infty$ *then* $\mathcal{R} = \mathrm{End}(L)$.

Now let $(\rho, L)$ be a locally regular representation of $G$ with $\dim L$ countable. Assume that $\mathcal{R} \subset \mathrm{End}(L)$ satisfies conditions (i) and (ii) stated before Theorem 1.4.

**Lemma 2.4.** *Let $X \subset L$ be a finite-dimensional $G$ invariant subspace. Then $\mathcal{R}^G|_X = \mathrm{Hom}_G(X, L)$.*

[a]See [16, Theorem 4.5.12] for the case that $\mathcal{R}$ is a graded algebra; the generalization presented here is due to Agricola [1].

2.2. **Proof of Duality Theorem.** Take $\lambda \in \mathrm{Spec}(\rho)$ and let $Z_\lambda \subset L_{(\lambda)}$ be any irreducible $G$-submodule. Given $f \in L$, we denote by $U_f = \mathcal{R}^G f$ the cyclic $\mathcal{R}^G$ module generated by $f$. We write $\mathbb{C}[G]$ for the *group algebra* of $G$ (the formal finite linear combinations of the elements of $G$).

**(a)** *If $0 \neq M \subset L_{(\lambda)}$ is an $\mathcal{R}^G$-module, then $M \cap Z_\lambda \neq 0$.*

To verify this, take $0 \neq m \in M$ and set $X = \mathrm{Span}\{\rho(G)m\}$. Then $\dim X < \infty$ and $X \subset L_{(\lambda)}$. Hence there exists $T \in \mathrm{Hom}_G(X, Z_\lambda)$ with $Tm \neq 0$. By Lemma 2.4 there exists $r \in \mathcal{R}^G$ with $r|_X = T$. Then $rm = Tm \in M \cap Z_\lambda$.

**(b)** *If $0 \neq f \in Z_\lambda$ then $U_f \cap Z_\lambda = \mathbb{C}f$.*

Take $u = rf \in U_f \cap Z_\lambda$. Since $Z_\lambda = \mathrm{Span}\{\rho(G)f\}$, we have

$$rZ_\lambda = \mathrm{Span}\{\rho(G)rf\} = \mathrm{Span}\{\rho(G)u\} \subset Z_\lambda$$

Thus $r|_{Z_\lambda} \in \mathrm{End}_G(Z_\lambda) = \mathbb{C}I$ by Schur's Lemma. So $u = r \cdot f \in \mathbb{C}f$, proving **(b)**.

**(c)** *If $f \in Z_\lambda$ is nonzero, then $U_f$ is an irreducible $\mathcal{R}^G$-module.*

Indeed, if $0 \neq M \subset U_f$ is an $\mathcal{R}^G$-submodule, then $0 \neq M \cap Z_\lambda \subset \mathbb{C}f$ by **(a)** and **(b)**. Thus $f \in M$ and hence $M = U_f$, which proves **(c)**.

**(d)** *Let $f_1, \ldots, f_d$ be a basis of $Z_\lambda$. Set $M_i = U_{f_i}$. Then the sum $\sum_{i=1}^d M_i$ is direct and $M_i \cong M_j$ as $\mathcal{R}^G$ modules.*

We have $\mathrm{Span}\{\rho(g)|_{Z_\lambda} : g \in G\} = \mathrm{End}(Z_\lambda)$ by Corollary 2.3. Thus for each $i$ there exists an element $u_i \in \mathbb{C}[G]$ such that $\rho(u_i)f_j = \delta_{ij} f_j$. Suppose $m_i \in M_i$ and $\sum_i m_i = 0$. There exist $r_i \in \mathcal{R}^G$ so that $m_i = r_i f_i$. Hence

$$0 = \rho(u_j)\Big(\sum_i m_i\Big) = \sum_i \rho(u_j) r_i f_i = \sum_i r_i \rho(u_j) f_i = r_j f_j = m_j$$

for $j = 1, \ldots, d$. This proves the first statement of **(d)**. For the second, apply Corollary 2.3 again to obtain $u_{ji} \in \mathbb{C}[G]$ such that $\rho(u_{ji})f_i = f_j$. Since $M_i$ and $M_j$ are irreducible by **(c)**, the map $\rho(u_{ji}) : M_i \to M_j$ is an $\mathcal{R}^G$-module isomorphism, by Schur's Lemma.

**(e)** *Let $M_i$ be as in* **(d)**. *Then $L_{(\lambda)} = \bigoplus_{i=1}^d M_i$.*

Recall that $L_{(\lambda)}$ is the sum of all irreducible $G$-submodules of $L$ that are in the class $\lambda$. Thus it is enough to show that if $W_\lambda$ is such a submodule then

$$W_\lambda \subset \bigoplus_{i=1}^{d} M_i \,. \tag{2.1}$$

Take a $G$ isomorphism $T : Z_\lambda \to W_\lambda$. Then Lemma 2.4 furnishes $r \in \mathcal{R}^G$ such that $r = T$ on $Z_\lambda$. Hence $W_\lambda$ satisfies (2.1), which proves **(e)**.

The first assertion of Theorem 1.4 now follows from **(c)**, **(d)**, and **(e)**. To prove the second assertion, it suffices to prove the following.

**(f)** *Let $f_\lambda$ and $f_\mu$ be nonzero vectors in irreducible $G$ subspaces $Z_\lambda$ and $Z_\mu$. Suppose $U_{f_\lambda} \cong U_{f_\mu}$ as $\mathcal{R}^G$-modules. Then $\lambda = \mu$.*

Let $T : U_{f_\lambda} \to U_{f_\mu}$ be an $\mathcal{R}^G$-module isomorphism. Let $X$ be a finite-dimensional $G$-invariant subspace containing $f_\lambda$ and $Tf_\lambda$. There is a projection operator $P_\lambda : X \to L_{(\lambda)}$ onto the $\lambda$-isotypic component of $L$, and Lemma 2.4 furnishes $r \in \mathcal{R}^G$ such that $r|_X = P_\lambda$. Thus $r \cdot f_\lambda = f_\lambda$ so we have

$$Tf_\lambda = Trf_\lambda = rTf_\lambda = P_\lambda T f_\lambda \in L_{(\lambda)}.$$

Since $T$ is an $\mathcal{R}^G$ module isomorphism, it follows that $U_{f_\mu} \subset L_{(\lambda)}$. Hence $f_\mu \in L_{(\lambda)}$, and so we conclude that $\mu = \lambda$. $\square$

2.3. **Examples.**

**1. (Product Groups)** Let $H$ and $K$ be reductive complex algebraic groups, and let $G = H \times K$ be the direct product algebraic group, where $\mathrm{Aff}(G) \cong \mathrm{Aff}(H) \otimes \mathrm{Aff}(K)$ under the natural pointwise multiplication map. We can use the duality theorem to prove that $\widehat{G} = \widehat{H} \times \widehat{K}$: *Every irreducible regular representation* $(L, \rho)$ *of* $G$ *is given by*

$$L = M \otimes N, \quad \rho(h,k) = \sigma(h) \otimes \tau(k) \quad \text{for } h \in H \text{ and } k \in K \tag{2.2}$$

*where* $(\sigma, M)$ *is an irreducible representation of* $H$ *and* $(\tau, N)$ *is an irreducible representation* $K$. To prove this, suppose first that $(\rho, L)$ is defined by (2.2). Then Corollary 2.3 implies that $\mathrm{End}(L)$ is spanned by the transformations $\{\rho(h,k) : h \in H, k \in K\}$ and hence $\mathrm{End}_G(L) = \mathbb{C}I$, showing that $L$ is irreducible.

Conversely, given an irreducible regular representation $(\rho, L)$ of $G$, use Theorem 1.4 (with $\mathcal{R} = \mathrm{End}(L)$) to decompose $L$ as a $K$-module:

$$L = \bigoplus_{\lambda \in \widehat{K}} E^\lambda \otimes F^\lambda \,. \tag{2.3}$$

Set $\sigma(h) = \rho(h,1)$ and $\tau(k) = \rho(1,k)$. Since $\sigma(h)$ commututes with $\tau(k)$, $H$ acts on each $E^\lambda$ by some representation $\mu^\lambda$. We claim that $E^\lambda$ is irreducible under $H$. To prove this, note that

$$\mathrm{End}_K(L) \cong \bigoplus_{\lambda \in \widehat{K}} \mathrm{End}(E^\lambda) \otimes I \,. \tag{2.4}$$

Given $T \in \mathrm{End}_K(L)$, we know by Corollary 2.3 that $T$ is a linear combination of the transformations $\sigma(h)\tau(k)$. Under the isomorphism (2.4) the $K$-invariant transformations only act on $E^\lambda$. This proves that $\mathrm{End}_K(L)$ is spanned by $\{\sigma(h) : h \in H\}$, and hence $E^\lambda$ is irreducible under $H$ by Theorem 1.4. Thus each summand in (2.3) is an irreducible $G$ module, by the earlier argument, so there can be only one summand.

**2. (Multiplicity-free Representations of Product Groups)** Suppose $(\rho, L)$ is any locally regular representation of $G$ that is multiplicity-free. By Example **1.** the isotypic decomposition of $L$ under $H \times K$ is of the form

$$L = \bigoplus_{(\alpha,\,\beta) \in \Lambda} E^\alpha \otimes F^\beta \tag{2.5}$$

where $\Lambda \subset \widehat{H} \times \widehat{K}$ and $E^\alpha$ is the irreducible $H$-module of type $\alpha$, while $F^\beta$ is the irreducible $K$-module of type $\beta$. Set $\sigma = \rho|_H$ and $\tau = \rho|_K$. Then $\mathrm{Spec}(\sigma)$ is the projection $\Lambda \to \widehat{H}$, whereas $\mathrm{Spec}(\tau)$ is the projection $\Lambda \to \widehat{K}$. In general $\Lambda$ is not determined by these projections. If both of these projections are injective, we say that the representation $\rho$ sets up a *duality correspondence* between $\mathrm{Spec}(\sigma)$ and $\mathrm{Spec}(\tau)$. Clearly such representations of $G$ must be very special, and in these lectures they will play an important role. The next example is the most familiar of them.

**3. (Two-sided group action)** Let $K$ be any reductive complex algebraic group. Set $G = K \times K$ and $L = \mathrm{Aff}(K)$. Define the representation $\rho$ of $G$ on $L$ by

$$\rho(x,y)f(k) = f(x^{-1}ky) \quad \text{for } k, x, y \in K.$$

From Example **1.** we know that $\widehat{G} = \widehat{K} \times \widehat{K}$. Consider $\mathrm{Aff}(K)$ as a $K$-module relative to the *right translation* action $\rho(1,k)$ and apply Theorem 1.3:

$$\mathrm{Aff}(K) = \bigoplus_{\lambda \in \widehat{K}} E^\lambda \otimes F^\lambda \,. \tag{2.6}$$

Here $K$ acts on $E^\lambda = \mathrm{Hom}_K(F^\lambda, \mathrm{Aff}(K))$ by $\rho(k,1) \circ T$, where $T : F^\lambda \to \mathrm{Aff}(K)$ intertwines the action of $K$ on $F^\lambda$ with the right translation action of $K$ on $\mathrm{Aff}(K)$.

We claim that $E^\lambda \cong F^{\lambda^*}$. To prove this, define a map $E^\lambda \to F^{\lambda^*}$ (a special case of *Frobenius reciprocity*) by

$$T \mapsto \widehat{T} \in F^{\lambda^*}, \quad \langle \widehat{T}, v \rangle = (Tv)(1) \quad \text{for } v \in F^\lambda.$$

This map obviously intertwines the action of $K$. It is injective, since $(Tv)(1) = 0$ for all $v \in F^\lambda$ implies

$$(Tv)(k) = (T\pi^\lambda(k)v)(1) = 0 \quad \text{for all } k \in K,$$

and hence $T = 0$. The map is surjective, since $v^* \in F^{\lambda^*}$ defines $T \in E^\lambda$ by

$$(Tv)(k) = \langle v^*, \pi^\lambda(k)v \rangle.$$

Clearly $\widehat{T} = v^*$. Thus the decomposition (2.6), relative to the action of $K \times K$, is

$$\mathrm{Aff}(K) \cong \bigoplus_{\lambda \in \widehat{K}} F^{\lambda^*} \otimes F^{\lambda} \cong \bigoplus_{\lambda \in \widehat{K}} \mathrm{End}(F^{\lambda}) .$$

This shows that $\mathrm{Aff}(K)$ is *multiplicity free* as a representation of $K \times K$ and there is a duality correspondence $\lambda \longleftrightarrow \lambda^*$.

**4. (Harmonics on the zero-sphere)** Let $G = \mathrm{O}(1) = \{\pm 1\}$ acting on $\mathbb{C}$, and take $L = \mathcal{P}(\mathbb{C})$. In this case

$$\widehat{G} = \{F^+,\, F^-\} \quad \text{(trivial, signum)}.$$

The $G$-isotypic decomposition of $L$ is thus

$$L = L^+ \oplus L^- \quad \text{(even polynomials} \oplus \text{odd polynomials)}$$

and each component has infinite multiplicity. We apply the duality philosophy to explain the multiplicities by finding operators on $L$ that commute with $G$. Let $\mathcal{PD}(\mathbb{C})$ be the polynomial coefficient differential operators on $\mathcal{P}(\mathbb{C})$. Then one has

**(a)** *The operators $\Delta = (d/dx)^2$, multiplication by $x^2$, and $x(d/dx) + 1/2$* (shifted Euler operator) *commute with $G$ and span a Lie algebra $\mathfrak{g}' \cong \mathfrak{sl}(2, \mathbb{C})$ in $\mathcal{PD}(\mathbb{C})$.*

**(b)** *The Lie algebra $\mathfrak{g}'$ generates the commutant $\mathcal{PD}(\mathbb{C})^G$ of $G$.*

The proof of **(a)** is an easy calculation. The proof of **(b)** follows by considering the *symbol* $f(x, \xi)$ of a differential operator and using the fact that the algebra of $G$-invariant polynomials in $(x, \xi)$ is generated by the quadratic polynomials $x^2$, $x\xi$, and $\xi^2$.

We define the *G-harmonic polynomials*

$$\mathcal{H} = \mathrm{Ker}(\Delta) = (\mathbb{C} \cdot 1) \oplus (\mathbb{C} \cdot x).$$

Since $\Delta$ commutes with $G$, we have $G \cdot \mathcal{H} = \mathcal{H}$. Also $\mathcal{H}$ is multiplicity-free as a $G$-module. Let

$$\mathcal{I} = \mathcal{P}(\mathbb{C})^G = \mathbb{C}[x^2]$$

(the $G$-invariant polynomials). Then we have the *Invariant-Harmonic Decomposition:*

$$\mathcal{P}(\mathbb{C}) = E^+ \oplus E^- \cong \mathcal{I} \otimes \mathcal{H}$$

where $E^+ = \mathbb{C}[x^2] \cdot 1, \quad E^- = \mathbb{C}[x^2] \cdot x$. We view this decomposition from the perspective of duality as follows:

- $E^+$ is an irreducible $\mathfrak{g}'$ module generated by 1.
- $E^-$ is an irreducible $\mathfrak{g}'$ module generated by $x$.
- $\mathcal{P}(\mathbb{C})$ is multiplicity-free as module for $\mathfrak{g}' \times G$:

$$\mathcal{P}(\mathbb{C}) = (E^+ \otimes F^+) \oplus (E^- \otimes F^-)$$

From the algebraic point of view, we now have a complete picture of $\mathcal{P}(\mathbb{C})$ as a module for $G$ and $\mathfrak{g}'$. However, there is much more that can be seen on the analytical side. There is a pre-Hilbert space structure on $\mathcal{P}(\mathbb{C})$ given by the *Fischer inner product*:

$$\langle f \mid g \rangle = \partial(f) g^*(0) = \int_{\mathbb{C}} f(x)\overline{g(x)}\, d\mu(x)$$

(where $d\mu(x)$ is normalized Gaussian measure on $\mathbb{C}$). We define the Bargmann-Fock space $\mathbf{H}^2$ as the completion of $\mathcal{P}(\mathbb{C})$ in this norm. The elements of $\mathbf{H}^2$ are holomorphic functions on $\mathbb{C}$ that are square-integrable relative to Gaussian measure. Let

$$\mathfrak{g}'_0 = \{X \in \mathfrak{g}' \,:\, X \text{ is skew-Hermitian relative to } \langle \cdot \mid \cdot \rangle \,\}.$$

Then $\mathfrak{g}'_0$ is a real form of the Lie algebra $\mathfrak{g}'$ and is isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. Let $G' = \mathrm{SL}(2, \mathbb{R})$ and let $\widetilde{G}'$ be the two-fold cover of $G'$. The analytic duality correspondence between $G$ and $\widetilde{G}'$ is the following.

**Theorem 2.5.** *The representation of* $\mathfrak{g}_0'$ *on* $\mathcal{P}(\mathbb{C})$ *integrates to a unitary representation of* $\widetilde{G'}$ *on* $\mathbf{H}^2$ *(the* oscillator *or* metaplectic *representation). It decomposes under the action of* $\widetilde{G}' \times G$ *as a direct sum of irreducible Hilbert spaces*

$$(\mathbf{H}^2_+ \otimes F^+) \oplus (\mathbf{H}^2_- \otimes F^-) \quad \text{(multiplicity-free)}$$

This is a special case of *Howe duality* for unitary highest-weight representations. We will study it in full generality in later lectures.

## Lecture 3. Schur-Weyl Duality

3.1. **Commutant of** $\mathrm{GL}(n)$ **Action on Tensors.** Consider the action of $\mathrm{GL}(n,\mathbb{C})$ on $\bigotimes^k \mathbb{C}^n$ by the $k$th tensor power $\rho_k$ of its defining representation:

$$\rho_k(g)(v_1 \otimes \cdots \otimes v_k) = gv_1 \otimes \cdots \otimes gv_k \quad \text{for } v_i \in \mathbb{C}^n.$$

The symmetric group $\mathfrak{S}_k$ acts on $\bigotimes^k \mathbb{C}^n$ by permuting the tensor positions:

$$\sigma_k(s)(v_1 \otimes \cdots \otimes v_k) = v_{s^{-1}(1)} \otimes \cdots \otimes v_{s^{-1}(k)}$$

(the vector in position $i$ is moved to the vector in position $s(i)$). It is clear that $\sigma_k(s)\rho_k(g) = \rho_k(g)\sigma_k(s)$ for all $g \in \mathrm{GL}(n,\mathbb{C})$ and $s \in \mathfrak{S}_k$.

**Proposition 3.1** (Schur)**.** *Any linear transformation* $B$ *on* $\bigotimes^k \mathbb{C}^n$ *that commutes with* $\sigma_k(\mathfrak{S}_k)$ *is a linear combination of the transformations* $\rho_k(g)$, $g \in \mathrm{GL}(n,\mathbb{C})$.

Applying Proposition 3.1 and Theorem 1.4, we obtain a preliminary version of *Schur-Weyl duality:*

**Corollary 3.2.** *There are irreducible, mutually inequivalent* $\mathfrak{S}_k$ *modules* $E^\lambda$ *and irreducible, mutually inequivalent* $\mathrm{GL}(n,\mathbb{C})$ *modules* $F^\lambda$ *so that*

$$\bigotimes^k \mathbb{C}^n \cong \bigoplus_{\lambda \in \mathrm{Spec}(\rho_k)} E^\lambda \otimes F^\lambda$$

*as a representation of* $\mathfrak{S}_k \times \mathrm{GL}(n,\mathbb{C})$. *The representation* $E^\lambda$ *uniquely determines* $F^\lambda$ *and conversely.*

3.2. **Highest Weight Theory.** To make Schur-Weyl duality an effective tool, we will construct the irreducible regular representations of $\mathrm{GL}(n,\mathbb{C})$ by the *Theorem of the Highest Weight.* We give details for $\mathrm{GL}(n,\mathbb{C})$; analogous results hold for any complex reductive algebraic group (see [16, Chap. 5]). The starting point is the *Gauss decomposition.* Let $H$ be the subgroup of diagonal matrices, $N$ the subgroup of upper-triangular unipotent matrices (all diagonal entries 1), and $\bar{N}$ the subgroup of lower triangular unipotent matrices. Then $\bar{N}HN$ is a Zariski-dense open subset of $G$, and a *generic* element $g \in G$ has a unique factorization $g = \bar{n}hn$.[b] Thus a regular representation of $G$ is completely determined by its restriction to the subgroups $\bar{N}$, $H$, and $N$.

The subgroup $H$ is a *maximal algebraic torus* in $G$. In particular, it is a reductive complex algebraic group. The irreducible representations of $H$ are one-dimensional and given by

$$h = \mathrm{diag}[x_1, \ldots, x_n] \mapsto h^\mu = x_1^{m_1} \cdots x_n^{m_n}, \quad \text{where } \mu = [m_1, \ldots, m_n] \in \mathbb{Z}^n.$$

Thus we may identify $\widehat{H}$ with $\mathbb{Z}^n$. If $(\rho, V)$ is a regular representation of $G$, then the restriction of $\rho$ to $H$ decomposes into *weight spaces:*

$$V = \bigoplus_{\mu \in \Phi(V)} V(\mu), \quad \text{where } V(\mu) \neq 0 \text{ and } \rho(h)v = h^\mu v \text{ for } v \in V(\mu).$$

We call $\Phi(V) \subset \widehat{H}$ the set of *weights* of $V$.

Let $\mathrm{Norm}_G(H)$ be the *normalizer* of $H$ in $G$ ($HgH = gH$), and $W = \mathrm{Norm}_G(H)/H$ the *Weyl group* of $G$. The elements of $W$ permute the weight spaces and the weights of $V$. In this case, $W \cong \mathfrak{S}_n$ may be identified with the group of *permutation matrices* in $G$, and the action of $W$ on $H$ and $\widehat{H}$ is by the usual permutation of coordinates. Every $W$ orbit in $\widehat{H}$ contains a unique *dominant weight*

$$\mu = [m_1, \ldots, m_n], \qquad m_1 \geq m_2 \geq \cdots \geq m_n\,.$$

[b]The precise condition from linear algebra is that the principal minors $\Delta_i(g) \neq 0$ for $i = 1, 2, ..., n$.

We denote by $\mathbb{Z}^n_{++}$ the set of all such $\mu \in \mathbb{Z}^n$.

**Examples**

**1.** Let $V = \mathbb{C}^n$ be the *defining representation* of $G$. Then

$$\Phi(V) = \{\varepsilon_1, \ldots, \varepsilon_n\}, \quad \text{where } \varepsilon(h) = x_i \text{ for } h = \mathrm{diag}[x_1, \ldots, x_n]$$

Here $\Phi(V) = W \cdot \varepsilon_1$ is a single $W$ orbit with dominant weight $\varepsilon_1$.

**2.** Let $V = \bigotimes^k \mathbb{C}^n$. The basis $\{e_I\}$ used in the proof of Proposition 3.1 diagonalizes $\rho_k(H)$. For an index $I = [i_1, \ldots, i_k]$, with $1 \le i_j \le n$, define

$$\mu_I = [\mu_1, \ldots, \mu_n], \qquad \text{where } \mu_p = \#\{j \,:\, i_j = p\}.$$

Then $\rho_k(h)e_I = h^{\mu_I} e_I$ for $h \in H$. Hence for $\lambda \in \hat{H}$,

$$V(\lambda) = \mathrm{Span}\{e_I \,:\, \mu_I = \lambda\}.$$

In particular, $V(\lambda) \neq 0$ if and only if $\lambda_i \ge 0$ for $i = 1, \ldots, n$ and $|\lambda| = k$, where $|\lambda| = \lambda_1 + \cdots + \lambda_n$. Thus $\Phi(\bigotimes^k \mathbb{C}^n) = W \cdot \mathrm{Par}(k, n)$, where $\mathrm{Par}(k, n)$ is the set of all partitions of $k$ with at most $n$ parts. Each such partition defines a dominant weight $\mu$ of $H$ such that $h \mapsto h^\mu$ is a *polynomial function* on $H$ (no negative powers of the coordinates $x_i$).

**3.** Let $\mathfrak{g} = \mathrm{Lie}(G) = M_n(\mathbb{C})$ be the Lie algebra of $G$, and let $\mathrm{Ad}(g)x = gxg^{-1}$ be the adjoint representation. The weights are 0 and $\{\varepsilon_i - \varepsilon_j \,:\, 1 \le i \ne j \le n\}$. We call the nonzero weights the *roots* of $\mathfrak{h}$ on $\mathfrak{g}$. The corresponding *root spaces* are

$$\mathfrak{g}_0 = \mathfrak{h} = \mathrm{Lie}(H), \quad \mathfrak{g}_{\varepsilon_i - \varepsilon_j} = \mathbb{C}E_{ij}$$

where $E_{ij}$ is the usual elementary matrix with 1 in position $(i, j)$ and zero elsewhere. If $\alpha = \varepsilon_i - \varepsilon_j$, then we say $\alpha > 0$ if $i < j$ (so $E_{ij}$ is upper triangular) and $\alpha < 0$ if $i > j$. We denote the set of positive roots by $\Phi^+$ and the set of negative roots by $\Phi^-$. Thus

$$\mathfrak{n} = \mathrm{Lie}(N) = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha, \qquad \bar{\mathfrak{n}} = \mathrm{Lie}(\bar{N}) = \bigoplus_{\alpha \in \Phi^-} \mathfrak{g}_\alpha.$$

The Lie algebra (additive) version of the Gauss decomposition is the *triangular decomposition* $\mathfrak{g} = \bar{\mathfrak{n}} \oplus \mathfrak{h} \oplus \mathfrak{n}$.

If $(\rho, V)$ is any regular representation of $G$, then there is an associated Lie algebra representation $d\rho$ of $\mathfrak{g}$ defined by

$$d\rho(X)v = \frac{d}{dt}\rho(\exp tX)v\bigg|_{t=0}$$

Clearly $\rho(g)d\rho(X)v = d\rho(\mathrm{Ad}(g)X)\rho(g)v$. Hence if $h \in H$, $E_\alpha \in \mathfrak{g}_\alpha$ and $v \in V(\mu)$, then

$$\rho(h)d\rho(E_\alpha)v = d\rho(\mathrm{Ad}(h)E_\alpha)\rho(h)v = h^{\alpha+\mu} d\rho(E_\alpha)v.$$

This shows that $d\rho(\mathfrak{g}_\alpha)V(\mu) \subset V(\mu + \alpha)$. Thus

$$d\rho(\mathfrak{n})V(\mu) \subset \bigoplus_{\lambda \in \mu + \Phi^+} V(\lambda). \tag{3.1}$$

We call $\mu \in \Phi(V)$ an $N$-*extreme* weight if $\mu + \alpha \notin \Phi(V)$ for all $\alpha \in \Phi^+$.

**Theorem 3.3.** *Let $(\rho, V)$ be an irreducible regular representation of $G$. Then there is a unique $N$-extreme weight $\mu_0 \in \Phi(V)$. This weight is dominant, the weight space $V(\mu_0) = V^N$ (the $N$-fixed vectors in $V$), and* $\dim V^N = 1$.

We call $\mu_0$ the *highest weight* of the representation $(\rho, V)$. We next show that $\mu_0$ determines $(\rho, V)$ uniquely up to isomorphism. Let $s_0 \in W$ be the permutation $[1, 2, \ldots, n] \mapsto [n, \ldots, 2, 1]$. Then $\mathrm{Ad}(s_0)N = \bar{N}$ and $s_0 \cdot \Phi^+ = \Phi^-$ (this last property uniquely characterizes $s_0$). Since the weights and weight multiplicities of $(\rho, V)$ are invariant under the Weyl group, we see from Theorem 3.3 that $\Phi(V)$ has a unique minimal element $-s_0\mu_0$ (the *lowest weight*), and $V(-s_0\mu_0) = V^{\bar{N}}$. The natural bilinear form on $V \times V^*$ is invariant under $H$, so its restriction to $V(\lambda) \times V^*(-\lambda)$ is nondegenerate. Thus $-\mu_0$ is the lowest weight of $V^*$.

**Theorem 3.4.** *Let $(\rho, V)$ and $(\sigma, U)$ be irreducible regular representations of $G$ with the same highest weight $\lambda$. If $v_0 \in V$ and $u_0 \in U$ are highest weight vectors, then there exists a unique $G$-isomorphism $T : U \to V$ such that $Tu_0 = v_0$. Thus $(\rho, V)$ is uniquely determined by its highest weight.*

For applications to duality we will need the following sharpening of Theorem 3.3.

**Theorem 3.5.** *Let $(\rho, L)$ be any regular representation of $G$. Suppose that $0 \neq v_0 \in L(\lambda)^N$. Then the subspace $V = \mathrm{Span}\{\rho(G)v_0\}$ of $L$ is an irreducible $G$-module with highest weight $\lambda$.*

**Corollary 3.6.** *For every dominant weight $\lambda$ of $H$ there exists an irreducible representation $(\pi^\lambda, F^\lambda)$ with highest weight $\lambda$.*

We shall refer to Theorems 3.3, 3.4, and 3.5 and Corollary 3.6 collectively as the *Theorem of the Highest Weight*.

3.3. **Duality and $N$-fixed Vectors.** Let $(\rho, L)$ be any regular $G$-module. By the Theorem of the Highest Weight we can identify $\mathrm{Spec}(\rho)$ with the set of dominant weights $\lambda$ such that $L(\lambda)^N \neq 0$. For $\lambda \in \mathrm{Spec}(\rho)$ set $E^\lambda = L(\lambda)^N$. This space is invariant under the commuting algebra $\mathrm{End}_G(L)$.

**Theorem 3.7.** *Under the joint action of $G$ and $\mathrm{End}_G(L)$ the space $L$ decomposes as*

$$L \cong \bigoplus_{\lambda \in \mathrm{Spec}(\rho)} E^\lambda \otimes F^\lambda. \tag{3.2}$$

*Furthermore, $E^\lambda$ is an irreducible module for $\mathrm{End}_G(L)$, and distinct values of $\lambda$ give inequivalent modules for $\mathrm{End}_G(L)$.*

We can now give a more precise version of Schur-Weyl duality (Corollary 3.2). A regular representation $\pi$ of $G$ is said to be *polynomial* if the matrix entries of $\pi$ are polynomial functions on $G$ (with no negative powers of $\det(g)$). When $\pi = \pi^\lambda$ is irreducible, it is a polynomial representation if and only if $\lambda_n \geq 0$. In this case $\lambda$ corresponds to a partition of $|\lambda|$ with at most $n$ parts.

**Theorem 3.8** (Schur-Weyl Duality)**.** *For $\lambda \in \mathrm{Par}(k, n)$ let $(\sigma^\lambda, E^\lambda)$ be the representation of $\mathfrak{S}_k$ on the space of $N$-fixed $k$-tensors of weight $\lambda$, where $N$ is the upper triangular unipotent subgroup of $\mathrm{GL}(n, \mathbb{C})$. Let $(\pi^\lambda, F^\lambda_{(n)})$ be the irreducible representation of $\mathrm{GL}(n, \mathbb{C})$ with highest weight $\lambda$. Then $E^\lambda$ is an irreducible $\mathfrak{S}_k$ module. Under the action of $\mathfrak{S}_k \times \mathrm{GL}(n, \mathbb{C})$, the space of $k$-tensors over $\mathbb{C}^n$ decomposes as*

$$\bigotimes^k \mathbb{C}^n \cong \bigoplus_{\lambda \in \mathrm{Par}(k,n)} E^\lambda \otimes F^\lambda_{(n)}.$$

*The representations $E^\lambda$ are mutually inequivalent, and when $n \geq k$ they give all the irreducible representations of $\mathfrak{S}_k$. Furthermore, every irreducible polynomial representation of $\mathrm{GL}(n, \mathbb{C})$ occurs in $(\mathbb{C}^n)^{\otimes k}$ for some $k$.*

**Examples.**

**1.** The group $\mathfrak{S}_k$ has two one-dimensional representations: the *trivial* representation and the *sign* representation. The corresponding subspaces of $\bigotimes^k \mathbb{C}^n$ are the *symmetric tensors* $S^k(\mathbb{C}^n)$ and (if $n \geq k$) the *skew symmetric tensors* $\bigwedge^k \mathbb{C}^n$. Hence these subspaces must be irreducible $\mathrm{GL}(n, \mathbb{C})$ modules, by Schur-Weyl duality (this is also easy to verify directly). The symmetric tensor $e_1^{\otimes k}$ is $N$-fixed with weight $k\varepsilon_1$, while the skew-symmetric tensor $e_1 \wedge \cdots \wedge e_k$ is $N$-fixed with weight $\varepsilon_1 + \cdots + \varepsilon_k$ (when $k \leq n$). Thus in the duality correspondence,

$$\begin{aligned}
(\text{trivial}, \mathbb{C}) = (\sigma^{[k]}, E^{[k]}) \quad &\longleftrightarrow \quad (\pi^{[k]}, S^k(\mathbb{C}^n)) \\
(\text{sgn}, \mathbb{C}) = (\sigma^{[1^k]}, E^{[1^k]}) \quad &\longleftrightarrow \quad (\pi^{[1^k]}, \textstyle\bigwedge^k \mathbb{C}^n) \quad \text{if } n \geq k.
\end{aligned}$$

When $k = 2$ and $n \geq 2$ this gives the complete decomposition of $\bigotimes^2 \mathbb{C}^n$: under $\mathfrak{S}_2 \times \mathrm{GL}(n, \mathbb{C})$:

$$\mathbb{C}^n \otimes \mathbb{C}^n \cong \left\{ E^{[2,0]} \otimes S^2(\mathbb{C}^n) \right\} \oplus \left\{ E^{[1,1]} \otimes \bigwedge^2 \mathbb{C}^n \right\}.$$

**2.** Consider $\bigotimes^3 \mathbb{C}^n$ for $n \geq 3$. There are three partitions of 3, giving the decomposition

$$\bigotimes^3 \mathbb{C}^n \cong \Big\{E^{[3,0]} \otimes S^3(\mathbb{C}^n)\Big\} \oplus \Big\{E^{[2,1]} \otimes F^{[2,1]}\Big\} \oplus \Big\{E^{[1,1,1]} \otimes \bigwedge{}^3\mathbb{C}^n\Big\}$$

under $\mathfrak{S}_3 \times \mathrm{GL}(n, \mathbb{C})$. Here the representation $E^{[2,1]}$ of $\mathfrak{S}_3$ is the two-dimensional *standard representation* on $\mathbb{C}^3/\mathbb{C}[1,1,1]$.

We can view Schur-Weyl Duality as a method to construct representations of $G' = \mathfrak{S}_k$ from representations of $G = \mathrm{GL}(n, \mathbb{C})$ via the Theorem of the Highest Weight. Here we take the representations of $G$ as the known objects, and the representations of $G'$ as the unknown objects.[c] The relative size of $n$ (the *rank* of $G$) and $k$ then determines which representations of $G'$ we get this way.

$n \geq k$**:** All partitions of $k$ have at most $k$ parts, so all representations of $\mathfrak{S}_k$ occur in $\bigotimes^k \mathbb{C}^n$ in this case.

$n \leq k$**:** Only those representations of $\mathfrak{S}_k$ occur in $\bigotimes^k \mathbb{C}^n$ that correspond to partitions of $k$ with at most $n$ parts.

To make this method effective, we will develop *character formulas* for the representations of $\mathfrak{S}_k$ in the next two lectures, based on the celebrated *Weyl character formula* for $\mathrm{GL}(n, \mathbb{C})$.

## Lecture 4. Commutant Character Formulas

4.1. **Characters.** Let $G$ be a connected complex reductive algebraic group. Then $G$ contains a maximal algebraic torus $H$ and a maximal connected solvable subgroup $HN$ (semidirect product), where $N$ is the unipotent radical of $HN$. In fact, one can always embed $G$ into $\mathrm{GL}(n, \mathbb{C})$ so that $H$ consists of the diagonal matrices in $G$, and $N$ the upper-triangular unipotent matrices in $G$, just as in the case of $\mathrm{GL}(n, \mathbb{C})$ treated in Lecture 3. Let $\mathfrak{h} = \mathrm{Lie}(H)$ and $\mathfrak{n} = \mathrm{Lie}(N)$.

The irreducible representations of the torus $H$ are given by $h \mapsto h^\lambda$, where $\lambda$ is in the *weight lattice* $P \subset \mathfrak{h}^*$ of $H$. By the Theorem of the Highest Weight (which is proved for $G$ along the same lines as in Lecture 3 for $\mathrm{GL}(n, \mathbb{C})$), the irreducible regular representations of $G$ are parameterized by the set $P_{++}$ of *dominant weights* determined by the choice of $N$. For $\lambda \in P_{++}$ let $(\pi^\lambda, F^\lambda)$ be the irreducible representation of $G$ with highest weight $\lambda$.

Let $(\pi, V)$ be a finite-dimensional rational representation of $G$. Set $\mathcal{B} = \mathrm{End}_G(V)$. From Theorem 3.7 $V$ decomposes under the joint action of $G$ and $\mathcal{B}$ into a multiplicity-free direct sum

$$V \cong \bigoplus_{\lambda \in \mathrm{Spec}(\pi)} E^\lambda \otimes F^\lambda \tag{4.1}$$

Here $g \in G$ acts by $1 \otimes \pi^\lambda(g)$ and $b \in \mathcal{B}$ acts by $\sigma^\lambda(b) \otimes 1$ on the summands in (4.1). We may take $E^\lambda = V(\lambda)^N$ (the space of $N$-fixed vectors of weight $\lambda$ in $V$) with $\sigma^\lambda(b)$ the natural action of $b \in \mathcal{B}$ on this space.

Finding the spaces $V(\lambda)^N$ explicitly is usually difficult. An easier problem is to calculate characters. For $\lambda \in P_{++}$ we write

$$\chi_\lambda(b) = \mathrm{tr}(\sigma^\lambda(b)) = \mathrm{tr}(b|_{V(\lambda)^N}), \quad \text{for } b \in \mathcal{B}.$$

4.2. **Frobenius and Determinant Character Formulas.** We now obtain two formulas for the characters $\chi_\lambda$ that only involve the full $H$-weight spaces in $V$. Let $\Phi^+$ be the weights of $\mathrm{Ad}(H)$ on $\mathfrak{n}$ and $\rho = \frac{1}{2}\sum_{\alpha \in \Phi^+} \alpha$. Let $W = \mathrm{Norm}_G(H)/H$ be the Weyl group of $(G, H)$. Set

$$D(h) = \sum_{s \in W} \mathrm{sgn}(s)\, h^{s \cdot \rho} \qquad \text{for } h \in H$$

(the Weyl denominator). Here $s \mapsto \mathrm{sgn}(s) = \det(\mathrm{Ad}(s)|_{\mathfrak{h}})$ is the usual signum character on $W$.

**Theorem 4.1** (Generalized Frobenius Formula)**.** *For $\lambda \in P_{++}$ and $b \in \mathcal{B}$ one has*

$$\chi_\lambda(b) = \textit{coefficient of } h^{\lambda+\rho} \textit{ in } D(h)\,\mathrm{tr}_V(\pi(h)b) \tag{4.2}$$

*(where $h \in H$).*

[c] The representations of $\mathfrak{S}_k$ can be constructed directly by group-theoretic and combinatorial methods. Special elements of the group algebra $\mathbb{C}[\mathfrak{S}_k]$ (*Young symmetrizers*) project tensor space onto irreducible representations of $\mathrm{GL}(n, \mathbb{C})$ – the so-called *Weyl modules*–see [16, §9.3].

**Theorem 4.2** (Generalized Determinant Formula)**.** *For $\lambda \in P_{++}$ and $b \in \mathcal{B}$ one has*

$$\chi_\lambda(b) = \sum_{s\in W} \operatorname{sgn}(s) \operatorname{tr}_{V(\lambda+\rho-s\cdot\rho)}(b). \tag{4.3}$$

*In particular,*

$$\dim E^\lambda = \sum_{s\in W} \operatorname{sgn}(s) \dim V(\lambda+\rho-s\cdot\rho). \tag{4.4}$$

4.3. **Proof of Frobenius Character Formula.** For $\lambda \in P_{++}$ we write

$$\chi^\lambda(g) = \operatorname{tr}(\pi^\lambda(g)) \quad \text{for } g \in G.$$

(the character of the representation with highest weight $\lambda$). We note from (4.1) that

$$\operatorname{tr}_V(\pi(g)b) = \sum_{\lambda\in\operatorname{Spec}(\pi)} \chi^\lambda(g)\,\chi_\lambda(b) \qquad \text{for } g \in G \text{ and } b \in \mathcal{B} \tag{4.5}$$

By the Weyl character formula (WCF), we have

$$D(h)\,\chi^\lambda(h) = \sum_{s\in W} \operatorname{sgn}(s)\, h^{s\cdot(\lambda+\rho)} \qquad \text{for } h \in H.$$

Using the WCF in (4.5) we can write

$$D(h)\operatorname{tr}_V(\pi(h)b) = \sum_{\lambda\in\operatorname{Spec}(\pi)} \sum_{s\in W} \operatorname{sgn}(s)\,\chi_\lambda(b)\,h^{s\cdot(\lambda+\rho)} \tag{4.6}$$

Due to the shift by $\rho$, the map $(s,\lambda) \mapsto s\cdot(\lambda+\rho)$ from $W \times P_{++} \to P$ is injective[d]. Hence the character $h \mapsto h^{\lambda+\rho}$ only occurs once in (4.6), and has coefficient $\chi_\lambda(b)$, as claimed. □

4.4. **Proof of Determinant Character Formula.** For the proof of Theorem 4.2, we need the following consequence of the WCF (which is, in fact, equivalent to the WCF).

**Lemma 4.3.** *Let $m_\lambda(\mu) = \dim F^\lambda(\mu)$ for $\mu \in P$ and $\lambda \in P_{++}$ (the* multiplicity *of the weight $\mu$ in $F^\lambda$). Then for $\lambda, \mu \in P_{++}$ one has*

$$\sum_{s\in W} \operatorname{sgn}(s)\, m_\mu(\lambda+\rho-s\cdot\rho) = \delta_{\lambda\mu}.$$

**Proposition 4.4** (Outer Multiplicity Formula)**.** *Let $L$ be any regular $G$ module. For $\lambda \in P_{++}$ let* $\operatorname{mult}_L(\lambda)$ *be the multiplicity of the representation $F^\lambda$. Then*

$$\operatorname{mult}_L(\lambda) = \sum_{s\in W} \operatorname{sgn}(s) \dim L(\lambda+\rho-s\cdot\rho) \tag{4.7}$$

## Lecture 5. Character Formulas for Schur-Weyl Duality

5.1. **Frobenius Formula for $\mathfrak{S}_k$ Characters.** We now apply the commutant character formulas to the Schur-Weyl duality between $G = \mathrm{GL}(n,\mathbb{C})$ and $\mathfrak{S}_k$, both acting on $V = \bigotimes^k \mathbb{C}^n$. Recall that the conjugacy classes in $\mathfrak{S}_k$ are described by cycle lengths. We denote by $C(1^{r_1}2^{r_2}\cdots k^{r_k})$ the class of elements with $r_j$ cycles of length $j$, where $r_1 + 2r_2 + 3r_3 + \cdots = k$. A permutation $s$ in this class has $r_1$ fixed points, $r_2$ transpositions, and so on. To apply Theorem 4.1 in this context, we need to calculate the polynomial

$$h \mapsto \operatorname{tr}_V(\rho_k(h)\sigma_k(s)), \qquad \text{where } h = \operatorname{diag}[x_1,\ldots,x_n].$$

Recall that the tensors $\{e_I\}$ give a basis for $V$. The action of $h \in H$ and $s \in \mathfrak{S}_k$ is

$$\rho_k(h)e_I = x^{\mu(I)}e_I, \qquad \sigma_k(s)e_I = e_{s\cdot I}$$

where $\mu(I) = [\mu_1,\cdots,\mu_n]$ with $\mu_j = \#\{p : i_p = j\}$ and

$$s\cdot I = [i_{s^{-1}(1)},\ldots,i_{s^{-1}(k)}].$$

Since $\sigma(s)$ permutes the basis $\{e_I\}$ and each $e_I$ is a weight vector for $H$, it follows that

$$\operatorname{tr}_V(\rho_k(h)\sigma_k(s)) = \operatorname{tr}_{F_s}(\rho_k(h)) \qquad \text{for } h \in H, \tag{5.1}$$

[d] in the case of $\mathrm{GL}(n,\mathbb{C})$ the partition $\lambda+\rho$ has all parts of *different* sizes

where $F_s = \mathrm{Span}\{e_I \,:\, s \cdot I = I\}$. Let $V_j = \mathrm{Span}\{e_1^{\otimes j}, e_2^{\otimes j}, \cdots, e_n^{\otimes j}\} \subset \bigotimes^j \mathbb{C}^n$.

**Lemma 5.1.** *If $s \in C(1^{r_1} 2^{r_2} \cdots k^{r_k})$ then*

$$F_s \cong V_1^{\otimes r_1} \otimes V_2^{\otimes r_2} \otimes \cdots \otimes V_k^{\otimes r_k}$$

*as an $H$-module.*

For $h = \mathrm{diag}[x_1, \ldots, x_n]$ define

$$p_j(x) = \mathrm{tr}_{V_j}(\rho_k(h)) = x_1^j + \cdots + x_n^j \qquad (j\text{th power sum}).$$

Then Lemma 5.1 implies that

$$\mathrm{tr}_{F_s}(\rho_k(h)) = \prod_{j=1}^{k} \mathrm{tr}_{V_j}(\rho_k(h))^{r_j} = \prod_{j=1}^{k} p_j(x)^{r_j}.$$

Hence from Theorem 4.1 and (5.1) we obtain the *Frobenius character formula*:

**Theorem 5.2.** *Let $s \in C(1^{r_1} 2^{r_2} \cdots k^{r_k})$ and $\lambda \in \mathrm{Par}(k, n)$. Then*

$$\chi_\lambda(s) = \textit{coefficient of } x^{\lambda+\rho_{[n]}} \textit{ in } D_n(x)\Big\{\prod_{j=1}^{k} p_j(x)^{r_j}\Big\}$$

*where $\rho_{[n]} = [n-1, n-2, \ldots, 1, 0]$ and $D_n(x) = \prod_{1 \le i < j \le n}(x_i - x_j)$.*

**Examples**

**1.** Suppose $s = (1, 2, \ldots, m)(m+1) \cdots (k)$ is a single $m$-cycle with $k - m$ fixed points. Then

$$\chi_\lambda(s) = \text{ coefficient of } x^{\lambda+\rho_{[n]}} \text{ in } \ (x_1 + \cdots + x_n)^{k-m}(x_1^m + \cdots + x_n^m) \prod_{1 \le i < j \le n}(x_i - x_j).$$

We call a monomial $x_1^{a_1} \cdots x_n^{a_n}$ *strictly dominant* if $a_1 > a_2 > \cdots > a_n$. For partitions $\lambda$ with two parts and cycles of maximum length $m = k$, the strictly dominant terms in this formula are $x_1^{k+1} - x_1^k x_2$. Hence for $s = (1, 2, \ldots, k)$,

$$\chi_\lambda(s) = \begin{cases} -1 & \text{for } \lambda = [k-1, 1], \\ \ \ 0 & \text{for } \lambda = [k-j, j] \text{ with } j > 1. \end{cases}$$

**2.** Consider the group $\mathfrak{S}_3$, which has three conjugacy classes: $C(1^3) = \{\text{identity}\}$, $C(1^1 2^1) = \{(12), (13), (23)\}$ and $C(3^1) = \{(123), (132)\}$. As we noted at the end of Lecture 3, the three representations of $\mathfrak{S}_3$ are $\sigma^{[3]}$ (the trivial representation), $\sigma^{[2,1]}$ (the two-dimensional standard representation), and $\sigma^{[1,1,1]}$ (the signum representation). To calculate the character of $\sigma^{[2,1]}$ by the Frobenius formula, we let $x = [x_1, x_2]$ and expand the polynomials

$$\begin{aligned} D_2(x)p_1(x)^3 &= x_1^4 + 2x_1^3 x_2 + \cdots \\ D_2(x)p_1(x)p_2(x) &= x_1^4 + \cdots \\ D_2(x)p_3(x) &= x_1^4 - x_1^3 x_2 + \cdots \end{aligned}$$

where $\cdots$ indicates non-dominant terms. By Theorem 4.1 the coefficients of the dominant terms in these formulas furnish the entries in the character table for $\mathfrak{S}_3$. We write $\chi_\lambda$ for the character of the representation $\sigma^\lambda$. For example, when $\lambda = [2, 1]$ we have $\lambda + \rho = [3, 1]$, so the coefficient of $x_1^3 x_2$ in $D_2(x)p_3(x)$ gives the value of $\chi_{[2,1]}$ on the conjugacy class $C(3^1)$. Table I gives all the characters, where the top row indicates the number of elements in each conjugacy class, and the rows in the table give the character values for each irreducible representation.

Table I: Character Table of $\mathfrak{S}_3$

| # elements: | 1 | 3 | 2 |
|---|---|---|---|
| conjugacy class: | $C(1^3)$ | $C(1^1 2^1)$ | $C(3^1)$ |
| $\chi^{[3]}$ | 1 | 1 | 1 |
| $\chi^{[2,1]}$ | 2 | 0 | $-1$ |
| $\chi^{[1,1,1]}$ | 1 | $-1$ | 1 |

5.2. **Determinant Formula for $\mathfrak{S}_k$ Characters.** We next apply Theorem 4.2 to obtain an alternating sum formula for the characters of $\mathfrak{S}_k$. For this, we need to identify the weight spaces $V(\nu)$ as $\mathfrak{S}_k$-modules. Here $\nu = [\nu_1, \ldots, \nu_n]$ with $\nu_i \geq 0$ and $\nu_1 + \cdots + \nu_n = k$. We have already seen that $V(\nu) = \mathrm{Span}\{e_I : \mu(I) = \nu\}$.

**Lemma 5.3.** *Let $\mathfrak{S}_\nu = \mathfrak{S}_{\nu_1} \times \cdots \times \mathfrak{S}_{\nu_n} \subset \mathfrak{S}_k$. Then $V(\nu) \cong \mathbb{C}[\mathfrak{S}_k/\mathfrak{S}_\nu]$ as a $\mathfrak{S}_k$-module.*

From the lemma we see that $\sigma_k(s)$ acts as a permutation matrix on $V(\nu)$, and hence

$$\mathrm{tr}_{V(\nu)}(\sigma_k(s)) = \#\{\text{fixed points of } s \text{ on } \mathfrak{S}_k/\mathfrak{S}_\nu\}.$$

The Weyl group for $G$ is $\mathfrak{S}_n$ and acts on the weight lattice $P$ as permutations of the coordinates of the weights. Applying Theorem 4.2 and using Lemma 5.3, we obtain the following character formula.

**Theorem 5.4.** *Let $\lambda \in \mathrm{Par}(k, n)$ and $s \in \mathfrak{S}_k$. Then*

$$\chi_\lambda(s) = \sum_t \mathrm{sgn}(t)\ \#\{\textit{fixed points of } s \textit{ on } \mathfrak{S}_k/\mathfrak{S}_{\lambda+\rho_{[n]}-t\cdot\rho_{[n]}}\}.$$

*Here the sum is over all $t \in \mathfrak{S}_n$ such that all the coordinates of $\lambda + \rho_{[n]} - t \cdot \rho_{[n]}$ are non-negative. In particular,*

$$\dim E^\lambda = \sum_{t \in \mathfrak{S}_n} \mathrm{sgn}(t) \binom{k}{\lambda + \rho_{[n]} - t \cdot \rho_{[n]}}.$$

In Theorem 5.4 $\rho_{[n]} = [n-1, n-2, \ldots, 1, 0]$ and

$$\binom{k}{\nu} = \frac{k!}{\nu!} \qquad (\text{where } \nu! = \nu_1! \cdots \nu_n!)$$

is the multinomial coefficient (with the usual convention that it is zero if any entry in $\nu$ is negative). The dimension formula can be written as a determinant and then reduced to Vandermonde form. This gives the following *product formula* for the dimension of the representation $E^\lambda$ that is analogous to the Weyl dimension formula for the representation $F^\lambda$.

**Corollary 5.5.** *Let $\lambda \in \mathrm{Par}(k, n)$. Then* $\dim E^\lambda = \dfrac{k!}{(\lambda + \rho_{[n]})!} D_n(\lambda + \rho_{[n]})$.

A partition $\lambda = [\lambda_1, \ldots, \lambda_n]$ can be represented in terms of its *Ferrers diagram*: a left-justified array of boxes, with $\lambda_i$ boxes in the $i$th row (counting from the top down). Each box in the diagram has a *hook length:* the total number of boxes to the right and below the given box (including the box itself). We can then fill each box with its hook length. For example, $\lambda = [4, 3, 1] \in \mathrm{Par}(8, 3)$ has Ferrers diagram and hook lengths

$$\begin{array}{|c|c|c|c|}\hline 6 & 4 & 3 & 1 \\ \hline 4 & 2 & 1 & \multicolumn{1}{c}{} \\ \cline{1-3} 1 & \multicolumn{3}{c}{} \\ \cline{1-1}\end{array}$$

From Corollary 5.5 one obtains (by induction on the number of columns of $\lambda$) the *Hook Length Formula*

$$\dim E^\lambda = \frac{k!}{\prod_{ij} h_{ij}(\lambda)} \tag{5.2}$$

where $h_{ij}(\lambda)$ is the hook length of the $ij$ box in the Ferrers diagram of $\lambda$. By way of comparison, the *Weyl Dimension Formula* for $\mathrm{GL}(n, \mathbb{C})$ can be written as

$$\dim F^\lambda_{(n)} = \frac{(\lambda + \rho_{[n]})!}{\rho_{[n]}! \prod_{ij} h_{ij}(\lambda)} \tag{5.3}$$

(see [16, §9.1.4, Ex. #9]). For example, for $\mathfrak{S}_8 \times \mathrm{GL}(3, \mathbb{C})$ acting on $\bigotimes^8 \mathbb{C}^3$ we have $\rho_{[3]} = [2, 1, 0]$ and

$$\dim E^{[4,3,1]} = \frac{8!}{6 \cdot 4 \cdot 4 \cdot 3 \cdot 2} = 70, \quad \dim F^{[4,3,1]}_{(3)} = \frac{6! \cdot 4!}{2!(6 \cdot 4 \cdot 4 \cdot 3 \cdot 2)} = 15.$$

Thus $E^{[4,3,1]} \otimes F^{[4,3,1]}_{(3)}$ is an irreducible subspace of $\bigotimes^8 \mathbb{C}^3$ of dimension $(70) \cdot (15)$.

5.3. **Schur-Weyl Duality and** $\mathrm{GL}(k)$**-**$\mathrm{GL}(n)$ **Duality.** There is another model for the irreducible representations of $\mathfrak{S}_k$ that comes from the identification of $\mathfrak{S}_k$ with the Weyl group of $\mathrm{GL}(k,\mathbb{C})$. Let $X = M_{k\times n}$ ($k \times n$ complex matrices) and let $\mathrm{GL}(k,\mathbb{C}) \times \mathrm{GL}(n,\mathbb{C})$ act on $\mathcal{P}(X)$ by

$$\rho(g_1, g_2)f(x) = f(g_1^t x g_2) \quad \text{for } g_1 \in \mathrm{GL}(k,\mathbb{C}) \text{ and } g_2 \in \mathrm{GL}(n,\mathbb{C}).$$

This representation is multiplicity free and decomposes as

$$\mathcal{P}(X) \cong \bigoplus_{\mu} F^{\mu}_{(k)} \otimes F^{\mu}_{(n)} \tag{5.4}$$

with the sum over all partitions $\mu$ with at most $\min\{k,n\}$ parts (see [16, §5.2.4] or the article by Benson-Ratcliff in this volume).

Let $H_k \subset \mathrm{GL}(k,\mathbb{C})$ be the maximal torus of diagonal matrices, and embed $\mathfrak{S}_k \subset \mathrm{GL}(k,\mathbb{C})$ as the permutation matrices. If we only consider the action of the subgroup $\mathrm{Norm}\,(H_k) \cong \mathfrak{S}_k \ltimes H_k$ of $\mathrm{GL}(k,\mathbb{C})$ on $X$ together with the right action of $\mathrm{GL}(n,\mathbb{C})$, then

$$X \cong \underbrace{(\mathbb{C}^n)^* \oplus \cdots \oplus (\mathbb{C}^n)^*}_{k \text{ summands}}.$$

Hence

$$\mathcal{P}(X) \cong \underbrace{S(\mathbb{C}^n) \otimes \cdots \otimes S(\mathbb{C}^n)}_{k \text{ factors}}$$

as a representation of $\mathrm{Norm}\,(H_k) \times \mathrm{GL}(n,\mathbb{C})$. Here $\mathfrak{S}_k$ acts by permuting the tensor factors, while $h = \mathrm{diag}[x_1, \ldots, x_k] \in H_k$ acts by multiplication by $x_j$ on the $j$th factor. The weight space decomposition of $\mathcal{P}(X)$ relative to the $H_k$ action is thus

$$\mathcal{P}(X)(\mu) \cong S^{m_1}(\mathbb{C}^n) \otimes \cdots \otimes S^{m_k}(\mathbb{C}^n) \quad \text{for } \mu = [m_1, \ldots, m_k]. \tag{5.5}$$

Here $\mathfrak{S}_k$ acts by permuting the factors in this decomposition while $\mathrm{GL}(n\,\mathbb{C})$ acts as usual on each copy of $\mathbb{C}^n$. In particular, the weight $\det_k = [1, 1, \ldots, 1]$ is fixed by $\mathfrak{S}_k$ and the corresponding weight space is

$$\mathcal{P}(X)(\det_k) \cong S^1(\mathbb{C}^n) \otimes \cdots \otimes S^1(\mathbb{C}^n) = (\mathbb{C}^n)^{\otimes k}$$

with the usual commuting actions of $\mathfrak{S}_k$ and $\mathrm{GL}(n,\mathbb{C})$. On the other hand, if we calculate this weight space using (5.4), we see that

$$(\mathbb{C}^n)^{\otimes k} \cong \bigoplus_{\lambda \in \mathrm{Par}(k,n)} F^{\lambda}_{(k)}(\det_k) \otimes F^{\lambda}_{(n)}$$

as a module for $\mathfrak{S}_k \times \mathrm{GL}(n,\mathbb{C})$. Invoking Theorem 3.8 we conclude: *For all* $\lambda \in \mathrm{Par}(k,n)$*,*

$$E^{\lambda} \cong F^{\lambda}_{(k)}(\det_k)$$

*as a* $\mathfrak{S}_k$ *module, with the action of* $\mathfrak{S}_k$ *coming from its embedding into* $\mathrm{GL}(k,\mathbb{C})$.

**Examples**

**1.** Take $\lambda = [1, \ldots, 1] \in \mathrm{Par}(k)$. Then the representation $F^{\lambda}_{(k)}$ of $\mathrm{GL}(k,\mathbb{C})$ is $\bigwedge^k \mathbb{C}^k$, on which $\mathrm{GL}(k,\mathbb{C})$ acts by $g \mapsto \det(g)$. This shows once again that $E^{\lambda}$ is the sgn representation of $\mathfrak{S}_k$.

**2.** Now take $\lambda = [k]$. Then the representation $F^{\lambda}_{(k)}$ of $\mathrm{GL}(k,\mathbb{C})$ is $S^k(\mathbb{C}^k) \cong \mathcal{P}^k((\mathbb{C}^k)^*)$. The $\det_k$ weight space is one-dimensional and spanned by the monomial $x_1 \cdots x_k$, which is fixed by $\mathfrak{S}_k$. Again we see that $E^{[k]}$ is the trivial representation of $\mathfrak{S}_k$.

## Lecture 6. Polynomial Invariants and FFT

6.1. **Invariant Polynomials.** Let $G$ be a reductive linear algebraic group. Recall from Lecture 1 that given a regular representation $(\pi, V)$ of $G$, we have a locally regular representation $\rho$ of $G$ as automorphisms of the commutative algebra $\mathcal{P}(V)$ of complex-valued polynomial functions on $V$:

$$\rho(g)f(v) = f(g^{-1}v) \quad \text{for } f \in \mathcal{P}(V) \text{ and } g \in G$$

(here we write $gv$ for $\pi(g)v$ when the action $\pi$ is clear from the context). Since $G$ acts by automorphisms of $\mathcal{P}(V)$, the space $\mathcal{J} = \mathcal{P}(V)^G$ of $G$-invariant polynomials is a subalgebra of $\mathcal{P}(V)$. Thus we can consider

$\mathcal{P}(V)$ as a module for $\mathcal{J}$ under the action of pointwise multiplication, which commutes with the $G$ action. Then in the isotypic decomposition

$$\mathcal{P}(V) = \bigoplus_{\lambda \in \widehat{G}} \mathcal{P}(V)_{(\lambda)}$$

each summand is invariant under $\mathcal{J}$. By Corollary 1.2 there is a projection $f \mapsto f^{\natural}$ from $\mathcal{P}(V)$ onto $\mathcal{J}$, with $\deg f^{\natural} \le \deg f$. If $f \in \mathcal{P}(V)$ and $\varphi \in \mathcal{J}$ then

$$(\varphi f)^{\natural} = \varphi f^{\natural} \tag{6.1}$$

(Decompose $f = f^{\natural} + \cdots$ into isotypic components; then $\varphi f = \varphi f^{\natural} + \cdots$ is the isotypic decomposition of $\varphi f$.)

**Theorem 6.1** (Hilbert-Hurwitz)**.** *$\mathcal{J}$ is finitely generated as an algebra over $\mathbb{C}$.*

We shall say that $\{\varphi_1, \ldots, \varphi_n\} \subset \mathcal{J}$ is a *basic set of $G$ invariants* if

(i) $\{\varphi_1, \ldots, \varphi_n\}$ generates $\mathcal{J}$ as an algebra over $\mathbb{C}$
(ii) each $\varphi_i$ is homogeneous (of some degree $d_i$)

and $n$ is as small as possible, subject to (i) and (ii). By Theorem 6.1 there always exists a basic set of invariants (the polynomials $\varphi_i$ are not unique but the set $\{d_1, \ldots, d_n\}$ of degrees is uniquely determined).

**Example.** Let $G = \mathfrak{S}_n$ and $V = \mathbb{C}^n$, with $G$ acting as permutations of the coordinates. Then

$$\rho(s)f(x_1, \ldots, x_n) = f(x_{s(1)}, \ldots, x_{s(n)}) \quad \text{for } f \in \mathbb{C}[x_1, \ldots, x_n] \text{ and } s \in \mathfrak{S}_n.$$

and $\mathcal{J}$ is the algebra of *symmetric functions* in $n$ variables. By the *fundamental theorem of symmetric functions* one has $\mathcal{J} = \mathbb{C}[\sigma_1, \ldots, \sigma_n]$, where

$$\sigma_p(x_1, \ldots, x_n) = \sum_{1 \le j_1 < \cdots < j_p \le n} x_{j_1} \cdots x_{j_p} \qquad (p\text{th elementary symmetric function})$$

Furthermore, the functions $\{\sigma_1, \ldots, \sigma_n\}$ are algebraically independent, so they give a basic set of invariants with degrees $d_p = p$. (The function $\sigma_p$ is the restriction to the diagonal matrices of the character of $\bigwedge^p \mathbb{C}^n$ as a representation of $\mathrm{GL}(n, \mathbb{C})$.)

6.2. **Invariants of Vectors and Covectors.** Take $G$ as a classical group ($\mathrm{GL}(n, \mathbb{C})$, $\mathrm{O}(\mathbb{C}^n, B)$, or $\mathrm{Sp}(\mathbb{C}^n, \Omega)$ with $n$ even) and $V = \mathbb{C}^n$ the *defining representation* of $G$. Let

$$V^m = \underbrace{V \oplus \cdots \oplus V}_{m \text{ column vectors}}, \qquad V^{*k} = \underbrace{V^* \oplus \cdots \oplus V^*}_{k \text{ row vectors}}$$

with the natural $G$ action on each summand. Then $\mathcal{P}(V^{*k} \oplus V^m)^G$ is the algebra of $G$-invariant polynomial functions of $k$ covectors and $m$ vectors. The *First Fundamental Theorem* (FFT) of invariant theory for the group $G$ gives an explicit description of sets of basic invariants (for all values of $k$ and $m$).

There is an alternate picture that reveals the *hidden symmetries* in this situation and gives an obvious algebra of $G$-invariant polynomials together with a set of quadratic generators. Namely, we have the $G$-isomorphisms

$$\begin{array}{rcll} V^{*k} & \cong & M_{k \times n} & \text{right } G \text{ action (matrix multiplication)} \\ V^m & \cong & M_{n \times m} & \text{left } G \text{ action (matrix multiplication)} \end{array}$$

where $M_{k \times n}$ is the vector space of $k \times n$ complex matrices. In this picture we see that the reductive group $L = \mathrm{GL}(k, \mathbb{C}) \times \mathrm{GL}(m, \mathbb{C})$ acts on $M_{k \times n} \oplus M_{n \times m}$ by

$$(a, b)(x \oplus y) = ax \oplus yb^{-1} \quad \text{for } a \in \mathrm{GL}(k, \mathbb{C}) \text{ and } b \in \mathrm{GL}(m, \mathbb{C}).$$

This action obviously commutes with the $G$ action. The induced action on functions makes $\mathcal{P}(M_{k \times n} \oplus M_{n \times m})^G$ into an $L$ module. Note that the maximal torus of diagonal matrices in $L$ acts in the original picture $V^{*k} \oplus V^m$ by scalar multiplication in each vector summand, while the Weyl group $\mathfrak{S}_k \times \mathfrak{S}_m$ of $L$ acts by permutation of positions of the summands.

Define the *multiplication map*

$$\mu : M_{k \times n} \oplus M_{n \times m} \to M_{k \times m} \qquad x \oplus y \mapsto xy \qquad \text{(matrix multiplication).}$$

Obviously $\mu(xg \oplus g^{-1}y) = \mu(x \oplus y)$ for all $g \in \mathrm{GL}(n, \mathbb{C})$. Hence we have an algebra homomorphism

$$\mu^* : \mathcal{P}(M_{k\times m}) \to \mathcal{P}(V^{*k} \oplus V^m)^{\mathrm{GL}(n,\mathbb{C})}, \qquad \mu^*(f)(x \oplus y) = f(xy).$$

In particular, if we take $f = x_{ij}$ (the $(i, j)$ matrix entry function on $M_{k\times m}$), then

$$\mu^*(x_{ij})(v_1^*, \cdots, v_k^*, v_1, \cdots, v_m) = \langle v_i^*, v_j \rangle$$

(the *contraction* of the $i$th covector with the $j$th vector).

There is a natural action of $L$ on $M_{k\times m}$ with $\mathrm{GL}(k, \mathbb{C})$ acting by left multiplication and $\mathrm{GL}(m, \mathbb{C})$ acting by right multiplication, Hence $L$ acts on $\mathcal{P}(M_{k\times m})^G$. The map $\mu$ intertwines the two $L$ actions.

6.3. **Polynomial FFT for** $\mathrm{GL}(n)$**.** The FFT for $\mathrm{GL}(n, \mathbb{C})$ is the assertion that the method just indicated to construct invariants furnishes the complete algebra of polynomial invariants.

**Theorem 6.2.** *Let $G = \mathrm{GL}(n, \mathbb{C})$. Then the map $\mu^*$ is surjective. Hence the $km$ quadratic polynomials $\phi_{ij} = \mu^*(x_{ij})$ with $1 \le i \le k$ and $1 \le j \le m$ give a set of basic invariants for $\mathcal{P}(M_{k\times n} \oplus M_{n\times m})^G$.*

After discussing tensor invariants in the next lecture we shall show there how this theorem is an immediate consequence of Proposition 3.1. At this point we observe that the image of $\mu$ consists of all $k \times m$ matrices $x$ with $\mathrm{rank}(x) \le \min(k, m, n)$. This gives rise to the following dichotomy:

**(1):** If $n \ge \min(k, m)$, then $\mu$ is surjective. Hence $\mu^*$ is injective and

$$\mathcal{P}(M_{k\times n} \oplus M_{n\times m})^{\mathrm{GL}(n,\mathbb{C})} \cong \mathcal{P}(M_{k\times m})$$

is a polynomial algebra with $km$ generators. One says that $M_{k\times m}$ is the *algebraic quotient* of $V^{*k} \oplus V^m$ by $\mathrm{GL}(n, \mathbb{C})$. The representation of $L$ on $\mathcal{P}(M_{k\times m})$ is multiplicity-free (see [16, Theorem 5.2.7] or the article by Benson-Ratcliff in this volume).

**(2):** If $n < \min(k, m)$ then $\mathrm{Ker}(\mu^*) \ne 0$. The group $L$ acts on $\mathrm{Ker}(\mu^*)$, and from the multiplicity-free decomposition of $\mathcal{P}(M_{k\times m})$ under $L$ one finds that $\mathrm{Ker}(\mu^*)$ is a *determinantal ideal* generated by $(n+1)\times(n+1)$ minors. Thus $\mathcal{P}(M_{k\times n} \oplus M_{n\times m})^{\mathrm{GL}(n,\mathbb{C})}$ is the algebra of regular functions on the associated *determinantal variety.* This is the *Second Fundamental Theorem* (SFT) for $\mathrm{GL}(n, \mathbb{C})$ invariants (see [16, Theorem 5.2.15] for the complete statement).

6.4. **Polynomial FFT for the Orthogonal Group.** We next consider the full *orthogonal group* relative to the bilinear form $B(x, y) = x^t y$ on $V = \mathbb{C}^n$:

$$G = \mathrm{O}(n, \mathbb{C}) = \{g \in \mathrm{GL}(n, \mathbb{C}) \ : \ g^t g = I\}$$

Since $V \cong V^*$ as a $G$-module, via the form $B$, it suffices to consider the invariants of $k$ vector arguments $\mathcal{P}(V^k)^G = \mathcal{P}(M_{n\times k})^G$, where $G$ acts on $M_{m\times k}$ by left multiplication. Define a map

$$\tau : M_{n\times k} \to SM_k \quad (k \times k \text{ symmetric matrices}), \quad \tau(x) = x^t x.$$

For $g \in G$ we have $\tau(gx) = x^t g^t g x = \tau(x)$. Hence

$$\tau^* : \mathcal{P}(SM_k) \to \mathcal{P}(V^k)^G$$

as in the case of $\mathrm{GL}(n, \mathbb{C})$. In particular, if we take $f = x_{ij}$ (the $(i, j)$ matrix entry function on $SM_k$), then

$$\tau^*(x_{ij})(v_1, \cdots, v_k) = v_i^t v_j$$

(the inner product of the $i$th and $j$th vectors).

The map $\tau$ intertwines the right action of the *hidden symmetry group* $L = \mathrm{GL}(k, \mathbb{C})$ on $M_{n\times k}$. Here the action of $L$ on $SM_k$ is given by $x \mapsto bxb^t$ (for $b \in L$).

**Theorem 6.3.** *Let $G = \mathrm{O}(n, \mathbb{C})$. Then the map $\tau^*$ is surjective. Hence the $k(k + 1)/2$ quadratic polynomials $\theta_{ij} = \tau^*(x_{ij})$ with $1 \le i \le j \le k$ give a set of basic invariants for $\mathcal{P}(M_{n\times k})^G$.*

*Proof for the case $n \ge k$:* There is a natural $G$-equivariant embedding $M_{n\times k} \subset M_{n\times n}$; just add $n - k$ columns of zeros on the right to make $x \in M_{n\times k}$ into an $n \times n$ matrix. Hence we may assume that $k = n$. Now see [16, Proposition 4.2.6] for the proof.[e] □

[e]The proof is by induction on $n$ and can be viewed as an algebraic group version of the $QR$ factorization for $M_n$ and the *Cholesky Decomposition* for $SM_n$. This result is associated with a particular *partial compactification* of the symmetric space $\mathrm{GL}(n, \mathbb{C})/\mathrm{O}(n, \mathbb{C})$.

We shall complete the proof for the general case $n < k$ after discussing tensor invariants in the next lecture. Here we observe that the image of $\tau$ consists of all $k \times k$ symmetric matrices $x$ with $\operatorname{rank}(x) \leq \min(k, n)$. This gives rise to the following dichotomy:

**(1):** If $n \geq k$, then $\tau$ is surjective. Hence $\tau^*$ is injective and $\mathcal{P}(M_{n\times k})^G \cong \mathcal{P}(SM_k)$ is a polynomial algebra with $k(k+1)/2$ generators. One says that $SM_k$ is the *algebraic quotient* of $M_{n\times k}$ by $\mathrm{O}(n, \mathbb{C})$. The representation of $L$ on $\mathcal{P}(M_{n\times k})^G$ is multiplicity-free (see [16, Theorem 5.2.9] or the article by Benson-Ratcliff in this volume).

**(2):** If $n < k$ then $\operatorname{Ker}(\tau^*) \neq 0$. From the multiplicity-free decomposition of $\mathcal{P}(SM_k)$ under $L$ one finds that $\operatorname{Ker}(\tau^*)$ is a *determinantal ideal* generated by $(n+1) \times (n+1)$ minors. Thus $\mathcal{P}(M_{n\times k})^G$ is the algebra of functions on the associated *symmetric determinantal variety*. This is the *Second Fundamental Theorem* (SFT) for $\mathrm{O}(n, \mathbb{C})$ invariants (see [16, Theorem 5.2.17] for the complete statement).

6.5. **Polynomial FFT for the Symplectic Group.** Now consider the *symplectic group* $G = \mathrm{Sp}(\mathbb{C}^n, \Omega)$, where $n = 2p$ is even and

$$\Omega(x, y) = x^t J y, \quad J = \begin{bmatrix} 0 & I_p \\ -I_p & 0 \end{bmatrix}$$

Here $I_p$ is the $p \times p$ identity matrix. Thus $G$ is the subgroup of $\mathrm{GL}(n, \mathbb{C})$ defined by $g^t J g = J$. Since $(\mathbb{C}^n)^* \cong \mathbb{C}^n$ via the form $\Omega$, it suffices to consider the invariants of $k$ vector arguments $\mathcal{P}(V^k)^G = \mathcal{P}(M_{n\times k})$. Define a map

$$\gamma : M_{n\times k} \to AM_k \quad (k \times k \text{ skew-symmetric matrices}), \quad \gamma(x) = x^t J x.$$

For $g \in G$ we have $\gamma(gx) = x^t g^t J g x = \gamma(x)$. Hence

$$\gamma^* : \mathcal{P}(AM_k) \to \mathcal{P}(V^k)^G$$

as in the case of $\mathrm{O}(n, \mathbb{C})$. In particular, if we take $f = x_{ij}$ (the $(i, j)$ matrix entry function on $AM_k$), then

$$\gamma^*(x_{ij})(v_1, \cdots, v_k) = \Omega(v_i, v_j)$$

(contraction of the $i$th and $j$th vectors by $\Omega$).

The map $\tau$ intertwines the right action of the *hidden symmetry group* $L = \mathrm{GL}(k, \mathbb{C})$ on $M_{n\times k}$ with the action of $L$ on $AM_k$ given by $x \mapsto bxb^t$ (for $b \in L$).

**Theorem 6.4.** *Let $G = \mathrm{Sp}(\mathbb{C}^n, \Omega)$. Then the map $\gamma^*$ is surjective. Hence the $k(k-1)/2$ quadratic polynomials $\omega_{ij} = \gamma^*(x_{ij})$ with $1 \leq i < j \leq k$ give a set of basic invariants for $\mathcal{P}(M_{n\times k})^G$.*

*Proof for the case $n \geq k$:* The same citation as for the orthogonal case. $\square$

We shall complete the proof for the general case $n < k$ after discussing tensor invariants in the next lecture. Here we observe that the image of $\gamma$ consists of all $k \times k$ skew-symmetric matrices $x$ with $\operatorname{rank}(x) \leq \min(k, n)$. This gives rise to the following dichotomy:

**(1):** If $n \geq k$, then $\gamma$ is surjective. Hence $\gamma^*$ is injective and $\mathcal{P}(M_{n\times k})^G \cong \mathcal{P}(AM_k)$ is a polynomial algebra with $k(k-1)/2$ generators. One says that $AM_k$ is the *algebraic quotient* of $M_{n\times k}$ by $\mathrm{Sp}(\mathbb{C}^n, \Omega)$. The representation of $L$ on $\mathcal{P}(M_{n\times k})^G$ is multiplicity-free (see [16, Theorem 5.2.11] or the article by Benson-Ratcliff in this volume). .

**(2):** If $n < k$ then $\operatorname{Ker}(\gamma^*) \neq 0$. From the multiplicity-free decomposition of $\mathcal{P}(AM_k)$ under $L$ one finds that $\operatorname{Ker}(\gamma^*)$ is generated by a set of *Pfaffian* polynomials of degree $n/2 + 1$. Thus $\mathcal{P}(M_{n\times k})^G$ is the algebra of functions on the associated *skew-symmetric Pfaffian variety*. This is the *Second Fundamental Theorem* (SFT) for $\mathrm{O}(n, \mathbb{C})$ invariants (see [16, Theorem 5.2.18] for the complete statement).

**Summary:** *For a classical group $G$* (general linear, orthogonal, symplectic), *the $G$-invariant polynomial functions of vectors and covectors are generated by all the possible $G$-invariant contractions of vectors and covectors.*

## Lecture 7. Tensor Invariants and Proof of FFT

7.1. **Tensor Invariants for** $\mathrm{GL}(V)$**.** We turn now from consideration of invariant polynomials to the general case of invariant tensors. Let $V = \mathbb{C}^n$ and consider the mixed tensor space $V^{\otimes m} \otimes V^{*\otimes k}$ as a $\mathrm{GL}(V)$ module. For $\zeta \in \mathbb{C}^\times$ the element $\zeta I_n$ of $\mathrm{GL}(V)$ acts by $\zeta^{m-k}$ on this space. Hence there are no nonzero $\mathrm{GL}(V)$ invariant tensors if $m \neq k$ and we can assume $m = k$. In this case

$$V^{\otimes k} \otimes V^{*\otimes k} \cong \mathrm{End}(V^{\otimes k}) \tag{7.1}$$

as a $\mathrm{GL}(V)$ module, and hence

$$(V^{\otimes k} \otimes V^{*\otimes k})^{\mathrm{GL(V)}} \cong \mathrm{End}_{\mathrm{GL}(V)}(V^{\otimes k}).$$

By Schur duality (Corollary 1.6 and Proposition 3.1) we know that $\mathrm{End}_{\mathrm{GL}(V)}(V^{\otimes k})$ is spanned by the transformations $\sigma_k(s)$, $s \in \mathfrak{S}_k$.

Let $\{e_1, \ldots, e_n\}$ be the standard basis for $\mathbb{C}^n$ and let $\{e_1^*, \ldots, e_n^*\}$ be the dual basis. For an index $I = [i_1, \ldots, i_k]$ with $1 \le i_p \le n$ we set $|I| = k$ and

$$e_I = e_{i_1} \otimes \cdots \otimes e_{i_k} \in V^{\otimes k}, \quad e_I^* = e_{i_1}^* \otimes \cdots \otimes e_{i_k}^* \in V^{*\otimes k}.$$

Recall from Lecture 3 that the action of $s \in \mathfrak{S}_k$ on $k$-tensors is $\sigma_k(s)e_I = e_{s\cdot I}$. Define

$$C_s = \sum_{|I|=k} e_{s\cdot I} \otimes e_I^*$$

Then $C_s$ corresponds to $\sigma_k(s)$ under the isomorphism (7.1). Thus we obtain the *First Fundamental Theorem of Tensor Invariants* for $\mathrm{GL}(V)$:

**Theorem 7.1.** *For $k \ge 1$ one has* $\quad (V^{\otimes k} \otimes V^{*\otimes k})^{\mathrm{GL}(V)} = \mathrm{Span}\{C_s \,:\, s \in \mathfrak{S}_k\}$.

The vector space $V^{\otimes k} \otimes V^{*\otimes k}$ is self-dual as a $\mathrm{GL}(V)$ module, and hence each of the mixed tensors $C_s$ can also be viewed as a linear functional. This gives the alternate version of the Tensor FFT for $\mathrm{GL}(V)$ in terms of *total contractions* of vectors with covectors:

**Corollary 7.2.** *The space of* $\mathrm{GL}(V)$*-invariant linear forms on* $V^{\otimes k} \otimes V^{*\otimes k}$ *is spanned by the contractions*

$$v_1 \otimes \cdots \otimes v_k \otimes v_1^* \otimes \cdots \otimes v_k^* \mapsto \prod_{j=1}^{k} \langle v_{s(j)}^*, \, v_j \rangle$$

*for $s \in \mathfrak{S}_k$.*

7.2. **Proof of Polynomial FFT for** $\mathrm{GL}(V)$**.** The polynomial form of the FFT for $\mathrm{GL}(V)$ is a consequence of Corollary 7.2. To prove this, we need to view $\mathrm{GL}(V)$-invariant polynomials as tensors with additional symmetries, as we did in Section 5.3. Let $\mathbb{T}_{k,m} = (\mathbb{C}^\times)^k \times (\mathbb{C}^\times)^m$ and write $t \in \mathbb{T}_{k,m}$ as $t = [x_1, \ldots x_k, y_1, \ldots, y_m]$. Denote the regular characters of $\mathbb{T}_{k,m}$ as

$$t \mapsto t^{[\mathbf{p},\mathbf{q}]} = \prod_{i=1}^{k} x_i^{p_i} \prod_{j=1}^{m} y_j^{q_j}$$

for $[\mathbf{p}, \mathbf{q}] \in \mathbb{Z}^k \times \mathbb{Z}^m$. Let $\mathbb{T}_{k,m}$ act on $z = [v_1, \ldots, v_k] \oplus [v_1^*, \ldots, v_m^*] \in V^k \oplus V^{*m}$ by

$$t \cdot z = [x_1 v_1, \ldots, x_k v_k] \oplus [y_1 v_1^*, \ldots, y_m v_m^*].$$

This action commutes with the $\mathrm{GL}(V)$-action on $V^k \oplus V^{*m}$, so $\mathrm{GL}(V)$ leaves invariant the weight spaces of $\mathbb{T}_{k,m}$ in $\mathcal{P}(V^k \oplus V^{*m})$. These weight spaces are described by the degrees of homogeneity of $f \in \mathcal{P}(V^k \oplus V^{*m})$ in $v_i$ and $v_j^*$ as follows. For $\mathbf{p} \in \mathbb{N}^k$ and $\mathbf{q} \in \mathbb{N}^m$ set

$$\mathcal{P}^{[\mathbf{p},\mathbf{q}]}(V^k \oplus V^{*m}) = \{f \in \mathcal{P}(V^k \oplus V^{*m}) \,:\, f(t \cdot z) = t^{[\mathbf{p},\mathbf{q}]} f(z)\}.$$

Then

$$\mathcal{P}(V^k \oplus V^{*m}) = \bigoplus_{\mathbf{p}\in\mathbb{N}^k} \bigoplus_{\mathbf{q}\in\mathbb{N}^m} \mathcal{P}^{[\mathbf{p},\mathbf{q}]}(V^k \oplus V^{*m}),$$

and this decomposition is $\mathrm{GL}(V)$-invariant. Thus

$$\mathcal{P}(V^k \oplus V^{*m})^G = \bigoplus_{\mathbf{p}\in\mathbb{N}^k} \bigoplus_{\mathbf{q}\in\mathbb{N}^m} \left[\mathcal{P}^{[\mathbf{p},\mathbf{q}]}(V^k \oplus V^{*m})\right]^{\mathrm{GL}(V)}. \tag{7.2}$$

We now give another realization of these weight spaces in terms of tensors. Given $\mathbf{p} \in \mathbb{N}^k$ and $\mathbf{q} \in \mathbb{N}^m$ we set

$$V^{*\otimes\mathbf{p}} \otimes V^{\otimes\mathbf{q}} = V^{*\otimes p_1} \otimes \cdots \otimes V^{*\otimes p_k} \otimes V^{\otimes q_1} \otimes \cdots \otimes V^{\otimes q_m}.$$

This space is isomorphic to $V^{*\otimes|\mathbf{p}|} \otimes V^{*\otimes|\mathbf{q}|}$ and is a $\mathrm{GL}(V)$ module with the usual action. Let $\mathfrak{S}_\mathbf{p} = \mathfrak{S}_{p_1} \times \cdots \times \mathfrak{S}_{p_k}$, with each factor acting as a group of permutations of the corresponding tensor factor in $V^{\otimes\mathbf{p}}$. This gives a representation of $\mathfrak{S}_\mathbf{p} \times \mathfrak{S}_\mathbf{q}$ on $V^{*\otimes\mathbf{p}} \otimes V^{\otimes\mathbf{q}}$ that commutes with the action of $\mathrm{GL}(V)$.

**Lemma 7.3.** *Let $\mathbf{p} \in \mathbb{N}^k$ and $\mathbf{q} \in \mathbb{N}^m$. There is a linear isomorphism*

$$\mathcal{P}^{[\mathbf{p},\mathbf{q}]}(V^k \oplus V^{*m})^{\mathrm{GL}(V)} \cong \left[\left(V^{*\otimes|\mathbf{p}|} \otimes V^{\otimes|\mathbf{q}|}\right)^{\mathrm{GL}(V)}\right]^{\mathfrak{S}_\mathbf{p} \times \mathfrak{S}_\mathbf{q}}. \tag{7.3}$$

We now obtain the polynomial version of the First Fundamental Theorem for $\mathrm{GL}(V)$ from the tensor version. This theorem asserts that for each $\mathbf{p} \in \mathbb{N}^k$ and $\mathbf{q} \in \mathbb{N}^m$, the space

$$\mathcal{P}^{[\mathbf{p},\mathbf{q}]}(V^k \oplus V^{*m})^{\mathrm{GL}(V)}$$

is spanned by monomials of the form

$$\prod_{i=1}^{k}\prod_{j=1}^{m}\langle v_i, v_j^*\rangle^{r_{ij}} \tag{7.4}$$

for suitable choices of $k, m$ and $r_{ij}$. The subgroup $\mathbb{T} = \{\zeta I_V \,:\, \zeta \in \mathbb{C}^\times\}$ of $\mathrm{GL}(V)$ acts on $\mathcal{P}^{[\mathbf{p},\mathbf{q}]}(V^k \oplus V^{*m})$ by the character $\zeta \mapsto \zeta^{|\mathbf{q}|-|\mathbf{p}|}$, so we may assume that $|\mathbf{p}| = |\mathbf{q}| = r$, say. By Lemma 7.3,

$$\mathcal{P}^{[\mathbf{p},\mathbf{q}]}(V^k \oplus V^{*m})^{\mathrm{GL}(V)} \cong \left[\left(V^{*\otimes r} \otimes V^{\otimes r}\right)^{\mathrm{GL}(V)}\right]^{\mathfrak{S}_\mathbf{p} \times \mathfrak{S}_\mathbf{q}}. \tag{7.5}$$

From Theorem 7.1 we know that the space $(V^{*\otimes r} \otimes V^{\otimes r})^{\mathrm{GL}(V)}$ is spanned by the complete contractions $C_s$ for $s \in \mathfrak{S}_r$. Hence the right side of (7.5) is spanned by the tensors

$$\sum_{(g,h)\in\mathfrak{S}_\mathbf{p}\times\mathfrak{S}_\mathbf{q}} \sigma_r^*(g) \otimes \sigma_r(h) C_s$$

for $s \in \mathfrak{S}_r$. Under the isomorphism (7.5), the action of $\mathfrak{S}_\mathbf{p} \times \mathfrak{S}_\mathbf{q}$ disappears and these tensors correspond to the polynomials

$$\begin{aligned}
F_s(v_1, \ldots, v_k, v_1^*, \ldots, v_m^*) &= C_s(v_1^{\otimes p_1} \otimes \cdots \otimes v_k^{\otimes p_k} \otimes v_1^{*\otimes q_1} \otimes \cdots \otimes v_m^{*\otimes q_m}) \\
&= \langle v_1^{\otimes p_1} \otimes \cdots \otimes v_k^{\otimes p_k}, w_1^* \otimes \cdots \otimes w_r^*\rangle \\
&= \prod_{i=1}^{n}\langle w_i, w_i^*\rangle,
\end{aligned}$$

where each $w_i$ is $v_j$ for some $j$ and each $w_i^*$ is $v_{j'}^*$ for some $j'$ (depending on $s$). Obviously $F_s$ is of the form (7.4).

7.3. **Tensor Invariants for Orthogonal and Symplectic Groups.** Consider now a nondegenerate bilinear form $\omega$ on $V$, which we assume to be either symmetric or skew-symmetric. Let $G$ be the subgroup of $\mathrm{GL}(V)$ that preserves $\omega$ (so $G$ is either the full orthogonal group or the symplectic group). Any mixed tensor that is invariant under $\mathrm{GL}(V)$ is also invariant under $G$, of course. To find additional tensor invariants, we can use the $G$-module isomorphism $V \cong V^*$ furnished by $\omega$ to restrict attention to $V^{*\otimes k}$. Furthermore, $(V^{*\otimes k})^G = 0$ if $k$ is odd, since $-I \in G$. Hence we need only find a linear basis for $(V^{*\otimes 2k})^G$.

The given form $\omega \in (V^{*\otimes 2})^G$ by definition. Since $G$ preserves tensor multiplication, it follows that

$$\theta_k = \omega^{\otimes k} \in (V^{*\otimes 2k})^G.$$

The representation $\sigma_{2k}$ of $\mathfrak{S}_{2k}$ on $V^{*\otimes 2k}$ commutes with the action of $G$, of course, so the tensors $\sigma_{2k}(s)\theta_k$ are also $G$ invariant, for every $s \in \mathfrak{S}_{2k}$.

**Theorem 7.4.** *For all integers $k \geq 1$ one has $(V^{*\otimes 2k})^G = \mathrm{Span}\{\sigma_{2k}(s)\theta_k \,:\, s \in \mathfrak{S}_{2k}\}$.*

Because of the symmetries of the tensor $\theta_k$ under the action of $\mathfrak{S}_{2k}$, there are redundancies in the spanning set of Theorem 7.4. A labeling that factors out these symmetries is the following, which we will also use in Lecture 13. Define a *two-partition* of the set $\{1, \ldots, 2k\}$ to be any set of $k$ pairs $\xi = \{\{i_1, j_1\}, \ldots, \{i_k, j_k\}\}$ such that $\{i_1, j_1, \ldots, i_k, j_k\}$ consists of the integers $1, \ldots, 2k$. Denote the set of all 2-partitions of $k$ by $\Xi_k$. For $\xi \in \Xi_k$ define the *complete contraction*

$$\lambda_\xi(v_1 \otimes \cdots \otimes v_{2k}) = \prod_{p=1}^{k} \omega(v_{i_p}, v_{j_p}).$$

(We label the pairs in $\xi$ so that $i_p < j_p$; then this definition has no sign ambiguity, even when $\omega$ is skew-symmetric.) The invariant tensors in Theorem 7.4 are just these contractions.

**Corollary 7.5.** *For all integers $k \geq 1$ one has* $(V^{*\otimes 2k})^G = \mathrm{Span}\{\lambda_\xi : \xi \in \Xi_k\}$.

*Proof of Theorem 7.4.* Following Attiyah-Bott-Patodi, we shift the action of $G$ from $V^{*\otimes 2k}$ to $\mathrm{End}V$ by a tensor algebra version of the classical *polarization operators*.[f] This transforms the space of $G$-invariant tensors into a different space of $\mathrm{GL}(V)$-invariant tensors built from $G$-invariant polynomials on $\mathrm{End}(V)$. The results of Lecture 6 allow us to express these $G$-invariant polynomials as covariant tensors with no further $G$-invariance condition. By this means each $G$-invariant tensor gives rise to a unique mixed $\mathrm{GL}(V)$-invariant tensor. But we know that all such tensors are linear combinations of complete vector-covector contractions. Finally, specializing the polarization variables, we find that the original $G$-invariant tensor is in the span of the complete contractions relative to the form $\omega$.

In more detail, given $\lambda \in V^{*\otimes m}$ we define $\Phi_\lambda \in \mathcal{P}^k(\mathrm{End}V) \otimes V^{*\otimes m}$ by

$$\Phi_\lambda(X, w) = \langle \lambda,\, X^{\otimes m} w \rangle \quad \text{for } X \in \mathrm{End}V \text{ and } w \in V^{\otimes m}.$$

Since $\langle \lambda, w \rangle = \Phi_\lambda(I, w)$, we see that the map $\lambda \mapsto \Phi_\lambda$ is injective.

Let $G \subset \mathrm{GL}(V)$ be any subgroup, for the moment. Let $G$ act on $\mathcal{P}^m(\mathrm{End}V) \otimes V^{*\otimes m}$ by *left multiplication* on $\mathrm{End}(V)$ only (no action on $V^{*\otimes m}$). Let $\mathrm{GL}(V)$ act by *right multiplication* on the $\mathrm{End}V$ factor and in the usual way on $V^{*\otimes m}$. Since these actions of $G$ and $\mathrm{GL}(V)$ mutually commute, we obtain a representation of the product group $G \times \mathrm{GL}(V)$. In particular,

$$(g, h) \cdot \Phi_\lambda(X, w) = \langle \lambda,\, (g^{-1}Xh)^{\otimes m} h^{-1} \cdot w \rangle = \Phi_{g \cdot \lambda}(X, w)$$

for $g \in G, h \in \mathrm{GL}(V)$. Hence $\Phi_\lambda$ is automatically invariant under $\mathrm{GL}(V)$ for any $\lambda$, while if $\lambda$ is $G$-invariant, then so is $\Phi_\lambda$. Conversely, if $\Phi \in \mathcal{P}^m(\mathrm{End}V) \otimes V^{*\otimes m}$ is invariant under $G \times \mathrm{GL}(V)$, then $\Phi$ is determined by the linear functional $\lambda : w \mapsto \Phi(I, w)$ since $\Phi(h, w) = \Phi(I, h \cdot w)$ for $h \in \mathrm{GL}(V)$ and $\mathrm{GL}(V)$ is dense in $\mathrm{End}V$. Furthermore, for $g \in G$ we have

$$\Phi(I, w) = \Phi(gI, w) = \Phi(Ig, w) = \Phi(I, g \cdot w)$$

(here we have used the inclusion $G \subset \mathrm{GL}(V)$ to pass from the left action of $G$ on $\mathrm{End}V$ to the right action of $\mathrm{GL}(V)$ on $\mathrm{End}V$). Hence $\Phi = \Phi_\lambda$ with $\lambda \in (V^{*\otimes m})^G$. The map $\lambda \mapsto \Phi_\lambda$ thus gives a linear isomorphism

$$(V^{*\otimes m})^G \cong \left[\mathcal{P}^m(\mathrm{End}V)^{L(G)} \otimes V^{*\otimes m}\right]^{\mathrm{GL}(V)} \tag{7.6}$$

where $L(G)$ denotes the left-multiplication action of $G$ on $\mathrm{End}(V)$.

Let $\lambda \in \left[V^{*\otimes 2k}\right]^G$. Then by (7.6) with $m = 2k$, we have

$$\Phi_\lambda \in \left[\mathcal{P}^{2k}(\mathrm{End}V)^G \otimes V^{*\otimes 2k}\right]^{\mathrm{GL}(V)}.$$

and $\lambda = \Phi_\lambda(I)$. By Theorems 6.3 and 6.4 (which we have proved in the case $k = \dim V$), there is a polynomial $F_\lambda$ on $SM_n \times V^{\otimes 2k}$ when $G = \mathrm{O}(V)$ or on $AM_n \times V^{\otimes 2k}$ when $G = \mathrm{Sp}(V)$, so that for all $X \in M_n$ and $w \in V^{\otimes 2k}$,

$$\Phi_\lambda(X, w) = \begin{cases} F_\lambda(X^t X,\, w) & \text{when } G = \mathrm{O}(V), \\ F_\lambda(X^t J_n X,\, w) & \text{when } G = \mathrm{Sp}(V). \end{cases}$$

[f]This can be viewed as *unseparation of variables*, and is another instance of a hidden symmetry.

We view $F_\lambda$ as an element of $\mathcal{P}^k(SM_n) \otimes V^{*\otimes 2k}$ (*resp.* of $\mathcal{P}^k(AM_n) \otimes V^{*\otimes 2k}$). Note that

$$\langle \lambda, w\rangle = \Phi_\lambda(I_n, w) = \begin{cases} F_\lambda(I_n,\, w) & \text{when } G = \mathrm{O}(V), \\ F_\lambda(J_n,\, w) & \text{when } G = \mathrm{Sp}(V). \end{cases}$$

The next step is to translate the $\mathrm{GL}(V)$-invariance of $\Phi_\lambda$ into an appropriate invariance property of $F_\lambda$. The map

$$\Theta : M_n \to V^* \otimes V^*, \quad \Theta(x) = \sum_{i,j} x_{ij}\, e_i^* \otimes e_j^*$$

furnishes $\mathrm{GL}(V)$-module isomorphisms $AM_n \cong \bigwedge^2 V^*$ and $SM_n \cong S^2V^*$. Hence

$$\mathcal{P}^k(AM_n) \cong S^k(\textstyle\bigwedge^2 V), \qquad \mathcal{P}^k(SM_n) \cong S^k(S^2V)$$

Thus there is a $\mathrm{GL}(V)$-invariant tensor $C \in V^{\otimes 2k} \otimes V^{*\otimes 2k}$ so that

$$F_\lambda(A, w) = \langle A^{\otimes k} \otimes w,\, C\rangle$$

for $w \in V^{\otimes 2k}$ and $A$ in either $S^2V^*$ or $\bigwedge^2 V^*$. By the tensor FFT for $\mathrm{GL}(V)$ (tensor form) we may assume that $C$ is a *complete contraction* :

$$C = \sum_{|I|=2k} e^*_{s\cdot I} \otimes e_I$$

for some $s \in \mathfrak{S}_{2k}$. When $G = \mathrm{O}(V)$ we take $A = I_n$ to recover the original $G$-invariant tensor $\lambda$ as

$$\begin{aligned} \langle \lambda, w\rangle &= F_\lambda(I_n, w) = \langle \Theta(I_n)^{\otimes k} \otimes w,\, C\rangle \\ &= \sum_{|I|=2k} \langle e_I,\, \Theta(I_n)^{\otimes k}\rangle \langle w,\, e^*_{s\cdot I}\rangle = \langle \sigma_{2k}(s)\Theta(I_n)^{\otimes k},\, w\rangle. \end{aligned}$$

When $G = \mathrm{Sp}(V)$, we likewise take $A = J_n$ to get $\langle \lambda, w\rangle = \langle \sigma_{2k}(s)\Theta(J_n)^{\otimes k},\, w\rangle$. Since

$$\theta_k^* = \begin{cases} \Theta(I_n)^{\otimes k} & \text{when } G = \mathrm{O}(V) \\ \Theta(J_n)^{\otimes k} & \text{when } G = \mathrm{Sp}(V), \end{cases}$$

we conclude that $\lambda = \sigma_{2k}(s)\theta_k^*$.

7.4. **Proof of Polynomial FFT for Orthogonal and Symplectic Groups.** We finally complete the proof of the FFT for the action of $G = \mathrm{O}(V)$ or $G = \mathrm{Sp}(V)$ on $\mathcal{P}(V)$, using an argument similar to the case of $\mathrm{GL}(V)$ to deduce the polynomial version of the FFT from the tensor version. Let $\mathbf{p} \in \mathbb{N}^m$. Since $-I \in G$ and acts by $(-1)^{|\mathbf{p}|}$ on $\mathcal{P}^{[\mathbf{p}]}(V^m)$, we may assume that $|\mathbf{p}| = 2k$. We now show that the space $\mathcal{P}^{[\mathbf{p}]}(V^m)^G$ is spanned by monomials

$$\varphi(v_1, \ldots, v_m) = \prod_{i,j=1}^{m} \omega(v_i, v_j)^{r_{ij}} \tag{7.7}$$

of weight $\mathbf{p}$. This will prove the FFT (polynomial version) for $G$.

By Lemma 7.3,

$$\mathcal{P}^{[\mathbf{p}]}(V^m)^G \cong \left[\left(V^{*\otimes 2k}\right)^G\right]^{\mathfrak{S}_\mathbf{p}}. \tag{7.8}$$

The space $(V^{*\otimes 2k})^G$ is spanned by the tensors $\sigma^*_{2k}(s)\theta^*_k$ for $s \in \mathfrak{S}_{2k}$ (Theorem 7.4). Hence the right side of (7.8) is spanned by the tensors

$$\sum_{t \in \mathfrak{S}_\mathbf{p}} \sigma^*_{2k}(ts)\theta^*_k$$

for $s \in \mathfrak{S}_{2k}$. Under the isomorphism (7.8), the action of $\mathfrak{S}_\mathbf{p}$ disappears and these tensors correspond to the polynomials

$$F_s(v_1, \ldots, v_m) = \sigma^*_{2k}(s)\theta^*_k(v_1^{\otimes p_1} \otimes \cdots \otimes v_m^{\otimes p_m}) = \prod_{i=1}^{k} \omega(u_i, u_{k+i}),$$

where each $u_i$ is $v_j$ for some $j$ (depending on $s$). Thus $F_s$ is of the form (7.7).

## Lecture 8. Weyl Algebra and Howe Duality

8.1. **Duality in the Weyl Algebra.** We shall now apply the general duality theorem from Lecture 1 to the following situation. Let $V$ be an $n$-dimensional vector space over $\mathbb{C}$ and let $x_1, \dots, x_n$ be coordinates on $V$ relative to a basis $\{e_1, \dots, e_n\}$. Let $\xi_1, \dots, \xi_n$ be the coordinates for $V^*$ relative to the dual basis $\{e_1^*, \dots, e_n^*\}$. We denote by $\mathcal{PD}(V)$ the algebra of *polynomial coefficient differential operators* on $V$. This is the subalgebra of $\mathrm{End}(\mathcal{P}(V))$ generated (as an associative algebra) by the operators

$$D_i = \frac{\partial}{\partial x_i}, \quad M_i = \text{multiplication by } x_i \quad (i = 1, \dots, n).$$

Since $(\partial/\partial x_i)(x_j f) = (\partial x_j/\partial x_i) f + x_j(\partial f/\partial x_i)$ for $f \in \mathcal{P}(V)$, these operators satisfy the *Heisenberg commutation relations*

$$[D_i,\, M_j] = \delta_{ij} I \quad \text{for } i, j = 1, \dots, n. \tag{8.1}$$

(the algebra $\mathcal{PD}(V)$ is often called the *Weyl algebra*).

Define $\mathcal{PD}_0(V) = \mathbb{C}I$ and for $k \geq 1$ let $\mathcal{PD}_k(V)$ be the linear span of all products of $k$ or fewer operators from the generating set $\{D_1, \dots, D_n, M_1, \dots, M_n\}$. This defines an increasing filtration of the algebra $\mathcal{PD}(V)$:

$$\mathcal{PD}_0(V) \subset \cdots \subset \mathcal{PD}_k(V) \subset \mathcal{PD}_{k+1}(V) \subset \cdots \quad \text{with} \bigcup_{k \geq 0} \mathcal{PD}_k(V) = \mathcal{PD}(V)$$

$$\mathcal{PD}_k(V) \cdot \mathcal{PD}_m(V) \subset \mathcal{PD}_{k+m}(V).$$

Let $\mathrm{Gr}(\mathcal{PD}(V)) = \bigoplus_{k \geq 0} \mathrm{Gr}^k(\mathcal{PD}(V))$ be the associated graded algebra. If $T \in \mathcal{PD}(V)$ then we say $T$ has *filtration degree* $k$ if $T \in \mathcal{PD}_k(V)$ but $T \notin \mathcal{PD}_{k-1}(V)$, and we write $\deg(T) = k$. We write

$$\mathrm{Gr}(T) = T + \mathcal{PD}_{k-1}(V) \in \mathrm{Gr}^k(\mathcal{PD}(V))$$

when $\deg(T) = k$. The map $T \mapsto \mathrm{Gr}(T)$ is a linear isomorphism (but not an algebra homomorphism) from $\mathcal{PD}(V)$ to $\mathrm{Gr}(\mathcal{PD}(V))$. From (8.1) it is easily verified that

$$\deg(M^\alpha D^\beta) = |\alpha| + |\beta| \quad \text{for } \alpha, \beta \in \mathbb{N}^n$$

and the set of operators $\{M^\alpha D^\beta : \alpha, \beta \in \mathbb{N}^n\}$ is a (vector-space) basis for $\mathcal{PD}(V)$, where we write

$$M^\alpha = M_1^{\alpha_1} \cdots M_n^{\alpha_n}, \qquad D^\beta = D_1^{\beta_1} \cdots D_n^{\beta_n}.$$

Let $\rho$ be the representation of $\mathrm{GL}(V)$ on $\mathcal{P}(V)$ with

$$\rho(g) f(x) = f(g^{-1}x) \quad \text{for } f \in \mathcal{P}(V).$$

We view $\mathcal{PD}(V)$ as a $\mathrm{GL}(V)$-module relative to the action

$$g \cdot T = \rho(g) T \rho(g^{-1}) \quad \text{for } T \in \mathcal{PD}(V),\ g \in \mathrm{GL}(V).$$

For $g \in \mathrm{GL}(V)$ with matrix $[g_{ij}]$ relative to the basis $\{e_1, \dots, e_n\}$, we calculate that

$$\rho(g) D_j \rho(g^{-1}) = \sum_{i=1}^n g_{ij} D_i, \qquad \rho(g) M_i \rho(g^{-1}) = \sum_{j=1}^n g_{ij} M_j. \tag{8.2}$$

The set $\{\mathrm{Gr}(M^\alpha D^\beta) : |\alpha| + |\beta| = k\}$ is a basis for $\mathrm{Gr}^k(\mathcal{PD}(V))$. Thus the nonzero operators of filtration degree $k$ are those of the form

$$T = \sum_{|\alpha|+|\beta| \leq k} c_{\alpha\beta} M^\alpha D^\beta \tag{8.3}$$

with $c_{\alpha\beta} \neq 0$ for some pair $\alpha, \beta$ with $|\alpha| + |\beta| = k$ (note that the filtration degree of $T$ is generally larger than the order of $T$ as a differential operator). If $T$ in (8.3) has filtration degree $k$ then we define the *symbol* of $T$ to be the polynomial $\sigma(T) \in \mathcal{P}^k(V \oplus V^*)$ given by

$$\sigma(T) = \sum_{|\alpha|+|\beta|=k} c_{\alpha\beta} x^\alpha \xi^\beta.$$

**Lemma 8.1.** *The symbol map gives a linear isomorphism* $\mathcal{PD}(V) \cong \mathcal{P}(V \oplus V^*)$ *as* $\mathrm{GL}(V)$*-modules.*

We can now obtain the general *Weyl algebra duality theorem:*

**Theorem 8.2.** *Let $G$ be a reductive algebraic group acting regularly on $V$. Then there is a multiplicity-free decomposition*

$$\mathcal{P}(V) \cong \bigoplus_{\lambda \in \Sigma(V)} E^{\lambda} \otimes F^{\lambda}, \tag{8.4}$$

*as a module under the joint actions of $\mathcal{PD}(V)^G$ and $\mathbb{C}[G]$. Here $\Sigma(V) \subset \widehat{G}$, $F^{\lambda}$ is an irreducible regular $G$-module of type $\lambda$, and $E^{\lambda}$ is an irreducible module for $\mathcal{PD}(V)^G$ that uniquely determines $\lambda$.*

To use Theorem 8.2 effectively for a particular $G$-module $V$ we need a more explicit description of the algebra $\mathcal{PD}(V)^G$. The following result is a first step in that direction.

**Theorem 8.3.** *Let $\{\psi_1, \ldots, \psi_r\}$ generate the algebra $\mathcal{P}(V \oplus V^*)^G$ . Suppose $T_j \in \mathcal{PD}(V)^G$ are such that $\sigma(T_j) = \psi_j$ for $j = 1, \ldots, r$. Then $\{T_1, \ldots, T_r\}$ generates the algebra $\mathcal{PD}(V)^G$.*

**Corollary 8.4.** (Notation as in Theorem 8.3) *Suppose $T_1, \ldots T_r$ can be chosen so that*

$$\mathfrak{g}' = \operatorname{Span}\{T_1, \ldots, T_r\}$$

*is a Lie subalgebra of $\mathcal{PD}(V)^G$. Then in the canonical decomposition* (8.4) *the spaces $E^{\lambda}$ are irreducible modules for the Lie algebra $\mathfrak{g}'$, and $\lambda$ is uniquely determined by the equivalence class of $E^{\lambda}$ as a $\mathfrak{g}'$-module. Hence there is a bijection (duality correspondence)*

$$\Sigma(V) \leftrightarrow \Lambda(V),$$

*where $\Lambda(V)$ is the set of irreducible representations of $\mathfrak{g}'$ that occur in $\mathcal{P}(V)$.*

*Remark.* Theorem 8.2 is also valid when $V$ is any smooth connected affine $G$-variety. Here we take $\mathcal{R} = \mathbb{D}(V)$ to be the ring of *algebraic differential operators* on $V$ and use Theorem 1.4.[g] The algebra $\mathbb{D}(V)^G$ in this case ($G$ connected, reductive), has been studied by Knop [26]. He proves that its center $\mathfrak{Z}_G(V)$ is a polynomial ring in $\operatorname{rank}_G(V)$ generators, where $\operatorname{rank}_G(V) = \dim Bx - \dim Nx$ for a generic point $x \in V$ (here $B$ is a Borel subgroup of $G$ with nilradical $N$). Furthermore, $\mathbb{D}(V)^G$ is a free module over $\mathfrak{Z}_G(V)$ (this is a generalization of results of Kostant [27] for the case $V = G$, with $G$ acting by left multiplication). The representation theory of $\mathbb{D}(V)^G$ seems to be unknown, in general, although special cases have been studied by I. Agricola, F. Knop, T. Levasseur, G. Schwarz, J. Stafford, and others.

8.2. **Howe Duality for Orthogonal/Symplectic Groups.** We now determine the structure of $\mathcal{PD}(V)^G$ when $G$ is an orthogonal or symplectic group and $V$ is the sum of $n$ copies of the fundamental representation of $G$. Using the First Fundamental Theorem of classical invariant theory, we will show that the assumptions of Corollary 8.4 are satisfied. This will give the *Howe duality* between the (finite-dimensional) regular representations of $G$ occurring in $\mathcal{P}(V)$ and a set of irreducible representations of the *dual* Lie algebra $\mathfrak{g}'$.

Let $\omega$ be a nondegenerate bilinear form on $\mathbb{C}^k$ that is either symmetric or skew symmetric, and let $G \subset \mathrm{GL}(k, \mathbb{C})$ be the isometry group of $\omega$. Thus $G$ is the (complex) orthogonal group when $\omega$ is symmetric, and $G$ is the (complex) symplectic group when $\omega$ is skew (and $k$ even). Let $V = (\mathbb{C}^k)^n$. Then

$$\mathcal{P}(V \oplus V^*) = \mathcal{P}(\underbrace{\mathbb{C}^k \oplus \cdots \oplus \mathbb{C}^k}_{n \text{ copies}} \oplus \underbrace{(\mathbb{C}^k)^* \oplus \cdots \oplus (\mathbb{C}^k)^*}_{n \text{ copies}}).$$

Hence if $T \in \mathcal{PD}(V)$ then the symbol of $T$ is a polynomial function

$$f(\mathbf{x}_1, \ldots, \mathbf{x}_n, \xi_1, \ldots, \xi_n), \quad \text{where } \mathbf{x}_i \in \mathbb{C}^k, \quad \xi_j \in (\mathbb{C}^k)^*.$$

From Lecture 6 we know that the algebra of $G$-invariant polynomials $P(V \oplus V^*)^G$ is generated by three types of quadratic polynomials:

| | |
|---|---|
| evaluation of $\omega$ on two vectors: | $r_{ij}(\mathbf{x}_1, \ldots, \mathbf{x}_n, \xi_1, \ldots, \xi_n) = \omega(\mathbf{x}_i, \mathbf{x}_j)$ |
| evaluation of $\omega^*$ on two covectors: | $\rho_{ij}(\mathbf{x}_1, \ldots, \mathbf{x}_n, \xi_1, \ldots, \xi_n) = \omega^*(\xi_i, \xi_j)$ |
| contraction of vector-covector: | $c_{ij}(\mathbf{x}_1, \ldots, \mathbf{x}_n, \xi_1, \ldots, \xi_n) = \langle \mathbf{x}_i, \xi_j \rangle$ |

where $1 \leq i, j \leq n$ and $\omega^*$ is the form on $(\mathbb{C}^k)^*$ dual to $\omega$. There is a canonical $\mathrm{GL}(V)$-module isomorphism $\partial$ from $\mathcal{P}(V^*) \cong S(V)$ to the algebra of constant-coefficient differential operators on $V$. The linear span

[g] See [1]; the smoothness assumption on $V$ is essential here, since the action of $\mathbb{D}(V)$ on $\mathrm{Aff}(V)$ can fail to be irreducible when $V$ is not smooth.

of the quadratic invariant polynomials above furnish symbols for the following Lie algebras of $G$-invariant differential operators:

$$\begin{aligned} \mathfrak{p}_- &= \mathrm{Span}\{\text{ multiplication by } r_{ij} \;:\; 1 \le i,j \le n\} \\ \mathfrak{p}_+ &= \mathrm{Span}\{\text{ differentiation by } \Delta_{ij} = \partial(\rho_{ij}) \;:\; 1 \le i,j \le n\} \\ \mathfrak{k} &= \mathrm{Span}\{E_{ij} + \tfrac{k}{2}\delta_{ij} \;:\; 1 \le i,j \le n\} \end{aligned}$$

Here it is convenient to identify $V$ with $M_{n\times k}$, with $G$ acting by right multiplication. If $\mathbf{x}_i$ denotes the $i$th row of $x \in M_{n\times k}$, then

$$E_{ij} = \mathbf{x}_i \cdot \nabla_{\mathbf{x}_j} = \sum_{r=1}^{k} x_{ir} \frac{\partial}{\partial x_{jr}}.$$

The operators $E_{ij}$, which correspond to vector-covector contractions, commute with the right action of all of $\mathrm{GL}(k,\mathbb{C})$ ($E_{ij}$ is the classical *polarization operator*). Obviously $[\mathfrak{p}_-,\mathfrak{p}_-] = 0$ and $[\mathfrak{p}_+,\mathfrak{p}_+] = 0$. An easy calculation shows that

$$[\mathfrak{k},\mathfrak{p}_\pm] = \mathfrak{p}_\pm, \quad [\mathfrak{p}_-,\mathfrak{p}_+] = \mathfrak{k}.$$

The choice of shift $\frac{n}{2}\delta_{ij}$ for the operators in $\mathfrak{k}$ arises from the last commutation relation.

**Theorem 8.5.** *Set $\mathfrak{g}' = \mathfrak{p}_- + \mathfrak{k} + \mathfrak{p}_+$. Then $\mathfrak{g}'$ is a Lie algebra and it generates the associative algebra $\mathcal{PD}(M_{n\times k})^G$. Furthermore,*

$$\mathfrak{g}' \cong \begin{cases} \mathfrak{sp}(n,\mathbb{C}) & \text{when } \omega \text{ is symmetric} \\ \mathfrak{so}(2n,\mathbb{C}) & \text{when } \omega \text{ is skew} \end{cases}$$

*The subalgebra $\mathfrak{k} \cong \mathfrak{gl}(n,\mathbb{C})$ acts on $\mathcal{P}(M_{n\times k})$ by the differential of the representation*

$$\rho(g)f(x) = (\det g)^{-k/2} f(g^{-1}x)$$

*of $K = \mathrm{GL}(n,\mathbb{C})$ (replace $K$ by its two-fold cover if $k$ is odd).*

We call $\mathfrak{g}'$ the *Howe dual* of $\mathfrak{g} = \mathrm{Lie}(G)$ associated to the representation of $\mathfrak{g}$ on V. Notice that the correspondence between $\mathfrak{g}$ and $\mathfrak{g}'$ interchanges orthogonal and symplectic Lie algebras. There is an asymmetry between $\mathfrak{g}$ and $\mathfrak{g}'$, however. The action of $\mathfrak{g}$ on $\mathcal{P}(V)$ is by vector fields (corresponding to the representation of $G$ on $V$), whereas the subalgebras $\mathfrak{p}_\pm$ of $\mathfrak{g}'$ act by second-order differential operators and multiplication by quadratic polynomials, which do not come from a geometric action on $V$. We will show in Lecture 11 how to *exponentiate* the action of a real form of $\mathfrak{g}'$ on $\mathcal{P}(V)$ to a unitary representation of an associated real Lie group on a Hilbert-space completion of $\mathcal{P}(V)$.

### 8.3. **Howe Duality for** $\mathrm{GL}(k)$.

Now consider $G = \mathrm{GL}(k,\mathbb{C})$ acting on

$$V = \underbrace{\mathbb{C}^k \oplus \cdots \oplus \mathbb{C}^k}_{p \text{ copies}} \oplus \underbrace{(\mathbb{C}^k)^* \oplus \cdots \oplus (\mathbb{C}^k)^*}_{q \text{ copies}}$$

In this case the symbol of $T \in \mathcal{PD}(V)$ is a polynomial function

$$f(\mathbf{x}_1,\ldots,\mathbf{x}_p,\eta_1,\ldots,\eta_q,\xi_1,\ldots,\xi_p,\mathbf{y}_1,\ldots,\mathbf{y}_q),$$

where $[\mathbf{x}_1,\ldots,\mathbf{x}_p,\eta_1,\ldots\eta_q] \in V$ and $[\xi_1,\ldots,\xi_p,\mathbf{y}_1,\ldots\mathbf{y}_q] \in V^*$ ($\mathbf{x}_i$, $\mathbf{y}_j$ are vectors in $\mathbb{C}^k$ and $\xi_i$, $\eta_j$ are covectors in $(\mathbb{C}^k)^*$). Theorem 6.2 asserts that the algebra of $G$-invariant polynomials on $V \oplus V^*$ is generated by contractions of a vector with a covector. Now there are four possibilities for contractions:

| | | | |
|---|---|---|---|
| **(1)** | vector and covector in $V$: | $\langle \mathbf{x}_i,\, \eta_j\rangle$ | for $1 \le i \le p$ and $1 \le j \le q$ |
| **(2)** | vector and covector in $V^*$: | $\langle \mathbf{y}_j,\, \xi_i\rangle$ | for $1 \le i \le p$ and $1 \le j \le q$ |
| **(3)** | vector from $V$, covector from $V^*$: | $\langle \mathbf{x}_i,\, \xi_j\rangle$ | for $1 \le i,j \le p$ |
| **(4)** | covector from $V$, vector from $V^*$: | $\langle \mathbf{y}_i,\, \eta_j\rangle$ | for $1 \le i,j \le q$ |

We can identify $V$ with $M_{(p+q)\times k}$ if we make $g \in G$ act on the right by

$$\begin{bmatrix} x \\ \eta \end{bmatrix} \cdot g = \begin{bmatrix} x(g^t)^{-1} \\ \eta g \end{bmatrix} \quad \text{for } x \in M_{p\times k}, \quad \eta \in M_{q\times k}.$$

Here $\mathbf{x}_i$ is the $i$th row of $x$ and $\eta_j$ the $j$th row of $\eta$. Contractions of type **(1)** and **(2)** furnish symbols for the $G$-invariant operators

$$\begin{aligned}\mathfrak{p}_- &= \text{Span}\{\text{ multiplication by } r_{ij} = \langle \mathbf{x}_i, \eta_j\rangle\}\\ \mathfrak{p}_+ &= \text{Span}\{\text{ differentiation by } \Delta_{ij}\},\end{aligned}$$

where

$$\Delta_{ij} = \nabla_{\mathbf{x}_i} \cdot \nabla_{\eta_j} = \sum_{r=1}^{k} \frac{\partial}{\partial x_{ir}} \frac{\partial}{\partial \eta_{jr}} \qquad \text{for } 1 \le i \le p \text{ and } 1 \le j \le q.$$

The linear span of contractions of type **(3)** and **(4)** furnishes symbols for the $G$-invariant operators

$$\mathfrak{k} \;=\; \text{Span}\{E^{(x)}_{ij} + \tfrac{k}{2}\delta_{ij} \;:\; 1 \le i,j \le p\} \oplus \text{Span}\{E^{(\eta)}_{ij} + \tfrac{k}{2}\delta_{ij} \;:\; 1 \le i,j \le q\},$$

where $E^{(x)}_{ij}$ is the polarization operator for the $x$ variables and $E^{(\eta)}_{ij}$ for the $\eta$ variables. By the same argument as in Theorem 8.5 we conclude that $\mathcal{PD}(V)^G$ is generated by

$$\mathfrak{g}' = \mathfrak{p}_- + \mathfrak{k} + \mathfrak{p}_+\,.$$

These subalgebras have the commutation relations

$$[\mathfrak{k}, \mathfrak{p}_\pm] = \mathfrak{p}_\pm, \quad [\mathfrak{p}_-, \mathfrak{p}_+] \subset \mathfrak{k}.$$

In this case $\mathfrak{g}'$ is isomorphic to $\mathfrak{gl}(p+q, \mathbb{C})$ , with $\mathfrak{k} \cong \mathfrak{gl}(p, \mathbb{C}) \oplus \mathfrak{gl}(q, \mathbb{C})$. The action of $\mathfrak{k}$ on $\mathcal{P}(M_{p\times k} \oplus M_{q\times k})$ is the differential of the representation

$$\rho(g,h)f(x,\eta) = (\det g \det h)^{-k/2} f(g^{-1}x, h^{-1}\eta)$$

for $(g,h) \in K = \text{GL}(p,\mathbb{C}) \times \text{GL}(q,\mathbb{C})$. (We must replace $K$ by the two-fold covers of each factor when $n$ is odd).

## Lecture 9. Harmonic Duality

9.1. **Harmonic Polynomials.** Let $G$ be $\text{O}(\mathbb{C}^k, \omega)$, $\text{Sp}(\mathbb{C}^k, \omega)$, or $\text{GL}(k, \mathbb{C})$ acting on $V = M_{n\times k}$ on the right. In the case of $\text{GL}(\mathbb{C}^k)$ the first $p$ rows of $x \in M_{n\times k}$ transform as vectors, whereas the remaining $q$ rows transform as covectors. From Lecture 8 the *Howe dual* to $G$ is, respectively,

$$\mathfrak{g}' \cong \mathfrak{sp}(n, \mathbb{C}),\ \mathfrak{so}(2n, \mathbb{C}),\ \text{ or } \mathfrak{gl}(p+q, \mathbb{C}) \quad \text{with } p+q = n.$$

We will assume that $p > 0$ and $q > 0$ in the third case.[h] With $G$ fixed, the spectrum $\Sigma(V)$ of $G$ on $\mathcal{P}(V)$ only depends on $n$ (or the pair $p, q$ in the third case); we can thus denote it by $\Sigma(n)$ (or $\Sigma(p,q)$). From the dual point of view if we fix $\mathfrak{g}'$, then the set $\Lambda(V)$ of irreducible representations of $\mathfrak{g}'$ that occur in $\mathcal{P}(V)$ only depends on $k$; we can thus denote it as $\Lambda(k)$. The general duality theorem gives a bijection $\Sigma(n) \leftrightarrow \Lambda(k)$. We now show how to express this bijection in terms of *harmonic duality*.

In all cases there is a triangular decomposition

$$\mathfrak{g}' = \mathfrak{p}_- \oplus \mathfrak{k} \oplus \mathfrak{p}_+\,.$$

Here $\mathfrak{k}$ is the Lie algebra of the reductive group $K$ (a two-fold cover of $\text{GL}(n,\mathbb{C})$ or $\text{GL}(p,\mathbb{C}) \times \text{GL}(q,\mathbb{C})$ in general). The representation of $K$ on $\mathcal{P}(V)$ is the natural representation associated with the left multiplication action of $\text{GL}(n,\mathbb{C})$ on $V$ tensored with the one-dimensional representation

$$g \mapsto (\det g)^{-k/2} \quad \text{or} \quad (g,h) \mapsto (\det g \det h)^{-k/2}.$$

Let $\delta$ denote this character, viewed as a weight of the maximal torus of $K$. The subalgebra $\mathfrak{p}_-$ acts by multiplication by $G$-invariant quadratic polynomials, whereas $\mathfrak{p}_+$ acts by $G$-invariant constant-coefficient Laplace operators $\{\Delta_{ij}\}$.

We define the *$G$-harmonic polynomials* to be

$$\mathcal{H} = \mathcal{P}(V)^{\mathfrak{p}_+} = \bigcap\nolimits_{i,j} \text{Ker}(\Delta_{ij})$$

Since $\text{Ad}(K)\mathfrak{p}_+ = \mathfrak{p}_+$, the space $\mathcal{H}$ is invariant under the reductive group $K \times G$. In this lecture we will show that $\mathcal{H}$ gives a multiplicity-free duality pairing between irreducible representations of $K$ and $G$; furthermore, the decomposition of $\mathcal{H}$ generates the decomposition of $\mathcal{P}(V)$ under $\mathfrak{g}'$ and $G$.

[h] If $p = 0$ or $q = 0$ then $\mathfrak{k} = \mathfrak{g}'$ and the modules $E^\lambda \otimes F^\lambda$ that occur in the decomposition of $\mathcal{P}(V)$ are finite-dimensional. This is the well-known $\text{GL}(n)$-$\text{GL}(k)$ duality (see the lectures of Benson-Ratcliff in this volume).

Let $U(k) \subset \mathrm{GL}(k,\mathbb{C})$ denote the unitary group. Then $K_0 = K \cap U(n)$ is a compact real form of $K$. We assume that the bilinear form $\omega$ is chosen so that $G_0 = G \cap U(k)$ is a compact real form of $G$ and $\omega$ is real on $\mathbb{R}^k$. Define an inner product on $M_{n\times k}$ by

$$(x \mid y) = \mathrm{tr}(y^* x) \quad \text{for } x, y \in M_{n\times k} \quad (y^* = \bar{y}^t).$$

This inner product is invariant under $U(n) \times U(k)$, acting by left and right multiplication, hence it is invariant under $K_0 \times U_0$. We set $||x||^2 = (x \mid x)$.

Let $f(x) = \sum_\alpha c_\alpha x^\alpha$ be in $\mathcal{P}(V)$, where $\alpha$ is a multi-index and $x^\alpha = \prod_\alpha (x_{ij})^{\alpha_{ij}}$ as usual ($x_{ij}$ are the matrix entry functions on $M_{n\times k}$). Define the constant-coefficient differential operator

$$\partial(f) = \sum_\alpha c_\alpha \left(\frac{\partial}{\partial x}\right)^\alpha .$$

Then $f \mapsto \partial(f)$ is an algebra isomorphism from $\mathcal{P}(V)$ to the constant-coefficient differential operators on $V$ that is equivariant relative to the action of $U(k) \times U(n)$. Set $g^*(x) = \overline{g(\bar{x})}$ for $g \in \mathcal{P}(V)$. If $g(x) = \sum_\alpha d_\alpha x^\alpha$, then

$$(\partial(f) g^*)(0) = \sum_\alpha \alpha!\; c_\alpha \overline{d_\alpha}\,. \tag{9.1}$$

We define $\langle f \mid g\rangle = (\partial(f)g^*)(0)$. From (9.1) we see that this is a positive definite Hermitian inner product on $\mathcal{P}(V)$, called the *Fischer inner product*. We note that

$$\langle fg \mid h\rangle = \langle f \mid \partial(g^*)h\rangle \tag{9.2}$$

for all $f$, $g$, $h \in \mathcal{P}(V)$.

The Fischer inner product has the following analytic definition. Denote Lebesgue measure on $V$ by $d\lambda(z)$, where we identify $V$ with $\mathbb{R}^{2nk}$ via the real and imaginary parts of the matrix coordinates.

**Lemma 9.1.** *For $f, g \in \mathcal{P}(V)$ one has*

$$\langle f \mid g\rangle = \frac{1}{\pi^d}\int_V f(z)\overline{g(z)}\, e^{-||z||^2}\, d\lambda(z)$$

*($d = \dim_{\mathbb{C}} V = nk$).*

9.2. **Main Theorem.** We now apply the Weyl algebra theorem from Lecture 8 to obtain multiplicity-free decompositions of the harmonic polynomials and the entire space $\mathcal{P}(V)$.

**Theorem 9.2** (Harmonic Duality)**.**
(1) *The space $\mathcal{H}$ of $G$-harmonic polynomials on $V$ decomposes under $K \times G$ into mutually orthogonal subspaces (relative to the Fischer inner product) as*

$$\mathcal{H} = \bigoplus_{\sigma\in\Sigma(V)} \mathcal{E}^{\tau(\sigma)+\delta} \otimes \mathcal{F}^\sigma.$$

*Here $\Sigma(V) \subset \widehat{G}$ is the spectrum of $\mathcal{P}(V)$ as a $G$ module, $\mathcal{F}^\sigma \subset \mathcal{H}$ is an irreducible $G$-module of type $\sigma$, and $\mathcal{E}^{\tau(\sigma)+\delta} \subset \mathcal{H}$ is an irreducible finite-dimensional $K$-module with highest weight $\tau(\sigma)+\delta$. In particular, every irreducible representation of $G$ in $\mathcal{P}(V)$ is realized in the harmonic polynomials.*

(2) *Set $E^{\tau(\sigma)+\delta} = \mathcal{P}(V)^G \cdot \mathcal{E}^{\tau(\sigma)+\delta}$. Then $E^{\tau(\sigma)+\delta}$ is an irreducible $\mathfrak{g}'$ module and*

$$\mathcal{P}(V) = \bigoplus_{\sigma\in\Sigma(V)} E^{\tau(\sigma)+\delta} \otimes \mathcal{F}^\sigma$$

*is an orthogonal decomposition of $\mathcal{P}(V)$ (relative to the Fischer inner product) under the mutually commuting actions of $\mathfrak{g}'$ and $G$.*

(3) *The map $\sigma \mapsto \tau(\sigma)$ from $\Sigma(V) \to \widehat{K}$ is injective. Thus $\mathcal{H}$ is multiplicity-free as a $K \times G$ module.*

*Proof.* Since $\mathcal{H}$ is an invariant subspace for the reductive group $K \times G$, there is a subset $\Gamma \subset \widehat{K} \times \widehat{G}$ and multiplicity function $m : \Gamma \to \{1, 2, \ldots\}$ such that

$$\mathcal{H} \cong \bigoplus_{(\mu,\sigma)\in\Gamma} \mathbb{C}^{m(\mu,\sigma)} \otimes \mathcal{E}^{\mu+\delta} \otimes \mathcal{F}^\sigma \tag{9.3}$$

with $K \times G$ acting trivially on the multiplicity spaces $\mathbb{C}^{m(\mu,\sigma)}$. Indeed, we first consider $\mathcal{H}$ as a locally-finite $K \times G$-module relative to the natural left-right action on $V = M_{n\times k}$ (omitting the determinant twist from the $K$ representation) and use Proposition 1.1 and Example 1 in Section 2.3. Then tensor with the character $\det^{-k/2}$ of $K$ to shift the highest weights from $\mu$ to $\mu+\delta$. To prove that $\Gamma = \{(\tau(\sigma), \sigma) : \sigma \in \Sigma(V)\}$ and $m(\tau(\sigma), \sigma) = 1$, we need to examine the action of $\mathfrak{g}'$ on $\mathcal{P}(V)$ in more detail.

Let $\mathcal{J} = \mathcal{P}(V)^G$ be the $G$-invariant polynomials, and let $\mathcal{J}_j$ be the homogeneous polynomials of degree $j$ in $\mathcal{J}$. Then $\mathcal{J}_j = 0$ for $j$ odd, and $(\mathfrak{p}_-)^j$ acts by multiplication by $\mathcal{J}_{2j}$ on $\mathcal{P}(V)$. Since the bilinear form $\omega$ is real on $\mathbb{R}^k$, we have $\mathcal{J}^* = \mathcal{J}$ and $\mathcal{H}^* = \mathcal{H}$. Let $\mathcal{J}_+ = \{f \in \mathcal{J} : f(0) = 0\}$. We claim that

$$\mathcal{H}^\perp = \mathcal{J}_+ \cdot \mathcal{P}(V) \tag{9.4}$$

(orthogonal complement relative to the Fischer inner product). Indeed, if $f \in \mathcal{J}_+ \cdot \mathcal{P}(V)$ and $h \in \mathcal{H}$ then $\partial(f)h = 0$ by definition of $\mathcal{H}$, and thus $f \perp h$. Conversely, if $h \perp \mathcal{J}_+ \cdot \mathcal{P}(V)$ then for all $f \in \mathcal{P}(V)$ and $g \in \mathcal{J}_+$,

$$0 = \langle fg \mid h\rangle = \langle f \mid \partial(g^*)h\rangle.$$

Hence $\partial(g)h = 0$ for all $g \in \mathcal{J}_+$, so we have $h \in \mathcal{H}$.

We can now determine the general structure of the irreducible $\mathfrak{g}'$-modules in $\mathcal{P}(V)$. The commutation relations in $\mathfrak{g}'$ can be expressed as

$$\mathfrak{p}_+\mathfrak{p}_- \subset \mathfrak{p}_-\mathfrak{p}_+ + \mathfrak{k}, \qquad \mathfrak{k}\,\mathfrak{p}_- \subset \mathfrak{p}_-(\mathfrak{k}+1)$$

in the universal enveloping algebra $U(\mathfrak{g}')$. Hence by induction, one has

$$\mathfrak{p}_+(\mathfrak{p}_-)^m \subset (\mathfrak{p}_-)^m\mathfrak{p}_+ + (\mathfrak{p}_-)^{m-1}(\mathfrak{k}+1), \qquad \mathfrak{k}\,(\mathfrak{p}_-)^m \subset (\mathfrak{p}_-)^m(\mathfrak{k}+1) \tag{9.5}$$

for all integers $m \geq 1$. Thus if $Z \subset \mathcal{H}$ is any $\mathfrak{k}$-invariant linear subspace, then (9.5) implies that

$$\mathfrak{p}_+(\mathfrak{p}_-)^m \cdot Z \subset (\mathfrak{p}_-)^{m-1} \cdot Z \quad \text{and} \quad \mathfrak{k}\,(\mathfrak{p}_-)^m \cdot Z \subset (\mathfrak{p}_-)^m \cdot Z \tag{9.6}$$

for all $m \geq 1$.

**(a)** *Let $\mathcal{E} \subset \mathcal{H}$ be any $\mathfrak{k}$-irreducible subspace. Set $E = \mathcal{J} \cdot \mathcal{E}$. Then $E$ is an irreducible $\mathfrak{g}'$-module and $\mathcal{E} = E \cap \mathcal{H}$.*

Indeed, (9.6) implies that $E$ is invariant under $\mathfrak{g}'$. Also every $f \in E$ is of the form

$$f = \sum_{j=0}^{m} g_j h_j \quad \text{where } 0 \neq g_j \in \mathcal{J}_{2j} \text{ and } h_j \in \mathcal{E}. \tag{9.7}$$

Suppose $F \subset E$ is a nonzero $\mathfrak{g}'$-invariant subspace. Take $f \in F$ so that the integer $m$ in (9.7) is minimal. Then (9.6) implies that $\mathfrak{p}_+ f = 0$. Hence $f \in \mathcal{H}$. Thus

$$\sum_{j=1}^{m} g_j h_j = f - g_0 h_0 \in \mathcal{H}.$$

Since the left side is in $\mathcal{J}_+ \cdot \mathcal{P}(V)$, it must be zero by (9.4). Hence we conclude that $f \in \mathcal{E}$. But $\mathfrak{k}$ acts irreducibly on $\mathcal{E}$, so $U(\mathfrak{k})f = \mathcal{E}$ and thus $F = E$. The same argument shows that $E \cap \mathcal{H} = \mathcal{E}$, completing the proof of **(a)**.

**(b)** *Let $E \subset \mathcal{P}(V)$ be an irreducible $\mathfrak{g}'$-module. Set $\mathcal{E} = E \cap \mathcal{H}$. Then $\mathcal{E}$ is an irreducible $\mathfrak{k}$-module and $E = \mathcal{J} \cdot \mathcal{E}$.*

Note that the action of $\mathfrak{p}_+$ on $\mathcal{P}(V)$ lowers the degree of polynomials, so $\mathcal{E} \neq 0$. If $0 \neq \mathcal{F} \subset \mathcal{E}$ were a proper $\mathfrak{k}$-submodule, then $\mathcal{J} \cdot \mathcal{F} \subset E$ would be a proper irreducible $\mathfrak{g}'$-submodule by **(a)**. Hence $\mathcal{E}$ must be irreducible as a $\mathfrak{k}$-module and $E = \mathcal{J} \cdot \mathcal{E}$, proving **(b)**.

**(c)** *Let $\mathcal{E}$ and $\mathcal{F}$ be $\mathfrak{k}$-invariant subspaces of $\mathcal{H}$. Assume that $\mathcal{E} \perp \mathcal{F}$ (relative to the Fischer inner product). Set $E = \mathcal{J} \cdot \mathcal{E}$ and $F = \mathcal{J} \cdot \mathcal{F}$. Then $E \perp F$.*

By (9.4) we have the orthogonal decompositions

$$E = \mathcal{E} \oplus \mathcal{J}_+ \cdot \mathcal{E} \qquad F = \mathcal{F} \oplus \mathcal{J}_+ \cdot \mathcal{F}.$$

Thus $E \perp \mathcal{F}$ and $F \perp \mathcal{E}$, so we only need to verify that $\mathcal{J}_+ \cdot \mathcal{E} \perp \mathcal{J}_+ \cdot \mathcal{F}$. Now

$$\langle \mathcal{J}_+ \cdot \mathcal{E} \mid \mathcal{J}_+ \cdot \mathcal{F}\rangle = \langle \mathcal{E} \mid \partial(\mathcal{J}_+)\mathcal{J}_+ \cdot \mathcal{F}\rangle$$

But $\partial(\mathcal{J}_+)\mathcal{J}_+ \cdot \mathcal{F} \subset F$ since $\mathcal{F}$ is $\mathfrak{k}$-invariant. Hence $\mathcal{E} \perp \partial(\mathcal{J}_+)\mathcal{J}_+ \cdot \mathcal{F}$, proving **(c)**.

We now complete the proof of the theorem. It is clear from the integral formula for the Fischer inner product (Lemma 9.1) that $G_0$ and $K_0$ act by unitary operators on $\mathcal{P}(V)$, hence the decomposition (9.3) of $\mathcal{H}$ is orthogonal relative to the Fischer inner product because $G_0$ and $K_0$ have the same finite-dimensional invariant subspaces in $\mathcal{P}(V)$ as $G$ and $K$, respectively (see [16, §2.4.4]). Also, since $K$ is connected, a finite-dimensional subspace of $\mathcal{P}(V)$ is invariant under $K$ if and only if it is invariant under $\mathfrak{k}$.

Let $(\mu, \sigma) \in \Gamma$ occur in (9.3). By **(a)**, **(b)** and Theorem 3.4 we know that the irreducible $\mathfrak{g}'$-module $E^{\mu+\delta} = \mathcal{J} \cdot \mathcal{E}^{\mu+\delta}$ uniquely determines $\mu$. On the other hand, Theorem 8.2 asserts that $\mathcal{P}(V)$ is semisimple as a $\mathfrak{g}'$-module, with the $\mathfrak{g}'$ multiplicity spaces being irreducible regular $G$-modules corresponding bijectively to the associated $\mathfrak{g}'$-modules. Hence $\Gamma$ is determined by its projection onto $\widehat{G}$. If we call this projection $\Sigma$ and write the elements of $\Gamma$ as $(\tau(\sigma), \sigma)$, then the map $\sigma \mapsto \tau(\sigma)$ is injective. The multiplicities $m(\tau(\sigma), \sigma) = 1$ for all $\sigma \in \Sigma$, since otherwise **(a)** and **(c)** would imply that $\mathcal{F}^\sigma$ is paired with more than one copy of an irreducible $\mathfrak{g}'$ module, contradicting Theorem 8.2. Finally, **(b)** implies that $\Sigma = \Sigma(V)$, since $\mathcal{P}(V)$ is semisimple as a $\mathfrak{g}'$-module. $\square$

*Remarks.* **1.** Theorem 9.2 was obtained by Howe in his influential paper [21] (which circulated as a preprint for more than a decade); his proof used an argument based on a filtration by finite-dimensional subspaces and the classical double commutant theorem, instead of Theorem 8.2. Knowing that the decomposition of the harmonics is multiplicity free simplifies the task of finding harmonic highest weight vectors, as we will see in Lecture 10.

**2.** The shift by $\delta$ in the highest weights for $K$ in the harmonic decomposition would appear to be a minor nuisance. In fact, it plays an important analytic role. For $T \in \mathcal{PD}(V)$ let $T^*$ denote the adjoint of $T$ relative to the Fischer inner product:

$$\langle Tf \mid g \rangle = \langle f \mid T^* g \rangle.$$

If we write $T$ in polarized form as $T = \sum_j f_j \partial(g_j)$ with $f_j,\, g_j \in \mathcal{P}(V)$, then we see from (9.2) that $T^* = \sum_j g_j^* \partial(f_j^*)$. Hence $(\mathfrak{p}_\pm)^* = \mathfrak{p}_\mp$ and $\mathfrak{k}^* = \mathfrak{k}$. It follows that

$$\mathfrak{g}_0' := \{T \in \mathfrak{g}' \,:\, T^* = -T\}$$

is a real form of $\mathfrak{g}'$. We will show in Lecture 11 that the irreducible representation of $\mathfrak{g}_0'$ on the space $E^\lambda$, with $\lambda = \tau(\sigma) + \delta$, can be integrated to a unitary representation $\pi^\lambda$ of a (non-compact) real group $G_0'$ with Lie algebra $\mathfrak{g}_0'$. The shift by $\delta$ controls the rate of decay at infinity on $G_0'$ of the matrix entries of $\pi^\lambda$. We will show in Lecture 12 that for $k$ large enough (relative to $n$), the representations $\pi^\lambda$ are square-integrable (recall that $\delta = k\delta_0$, where $\delta_0$ is a fixed weight of $\mathfrak{g}'$).

**3.** The injective map $\sigma \mapsto \pi^{\tau(\sigma)+\delta}$ is called the *theta-correspondence* (more precisely, the *local* theta-correspondence over $\mathbb{R}$) because of the connection between the oscillator representation and theta-functions (see [5]). There are many recent papers devoted to the problem of understanding the theta-correspondence from a geometric orbit perspective (see [19, Ch. 12] for a survey).

## Lecture 10. Decomposition of Harmonic Polynomials

We now turn to the explicit determination of the harmonic duality from Lecture 9 when $G$ is the orthogonal group and $\mathfrak{g}'$ the symplectic Lie algebra (for the other two cases, when $G$ is the symplectic or general linear group, see [23] and [6]). It is convenient to take $G$ as the orthogonal group $\mathrm{O}(\mathbb{C}^k, \omega)$ for the symmetric form $\omega(x, y) = x^t C_k y$ on $\mathbb{C}^k$, where

$$C_k = \begin{bmatrix} 0 & I_l \\ I_l & 0 \end{bmatrix} \quad \text{when } k = 2l, \quad C_k = \begin{bmatrix} 0 & I_l & 0 \\ I_l & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{when } k = 2l+1.$$

Here $I_l$ denotes the $l \times l$ identity matrix. This choice of $\omega$ ensures that the diagonal matrices in $G$ give a maximal torus. Also $G$ is a *self-adjoint* matrix group (invariant under $g \mapsto g^*$), so the subgroup $G_0 = G \cap U(k)$ is a compact real form of $G$, and $\omega$ is real on the real matrices, as we assumed in Lecture 8.

In accordance with the block decomposition of $C_k$, we write elements $z \in M_{n\times k}$ as

$$z = [\,x \quad y\,] \quad \text{when } k = 2l, \qquad z = [\,x \quad y \quad t\,] \quad \text{when } k = 2l+1, \tag{10.1}$$

where $x, y \in M_{n\times l}$ and $t \in \mathbb{C}^n$. Define the map

$$\beta : M_{n\times k} \to SM_n, \qquad \beta(z) = zC_kz^t.$$

From Theorem 6.3 we know that the algebra of $G$-invariant polynomials on $M_{n\times k}$ (relative to right $G$-multiplication) is generated by the matrix entries of $\beta$:

$$\beta(z)_{pq} = \begin{cases} \sum_{s=1}^{l} (y_{ps}\, x_{qs} + x_{ps}\, y_{qs}) & \text{when } k = 2l, \\[2ex] \sum_{s=1}^{l} (y_{ps}\, x_{qs} + x_{ps}\, y_{qs}) + t_pt_q & \text{when } k = 2l+1. \end{cases} \tag{10.2}$$

We denote by $\Delta_{pq} = \partial(\beta_{pq})$ the corresponding constant-coefficient differential operators, as in Section 9.1.

The space of *$G$-harmonic polynomials* is

$$\mathcal{H} = \{ f \in \mathcal{P}(M_{n\times k}) \,:\, \Delta_{pq} f = 0 \quad \text{for } 1 \le p, q \le n \}.$$

Denote by $\mathcal{H}^{(j)}$ the $G$-harmonic polynomials that are homogeneous of degree $j$. The space $\mathcal{H}$ is invariant under $\mathrm{GL}(n,\mathbb{C}) \times G$ with the action

$$\pi(h,g)f(z) = f(h^{-1}zg) \quad \text{for } h \in \mathrm{GL}(n,\mathbb{C}) \text{ and } g \in G.$$

(Note that we have omitted the factor $(\det h)^{-k/2}$ that occurs in Theorem 8.5, so $\pi$ is single-valued on $\mathrm{GL}(n,\mathbb{C})$ even when $k$ is odd). From Theorem 9.2 we know that $\mathcal{H}$ decomposes under the representation $\pi$ as a multiplicity-free direct sum[i]

$$\mathcal{H} = \bigoplus_{\sigma\in\Sigma} \mathcal{E}^{\tau(\sigma)} \otimes \mathcal{F}^{\sigma}.$$

We will now determine $\Sigma$ and the duality correspondence $\sigma \mapsto \tau(\sigma)$. The key point is to find generators for the algebra $\mathcal{H}^{N_n\times N}$ of harmonic highest weight vectors, relative to a Borel subgroup $B_n \times B \subset \mathrm{GL}(n,\mathbb{C}) \times G$. Here $B_n = D_nN_n$ is the upper-triangular subgroup of $\mathrm{GL}(n,\mathbb{C})$ ($D_n$ the diagonal matrices, $N_n$ the unipotent upper-triangular matrices), and $B = HN$ is a Borel subgroup of $G$. The fact that $\mathcal{H}$ is multiplicity-free under $\mathrm{GL}(n,\mathbb{C}) \times G$ will play a crucial role.

*Notation:* We denote by $\varepsilon_j$ the character $\mathrm{diag}[\,a_1, \dots, a_n\,] \mapsto a_j$ of $D_n$. We write $\mathbb{N}^p_{++}$ for the integer $p$-tuples $\lambda = [\,m_1, \dots, m_p\,]$ with $m_1 \ge m_2 \ge \cdots \ge m_p \ge 0$. Set $|\lambda| = m_1 + \cdots + m_p$ and define the *depth* of $\lambda$ to be the smallest integer $i$ such that $m_i > 0$ (if $\lambda = 0$, set $\mathrm{depth}(0) = 0$).

10.1. $\mathrm{O}(k)$ **Harmonics ($k$ odd).** Assume that $k = 2l+1$ is odd. Then $G = G^\circ \times \{\pm I\}$, where $G^\circ = \mathrm{SO}(\mathbb{C}^k, \omega)$ is the identity component of $G$. We fix the Borel subgroup $B = HN \subset G^\circ$ as follows. The maximal torus $H$ consists of the diagonal matrices

$$h = \mathrm{diag}[\,x_1, \dots, x_l, x_1^{-1}, \dots, x_l^{-1}, 1\,], \quad x_i \in \mathbb{C}^\times.$$

The unipotent radical $N$ has Lie algebra $\mathfrak{n}$ consisting of the matrices with block decomposition

$$\begin{bmatrix} a & b & c \\ 0 & -a^t & 0 \\ 0 & -c^t & 0 \end{bmatrix}, \quad a \in M_{l\times l} \text{ strictly upper-triangular}, \ b = -b^t \in M_{l\times l},\ c \in \mathbb{C}^l. \tag{10.3}$$

The weights of $H$ are parameterized by $\mathbb{Z}^l$. For $h \in H$ and $\lambda = [\,m_1, \dots, m_l\,] \in \mathbb{Z}^l$ we set $h^\lambda = x_1^{m_1} \cdots x_l^{m_l}$ for the corresponding character of $H$.

The irreducible representations of $G$ remain irreducible on restriction to $G^\circ$ and $\widehat{G}$ is parameterized as $\{\pi^{\lambda,\epsilon}\}$, where $\lambda \in \mathbb{N}^l_{++}$ is the highest weight for $G^\circ$, $\epsilon = \pm 1$, and $\pi^{\lambda,\epsilon}(-I) = \epsilon(-I)^{|\lambda|}$. Thus $\widehat{G} = \widehat{G}_1 \cup \widehat{G}_{-1}$, where

$$\widehat{G}_1 = \{ (\lambda, 1) \,:\, \lambda \in \mathbb{N}^l_{++} \}, \qquad \widehat{G}_{-1} = \{ (\lambda, -1) \,:\, \lambda \in \mathbb{N}^l_{++} \},$$

(see [16, §5.2.2]).

[i]The irreducible $\mathrm{GL}(n,\mathbb{C})$-module $\mathcal{E}^{\tau(\sigma)} \cong F^{\tau(\sigma)}_{(n)}$, in the notation of Lecture 3.

**Theorem 10.1.** *($G = \mathrm{O}(\mathbb{C}^k, \omega)$, $k = 2l+1$) Let $\Sigma$ be the spectrum of $G$ on the $G$-harmonic polynomials $\mathcal{H} \subset \mathcal{P}(M_{n\times k})$.*

(a) *Assume $k \le n$. Then $\Sigma = \widehat{G}$ and hence $\Sigma$ does not depend on $n$ ($G$-stable range).*

(b) *Assume $l < n < k$. Then $\widehat{G}_1 \subset \Sigma$ and*

$$\Sigma \cap \widehat{G}_{-1} = \{(\lambda, -1) \,:\, k - n \le \mathrm{depth}(\lambda) \le l\,\}$$

*(unstable range: $\Sigma$ depends on $k$ and $n$).*

(c) *Assume $n \le l$. Then $\Sigma \cap \widehat{G}_{-1} = \emptyset$ and*

$$\Sigma \cap \widehat{G}_1 = \{(\lambda, 1) \,:\, \mathrm{depth}(\lambda) \le n\,\}.$$

*Thus $\Sigma$ does not depend on $k$ ($\mathrm{GL}(n)$-stable range).*

*The duality correspondence is given as follows: Let $\lambda = [\,m_1\,, \ldots, m_d\,, 0, \ldots, 0\,] \in \mathbb{N}^l_{++}$ have depth $d$ with $0 \le d \le \min\{l, n\}$. Then*

$$\tau(\sigma) = \begin{cases} [\,\underbrace{0, \ldots, 0}_{n-d}, -m_d, \ldots, -m_1\,] & \text{for } \ \sigma = (\lambda, 1) \in \Sigma \cap \widehat{G}_1, \\[2ex] [\,\underbrace{0, \ldots, 0}_{n-k+d}, \underbrace{-1, \ldots, -1}_{k-2d}, -m_d, \ldots, -m_1\,] & \text{for } \ \sigma = (\lambda, -1) \in \Sigma \cap \widehat{G}_{-1} \end{cases}.$$

**Remark.** The parameter $\epsilon$ for the representation $\pi^{\lambda,\epsilon}$ is determined by the corresponding $\mathrm{GL}(n)$ highest weight $\tau(\lambda, \epsilon)$, since left multiplication by $-I_n$ on $M_{n\times k}$ is the same as right multiplication by $-I_k$. Hence $\mathcal{H}$ is also multiplicity-free as a module for $\mathrm{GL}(n, \mathbb{C}) \times G^\circ$ (this property will be used in the proof).

The first step in the proof of Theorem 10.1 is to find a set of generators for the joint eigenfunctions of $B_n \times B$ in $\mathcal{H}$. Just as in the case of $\mathrm{GL}(n) \times \mathrm{GL}(k)$ duality (see the article by Benson-Ratcliff), the general strategy is to take appropriate minor determinants. By (10.2) the operators $\Delta_{pq}$ are given in coordinates as

$$\Delta_{pq} = \sum_{s=1}^{l} \left( \frac{\partial}{\partial y_{ps}} \frac{\partial}{\partial x_{qs}} + \frac{\partial}{\partial x_{ps}} \frac{\partial}{\partial y_{qs}} \right) + \frac{\partial^2}{\partial t_p t_q}\,. \tag{10.4}$$

The minors of $z = [\,x \quad y \quad t\,]$ are linear functions of each column of the matrix components $x, y, t$. If the minors are chosen to depend only on $x$ or to be linear in $t$, then they will obviously be harmonic. If they depend on both $x$ and $y$, then interchanging an $x$ column for a $y$ column will change the sign of the minor but not change the action of the operators $\Delta_{pq} = \Delta_{qp}$, so once again the minor will be harmonic.

We now proceed to carry out this program. Let $p \le n$ and $q \le l$. For $u \in M_{n\times l}$ define $p \times q$ submatrices $L_{p,q}(u)$ and $R_{p,q}(u)$ of $u$ by

$$u = \begin{bmatrix} * & * \\ L_{p,q}(u) & * \end{bmatrix} = \begin{bmatrix} * & * \\ * & R_{p,q}(u) \end{bmatrix}$$

For $t \in \mathbb{C}^n$ and $j \le n$ define

$$t_{(j)} = \begin{bmatrix} t_{n-j+1} \\ \vdots \\ t_n \end{bmatrix} \in \mathbb{C}^{n-j}$$

(the bottom $j$ entries of $t$).

Let $z = [\ x \quad y \quad t\ ] \in M_{n\times k}$ as in (10.1). Define

$$f_j(z) = \det L_{j,j}(x) \quad \text{for } 1 \le j \le \min\{l, n\}.$$

If $n \ge l + 1$ then we also define

$$g_j(z) = \begin{cases} \det \left[\ L_{j,l}(x) \quad t_{(j)}\ \right] & \text{for } j = l + 1, \\[2ex] \det \left[\ L_{j,l}(x) \quad R_{j,\,j-l-1}(y) \quad t_{(j)}\ \right] & \text{for } l + 2 \le j \le \min\{n, k\}. \end{cases}$$

**Lemma 10.2.**
(a) *Let $1 \le j \le \min\{l, n\}$. Then $f_j \in \mathcal{H}^{(j)}$ and $f_j$ is a $B_n \times B$ eigenfunction of weight $(\mu, \nu)$, where*

$$\mu = -\varepsilon_{n-j+1} - \cdots - \varepsilon_n \quad \textit{and} \quad \nu = \varepsilon_1 + \cdots + \varepsilon_j\,.$$

(b) *Assume $n > l$ and let $l+1 \le j \le \min\{n, k\}$. Then $g_j \in \mathcal{H}^{(j)}$ and $g_j$ is a $B_n \times B$ eigenfunction of weight $(\mu, \gamma)$, where*

$$\mu = -\varepsilon_{n-j+1} - \cdots - \varepsilon_n \quad \textit{and} \quad \gamma = \varepsilon_1 + \cdots + \varepsilon_{k-j}$$

*(here $\gamma = 0$ if $j = k$).*

**Corollary 10.3.** *Let $\mathbf{m} = [\,m_1\,, \ldots, m_r\,] \in \mathbb{N}^r_{++}$, where $r = \min\{l, n\}$. Assume that $\mathbf{m}$ has depth $d$ and set $\lambda = [\,\mathbf{m}, 0, \ldots, 0\,] \in \mathbb{N}^l_{++}$. Define $\varphi_{\mathbf{m}} = f_1^{m_1 - m_2} \cdots f_{d-1}^{m_{d-1} - m_d}\, f_d^{m_d}$ (when $\mathbf{m} = 0$ set $\varphi_0(z) = 1$).*

(a) *$\varphi_{\mathbf{m}}$ is a $G$-harmonic polynomial, homogeneous of degree $|m|$. Thus $\varphi_{\mathbf{m}}(-z) = (-1)^{|\mathbf{m}|}\varphi_{\mathbf{m}}(z)$ for $z \in M_{n\times k}$. Furthermore, $\varphi_{\mathbf{m}}$ is a $B_n \times B$ eigenfunction of weight $(\alpha, \lambda)$, where*

$$\alpha = [\,\underbrace{0, \ldots, 0}_{n-d}\,, -m_d\,, \ldots, -m_1\,]$$

*(when $\mathbf{m} = 0$ take $\alpha = 0$).*

(b) *Suppose $n > l$ and $n - k + d \ge 0$. For $\mathbf{m} \ne 0$, define $\psi_{\mathbf{m}} = \varphi_{\mathbf{m}}\, g_{k-d}/f_d$ (when $\mathbf{m} = 0$ set $\psi_0 = g_k$). Then $\psi_{\mathbf{m}}$ is a $G$-harmonic polynomial, homogeneous of degree $|m| + k - 2d$. Thus $\psi_{\mathbf{m}}(-z) = -(-1)^{|\mathbf{m}|}\psi_{\mathbf{m}}(z)$ for $z \in M_{n\times k}$. Furthermore, $\psi_{\mathbf{m}}$ is a $B_n \times B$ eigenfunction of weight $(\beta, \lambda)$, where*

$$\beta = [\,\underbrace{0, \ldots, 0}_{n-k+d}\,, \underbrace{-1, \cdots, -1}_{k-2d}, -m_d\,, \ldots, -m_1\,].$$

*(When $\mathbf{m} = 0$ take $\beta = [\,\underbrace{0, \ldots, 0}_{n-k}\,, \underbrace{-1, \cdots, -1}_{k}\,]$.)*

Now we turn to the proof of Theorem 10.1. A $B_n \times B$ joint eigenfunction generates an irreducible subspace under the action of $\mathrm{GL}(n) \times G^\circ$ by Theorem 3.5. Since the space of harmonic polynomials on $M_{n\times k}$ is a multiplicity-free $\mathrm{GL}(k, \mathbb{C}) \times G^\circ$ module, it follows that a $B_k \times B$ eigenfunction is uniquely determined (up to a scalar multiple) by its weight and parity. If $\mathbf{m} \in \mathbb{N}^l_{++}$ has depth $d \le n$, then from Corollary 10.3 we see that the right translates of $\varphi_{\mathbf{m}}$ under $G$ span an irreducible space of type $(\mathbf{m}, 1)$, while the right translates of $\psi_{\mathbf{m}}$ under $G$ span an irreducible space of type $(\mathbf{m}, -1)$. When $k \le n$, then the conditions $n > l$ and $n - k + d \ge 0$ in part (b) of Corollary 10.3 are automatic. Thus every irreducible representation of $G$ occurs in $\mathcal{H}$ in this case, as asserted in part (a) of the theorem.

To prove parts (b) and (c) of the theorem, assume that $k > n$. Let $f \in \mathcal{H}$ be a $B_n \times B$ eigenfunction. Define a polynomial $\widetilde{f}$ on $M_{k\times k}$ by

$$\widetilde{f}\left(\begin{bmatrix} z' \\ z'' \end{bmatrix}\right) = f(z'') \quad \text{for } z' \in M_{(k-n)\times k} \text{ and } z'' \in M_{n\times k}.$$

We claim that $\widetilde{f}$ is $G$-harmonic. Indeed, if $\min\{p, q\} \le k - n$ then $\Delta_{pq}\widetilde{f} = 0$ since $\widetilde{f}$ does not depend on the variables $z_{pq}$ for $p \le k - n$. On the other hand, if $\min\{p, q\} > k - n$ then

$$\Delta_{pq}\widetilde{f}(z) = \Delta_{p'q'} f(z'') = 0$$

(where $p' = p - k + n$ and $q' = j - k + n$), since $f$ is $G$-harmonic. To see that $\widetilde{f}$ is a $B_k \times B$ eigenfunction, write $b \in B_k$ as

$$b = \begin{bmatrix} \alpha & \beta \\ 0 & \delta \end{bmatrix}, \quad \text{where } \alpha \in B_{k-n},\ \beta \in M_{(k-n)\times n},\ \delta \in B_n.$$

Then $\widetilde{f}(b^{-1}zb') = f(\delta^{-1}z''b')$ for $b' \in B$. Since $f$ is $B_n \times B$ eigenfunction, it follows that $\widetilde{f}$ is a $B_k \times B$-eigenfunction. Furthermore this shows that $B_k$ weight $\mu$ of $\widetilde{f}$ is of the form

$$\mu = [\,\underbrace{0, \ldots, 0}_{k-n}\,, a_n\,, \ldots, a_1\,] \quad \text{with } a_n \ge \cdots \ge a_1,$$

Thus by part (a) we know that $\widetilde{f}$ is a multiple of either $\varphi_{\mathbf{m}}$ or $\psi_{\mathbf{m}}$ for some $\mathbf{m} \in \mathbb{N}^l_{++}$ of depth $d \le n$, since $f$ is homogeneous.

If $l < n < k$, then $\varphi_{\mathbf{m}}$ is defined for all $\mathbf{m} \in \mathbb{N}^l_{++}$, but $\psi_{\mathbf{m}}$ is only defined when the depth $d$ of $\mathbf{m}$ satisfies $k - n \le d \le l$. This implies part (b) of the theorem. If $n \le l$, then $\varphi_{\mathbf{m}}$ is defined for all $\mathbf{m}$ of depth $d \le n$, but in this case $\psi_{\mathbf{m}}$ is never defined. This implies part (c) of the theorem. The formula for the map $\tau$ follows from the formulas for $\alpha$ and $\beta$ in Corollary 10.3. □

10.2. $\mathrm{O}(k)$ **Harmonics ($k$ even).** We now assume that $k = 2l$ is even. We take the Borel subgroup $B \subset G$ whose Lie algebra consists of the matrices with block decomposition (block sizes $l \times l$)

$$\begin{bmatrix} a & b \\ 0 & -a^t \end{bmatrix} \qquad (a \text{ upper-triangular, } b^t = -b)$$

(If $k = 2$ then $b = 0$ and $B \cong \mathbb{C}^\times$). Let $N \subset B$ be the unipotent radical (the matrices as above with $a$ upper-triangular unipotent). Recall that

$$\mathrm{O}(\mathbb{C}^k, \omega) = G^\circ \rtimes \{I, s\}$$

where $G^\circ = \mathrm{SO}(\mathbb{C}^k, \omega)$ is the identity component and $s \in G$ is the reflection interchanging the basis vectors $e_l$ and $e_{2l}$ and fixing all other basis vectors $e_i$. Since $s$ normalizes $B$ it acts on the characters of $B$. Let $\lambda = [m_1, \ldots, m_l] \in \mathbb{N}^l_{++}$. If $m_l \neq 0$, then $s \cdot \lambda \neq \lambda$ (since $s$ changes $m_l$ to $-m_l$). In this case there is a unique irreducible $G$ representation $\pi^{\lambda,0}$ such that

$$\pi^{\lambda,0}|_{G^\circ} = \pi^\lambda \oplus \pi^{s\cdot\lambda}$$

(where $\pi^\mu$ denotes the irreducible $G^\circ$ representation with highest weight $\mu$). If $m_l = 0$, then there are two irreducible representations $\pi^{\lambda,\epsilon}$ $(\epsilon = \pm 1)$ of $G$ whose restriction to $G^\circ$ is $\pi^\lambda$. They are related by

$$\pi^{\lambda,\epsilon} = \det \otimes \pi^{\lambda,-\epsilon}$$

and labeled so that $\pi^{\lambda,\epsilon}(s)$ acts by $\epsilon$ on the $G^\circ$ highest weight vector (see [16, §5.2.2]). Thus $\widehat{G}$ can be written as a disjoint union $\widehat{G} = \widehat{G}_{-1} \cup \widehat{G}_0 \cup \widehat{G}_1$, where

$$\widehat{G}_{\pm 1} = \{\pi^{\lambda,\epsilon} : \mathrm{depth}(\lambda) < l,\ \epsilon = \pm 1\}, \qquad \widehat{G}_0 = \{\pi^{\lambda,\epsilon} : \mathrm{depth}(\lambda) = l,\ \epsilon = 0\}.$$

**Theorem 10.4.** *($G = \mathrm{O}(\mathbb{C}^k, \omega)$, $k = 2l$) Let $\Sigma$ be the spectrum of $G$ on the $G$-harmonic polynomials $\mathcal{H} \subset \mathcal{P}(M_{n\times k})$.*

(a) *Assume $k \le n$. Then $\Sigma = \widehat{G}$ and thus $\Sigma$ does not depend on $n$ ($G$-stable range).*

(b) *Assume $l < n < k$. Then $\widehat{G}_1 \cup \widehat{G}_0 \subset \Sigma$, whereas*

$$\Sigma \cap \widehat{G}_{-1} = \{(\lambda, -1) : k - n \le \mathrm{depth}(\lambda) < l\}$$

*(unstable range: $\Sigma$ depends on $k$ and $n$).*

(c) *Assume $n = l$. Then $\Sigma = \widehat{G}_1 \cup \widehat{G}_0$.*

(d) *Assume $n < l$. Then $\Sigma = \{(\lambda, 1) : \mathrm{depth}(\lambda) \le n\} \subset \widehat{G}_1$.*

*Thus $\Sigma$ does not depend on $k$ when $n \le l$ ($\mathrm{GL}(n)$-stable range). The duality correspondence is given as follows: Let $\lambda = [m_1, \ldots, m_d, 0, \ldots, 0] \in \mathbb{N}^l_{++}$ have depth $d$ with $1 \le d \le \min\{l, n\}$. Then*

$$\tau(\sigma) = \begin{cases} [\underbrace{0, \ldots, 0}_{n-d}, -m_d, \ldots, -m_1] & \text{for } \sigma = (\lambda, \epsilon) \in \Sigma \cap (\widehat{G}_1 \cup \widehat{G}_0), \\[2ex] [\underbrace{0, \ldots, 0}_{n-k+d}, \underbrace{-1, \ldots, -1}_{k-2d}, -m_d, \ldots, -m_1] & \text{for } \sigma = (\lambda, -1) \in \Sigma \cap \widehat{G}_{-1}. \end{cases}$$

To prove the theorem, we will find a set of generators for the joint eigenfunctions of $B_n \times B$ in $\mathcal{H}$. For $u \in M_{n\times l}$, $p \le n$, and $q \le l$, define matrices $L_{p,q}(u)$ and $R_{p,q}(u)$ as in the proof of Theorem 10.1. In this case we write $z = [\ x \quad y\ ]$ as in (10.1) and we define

$$f_j(z) = \det L_{j,j}(x) \quad \text{for } 1 \le j \le \min\{l, n\}.$$

If $n \ge l + 1$ then we also define

$$g_j(z) = \det \left[\ L_{j,l}(x) \quad R_{j,j-l}(y)\ \right] \quad \text{for } l + 1 \le j \le \min\{k, n\}.$$

**Lemma 10.5.** (a) *Let* $1 \le j \le \min\{l, n\}$. *Then* $f_j \in \mathcal{H}^{(j)}$ *and* $f_j$ *is a* $B_n \times B$ *eigenfunction of weight* $(\mu, \nu)$, *where*

$$\mu = -\varepsilon_{n-j+1} - \cdots - \varepsilon_n \quad \text{and} \quad \nu = \varepsilon_1 + \cdots + \varepsilon_j\,.$$

*Furthermore* $f_j(zs) = f_j(z)$ *if* $j < l$, *where* $s \in G$ *is the reflection* $e_l \leftrightarrow e_{2l}$.

(b) *Suppose* $n \ge l+1$ *and take* $l+1 \le j \le \min\{n, k\}$. *Then* $g_j \in \mathcal{H}^{(j)}$ *and* $g_j$ *is a* $B_n \times B$ *eigenfunction of weight* $(\mu, \gamma)$, *where*

$$\mu = -\varepsilon_{n-j+1} - \cdots - \varepsilon_n \quad \text{and} \quad \gamma = \varepsilon_1 + \cdots + \varepsilon_{k-j}$$

*(here* $\gamma = 0$ *if* $j = k$*). Furthermore* $g_j(zs) = -g_j(z)$, *with* $s \in G$ *as in (a).*

**Corollary 10.6.** *Let* $\mathbf{m} = [\, m_1, \dots, m_r\,] \in \mathbb{N}^r_{++}$, *where* $r = \min\{l, n\}$. *Assume* $\mathbf{m}$ *has depth* $d$ *and set* $\lambda = [\,\mathbf{m}, 0, \dots, 0\,] \in \mathbb{N}^l_{++}$. *Define* $\varphi_{\mathbf{m}} = f_1^{m_1 - m_2} \cdots f_{d-1}^{m_{d-1} - m_d}\, f_d^{m_d}$ *(when* $\mathbf{m} = 0$ *set* $\varphi_0 = 1$*).*

(a) $\varphi_{\mathbf{m}}$ *is a* $G$*-harmonic polynomial, homogeneous of degree* $|\mathbf{m}|$. *Furthermore,* $\varphi_{\mathbf{m}}$ *is a* $B_n \times B$ *eigenfunction of weight* $(\alpha, \lambda)$, *where*

$$\alpha = [\underbrace{0, \dots, 0}_{n-d}, -m_d, \dots, -m_1\,].$$

*(when* $\mathbf{m} = 0$ *take* $\alpha = 0$*). Set* $\varphi^s_{\mathbf{m}}(z) = \varphi_{\mathbf{m}}(zs)$. *Then* $\varphi^s_{\mathbf{m}} = \varphi_{\mathbf{m}}$ *when* $d < l$.

(b) *Let* $n > l$. *If* $d < l$ *and* $n - k + d \ge 0$ *define* $\psi_{\mathbf{m}} = \varphi_{\mathbf{m}}\, g_{k-d}/f_d$ *(when* $\mathbf{m} = 0$ *set* $\psi_0 = g_k$*). Then* $\psi_{\mathbf{m}}$ *is a* $G$*-harmonic polynomial, homogeneous of degree* $|m| + k - 2d$. *Furthermore* $\psi_{\mathbf{m}}$ *is a* $B_n \times B$ *eigenfunction of weight* $(\beta, \lambda)$, *where*

$$\beta = [\underbrace{0, \dots, 0}_{n-k+d}, \underbrace{-1, \cdots, -1}_{k-2d}, -m_d, \dots, -m_1\,]$$

*(when* $\mathbf{m} = 0$ *take* $\beta = [\underbrace{0, \dots, 0}_{n-k}, \underbrace{-1, \cdots, -1}_{k}]$*). Set* $\psi^s_{\mathbf{m}}(z) = \psi_{\mathbf{m}}(zs)$. *Then* $\psi^s_{\mathbf{m}} = -\psi_{\mathbf{m}}$.

To prove Theorem 10.4, assume first that $n \ge k$. By Corollary 10.6 the functions $\varphi_{\mathbf{m}}$ are defined for all $\mathbf{m} \in \mathbb{N}^l_{++}$. If $\mathbf{m}$ has depth $l$ then the right $G$-translates of $\varphi_{\mathbf{m}}$ span an irreducible subspace of type $(\mathbf{m}, 0)$. If $\mathbf{m}$ has depth less than $l$ then $\psi_{\mathbf{m}}$ is also defined. In this case the right $G$-translates of $\varphi_{\mathbf{m}}$ span a $G$-irreducible subspace of type $(\mathbf{m}, 1)$, whereas the right $G$-translates of $\psi_{\mathbf{m}}$ span an irreducible subspace of type $(\mathbf{m}, -1)$. Thus we get all irreducible representations of $G$ in $\mathcal{H}$, as asserted in part (a) of the theorem.

The argument when $n < k$ proceeds as in the proof of Theorem 10.1 by lifting harmonic $B_n \times B$ eigenfunctions from $M_{n\times k}$ to $M_{k\times k}$. Note that $\varphi_{\mathbf{m}}$ is defined for all $\mathbf{m}$ of depth $d \le \min\{n, l\}$, whereas $\psi_{\mathbf{m}}$ is only defined when $n > l$ and $k - n \le d < l$. We omit the details. □

10.3. **Examples of Harmonic Decompositions. 1.** Assume $n \le l$ and $k = 2l + 1$ or $2l$, so that we are in case (c) of Theorem 10.1 or cases (c) and (d) of Theorem 10.4. The restrictions to $\mathrm{SO}(k)$ of the representations in $\Sigma$ are the *class* $n$ representations of $\mathrm{SO}(k)$–those that have a vector fixed under the subgroup $\mathrm{SO}(k - n)$. This follows from the branching law (see [16, §8.1]). In this case the harmonic polynomials on $M_{n\times k}$ decompose under $\mathrm{GL}(n) \times \mathrm{SO}(k)$ as

$$\mathcal{H} \cong \bigoplus_{\lambda \in \mathbb{N}^n_{++}} \mathcal{E}^{\lambda^{\check{}}} \otimes \mathcal{V}^{\lambda} \tag{10.5}$$

Here $\lambda^{\check{}} = [-m_n, \dots, -m_1]$ and $\mathcal{V}^{\lambda}$ is the irreducible $\mathrm{SO}(k)$ module with highest weight $\lambda$ (when $n = l$, $k = 2l$ is even and $m_l \ne 0$, then $\mathcal{V}^{\lambda}$ is the sum of the irreducible representations with highest weights $\lambda$ and $s\cdot\lambda$). For $n = 1$, (10.5) is the classical spherical harmonic decomposition and gives the decomposition of polynomials restricted to the sphere $\mathrm{SO}(k)/\mathrm{SO}(k-1)$. For $n > 1$ (10.5) gives the decomposition of polynomials restricted to the *Stiefel manifold* $\mathrm{SO}(k)/\mathrm{SO}(k - n)$. This decomposition was obtained by Gelbart [12] and Ton-That [33] before Kashiwara and Vergne [23] worked out the general case that we have presented here.

**2.** Now assume $l < n < k$, so that we are in case (b) of Theorems 10.1 and 10.4. The decomposition of the harmonics in this case was obtained by Strichartz [30]. For example, let $n = 2$ and $k = 3$. Then we have the decomposition

$$\mathcal{H} = \Big\{ \bigoplus_{m\geq 0} \mathcal{E}^{[0,-m]} \otimes \mathcal{V}^{[m]} \Big\} \oplus \Big\{ \bigoplus_{m\geq 1} \mathcal{E}^{[-1,-m]} \otimes \mathcal{V}^{[m]} \Big\}$$

of the harmonic polynomials on $M_{2\times 3}$. Here $\mathcal{V}^{[m]}$ denotes the irreducible SO(3) representation with highest weight $m\varepsilon_1$. The $B_2 \times B$ harmonic eigenfunction $\varphi_{(m)}(z) = x_2^m$ generates the summand $\mathcal{E}^{[0,-m]} \otimes \mathcal{V}^{[m]}$. The $B_2 \times B$ harmonic eigenfunction $\psi_{(m)}(z) = x_2^{m-1}(x_1t_2 - x_2t_1)$ generates the summand $\mathcal{E}^{[-1,-m]} \otimes \mathcal{V}^{[m]}$. Here we write

$$z = \begin{bmatrix} x_1 & y_1 & t_1 \\ x_2 & y_2 & t_2 \end{bmatrix}.$$

**3.** Let $n = 3$ and $k = 3$ so that we are in case (a) of Theorem 10.1. Then we have the decomposition

$$\mathcal{H} = \Big\{ \bigoplus_{m\geq 0} \mathcal{E}^{[0,0,-m]} \otimes \mathcal{V}^{[m]} \Big\} \oplus \Big\{ \mathcal{E}^{[-1,-1,-1]} \otimes \mathcal{V}^{[0]} \Big\} \oplus \Big\{ \bigoplus_{m\geq 1} \mathcal{E}^{[0,-1,-m]} \otimes \mathcal{V}^{[m]} \Big\}$$

of the harmonic polynomials on $M_{3\times 3}$ as a module for $\mathrm{GL}(3) \times \mathrm{SO}(3)$. The $B_3 \times B$ harmonic eigenfunction $\varphi_{(m)}(z) = x_3^m$ generates the summand $\mathcal{E}^{[0,0,-m]} \otimes \mathcal{V}^{[m]}$. The $B_3 \times B$ harmonic eigenfunction $\psi_{(m)}(z) = x_3^{m-1}(x_2t_3 - x_3t_2)$ generates the summand $\mathcal{E}^{[0,-1,-m]} \otimes \mathcal{V}^{[m]}$. Here we write

$$z = \begin{bmatrix} x_1 & y_1 & t_1 \\ x_2 & y_2 & t_2 \\ x_3 & y_3 & t_3 \end{bmatrix}.$$

For $m = 0$ the function $\psi_{(0)}(z) = \det z$ generates the one-dimensional summand $\mathcal{E}^{[-1,-1,-1]} \otimes \mathcal{V}^{[0]}$.

Let $\mathbb{C}^{(m)}$ denote the one-dimension representation $g \mapsto (\det g)^m$ of GL(3). Let $\varpi_1 = [1,0,0]$ and $\varpi_2 = [1,1,0]$. Then the GL(3) representations occurring in $\mathcal{H}$ are $\mathbb{C}^{(-1)}$,

$$\mathcal{E}^{[0,0,-m]} \cong \mathbb{C}^{(-m)} \otimes \mathcal{E}^{m\varpi_2}$$

for all $m \geq 0$, and

$$\mathcal{E}^{[0,-1,-m]} \cong \mathbb{C}^{(-m)} \otimes \mathcal{E}^{\varpi_1 + (m-1)\varpi_2}$$

for all $m \geq 1$.

## Lecture 11. Symplectic Group and Oscillator Representation

We now turn to the functional-analytic aspects of the harmonic duality decomposition in Theorem 9.2 (recall Example 4 in Section 2.3). If we replace the complex group $G$ by its compact real form $G_0 = G \cap \mathrm{U}(V)$ then the finite-dimensional representations $\mathcal{F}^\sigma$ remain irreducible under $G_0$ and the action of $G_0$ is unitary relative to the Fischer inner product.

We would like to have a similar picture for the dual representations $E^\lambda$ (where $\lambda = \tau(\sigma) + \delta$). At the Lie algebra level it is clear that to obtain a unitary representation, we should take the real form $\mathfrak{g}_0'$ of $\mathfrak{g}'$ that acts by skew-hermitian operators relative to the Fischer inner product. The analytic problem is to construct a unitary representation of an associated real Lie group $G_0'$ on the completion of $\mathcal{P}(V)$, and to describe its action on the Hilbert space completions of the infinite-dimensional spaces $E^\lambda$.

We will construct $G_0'$ as a subgroup of the *metaplectic group* $\mathrm{Mp}(nk, \mathbb{R})$ (the two-sheeted cover of the real symplectic group $\mathrm{Sp}(nk, \mathbb{R})$). The associated unitary representation will be the restriction to $G_0'$ of the *oscillator representation* of the metaplectic group. This representation already appears in the harmonic decomposition as a Lie algebra representation by elements of degree 2 in the Weyl algebra. However, when we try to exponentiate it to a unitary group representation, we encounter the conflict between the particle and the wave description of quantum mechanics; the representation has a simple description (the holomorphic model) relative to the maximal compact subgroup $K_0 \cong \mathrm{U}(n)$ of $\mathrm{Sp}(n, \mathbb{R})$, and another simple description (the real-wave model) relative to the maximal parabolic subgroup $P \cong \mathrm{GL}(n, \mathbb{R}) \ltimes SM_n(\mathbb{R})$ of $\mathrm{Sp}(n, \mathbb{R})$. In both descriptions $K_0 \cap P \cong \mathrm{O}(n)$ acts geometrically, but some of the remaining group elements act in a more subtle way. Thus it will be necessary to consider two matrix forms of the real symplectic group and the intertwining operator (the Bargmann-Segal transform) that relates the two versions of the oscillator representation.

11.1. **Real Symplectic Group.** Let $\mathrm{Sp}(n,\mathbb{C})$ be the subgroup of $\mathrm{GL}(2n,\mathbb{C})$ that preserves the skew-form

$$\Omega(x,y)=\sum_{i=1}^{n}(x_i y_{n+i}-x_{n+i}y_i)$$

on $\mathbb{C}^{2n}$. Thus $g\in \mathrm{Sp}(n,\mathbb{C})$ if and only if $g^t J_n g = J_n$, where $g^t$ denotes matrix transpose and $J_n$ is the matrix

$$J_n=\begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}.$$

We can also describe $\mathrm{Sp}(n,\mathbb{C})$ as the fixed-point group of the involution $\tau : g \mapsto J_n(g^t)^{-1}J_n^{-1}$ on $\mathrm{GL}(2n,\mathbb{C})$.

The Lie algebra $\mathfrak{sp}(n,\mathbb{C})$ of $\mathrm{Sp}(n,\mathbb{C})$ consists of all $X\in M_{2n}$ such that $J_nX+X^tJ_n=0$. These matrices have block form

$$X=\begin{bmatrix} A & B \\ C & -A^t \end{bmatrix} \quad \text{with } A\in M_n \text{ and } B,C\in SM_n.$$

Here we use the notation $M_n$ for the $n\times n$ complex matrices and $SM_n$ for the $n\times n$ symmetric complex matrices.

The *real symplectic group* $\mathrm{Sp}(n,\mathbb{R})=\mathrm{Sp}(n,\mathbb{C})\cap \mathrm{GL}(2n,\mathbb{R})$. Its Lie algebra $\mathfrak{sp}(n,\mathbb{R})$ consists of all the real matrices in $\mathfrak{sp}(n,\mathbb{C})$.

*Maximal Compact Subgroup.* A fundamental technique for studying a unitary representation of a real reductive group such as $\mathrm{Sp}(n,\mathbb{R})$ is to restrict the representation to a maximal compact subgroup, under which the representation space decomposes as the (Hilbert-space) direct sum of multiples of irreducible (finite-dimensional) subspaces. The *real orthogonal group* $\mathrm{O}(k)\subset \mathrm{U}(k)$ is the subgroup of real unitary matrices. Since $\mathrm{Sp}(n,\mathbb{C})$ and $\mathrm{Sp}(n,\mathbb{R})$ are invariant under the map $g\mapsto g^*$, the groups

$$\mathrm{Sp}(n)=\mathrm{Sp}(n,\mathbb{C})\cap \mathrm{U}(2n) \quad \text{and} \quad \mathrm{Sp}(n,\mathbb{R})\cap \mathrm{U}(2n)=\mathrm{Sp}(n,\mathbb{R})\cap \mathrm{O}(2n)$$

are maximal compact subgroups of $\mathrm{Sp}(n,\mathbb{C})$ and $\mathrm{Sp}(n,\mathbb{R})$, respectively (see [25, Proposition 1.2]). The subgroup of diagonal matrices in $\mathrm{Sp}(n)$ is a maximal torus in $\mathrm{Sp}(n)$. However, the subgroup of diagonal matrices in $\mathrm{Sp}(n,\mathbb{R})\cap \mathrm{O}(2n)$ is finite and is *not* a maximal torus in $\mathrm{Sp}(n,\mathbb{R})$. Hence it is convenient to replace $\mathrm{Sp}(n,\mathbb{R})$ by an isomorphic real form $G_0$ so that the diagonal matrices in $G_0\cap \mathrm{U}(2n)$ comprise a maximal (compact) torus in $G_0$.

Define

$$I_{n,n}=\begin{bmatrix} I_n & 0 \\ 0 & -I_n \end{bmatrix}$$

and let $\sigma$ be the *conjugation* (conjugate-holomorphic involution) $\sigma(g)=I_{n,n}(g^*)^{-1}I_{n,n}$ on $\mathrm{GL}(2n,\mathbb{C})$. The fixed-point set of $\sigma$ is the real form $\mathrm{U}(n,n)$ of $\mathrm{GL}(2n,\mathbb{C})$. Set

$$K_n=I_{n,n}J_n=\begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix}.$$

Then $J_n^{-1}I_{n,n}=K_n$, so it follows that $\sigma\tau=\tau\sigma$. Hence $\sigma$ leaves $\mathrm{Sp}(n,\mathbb{C})$ invariant and its restriction to $\mathrm{Sp}(n,\mathbb{C})$ defines a conjugation of $\mathrm{Sp}(n,\mathbb{C})$ which we continue to denote as $\sigma$. If $g\in \mathrm{Sp}(n,\mathbb{C})$ then $\sigma(g)=\sigma\tau(g)=K_n\bar{g}K_n$. In terms of the $n\times n$ block decomposition, $\sigma$ acts by

$$\sigma\begin{bmatrix} A & B \\ C & D \end{bmatrix}=\begin{bmatrix} \bar{D} & \bar{C} \\ \bar{B} & \bar{A} \end{bmatrix}.$$

Define $G_0=\{g\in \mathrm{Sp}(n,\mathbb{C}) : \sigma(g)=g\}$. Then $G_0$ is a real form of $\mathrm{Sp}(n,\mathbb{C})$. Its Lie algebra $\mathfrak{g}_0=\mathrm{Lie}(G_0)$ consists of all matrices $X\in\mathfrak{sp}(n,\mathbb{C})$ such that $\sigma(X)=X$. In terms of the block decomposition, $\mathfrak{g}_0$ consists of the matrices

$$X=\begin{bmatrix} A & B \\ \bar{B} & \bar{A} \end{bmatrix}, \quad A^*=-A,\ B=B^t. \tag{11.1}$$

**Lemma 11.1.** *The subgroup $K_0 = G_0 \cap \mathrm{U}(2n)$ is a maximal compact subgroup of $G_0$ and consists of all matrices*

$$\begin{bmatrix} A & 0 \\ 0 & \bar{A} \end{bmatrix}, \quad \text{with } A \in \mathrm{U}(n).$$

*Hence $K_0 \cong \mathrm{U}(n)$ and the subgroup of diagonal matrices in $K_0$ is a maximal compact torus of $G_0$.*

Define an involution $\theta$ on $\mathrm{Sp}(n, \mathbb{C})$ by

$$\theta(g) = I_{n,n} g I_{n,n}$$

(note that $\theta$ is an inner automorphism of $\mathrm{Sp}(n, \mathbb{C})$). If $g \in G_0$ then $(g^t)^{-1} = J_n g J_n^{-1}$ and $\bar{g} = K_n g K_n$. Hence $(g^*)^{-1} = J_n \bar{g} J_n^{-1} = J_n K_n g K_n J_n^{-1}$. Since $J_n K_n = I_{n,n}$, it follows that

$$\theta(g) = (g^*)^{-1} \quad \text{for } g \in G_0.$$

Thus the maximal compact subgroup $K_0$ is the fixed-point set of $\theta$ in $G_0$. Its complexification is

$$K = \{g \in \mathrm{Sp}(n, \mathbb{C}) \,:\, \theta(g) = g\}.$$

Note that if $g \in \mathrm{GL}(2n, \mathbb{C})$, then $\theta(g) = g$ if and only if

$$g = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}, \quad a, d \in \mathrm{GL}(n, \mathbb{C}).$$

If $g \in K$, then in this block decomposition $d = (a^t)^{-1}$. Hence $K \cong \mathrm{GL}(n, \mathbb{C})$ via the homomorphism

$$a \mapsto \begin{bmatrix} a & 0 \\ 0 & (a^t)^{-1} \end{bmatrix}.$$

The complexification of the Lie algebra $\mathfrak{k}_0$ of $K_0$ is the Lie algebra $\mathfrak{k} \cong \mathfrak{gl}(n, \mathbb{C})$ of $K$.

The involution $\theta$ gives a decomposition of $\mathfrak{sp}(n, \mathbb{C})$. The $+1$ eigenspace of $\theta$ on $\mathfrak{sp}(n, \mathbb{C})$ is $\mathfrak{k}$, whereas the $-1$ eigenspace is

$$\mathfrak{p} = \left\{ \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \,:\, B, C \in SM_n \right\}.$$

We have $\mathfrak{sp}(n, \mathbb{C}) = \mathfrak{k} \oplus \mathfrak{p}$ with commutation relations

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}.$$

The center of $\mathfrak{k}$ is spanned by $I_{n,n}$ and $\mathfrak{k} = \mathbb{C} I_{n,n} \oplus [\mathfrak{k}, \mathfrak{k}]$, with the derived algebra $[\mathfrak{k}, \mathfrak{k}] \cong \mathfrak{sl}(n, \mathbb{C})$. The $\pm 1$ eigenspaces of $\mathrm{ad} I_{n,n}$ on $\mathfrak{p}$ are

$$\mathfrak{p}_+ = \left\{ \begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix} \,:\, B \in SM_n \right\}, \quad \mathfrak{p}_- = \left\{ \begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix} \,:\, C \in SM_n \right\}.$$

These subspaces are invariant under $\mathfrak{k}$ and have the commutation relations

$$[\mathfrak{p}_+, \mathfrak{p}_+] = 0, \quad [\mathfrak{p}_-, \mathfrak{p}_-] = 0, \quad [\mathfrak{p}_+, \mathfrak{p}_-] \subset \mathfrak{k}.$$

Thus there is a triangular decomposition

$$\mathfrak{sp}(n, \mathbb{C}) = \mathfrak{p}_- \oplus \mathfrak{k} \oplus \mathfrak{p}_+$$

(as we already noted in Lecture 8). The conjugation $\sigma$ interchanges $\mathfrak{p}_+$ and $\mathfrak{p}_-$, since

$$\sigma\left( \begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 \\ \bar{B} & 0 \end{bmatrix}$$

for $B \in SM_n$.

We can describe these decompositions in terms of root spaces as follows (see [16, §2.3.1]). The complexification $T$ of $T_0$ is a maximal (algebraic) torus in $\mathrm{Sp}(n, \mathbb{C})$ and has Lie algebra

$$\mathfrak{t} = \{X = \mathrm{diag}[x_1, \ldots, x_n, -x_1, \ldots, -x_n] \,:\, x_j \in \mathbb{C}\}.$$

The set of roots $\Phi = \Phi(\mathfrak{g}, \mathfrak{t})$ of $\mathfrak{t}$ on $\mathfrak{g}$ is $\pm\varepsilon_i \pm \varepsilon_j$ for $1 \le i, j \le n$, where $\varepsilon_i(X) = x_i$ for $X \in \mathfrak{t}$ as above. We have $\Phi = \Phi_c \cup \Phi_n$, where

$$\Phi_c = \{\pm(\varepsilon_i - \varepsilon_j) \,:\, 1 \le i < j \le n\}$$

is the set of *compact* roots (the roots of $\mathfrak{t}$ on $\mathfrak{k}$) and

$$\Phi_n = \{\pm(\varepsilon_i + \varepsilon_j) \,:\, 1 \le i \le j \le n\}$$

is the set of *noncompact* roots (the roots of $\mathfrak{t}$ on $\mathfrak{p}$).

Take the set of positive roots $\Phi^+$ to be $\varepsilon_i \pm \varepsilon_j$ for $1 \le i \le j \le n$, and let $\Phi_c^+$ (*respectively* $\Phi_n^+$) be the positive compact (*respectively* noncompact) roots. Then

$$\mathfrak{k} = \mathfrak{t} + \sum_{\alpha\in\Phi_c} \mathfrak{g}_\alpha \qquad \mathfrak{p}_\pm = \sum_{\beta\in\Phi_n^+} \mathfrak{g}_{\pm\beta}$$

The simple roots in $\Phi^+$ are $\alpha_1, \dots, \alpha_n$, where

$$\alpha_i = \varepsilon_i - \varepsilon_{i+1} \quad \text{for } i = 1, \dots, n-1 \text{ and } \alpha_n = 2\varepsilon_n.$$

The unique simple non-compact root is the long root $\alpha_n$, and the highest root is $\gamma = 2\varepsilon_1$ (it is noncompact). Let $\rho$ be one-half the sum of the positive roots. Then

$$\rho = n\varepsilon_1 + (n-1)\varepsilon_2 + \cdots + \varepsilon_n. \tag{11.2}$$

*Cayley Transform.* We now show that the group $G_0 = \mathrm{Sp}(n,\mathbb{C}) \cap \mathrm{U}(n,n)$ is conjugate to $\mathrm{Sp}(n,\mathbb{R})$ within $\mathrm{Sp}(n,\mathbb{C})$. To understand this in terms of the adjoint representation of $\mathfrak{sp}(n,\mathbb{C})$, consider first the case $n = 1$ (recall that $\mathrm{Sp}(1,\mathbb{C}) = \mathrm{SL}(2,\mathbb{C})$). Set

$$\mathbf{k} = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}, \quad \mathbf{x} = \frac{1}{2}\begin{bmatrix} 1 & -i \\ -i & -1 \end{bmatrix}, \quad \mathbf{y} = \frac{1}{2}\begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix}.$$

where $i$ is a fixed choice of $\sqrt{-1}$. Then $[\mathbf{k},\mathbf{x}] = 2\mathbf{x}$, $[\mathbf{k},\mathbf{y}] = -2\mathbf{y}$, $[\mathbf{x},\mathbf{y}] = \mathbf{k}$, so $\{\mathbf{x},\mathbf{y},\mathbf{k}\}$ is a TDS (three-dimensional simple) triple. Furthermore, the one-parameter subgroup

$$t \mapsto \exp(it\mathbf{k}) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}, \quad t \in \mathbb{R},$$

is a maximal compact torus in $\mathrm{SL}(2,\mathbb{R})$. Let

$$\mathbf{h} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \mathbf{e} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{f} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

be the standard TDS in $\mathfrak{sl}(2,\mathbb{C})$. We can conjugate $\{\mathbf{x},\mathbf{y},\mathbf{k}\}$ to $\{\mathbf{e},\mathbf{f},\mathbf{h}\}$ as follows: Since $\mathbf{x}+\mathbf{y} = \mathbf{h}$, we have

$$(\mathrm{ad}(\mathbf{y}-\mathbf{x}))(\mathbf{k}) = 2\mathbf{h}, \quad (\mathrm{ad}(\mathbf{y}-\mathbf{x}))(\mathbf{h}) = -2\mathbf{k},$$

and so

$$e^{t\,\mathrm{ad}(\mathbf{y}-\mathbf{x})}\mathbf{k} = (\cos 2t)\mathbf{h} + (\sin 2t)\mathbf{k}, \quad \text{for } t \in \mathbb{C}.$$

Setting $t = \pi/4$, we obtain

$$e^{(\pi/4)\mathrm{ad}(\mathbf{y}-\mathbf{x})}\mathbf{k} = \mathbf{h}.$$

Since $(\mathbf{y}-\mathbf{x})^2 = -I$, we have

$$\exp[t(\mathbf{y}-\mathbf{x})] = (\cos t)I + (\sin t)(\mathbf{y}-\mathbf{x}), \quad \text{for } t \in \mathbb{C}.$$

Define

$$\mathbf{c} = \exp[\frac{\pi}{4}(\mathbf{y}-\mathbf{x})] = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}.$$

Then $\mathbf{c}(i\mathbf{k})\mathbf{c}^{-1} = i\mathbf{h}$. Thus $\mathbf{c}$ conjugates the compact torus in $\mathrm{SL}(2,\mathbb{R})$ generated by $i\mathbf{k}$ to the compact torus in $G_0$ generated by $i\mathbf{h}$:

$$\mathbf{c}\begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}\mathbf{c}^{-1} = \begin{bmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{bmatrix}, \quad t \in \mathbb{R},$$

The automorphism $g \mapsto \mathbf{c}g\mathbf{c}^{-1}$ is called the *Cayley transform.*

A similar construction works in $\mathrm{Sp}(n,\mathbb{C})$ (and for other real semisimple Lie groups). Set

$$\mathbf{c} = \frac{1}{\sqrt{2}}\begin{bmatrix} I_n & iI_n \\ iI_n & I_n \end{bmatrix}.$$

Then $\mathbf{c} \in \mathrm{Sp}(n)$ and $\mathbf{c}^{-1} = \bar{\mathbf{c}}$.

**Lemma 11.2.** *Let* $G_0 = \mathrm{U}(n,n) \cap \mathrm{Sp}(n,\mathbb{C})$. *Then* $\mathbf{c}^{-1}G_0\mathbf{c} = \mathrm{Sp}(n,\mathbb{R})$.

*Maximal Parabolic Subgroup.* Let $P$ be the subgroup of $\mathrm{Sp}(n,\mathbb{R})$ consisting of the matrices

$$\begin{bmatrix} A & B \\ 0 & (A^t)^{-1} \end{bmatrix}, \quad A \in \mathrm{GL}(n,\mathbb{R}),\ B \in SM_n(\mathbb{R}),$$

where $SM_n(\mathbb{R})$ denotes the real $n \times n$ symmetric matrices. The group $P$ is a *maximal parabolic subgroup* of $\mathrm{Sp}(n,\mathbb{R})$. It has the structure of a semidirect product $MN$, where $M$ consists of the block-diagonal matrices

$$\begin{bmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{bmatrix}, \quad A \in \mathrm{GL}(n,\mathbb{R})$$

and $N$ consists of the matrices

$$\begin{bmatrix} I_n & B \\ 0 & I_n \end{bmatrix}, \quad B \in SM_n(\mathbb{R}).$$

Thus as Lie groups $M \cong \mathrm{GL}(n,\mathbb{R})$ and $N \cong SM_n(\mathbb{R})$ (an abelian group). The group $K_0 \cap P$ consists of all matrices

$$\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}, \quad A \in \mathrm{O}(n)$$

where $\mathrm{O}(n) = \{g \in \mathrm{GL}(n,\mathbb{R}) : g^t g = I_n\}$ is the usual real orthogonal group. Define $N^- = N^t$. Thus $N^-$ is the group of matrices

$$\begin{bmatrix} I_n & 0 \\ C & I_n \end{bmatrix}, \quad C \in SM_n(\mathbb{R}).$$

Then $P^- = MN^-$ is the *opposite parabolic* subgroup to $P$ and $P \cap P^- = M$. Note that

$$J_n \begin{bmatrix} I_n & B \\ 0 & I_n \end{bmatrix} J_n^{-1} = \begin{bmatrix} I_n & 0 \\ -B & I_n \end{bmatrix} \quad \text{and } J_n \begin{bmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{bmatrix} J_n^{-1} = \begin{bmatrix} (A^t)^{-1} & 0 \\ 0 & A \end{bmatrix}.$$

Thus $P^- = J_n P J_n^{-1}$ is conjugate to $P$ in $\mathrm{Sp}(n,\mathbb{R})$.

Let

$$g = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

be in $\mathrm{Sp}(n,\mathbb{R})$. If $\det A \neq 0$, then $D = (A^t)^{-1} + CB$ (this follows from $g^t J_n g = J_n$). Hence we can factor $g$ as

$$g = \begin{bmatrix} I_n & 0 \\ C & I_n \end{bmatrix} \begin{bmatrix} A & B \\ 0 & (A^t)^{-1} \end{bmatrix} \in N^- MN.$$

Thus the subset $N^-MN$ is open and dense in $\mathrm{Sp}(n,\mathbb{R})$. This shows that a continuous representation $\pi$ of $\mathrm{Sp}(n,\mathbb{R})$ is uniquely determined by its restriction to the subgroup $P$ together with the single operator $\pi(J_n)$.

11.2. **Holomorphic (coherent-state) Model for Oscillator Representation.** We write $\mathbb{H}^2(\mathbb{C}^n) = \mathbb{H}^2(\mathbb{C}^n, e^{-||z||^2} d\lambda(z))$ for the Hilbert-space completion of $\mathcal{P}(\mathbb{C}^n)$ relative to the Fischer inner product introduced in Lecture 9. The elements of this space are naturally identified with holomorphic functions $f$ on $\mathbb{C}^n$ such that

$$\int_{\mathbb{C}^n} |f(z)|^2 \, e^{-||z||^2} d\lambda(z) < \infty.$$

For each $w \in \mathbb{C}^n$ the function $K_w(z) = e^{\langle z|w \rangle}$ is in $\mathbb{H}^2(\mathbb{C}^n)$ and plays the role of *reproducing kernel* for this space:

$$f(w) = \langle f \mid K_w \rangle \quad \text{for } f \in \mathbb{H}^2(\mathbb{C}^n)$$

(see [7, Prop. XI.1.1]).

Define the *annihilation operators* $A_j$ and *creation operators* $A_j^\dagger$ by

$$A_j f(z) = \frac{\partial}{\partial z_j} f(z), \quad A_j^\dagger f(z) = z_j f(z) \quad \text{for } f \in \mathcal{P}(\mathbb{C}^n).$$

These operators satisfy the commutation relations $[A_i, A_j^\dagger] = \delta_{ij} I$. The are mutually adjoint relative to the Fischer inner product:

$$\langle A_j \varphi \mid \psi \rangle = \langle \varphi \mid A_j^\dagger \psi \rangle \tag{11.3}$$

for $\varphi, \psi \in \mathcal{P}(\mathbb{C}^n)$. The operators $\{A_1, \ldots, A_n, A_1^\dagger, \ldots, A_n^\dagger\}$ generate the Weyl algebra $\mathcal{PD}(\mathbb{C}^n)$.

Define a representation $\varpi$ of $\mathfrak{sp}(n,\mathbb{C})$ on $\mathcal{P}(\mathbb{C}^n)$ as follows. For $b \in SM_n$ let $Q_b(z) = z^t b z$ be the quadratic form on $\mathbb{C}^n$ defined by $b$. Let

$$X = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \in \mathfrak{p}_+, \quad Y = \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix} \in \mathfrak{p}_-,$$

where $b, c \in SM_n$. Define operators $\varpi(X)$ and $\varpi(Y)$ on $\mathcal{P}(\mathbb{C}^n)$ by

$$\varpi(X)f(z) = \frac{1}{2i}\partial(Q_b)f(z), \quad \varpi(Y)f(z) = \frac{1}{2i}Q_c(z)f(z)$$

where $i = \sqrt{-1}$ (recall the map $Q \mapsto \partial(Q)$ from $\mathcal{P}(\mathbb{C}^n)$ to constant-coefficient differential operators on $\mathcal{P}(\mathbb{C}^n)$ that was introduced in the proof of Theorem 9.2). We calculate that

$$[\varpi(X), \varpi(Y)] = -\sum_{j,k}(bc)_{kj}\, z_j \frac{\partial}{\partial z_k} - \frac{1}{2}\mathrm{tr}(bc).$$

Let $H = \begin{bmatrix} h & 0 \\ 0 & -h^t \end{bmatrix} \in \mathfrak{k}$, where $h \in M_n$. Define the operator $\varpi(H)$ by

$$\varpi(H) = -\sum_{j,k} h_{kj} z_j \frac{\partial}{\partial z_k} - \frac{1}{2}\mathrm{tr}(h)$$

Since $[X, Y] = \begin{bmatrix} bc & 0 \\ 0 & -cb \end{bmatrix}$, we see that $[\varpi(X), \varpi(Y)] = \varpi([X, Y])$. One calculates that

$$[\varpi(H), \varpi(X)] = \varpi([H, X]), \quad [\varpi(H), \varpi(Y)] = \varpi([H, Y])$$

(note that $H \mapsto \varpi(H)$ is the standard representation of $\mathfrak{k} \cong \mathfrak{gl}(n,\mathbb{C})$ on $\mathcal{P}(\mathbb{C}^n)$ tensored with the one-dimensional representation $H \mapsto -\frac{1}{2}\mathrm{tr}(h)$). Thus $\varpi$ is a representation of $\mathfrak{sp}(n,\mathbb{C})$ on $\mathcal{P}(\mathbb{C}^n)$ (this is the representation of $\mathfrak{g}' = \mathfrak{sp}(n,\mathbb{C})$ in Theorem 8.5 for the case $G = \mathrm{O}(1,\mathbb{C})$).

We can write the representation $\varpi$ in terms of the annihilation and creation operators as

$$\varpi(X) = \frac{1}{2i}\sum_{j,k} b_{jk} A_j A_k, \quad \varpi(Y) = \frac{1}{2i}\sum_{j,k} c_{jk} A_j^\dagger A_k^\dagger$$

$$\varpi(H) = -\frac{1}{2}\sum_{j,k} h_{jk}(A_j A_k^\dagger + A_k^\dagger A_j)$$

for $X, Y, H$ as above. Now take $Y = \sigma(X)$, where $\sigma$ is the conjugation defining the real form $\mathfrak{g}_0$ of $\mathfrak{sp}(n,\mathbb{C})$. Then $c_{jk} = \bar{b}_{jk}$, so from (11.3) we see that

$$\langle \varpi(X)\varphi \mid \psi \rangle = -\langle \varphi \mid \varpi(\sigma(X))\psi \rangle \tag{11.4}$$

for all $X \in \mathfrak{p}_+$ and $\varphi, \psi \in \mathcal{P}(\mathbb{C}^n)$. Since $\sigma(\mathfrak{p}_\pm) = \mathfrak{p}_\mp$ and $\mathfrak{p}_+, \mathfrak{p}_-$ generate $\mathfrak{sp}(n,\mathbb{C})$, it follows that relation (11.4) holds for all $X \in \mathfrak{sp}(n,\mathbb{C})$. Since $\mathfrak{g}_0$ is the fixed-point set of $\sigma$, we conclude that

$$\langle \varpi(Z)\varphi \mid \psi \rangle = -\langle \varphi \mid \varpi(Z)\psi \rangle \quad \text{for } Z \in \mathfrak{g}_0.$$

One says that $\varpi$ is a *unitarizable* representation of $\mathfrak{g}_0$.

By the Cartan decomposition $G_0 = K_0 \exp(\mathfrak{p}_0)$ the group $K_0$ is a topological retract of $G_0$. Since $\mathfrak{k}_0$ has a one-dimensional center, it follows that $G_0$ has an $m$-sheeted covering group for every integer $m$. Let $\gamma : \mathrm{Mp}(n,\mathbb{R}) \to G_0$ be the two-sheeted covering and let $\widetilde{K}_0 = \gamma^{-1}(K_0)$. Then $\widetilde{K}_0$ is a two-sheeted covering of $\mathrm{U}(n)$. If $\widetilde{k} \in \widetilde{K}_0$ and $z \in \mathbb{C}^n$ we set $\widetilde{k} \cdot z = uz$, where

$$\gamma(\widetilde{k}) = \begin{bmatrix} u & 0 \\ 0 & \bar{u} \end{bmatrix} \quad \text{with } u \in \mathrm{U}(n).$$

The function $\chi : \widetilde{k} \mapsto \det(u)^{-1/2}$ is a (single-valued) character of $\widetilde{K}_0$.

**Theorem 11.3.** *There is a unitary representation $\varpi$ of $\mathrm{Mp}(n,\mathbb{R})$ on $\mathbb{H}^2(\mathbb{C}^n)$ whose differential is the representation $\varpi$ of $\mathfrak{g}_0$. If $\widetilde{k} \in \widetilde{K}_0$ then*

$$\varpi(\widetilde{k})f(z) = \chi(\widetilde{k})f(\widetilde{k}^{-1} \cdot z) \quad \textit{for } f \in \mathbb{H}^2(\mathbb{C}^n).$$

*Proof.* Let
$$H_0 = \begin{bmatrix} iI_n & 0 \\ 0 & -iI_n \end{bmatrix} \in \mathfrak{k}_0.$$
Take the basis for $\varpi(\mathfrak{sp}(n,\mathbb{C}))$ to be the operators $A_iA_j$ and $A_i^\dagger A_j^\dagger$ for $1 \le i \le j \le n$ and $\frac{1}{2}(A_i^\dagger A_j + A_jA_i^\dagger)$ for $1 \le i, j \le n$. Define the seminorms $\rho_k$ on $\mathcal{P}(\mathbb{C}^n)$ in terms of this basis and the norm $||\varphi||$ as in Section 11.5.

**Lemma 11.4.** *The inequality*
$$\rho_1(\varphi) \le ||\varphi|| + ||\varpi(H_0)\varphi|| \tag{11.5}$$
*holds for all* $\varphi \in \mathcal{P}(\mathbb{C}^n)$.

*Proof of Theorem* 11.3: From Lemma 11.4 and Corollary 11.9 we obtain a unitary representation $\varpi$ of the universal covering group of $\mathrm{Sp}(n,\mathbb{R})$ whose differential is the representation $\varpi$. Now $H_0$ spans the center of $\mathfrak{k}_0$ and $\exp(2\pi H_0) = I$ in $\mathrm{Sp}(n,\mathbb{C})$. Since the one-parameter unitary group $\varpi(\exp(tH_0))$ acts by $e^{it(k+n/2)}$ on $\mathcal{P}^k(\mathbb{C}^n)$, we have $\varpi(\exp(4\pi H_0)) = I$. Hence the representation $\varpi$ descends to a single-valued representation of $\mathrm{Mp}(n,\mathbb{R})$. $\square$

We will call $\varpi$ the *oscillator representation* of $\mathrm{Sp}(n,\mathbb{R})$. We write $\varpi^{(n)}$ if the dependence on $n$ is not evident from the context. An explicit formula for $\varpi(g)$ as an integral operator was obtained by Bargmann (see [8, Theorem 4.37]).

11.3. **Bargmann-Segal Transform.** To obtain a realization of the oscillator representation in which the action of the maximal parabolic subgroup $P$ of $\mathrm{Sp}(n,\mathbb{R})$ is easily described, we use another representation of the creation and annihilation operators. Let $\mathcal{S}(\mathbb{R}^n)$ denote the Schwartz space of rapidly-decreasing smooth functions on $\mathbb{R}^n$. Define *creation* and *annihilation* operators
$$a_j^\dagger = \frac{1}{\sqrt{2}}\left(-\frac{\partial}{\partial x_j} + x_j\right), \qquad a_j = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_j} + x_j\right)$$
on $\mathcal{S}(\mathbb{R}^n)$ for $1 \le j \le n$. These operators satisfy the commutation relations $[a_i,\, a_j^\dagger] = \delta_{ij}I$ and they are mutually adjoint relative to the $L^2(\mathbb{R}^n,\, d\lambda(x))$ inner product. They leave invariant the space $\mathcal{P}(\mathbb{R}^n)$ of (complex-valued) polynomial functions on $\mathbb{R}^n$ and act irreducibly on this space.

Following [2] and [29], we construct a unitary operator
$$B : L^2(\mathbb{R}^n,\, d\lambda(x)) \to \mathbb{H}^2(\mathbb{C}^n,\, e^{-||z||^2} d\lambda(z))$$
that intertwines $a_j$ with $A_j$ and $a_j^\dagger$ with $A_j^\dagger$. Since the space $L^2(\mathbb{R}^n)$ (*respectively* $\mathbb{H}^2(\mathbb{C}^n)$) is the $n$-fold Hilbert-space tensor product of the space $L^2(\mathbb{R})$ (*respectively* $\mathbb{H}^2(\mathbb{C})$), it suffices to do the calculation for the case $n = 1$.

Because $\mathbb{H}^2$ has a reproducing kernel, any such operator $B$ will be given as
$$Bf(z) = \int_{-\infty}^{\infty} B(z,x)f(x)\,dx.$$
To intertwine the two pairs of creation-annihilation operators, the kernel $B(z,x)$ must be a holomorphic function of $z$ and smooth function of $x$ that satisfies
$$\begin{aligned} \frac{\partial}{\partial z}B(z,x) &= \frac{1}{\sqrt{2}}\left(-\frac{\partial}{\partial x} + x\right)B(z,x) \\ zB(z,x) &= \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x} + x\right)B(z,x) \end{aligned}$$
These equations imply that
$$\frac{\partial B}{\partial z} = (\sqrt{2}x - z)B \quad \text{and} \quad \frac{\partial B}{\partial x} = (\sqrt{2}z - x)B.$$
The solution is easily found to be
$$B(x,z) = C\exp\left\{\sqrt{2}xz - \frac{1}{2}(x^2+z^2)\right\}$$

with $C$ a constant. It remains to verify that the operator defined by this kernel is unitary (with appropriate choice of $C$). For this, take $p \in \mathcal{P}(\mathbb{R})$ and the normalized Gaussian (ground state)

$$\varphi_0(x) = \frac{1}{\sqrt{\pi}} \exp\left(-\frac{1}{2}x^2\right)$$

(note that $a_1\varphi_0 = 0$). Then

$$B(p\varphi_0)(z) = \frac{C}{\sqrt{\pi}} \int_{-\infty}^{\infty} p(x) \exp\left\{-x^2 + \sqrt{2}xz - \frac{1}{2}z^2\right\} dx.$$

Completing the square in the exponential and using the translation-invariance of the measure $dx$, we obtain

$$B(p\varphi_0)(z) = \frac{C}{\sqrt{\pi}} \int_{-\infty}^{\infty} p\left(x + \frac{z}{\sqrt{2}}\right) e^{-x^2}\, dx.$$

The right side of this equation is obviously a polynomial of the same degree as $p$, so it follows that $B$ is a bijection from $\mathcal{P}(\mathbb{R})\varphi_0$ onto $\mathcal{P}(\mathbb{C})$. Furthermore $B\varphi_0(z) = C$. Since $\varphi_0$ has $L^2$-norm 1 and the constant function 1 has $\mathbb{H}^2$-norm 1, we conclude that $C = 1$.

To complete the proof that $B$ is a unitary operator, we observe that $\mathcal{P}(\mathbb{R})\varphi_0$ is the cyclic space generated by $\varphi_0$ under the action of the operators $a_1$ and $a_1^\dagger$. Likewise $\mathcal{P}(\mathbb{C})$ is the cyclic space generated by the constant function 1 under the action of the operators $A_1$ and $A_1^\dagger$. The creation-annihilation operators act irreducibly on these spaces. Since $B^*$ intertwines the pair $\{a_1, a_1^\dagger\}$ with $\{A_1, A_1^\dagger\}$, it follows that $B^*B$ commutes with $a_1$ and $a_1^\dagger$ on $\mathcal{P}(\mathbb{R})\varphi_0$, while $BB^*$ commutes with $A_1$ and $A_1^\dagger$ on $\mathcal{P}(\mathbb{C})$. Thus $B^*B$ and $BB^*$ are both multiples of the identity by Lemma 2.1. Since we have normalized the kernel $B(x, z)$ so that $B$ carries the unit vector $\varphi_0$ to the unit vector 1, it follows that $B$ is unitary in the case $n = 1$. This implies that $B$ is unitary for any $n$ as remarked above.

For any integer $n \geq 1$ and polynomial $p \in \mathcal{P}(\mathbb{R}^n)$, define

$$B(\varphi_0 p)(z) = \pi^{-n/2} \int_{\mathbb{R}^n} p\left(x + \frac{z}{\sqrt{2}}\right) e^{-x^t x}\, d\lambda(x) \tag{11.6}$$

for $z \in \mathbb{C}^n$, where $\varphi_0(x) = \pi^{-n/2} \exp\left(-\frac{1}{2}x^t x\right)$ for $x \in \mathbb{R}^n$. Notice that if $g \in \mathrm{O}(n)$ then $\varphi_0(gx) = \varphi_0(x)$. Thus if we set $f_g(x) = f(g^{-1}x)$, then $B(f_g) = B(f)_g$ for $f = \varphi_0 p$ and $g \in \mathrm{O}(n)$. We have proved the following.

**Theorem 11.5** (Bargmann-Segal Transform)**.** *The operator $B$ maps the space $\mathcal{P}(\mathbb{R}^n)\varphi_0$ onto $\mathcal{P}(\mathbb{C}^n)$ bijectively. It extends to a unitary operator from $L^2(\mathbb{R}^n; dx)$ onto $\mathbb{H}^2(\mathbb{C}^n)$. If $f \in \mathcal{S}(\mathbb{R}^n)$ then*

$$Bf(z) = \int_{\mathbb{R}^n} f(x) \exp\left\{\sqrt{2}x^t z - \frac{1}{2}(x^t x + z^t z)\right\} d\lambda(x).$$

*Furthermore, $B$ intertwines the representations $f \mapsto f_g$ of $\mathrm{O}(n)$ on $L^2(\mathbb{R}^n; dx)$ and on $\mathbb{H}^2(\mathbb{C}^n)$. Also $Ba_jB^{-1} = A_j$ and $Ba_j^\dagger B^{-1} = A_j^\dagger$ on $\mathcal{P}(\mathbb{C}^n)$.*

**Remark.** Since $B$ is unitary, $B^{-1} = B^*$ is an integral operator on $\mathbb{H}^2(\mathbb{C}^n)$ with kernel $\overline{B(z, x)} = B(\bar{z}, x)$.

11.4. **Real (oscillatory-wave) Model for Oscillator Representation.** We now use the Bargmann-Segal transform to obtain the real-wave (Schrödinger) model of the oscillator representation.

Let $\gamma : \mathrm{Mp}(n, \mathbb{R}) \to \mathrm{Sp}(n, \mathbb{R})$ be the covering homomorphism and let $\mathbf{c} \in \mathrm{Sp}(n)$ be the Cayley transform. We define a unitary representation $\pi$ of $\mathrm{Mp}(n, \mathbb{R})$ on $L^2(\mathbb{R}^n)$ by

$$\pi(g) = B^{-1}\varpi(\mathbf{c}g\mathbf{c}^{-1})B \tag{11.7}$$

(here $\mathbf{c}g\mathbf{c}^{-1} \in \mathrm{Mp}(n, \mathbb{R})$ is the element such that $\gamma(\mathbf{c}g\mathbf{c}^{-1}) = \mathbf{c}\gamma(g)\mathbf{c}^{-1}$). For $f \in \mathcal{S}(\mathbb{R}^n)$ let $\mathcal{F}f$ be the Fourier transform

$$\mathcal{F}(f)(x) = \left(\frac{1}{\pi}\right)^{n/2} \int_{\mathbb{R}^n} e^{ix^t y} f(y)\, dy.$$

**Theorem 11.6.** *The action of $\pi(g)$ on $f \in \mathcal{S}(\mathbb{R}^n)$ is as follows:*

(1) *If* $\gamma(g) = \begin{bmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{bmatrix}$ *with* $A \in \mathrm{GL}(n, \mathbb{R})$ *then* $\pi(g)f(x) = (\det A)^{-1/2} f(A^{-1}x)$.

(2) *If* $\gamma(g) = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}$ *with* $b \in SM_n(\mathbb{R})$ *then* $\pi(g)f(x) = e^{-(i/2)x^t bx}\, f(x)$.

(3) *If* $\gamma(g) = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$ *with* $b \in SM_n(\mathbb{R})$ *then* $\mathcal{F}(\pi(g)f)(x) = e^{(i/2)x^t bx}\, \mathcal{F}f(x)$.

*These formulas uniquely determine $\pi$.*

*Proof.* For $X \in \mathfrak{sp}(n, \mathbb{R})$ write $d\pi(X) = B\varpi(\mathrm{Ad}(\mathbf{c})X)B^{-1}$ for the Lie algebra representation corresponding to $\pi$. Here the operator $d\pi(X)$ acts on $\mathcal{S}(\mathbb{R}^n)$.

(1): Let $X = \begin{bmatrix} h & 0 \\ 0 & -h \end{bmatrix}$ with $h = \mathrm{diag}[h_1, \ldots, h_n]$ and $h_j \in \mathbb{R}$. Then $\mathrm{Ad}(\mathbf{c})X = \begin{bmatrix} 0 & -ih \\ ih & 0 \end{bmatrix}$. Hence the formulas of Section 11.2 give

$$\begin{aligned} \varpi(\mathrm{Ad}(\mathbf{c})X) &= - \frac{1}{2}\sum_{j=1}^{n} h_j \left\{ (A_j)^2 - (A_j^\dagger)^2 \right\} \\ &= - \frac{1}{2}\sum_{j=1}^{n} h_j (A_j + A_j^\dagger)(A_j - A_j^\dagger) - \frac{1}{2}\mathrm{tr}(h). \end{aligned}$$

Now $B^{-1}A_jB = a_j$ and $B^{-1}A_j^\dagger B = a_j^\dagger$. Also we have

$$(a_j + a_j^\dagger)f(x) = \sqrt{2}x_j f(x), \qquad (a_j - a_j^\dagger)f(x) = \sqrt{2}\frac{\partial f}{\partial x_j}(x) \tag{11.8}$$

for $f \in \mathcal{S}(\mathbb{R}^n)$. Thus we obtain

$$d\pi(X) = -\sum_{j=1}^{n} h_j x_j \frac{\partial}{\partial x_j} - \frac{1}{2}\mathrm{tr}(h).$$

This shows that (1) is true for $A = \exp h$ (in this case $\det A > 0$ and there is no need to pass to the metaplectic group). If $A \in \mathrm{O}(n)$ then (1) holds for $g$, since $B$ intertwines the action of $\mathrm{O}(n)$ on $L^2(\mathbb{R}^n)$ and $\mathbb{H}^2(\mathbb{C}^n)$ (note that $\gamma(g)$ is in the maximal compact subgroup $K_0$ of $G_0$, so $\varpi(g)$ is described in Theorem 11.3). By the polar decomposition,

$$\mathrm{GL}(n, \mathbb{R}) = \mathrm{O}(n)(\exp \mathfrak{a})\mathrm{O}(n),$$

where $\mathfrak{a}$ is the subspace of real diagonal matrices. Hence (1) holds for all $A \in \mathrm{GL}(n, \mathbb{R})$.

(2): Let $X = \begin{bmatrix} 0 & 0 \\ b & 0 \end{bmatrix}$ with $b \in SM_n(\mathbb{R})$. Then $\mathrm{Ad}(\mathbf{c})X = \frac{1}{2}\begin{bmatrix} ib & b \\ b & -ib \end{bmatrix}$. Hence the formulas of Section 11.2 give

$$\begin{aligned} \varpi(\mathrm{Ad}(\mathbf{c})X) &= \frac{1}{4i}\sum_{j,k=1}^{n} b_{jk} \left\{ A_jA_k + A_j^\dagger A_k^\dagger + A_jA_k^\dagger + A_k^\dagger A_j \right\} \\ &= \frac{1}{4i}\sum_{j,k=1}^{n} b_{jk}(A_j + A_j^\dagger)(A_k + A_k^\dagger). \end{aligned}$$

Now applying the Bargmann-Segal transform and using (11.8), we obtain

$$d\pi(X)f(x) = \frac{1}{2i}\Big\{ \sum_{j,k=1}^{n} b_{jk}x_jx_k \Big\} f(x).$$

This proves (2).

(3): Let $X = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$ with $b \in SM_n(\mathbb{R})$. Then $\mathrm{Ad}(\mathbf{c})X = \frac{1}{2}\begin{bmatrix} -ib & b \\ b & ib \end{bmatrix}$. Thus

$$\begin{aligned}\varpi(\mathrm{Ad}(\mathbf{c})X) &= \frac{1}{4i}\sum_{j,k=1}^{n} b_{jk}\left\{A_jA_k + A_j^\dagger A_k^\dagger - A_jA_k^\dagger - A_k^\dagger A_j\right\} \\ &= \frac{1}{4i}\sum_{j,k=1}^{n} b_{jk}(A_j - A_j^\dagger)(A_k - A_k^\dagger).\end{aligned}$$

Applying the Bargmann-Segal transform and using (11.8) again, we obtain

$$d\pi(X)f(x) = \frac{1}{2i}\Big\{\sum_{j,k=1}^{n} b_{jk}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k}\Big\}f(x)$$

for $f \in \mathcal{S}(\mathbb{R}^n)$. Since $\mathcal{F}(\partial^2/\partial x_j\partial x_k)\mathcal{F}^{-1}$ is the operator of multiplication by $-x_jx_k$, this proves (3).

Formulas (1), (2), and (3) uniquely determine $\pi$ since $N^-MN$ is dense in $\mathrm{Sp}(n,\mathbb{R})$. □

11.5. **Analytic Vectors.** Here we present refinements of some results of Nelson [28] concerning exponentiation of Lie algebra representations, following the approach in [14]. Suppose $\mathfrak{g}_0$ is a real finite-dimensional Lie algebra, represented as skew-Hermitian (unbounded) operators on a complex inner product space $\mathcal{V}$ (not assumed complete). Let $\mathfrak{g}$ be the complexification of $\mathfrak{g}_0$. Then $X \mapsto X^*$ (the Hermitian adjoint of $X$ relative to the inner product on $\mathcal{V}$) is a conjugate-linear anti-automorphism of $\mathfrak{g}$ such that $X^* = -X$ for $X \in \mathfrak{g}_0$.

Fix a basis $\{X_1, \ldots, X_d\}$ for $\mathfrak{g}$. Define seminorms $\rho_n$ on $\mathcal{V}$ by setting $\rho_0(v) = ||v||$ and

$$\rho_n(v) = \max_{i_1,\ldots,i_n} ||X_{i_1}\cdots X_{i_n}v||$$

for $n = 1, 2, \ldots$. Here $i_1, \ldots, i_n$ run over $1, 2, \ldots, d$ and $||u||$ denotes the norm of $u \in \mathcal{V}$. Let $V$ be the Hilbert-space completion of $\mathcal{V}$ relative to the norm $||v||$ and let $V^\infty$ be the completion of $\mathcal{V}$ relative to the family of seminorms $\{\rho_n\}$. Then the representation of $\mathfrak{g}$ on $\mathcal{V}$ extends continuously to a representation on $V^\infty$. The seminorm $\rho_n(v)$ is also defined for $v \in V^\infty$.

One says that $v \in V^\infty$ is an *analytic vector* for $\mathfrak{g}$ if there is an $r > 0$ such that

$$\sum_{n=0}^{\infty}\frac{r^n}{n!}\rho_n(v) < \infty. \tag{11.9}$$

Let $V_r^\omega \subset V^\infty$ be the subspace for which (11.9) holds. The space $V^\omega = \bigcup_{r>0} V_r^\omega$ of analytic vectors for $\mathfrak{g}$ is invariant under $\mathfrak{g}$.

**Theorem 11.7** (Nelson)**.** *Suppose there exists an $r > 0$ so that $V_r^\omega$ is dense in $V$. Then the representation of $\mathfrak{g}_0$ integrates to a strongly continuous unitary representation on $V$ of the simply-connected Lie group $\widetilde{G}_0$ with Lie algebra $\mathfrak{g}_0$.*

Suppose there is an element $H_0 \in \mathfrak{g}$ such that

$$\rho_1(v) \le ||v|| + ||H_0v|| \quad \text{for all } v \in \mathcal{V}. \tag{11.10}$$

Let $A = \max_{1\le i\le d} ||[H_0, X_i]||$ (the norm of $\mathrm{ad}H_0$ on $\mathfrak{g}$). Note that $A = 0$ if and only if $H_0$ is in the center of $\mathfrak{g}$, and this case is of no interest here. So we assume $A > 0$.

**Theorem 11.8.** *Every analytic vector for $H_0$ is an analytic vector for $\mathfrak{g}$. More precisely, if*

$$\sum_{n=0}^{\infty}\frac{s^n}{n!}||H_0^n v|| < \infty, \quad \textit{for some } s > 0, \tag{11.11}$$

*then $v \in V_r^\omega$ for all $r < \min\{A^{-1}, A^{-1}(1 - e^{-As})\}$.*

*Remark.* If $s$ can be arbitrarily large in (11.11) (one says that $v$ is an *entire vector* for $H_0$ in this case), then $v \in V_r^\omega$ for all $r < A^{-1}$. Note that this upper bound for $r$ is controlled by the non-commutativity of $\mathfrak{g}$ and it is finite if $A \neq 0$. In general $v$ is *not* an entire vector for $\mathfrak{g}$ (see [15] for more precise results along this line).

*Proof.* Let $Y_j$ be any of the basis elements $X_i$. Then the *a priori* estimate (11.10) implies that

$$||Y_{m+1}Y_m \cdots Y_1 v|| \leq ||Y_m \cdots Y_1 v|| + ||H_0 Y_m \cdots Y_1 v||$$

for all $v \in V^\infty$. Now

$$H_0 Y_m \cdots Y_1 = Y_m \cdots Y_1 H_0 + \sum_{k=1}^{m} Y_m \cdots Y_{k-1}[H_0, Y_k] Y_{k+1} \cdots Y_1,$$

and by definition of $\rho_m$ and the constant $A$ we have $||Y_m \cdots [H_0, Y_k] \cdots Y_1 v|| \leq A\rho_m(v)$. Hence

$$\begin{aligned} ||Y_{m+1} \cdots Y_1 v|| &\leq ||Y_m \cdots Y_1 H_0 v|| + (1 + mA)\rho_m(v) \\ &\leq \rho_m(H_0 v) + (1 + mA)\rho_m(v). \end{aligned}$$

Since this holds for any choice of $Y_1, \ldots, Y_{m+1}$, it implies

$$\rho_{m+1}(v) \leq \rho_m(H_0 v) + (1 + mA)\rho_m(v) \quad \text{for all } v \in \mathcal{V}. \tag{11.12}$$

Now fix $v \in V^\infty$ and set $a_{m,n} = \rho_m(H_0^n v)$. Replacing $v$ by $H_0^n v$ in (11.12), we see that the sequence $\{a_{m,n}\}$ satisfies the recursive inequalities

$$a_{m+1,n} \leq a_{m,n+1} + (1 + mA)a_{m,n}.$$

To estimate the rate of growth of $a_{m,n}$, we introduce the *majorant sequence* $b_{m,n}$ defined by $b_{0,n} = a_{0,n}$ for all $n$ and

$$b_{m+1,n} = b_{m,n+1} + (1 + mA)b_{m,n} \quad \text{for all } m \geq 0, n \geq 0.$$

Clearly $a_{m,n} \leq b_{m,n}$ for all $m, n$.

Consider the generating function

$$\varphi(x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{x^m}{m!} \frac{y^n}{n!} b_{m,n}.$$

The recursion for $b_{m,n}$ implies that (as a formal series)

$$(1 - Ax)\frac{\partial}{\partial x}\varphi(x, y) = \frac{\partial}{\partial y}\varphi(x, y) + \varphi(x, y).$$

We assume that the series $f(y) = \varphi(0, y)$ converges for $y = s$. The Cauchy problem for this analytic first-order p.d.e. is easily solved by the method of characteristics, and one obtains

$$\varphi(x, y) = (1 - Ax)^{-1/A} f(y - A^{-1}\log(1 - Ax)).$$

(the analytic solution must agree with the formal solution since the line $x = 0$ is non-characteristic). Setting $y = 0$, we see that the series for $\varphi(x, 0)$ converges absolutely for $|x| \leq r$ provided

$$r < \min\{A^{-1}, A^{-1}(1 - e^{-As})\}.$$

Since $a_{m,0} \leq b_{m,0}$ this proves the theorem. □

**Corollary 11.9.** *Suppose $H_0 \in \mathfrak{g}$ and $\mathcal{V}$ has an (algebraic) basis consisting of eigenvectors for $H_0$. Then $\mathcal{V} \subset V_r^\omega$ for all $r < A^{-1}$ and hence $V_r^\omega$ is dense in $V$ for all $r < A^{-1}$. Thus the representation of $\mathfrak{g}_0$ integrates to a strongly continuous unitary representation of $\widetilde{G}_0$ on $V$.*

*Proof.* If $H_0 v = \lambda v$, then the left side of (11.11) is $e^{s|\lambda|}$, and hence is finite for all $s > 0$. By the remark after Theorem (11.8) this implies that (11.9) holds for all $r < A^{-1}$. Now apply Theorem 11.7 □

## Lecture 12. Dual Pair $\mathrm{Sp}(n, \mathbf{R})$–$\mathrm{O}(k)$

The oscillator representation has many applications to analysis and physics (see [8] and [20], for example). Here we apply it in the context of unitary representation theory and highest weight representations (see [6] for more on this point). To determine which of the representations that occur in the decomposition of the oscillator representation are square-integrable, we apply Harish-Chandra's criterion to the explicit formula for the $\theta$-correspondence that we calculated in Theorems 10.1 and 10.4. In particular, we show that all the square-integrable highest-weight representations of $\mathrm{Sp}(n, \mathbb{R})$ occur in the duality correspondence with $\mathrm{O}(2n)$ (this was first proved by Gelbart [11]).

12.1. **Decomposition of $\mathbf{H}^2(M_{n\times k})$ under** $\mathrm{Mp}(n,\mathbb{R})\times\mathrm{O}(k)$**.** Let $G = \mathrm{O}(k,\mathbb{C}) = \{g \in \mathrm{GL}(k,\mathbb{C}) : gg^t = I\}$ and let $G' = \mathrm{Sp}(n,\mathbb{C}) \subset \mathrm{GL}(2n,\mathbb{C})$ be the symplectic group relative to the skew-form with matrix $J_n$ as in Section 11.1. Define a skew form $\Omega$ on $M_{2n\times k}$ by

$$\Omega(w,z) = \mathrm{tr}(w^t J_n z) \quad \text{for } w, z \in M_{2n\times k}\,.$$

Then $\Omega$ is nondegenerate. We embed $G'\times G$ into $\mathrm{Sp}(M_{2n\times k},\Omega)$ as follows. Let $g \in G$ and $h \in G'$. Then

$$\Omega(hwg, hzg) = \mathrm{tr}(w^t(h^t J_n h) z g g^t) = \Omega(w,z)$$

since $gg^t = I$ and $h^t J_n h = J_n$. Hence we have an injective regular homomorphism $L\times R : G'\times G \to \mathrm{Sp}(M_{2n\times k},\Omega)$ given by

$$R(g)z = zg^{-1}, \quad L(h)z = hz \quad \text{for } g \in G,\ h \in G',\ z \in M_{2n\times k}\,.$$

We identify $M_{2n\times k}$ with $\mathbb{C}^{2nk}$ by the map

$$z = [\mathbf{z}_1,\ldots,\mathbf{z}_k] \mapsto \tilde{z} = \begin{bmatrix} \mathbf{z}_1 \\ \vdots \\ \mathbf{z}_k \end{bmatrix} \in \mathbb{C}^{2nk}$$

where $\mathbf{z}_j \in \mathbb{C}^{2n}$ is the $j$th column of $z$. It is easy to check that

$$\Omega(z,w) = \tilde{z}^t J_{nk}\tilde{w},$$

so $\mathrm{Sp}(M_{2n\times k},\Omega)$ becomes $\mathrm{Sp}(nk,\mathbb{C})$ under this identification. Thus we will view $z$ either as a $2n\times k$ matrix or a vector in $\mathbb{C}^{2nk}$, whichever is more convenient for the calculation at hand.

Define a hermitian form on $M_{2n\times k}$ by

$$(z,w) = \mathrm{tr}(w^* I_{n,n} z),$$

where $I_{n,n}$ is the matrix in Section 11.1. We have $(z,w) = \tilde{w}^* I_{nk,nk}\tilde{z}$, so when $z \in M_{2n\times k}$ is identified with $\tilde{z} \in \mathbb{C}^{2nk}$, the form $(z,w)$ becomes the one used in Lecture 11 to define the group $U(nk,nk)$. Thus we will denote the isometry group of this form as $\mathrm{U}(nk,nk)$. If $g \in \mathrm{U}(n,n)$ then

$$(gz,gw) = \mathrm{tr}(w^* g^* I_{n,n} g z) = (z,w)$$

since $g^* I_{n,n} g = I_{n,n}$. Thus the left multiplication homomorphism $L : \mathrm{GL}(2n,\mathbb{C}) \to \mathrm{GL}(M_{2n\times k})$ carries $\mathrm{U}(n,n)$ into $\mathrm{U}(nk,nk)$. If $h \in \mathrm{U}(k)$ then

$$(zh,wh) = \mathrm{tr}(w^* I_{n,n} z h h^*) = (z,w).$$

Furthermore,

$$\begin{bmatrix} x \\ y \end{bmatrix} h = \begin{bmatrix} xh \\ yh \end{bmatrix}.$$

Hence the right multiplication homomorphism $R : \mathrm{GL}(k,\mathbb{C}) \to \mathrm{GL}(M_{2n\times k})$ carries $\mathrm{U}(k)$ into the maximal compact subgroup $\mathrm{U}(nk)\times\mathrm{U}(nk)$ of $\mathrm{U}(nk,nk)$.

Let $G_0 = G \cap \mathrm{U}(k) = \mathrm{O}(k)$ be the compact real form of $G$, and let $G_0' = G' \cap U(n,n) \cong \mathrm{Sp}(n,\mathbb{R})$ be the real form of $\mathrm{Sp}(n,\mathbb{C})$ as in Section 11.1. Let $K_0 = G_0' \cap \mathrm{U}(2n) \cong \mathrm{U}(n)$ be the maximal compact subgroup of $G_0'$. Then the embedding $L\times R : \mathrm{Sp}(n,\mathbb{C})\times\mathrm{O}(k,\mathbb{C}) \to \mathrm{Sp}(nk,\mathbb{C})$ gives an embedding of the real forms

$$G_0'\times G_0 \longrightarrow \mathrm{Sp}(nk,\mathbb{C})\cap\mathrm{U}(nk,nk) \cong \mathrm{Sp}(nk,\mathbb{R})$$

and carries the maximal compact subgroup $K_0\times G_0$ into the maximal compact subgroup $\mathrm{Sp}(nk,\mathbb{C})\cap(\mathrm{U}(nk)\times\mathrm{U}(nk))$ of $\mathrm{Sp}(nk,\mathbb{C})\cap\mathrm{U}(nk,nk)$. If $u \in \mathrm{U}(n)$ and $k_0 = \begin{bmatrix} u & 0 \\ 0 & \bar{u} \end{bmatrix}$ is the corresponding element of $K_0$, then the pair $(k_0,g) \in K_0\times G_0$ acts on $M_{2n\times k}$ by

$$L(k_0)R(g)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} uxg^t \\ \bar{u}yg^t \end{bmatrix}. \tag{12.1}$$

We now calculate the restriction of the oscillator representation $\varpi^{(nk)}$ to $L(K_0)\times R(G_0)$ in the holomorphic model on $\mathcal{P}(V)$, where $V = M_{n\times k}$. Let $(k_0,g) \in K_0\times G_0$. From (12.1) and Theorem 11.3 we see that

$$\varpi^{(nk)}(L(k_0)R(g))f(x) = (\det u)^{-k/2}(\det g)^{n/2} f(u^{-1}xg) \quad \text{for } f \in \mathcal{P}(V) \tag{12.2}$$

(note that the determinant of the map $x \mapsto u^{-1}xg$ is $(\det u)^{-k}(\det g)^n$). If $k$ and $n$ are both even, formula (12.2) defines a representation of $K_0 \times G_0$. For the general case, let $\widetilde{K}_0 \subset \mathrm{Mp}(n,\mathbb{R})$ be the two-sheeted cover of $K_0$ and let $\widetilde{G}_0$ be the lift of $R(G_0)$ to $\mathrm{Mp}(nk,\mathbb{R})$. Then (12.2) gives a single-valued unitary representation of $\widetilde{K}_0 \times \widetilde{G_0}$. In the following we shall simply drop the factor $(\det g)^{n/2}$ from (12.2) to make the representation single-valued on $G_0$. However, the factor $(\det u)^{-k/2}$ is essential for extending the representation from $\widetilde{K}_0$ to $\mathrm{Mp}(n,\mathbb{R})$. Let $\delta$ denote the differential of this character of $\widetilde{K_0}$.

Let $\mathfrak{g}_0'$ be the Lie algebra of $G_0'$, The complexification of $\mathfrak{g}_0'$ is $\mathfrak{g}' = \mathfrak{sp}(n,\mathbb{C})$. We now calculate the action of $\varpi^{(nk)}(L(\mathfrak{g}'))$. Let $X = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$ and $Y = \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix}$ with $b, c \in SM_n$. Then

$$L(X) = \begin{bmatrix} 0 & L(b) \\ 0 & 0 \end{bmatrix}, \quad L(Y) = \begin{bmatrix} 0 & 0 \\ L(c) & 0 \end{bmatrix},$$

(where $L(b)x = bx$ for $x \in M_{n\times k}$). The quadratic form on $V$ associated with $L(b)$ is

$$Q_{L(b)}(x) = x^t bx = \sum_{i,j} b_{ij}\Big\{\sum_{p=1}^{n} x_{ip}x_{jp}\Big\}.$$

Hence

$$\varpi^{(nk)}(L(X))f(x) = \frac{1}{2}\sum_{i,j} b_{ij}\Big\{\sum_{p=1}^{n}\frac{\partial^2}{\partial x_{ip}\partial x_{jp}}\Big\},$$

and $\varpi^{(nk)}(L(Y))$ is the operator of multiplication by $-\frac{1}{2}Q_{L(c)}$. Since $\mathfrak{p}_\pm$ generate $\mathfrak{g}'$, these operators determine $\varpi^{(nk)}(\mathfrak{g}')$. From Theorem 8.5 we conclude that the algebra $\mathcal{PD}(V)^G$ is generated by $\varpi^{(nk)}(\mathfrak{g}')$.

We recall some notation that was introduced earlier. Let $\mathcal{H}$ denote the space of $G$-harmonic polynomials on $M_{n\times k}$ and let $\Sigma \subset \widehat{G}$ be the spectrum of $G$ on $\mathcal{H}$. Let the map

$$\tau : \Sigma \to \Lambda \subset \mathbb{Z}^n_{++}$$

be as in Theorems 10.1 and 10.4. Let $\mathcal{F}^\sigma \subset \mathcal{H}$ be an irreducible $G$-module in the class $\sigma$. Let $\mathcal{E}^{\tau(\sigma)+\delta} \subset \mathcal{H}$ be the irreducible finite-dimensional $\widetilde{K}$-module with highest weight $\tau(\sigma)+\delta$, as in Theorem 9.2.

Let $V = M_{n\times k}$. Consider the unitary representation of $\mathrm{Mp}(n,\mathbb{R})\times \mathrm{O}(k)$ on $\mathbb{H}^2(V)$, where $\mathrm{Mp}(n,\mathbb{R})$ acts by the restriction of the oscillator representation $\varpi^{(nk)}$ and $\mathrm{O}(k)$ acts geometrically by right multiplication on $V$. For $\sigma \in \Sigma$ let $\mathbb{E}^{\tau(\sigma)+\delta}$ be the closure in $\mathbb{H}^2(V)$ of the $\mathfrak{g}'$-irreducible subspace $\mathcal{P}(V)^G \cdot \mathcal{E}^{\tau(\sigma)+\delta}$.

**Theorem 12.1.** *The spaces $\mathbb{E}^{\tau(\sigma)+\delta}$, for $\sigma \in \Sigma$, are irreducible and mutually inequivalent unitary representations of* $\mathrm{Mp}(n,\mathbb{R})$. *Furthermore, $\mathbb{H}^2(V)$ decomposes as a multiplicity-free Hilbert space orthogonal sum*

$$\mathbb{H}^2(V) = \bigoplus_{\sigma\in\Sigma} \mathbb{E}^{\tau(\sigma)+\delta} \otimes \mathcal{F}^\sigma \tag{12.3}$$

*under the action of* $\mathrm{Mp}(n,\mathbb{R}) \times \mathrm{O}(k)$.

*Remark.* When $k$ is even the character $u \mapsto (\det u)^{-k/2}$ occuring in the oscillator representation is well-defined on $U(n)$ and $\mathbb{E}^{\tau(\sigma)+\delta}$ gives an irreducible unitary representation of $\mathrm{Sp}(n,\mathbb{R})$.

12.2. **Square-integrable Representations of** $\mathrm{Sp}(n,\mathbb{R})$**.** The irreducible unitary representations of $\widetilde{G}_0 = \mathrm{Mp}(n,\mathbb{R})$ that occur in Theorem 12.1 are called *highest-weight* representations. Some of them also appear as discrete summands in the decomposition of the left regular representation of $\widetilde{G}_0$ on $L^2(\widetilde{G}_0)$ (these representations are called *square-integrable*). We now apply Harish-Chandra's criterion [17] to determine which of the representations $\mathbb{E}^{\tau(\sigma)+\delta}$ are square-integrable. It is convenient to give separate statements of the result depending on the parity of $k$.

**Theorem 12.2.** (notation of Theorem 10.1) *Let $k = 2l+1$ be odd. Let $\sigma \in \Sigma$.*

(a) *If $n > l+1$ then $\mathbb{E}^{\tau(\sigma)+\delta}$ is never square-integrable.*

(b) *If $n = l+1$ then $\mathbb{E}^{\tau(\sigma)+\delta}$ is square-integrable if and only if $\sigma = (\lambda, -1) \in \widehat{G}_{-1}$ and* $\mathrm{depth}(\lambda) = l$.

(c) *If $n \le l$ then $\mathbb{E}^{\tau(\sigma)+\delta}$ is square-integrable for all $\sigma \in \Sigma$.*

*Proof.* The general condition on the highest weight $\lambda$ for square-integrability is

$$\langle \lambda + \rho,\, \gamma\check{}\,\rangle < 0, \tag{12.4}$$

where $\rho$ is the one-half the sum of the positive roots and $\gamma\check{}$ is the coroot to the highest noncompact root $\gamma$. For $\mathfrak{sp}(n, \mathbb{R})$ we have $\rho = [n, n-1, \ldots, 2, 1]$ and $\gamma = 2\varepsilon_1$, so $\gamma\check{} = \varepsilon_1$ (see Section 11.1). We must check this condition when $\lambda = \tau(\sigma) + \delta$, with $\delta = [-k/2, \ldots, -k/2]$.

Let $\tau(\sigma)_1$ denote the first coordinate of $\tau(\sigma)$. Then $\langle \lambda + \rho, \gamma\check{}\,\rangle = \tau(\sigma)_1 - k/2 + n$. Since $k = 2l + 1$, the Harish-Chandra condition (12.4) is

$$\tau(\sigma)_1 < l + 1 - n. \tag{12.5}$$

Case (a): $n > l + 1$. The formulas for $\tau(\sigma)$ in Theorem 10.1 show that $\tau(\sigma)_1$ is either 0 or $-1$ in this case. But $l - n + 1 \le -1$, so (12.5) is never satisfied.

Case (b): $n = l + 1$. Now the right side of (12.5) is zero. The formulas for $\tau(\sigma)$ show that $\tau(\sigma)_1 < 0$ if and only if $\sigma = (\lambda, -1) \in \widehat{G}_{-1}$ with $d = l$.

Case (c): $n \le l$. Now the right side of (12.5) is positive. The formulas for $\tau(\sigma)$ show that $\tau(\sigma)_1 \le 0$ for all $\sigma \in \Sigma$, so (12.5) is always satisfied. □

**Theorem 12.3.** (notation of Theorem 10.4) *Let $k = 2l$ be even. Let $\sigma \in \Sigma$.*

(a) *If $n > l$ then $\mathbb{E}^{\tau(\sigma)+\delta}$ is never square-integrable.*

(b) *If $n = l$ then $\mathbb{E}^{\tau(\sigma)+\delta}$ is square-integrable if and only if $\sigma = (\lambda, 0) \in \widehat{G}_0$ and* $\operatorname{depth}(\lambda) = l$.

(c) *If $n < l$ then $\mathbb{E}^{\tau(\sigma)+\delta}$ is square-integrable for all $\sigma \in \Sigma$.*

*Proof.* When $k = 2l$ the Harish-Chandra condition (12.4) becomes

$$\tau(\sigma)_1 < l - n. \tag{12.6}$$

Case (a): $n > l$. The formulas for $\tau(\sigma)$ in Theorem 10.4 show that $\tau(\sigma)_1$ is either 0 or $-1$ in this case. But $l - n \le -1$, so (12.6) is never satisfied.

Case (b): $n = l$. Now the right side of (12.6) is zero. The formulas for $\tau(\sigma)$ show that $\tau(\sigma)_1 < 0$ if and only if $\sigma = (\lambda, 0) \in \widehat{G}_0$ and $\operatorname{depth}(\lambda) = l$.

Case (c): $n < l$. Now the right side of (12.6) is positive. The formulas for $\tau(\sigma)$ show that $\tau(\sigma)_1 \le 0$ for all $\sigma \in \Sigma$, so (12.6) is always satisfied. □

*Examples.*

**1.** Assume $k$ is even. Then the oscillator representation $\varpi^{(nk)}$ is single-valued on $\mathrm{Sp}(n, \mathbb{R})$. If $k \ge 2n$ then we see from the formula for the $\theta$-correspondence that every $\mathrm{GL}(n, \mathbb{C})$-highest weight $\lambda$ that satisfies the Harish-Chandra inequality is of the form $\tau(\sigma) + \delta$, for some $\sigma \in \Sigma$. Thus every highest-weight discrete-series representation of $\mathrm{Sp}(n, \mathbb{R})$ occurs in the reduction of $\varpi^{(nk)}$ in this case.

**2.** Let $\sigma$ be the trivial representation of $\mathrm{O}(k)$ (denoted by $\pi^{(0,1)}$ in Sections 10.1 and 10.2). Then $\sigma \in \Sigma$ for all $n \ge 1$ and $\tau(\sigma) = 0$, by Theorems 10.1 and 10.4. The representation, call it $\pi^+$, of $\mathrm{Mp}(n, \mathbb{R})$ that corresponds to $\sigma$ is square integrable if and only if $2n < k$. It occurs with multiplicity one in $\mathbb{H}^2(M_{n\times k})$, and has highest weight $\delta = [-k/2, \ldots, -k/2]$. This weight parameterizes the one-dimensional representation $g \mapsto \det(g)^{-k/2}$ of the maximal compact subgroup of $\mathrm{Mp}(n, \mathbb{R})$. Since $\mathcal{F}^\sigma$ consists of the constant functions, the space $\mathbb{H}^2_+ := \mathbb{E}^\delta$ of $\pi^+$ is the completion (in the Fischer norm) of the space $\mathcal{P}(M_{n\times k})^G$, where $G = \mathrm{O}(k, \mathbb{C})$.

**3.** Let $\sigma$ be the representation $g \mapsto \det(g)$ of $\mathrm{O}(k)$ (denoted by $\pi^{(0,-1)}$ in Sections 10.1 and 10.2). Then $\sigma \in \Sigma$ if and only if $n \ge k$. In this case

$$\tau(\sigma) = [\,\underbrace{0, \ldots, 0}_{n-k},\, \underbrace{-1, \ldots, -1}_{k}\,]$$

by Theorems 10.1 and 10.4. The representation, call it $\pi^-$, of $\mathrm{Mp}(n,\mathbb{R})$ that corresponds to $\sigma$ is never square integrable. It occurs with multiplicity one in $\mathbf{H}^2(M_{n\times k})$ when $n \geq k$, and it has highest weight

$$\lambda = [\underbrace{-k/2,\ldots,-k/2}_{n}] + [\underbrace{0,\ldots,0}_{n-k},\, \underbrace{-1,\ldots,-1}_{k}].$$

This is the highest weight of the representation $(\det)^{-k/2} \otimes \bigwedge^k(\mathbb{C}^n)^*$ of the maximal compact subgroup of $\mathrm{Mp}(n,\mathbb{R})$. In this case $\mathcal{F}^\sigma = \mathbb{C}g_k$, where $g_k(z)$ is the determinant of the bottom $k\times k$ block of $z \in M_{n\times k}$ when we take the orthogonal group $G = \mathrm{O}(\mathbb{C}^k,\omega)$ as in Lecture 10. Thus the space $\mathbb{H}^2_- := \mathbb{E}^\lambda$ of $\pi^-$ is the completion (in the Fischer norm) of the space $(\mathcal{P}(M_{n\times k})^G\, g_k) \otimes \bigwedge^k(\mathbb{C}^n)^*$. Note that for fixed $n$, one obtains a distinguished set of $n$ irreducible unitary highest-weight representations of $\mathrm{Mp}(n,\mathbb{R})$ this way by taking $k = 1,\ldots,n$.

**4.** By the harmonic duality theorem, $(\mathbb{H}^2_\pm, \pi^\pm)$ are the only irreducible $\mathrm{Mp}(n,\mathbb{R})$ modules that occur with multiplicity one in $\mathbb{H}^2(M_{n\times k})$. More details and other models for the representations $\mathbb{E}^{\tau(\sigma)+\delta}$ can be found in [6].

*Final Remarks.* In Schur-Weyl duality we took tensor powers of the representation of $\mathrm{GL}(n,\mathbb{C})$ on $\mathbb{C}^n$ (the representation of smallest dimension) to obtain all the irreducible finite-dimensional polynomial representations of $\mathrm{GL}(n,\mathbb{C})$. The two irreducible components $\pi^\pm$ of the oscillator representation on $\mathbb{H}^2(\mathbb{C}^n)$ are the smallest unitary highest-weight representations of $\mathrm{Mp}(n,\mathbb{R})$ in the sense of *Gelfand-Kirillov dimension* (see [32]). As we already noted in Section 11.3 the representation on $\mathbb{H}^2(M_{n\times k})$ is the $k$-fold tensor product of this representation:

$$\mathbb{H}^2(M_{n\times k}) = \bigotimes^k \mathbb{H}^2(\mathbb{C}^n) \quad \text{(Hilbert-space tensor product).} \tag{12.7}$$

Thus the action of the group $\mathrm{O}(k)$ on the right-side of (12.7) is another instance of a *hidden symmetry*.[j]

## Lecture 13. Brauer Algebra and Tensor Harmonics

In this final lecture we use duality to decompose the space of $k$-tensors under the action of the orthogonal or symplectic group $G$. This was first done by Brauer [4], who determined the generators and relations of the $G$-centralizer algebra. The complication here is that this algebra is not a group algebra (as was the case when $G = \mathrm{GL}(n,\mathbb{C})$). However, just as in the case of Howe duality, there is an analog of the *harmonic duality* of Lecture 9 in this situation. The centralizer algebra contains $\mathbb{C}[\mathfrak{S}_k]$ as a subalgebra, and there is a subspace of *harmonic tensors* (in Weyl's terminology *completely traceless*) which decomposes in a multiplicity-free way under the jointly commuting actions of $G$ and $\mathfrak{S}_k$. The full space of $k$-tensors then decomposes as the sum of spaces of *partially harmonic* tensors (see [16, §10.3], [9], and [10] for details).

13.1. **Centralizer Algebra and Brauer Diagrams.** Let $G$ be the full isometry group of a nondegenerate bilinear form $\omega$ on a finite-dimensional complex vector space $V$. We assume $\omega$ to be either symmetric or skew-symmetric. For $f \in V^*$ define $f^\flat \in V$ by

$$\omega(f^\flat, v) = \langle f, v\rangle \quad \text{for all } v \in V.$$

The map $f \mapsto f^\flat$ is then a $G$-isomorphism between $V^*$ and $V$. Define a $G$-module isomorphism $T : V^{*\otimes 2k} \to \mathrm{End}(V^{\otimes k})$ by

$$T(f_1 \otimes \cdots \otimes f_{2k})u = \omega(f_2^\flat \otimes f_4^\flat \otimes \cdots \otimes f_{2k}^\flat,\, u)\, f_1^\flat \otimes f_3^\flat \otimes \cdots \otimes f_{2k-1}^\flat \tag{13.1}$$

for $f_i \in V^*$ and $u \in V^{\otimes k}$. Here we have extended $\omega$ to a bilinear form on $V^{\otimes k}$ by

$$\omega(u_1 \otimes \cdots \otimes u_k,\, v_1 \otimes \cdots \otimes v_k) = \prod_{i=1}^k \omega(u_i, v_i) \quad \text{for } u_i, v_i \in V.$$

[j]The unitary representations that occur in the decomposition of this tensor product are the mathematical analog of the elementary particles, some familiar and some exotic, that physicists create by high-energy collisions of the basic particles.

**Theorem 13.1.** *Let $\Xi_k$ be the set of two-partitions of $\{1, \ldots, 2k\}$. For $\xi \in \Xi_k$ let $\lambda_\xi \in (V^{*\otimes 2k})^G$ be the corresponding complete contraction. Then*

$$\mathrm{End}_G(V^{\otimes k}) = \mathrm{Span}\{T(\lambda_\xi) : \xi \in \Xi_k\}.$$

Theorem 13.1 only gives a spanning set for the centralizer algebra $\mathrm{End}_G(V^{\otimes k})$ as a vector space. To describe the multiplicative structure of this algebra it is convenient to introduce a graphic presentation of the set of two-partitions. We display the set $\{1, 2, \ldots, 2k\}$ as an array of two rows of $k$ labeled dots, with the dots in the top row labeled $1, 3, \ldots, 2k-1$ from left to right, and the dots in the bottom row labeled $2, 4, \ldots, 2k$. Consider the set $X_k$ of all (unoriented) graphs whose vertices are the two rows of dots, and such that each dot is connected with exactly one other dot by an edge. (A dot in the top row can be connected either with another dot in the top row or with a dot in the bottom row.) An example with $k = 5$ is shown in Figure 1. We call an element of $X_k$ a *Brauer diagram.*[k] Thus we can identify the

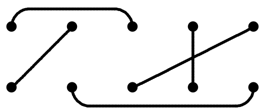
$x_1 =$ 

FIGURE 1. A Brauer Diagram.

set $\Xi_k$ of two-partitions with $X_k$; if $\xi \in \Xi_k$ corresponds to the Brauer diagram $x \in X_k$, we shall write $\lambda_x$ for the complete contraction $\lambda_\xi$.

The group $\mathfrak{S}_{2k}$ acts transitively on $X_k$ by permuting the dots according to their labels. If $x \in X_k$ and $s \in \mathfrak{S}_{2k}$ then $s \cdot x$ is the graph obtained by permuting the dots by $s$ and maintaining the edge connections (dot $s(i)$ is connected to dot $s(j)$ in $s \cdot x$ if and only if dot $i$ is connected to dot $j$ in $x$). Clearly

$$\sigma_{2k}^*(s)\lambda_x = \lambda_{s\cdot x} \quad \text{for } s \in \mathfrak{S}_{2k} \text{ and } x \in X_k. \tag{13.2}$$

Here $\sigma_k$ denotes the representation of $\mathfrak{S}_k$ on $V^{\otimes k}$, as in Lecture 3, and $\sigma_{2k}^*$ is the contragredient representation on $V^{*\otimes 2k}$. Let $x_0$ be the graph with each dot in the top row connected with the dot below it (see Figure 2 for the case $k = 5$). Then the Brauer diagram $x_1$ in Figure 1 is $s \cdot x_0$ where $s \in \mathfrak{S}_{10}$ is the cyclic permutation (2594).

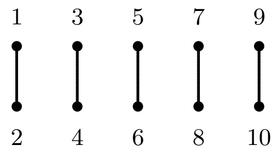
$x_0 =$ 


FIGURE 2. Basic Diagram with Labeled Dots.

Let $\tau : \mathfrak{S}_k \to \mathfrak{S}_{2k}$ be defined by $\tau(s)(2j-1) = 2s(j)-1$ and $\tau(s)(2j) = 2j$ for $j = 1, \ldots, k$. If $s \in \mathfrak{S}_k$, then $\tau(s)$ acts on a Brauer diagram by permuting the top row of dots according to $s$ while leaving each dot in the bottom row fixed. Clearly $\tau$ is an injective homomorphism, and from (13.1) and (13.2) we see that

$$\sigma_k(s)T(\lambda_x) = T(\lambda_{\tau(s)\cdot x}) \quad \text{for } s \in \mathfrak{S}_k \text{ and } x \in X_k. \tag{13.3}$$

For the basic diagram $x_0$ we have $\tau(s) \cdot x_0 = x_0$ if and only if $s$ is the identity, so the permutations in $\mathfrak{S}_k$ correspond to the diagrams in the orbit $\tau(\mathfrak{S}_k) \cdot x_0$ (these diagrams are just the *two-line* notation for a permutation). Hence by (13.3) the operators in $\mathrm{End}_G(V^{\otimes k})$ associated with the orbit of $x_0$ come from the natural action of $\mathfrak{S}_k$ on $V^{\otimes k}$. In particular, the basic diagram $x_0$ corresponding to the $G$-invariant tensor $\omega^{\otimes k}$ gives the identity operator on $V^{\otimes k}$.

The complete set of $\tau(\mathfrak{S}_k)$ orbits on $X_k$ can be described as follows. For $x \in X_k$ let $r$ be the number of edges in the diagram of $x$ that connect a dot in the top row with another dot in the top row (call such an edge a *top bar*). The bottom row of $x$ also must have $r$ such edges (call them *bottom bars*), and we call

[k]Kerov [24] uses the term *chip* because of the analogy with an integrated circuit chip, where the top row of dots are the input ports and the bottom row of dots the output ports. For a development of Kerov's approach, see [9] and [10].

$x$ an *r-bar diagram.* All diagrams in the $\tau(\mathfrak{S}_k)$-orbit of $x$ also have $r$ top bars, and there is a unique $z$ in this orbit with all its edges either horizontal or vertical (that is, if $z$ is considered as a two-partition of $2k$, then every odd-even pair $\{2i-1, 2j\}$ that occurs in $z$ has $i = j$). We will call such a Brauer diagram (or two-partition) *normalized.* The normalized diagrams give a set of representatives for the $\tau(\mathfrak{S}_k)$ orbits on $X_k$. For example, when $k = 3$ and $r = 1$ then there are three orbits of 1-bar diagrams, with normalized representatives indicated in Figure 3. These orbits correspond to the two-partitions

$$z_1 = \{\{1,2\},\{3,5\},\{4,6\}\}, \quad z_2 = \{\{1,5\},\{2,6\},\{3,4\}\}, \quad z_3 = \{\{1,3\},\{2,4\},\{5,6\}\}.$$

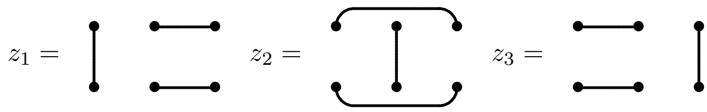



FIGURE 3. Normalized 1-Bar Brauer Diagrams ($k = 3$).

If $z$ is a normalized Brauer diagram, then for every top bar in $z$ joining the dots numbered $2i-1$ and $2j-1$ there is a corresponding bottom bar joining the dots numbered $2i$ and $2j$. We will say that $z$ contains an *$(i,j)$-bar* in this case (with the convention that $i < j$). For example, the normalized diagram in the orbit $\tau(\mathfrak{S}_5)x_1$ (with $x_1$ from Figure 1) is shown in Figure 4; it contains a $(2,5)$-bar.

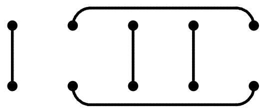


FIGURE 4. Normalized Brauer Diagram with $(2,5)$-Bar.

13.2. **Generators for the Centralizer Algebra.** A normalized Brauer diagram determines an element of the algebra $\mathrm{End}_G(V^{\otimes k})$ by Theorem 13.1. For example, the diagram shown in Figure 5 contains a single $(1,2)$-bar corresponding to the tensor $\sigma^*_{2k}(23)\omega^{\otimes k}$, where $(23)$ is the transposition $2 \leftrightarrow 3$. Since $\sigma^*_{2k}(23)\omega^{\otimes k} = (\sigma^*_4(23)\omega^{\otimes 2}) \otimes \omega^{\otimes(k-2)}$, we have

$$\begin{aligned} T(\sigma^*_{2k}(23)\omega^{\otimes k})\, v_1 \otimes v_2 \otimes u &= \Big\{ \sum_{p_2} \omega(v_1, f_{p_2})\omega(v_2, f^{p_2}) \Big\} \sum_{p_1} f_{p_1} \otimes f^{p_1} \otimes u \\ &= \omega(v_1, v_2)\, \theta \otimes u \end{aligned}$$

for $v_1, v_2 \in V$ and $u \in V^{\otimes(k-2)}$. Here $\{f_p\}$ and $\{f^p\}$ are bases for $V$ with $\omega(f_p, f^q) = \delta_{pq}$, and

$$\theta = \sum_p f_p \otimes f^p \in (V \otimes V)^G$$

is the tensor dual to $\omega$. Thus this 1-bar diagram gives an operator $\tau_{12} = T(\sigma_{2k}(23)\theta_k)$ which is the composition

$$V^{\otimes k} \xrightarrow{C_{12}} V^{\otimes(k-2)} \xrightarrow{D_{12}} V^{\otimes k},$$

with $C_{12}(v_1 \otimes v_2 \otimes u) = \omega(v_1, v_2)u$ a *contraction operator* (contract the first and second tensor positions by $\omega$) and $D_{12}(u) = \theta \otimes u$ an *expansion operator* (multiply on the left by $\theta$). These operators obviously intertwine the actions of $G$ on $V^{\otimes k}$ and $V^{\otimes(k-2)}$.

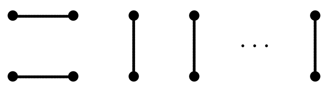


FIGURE 5. Brauer Diagram for $\tau_{12} = D_{12}C_{12}$.

In general, for any pair $1 \le i < j \le k$ we define the *ij-contraction operator* $C_{ij} : V^{\otimes k} \to V^{\otimes(k-2)}$ by

$$C_{ij}(v_1 \otimes \cdots \otimes v_k) = \omega(v_i, v_j)\, v_1 \otimes \cdots \otimes \widehat{v_i} \otimes \cdots \otimes \widehat{v_j} \otimes \cdots \otimes v_k$$

(omit $v_i$ and $v_j$ in tensor product) and the $ij$*-expansion operator* $D_{ij} : V^{\otimes(k-2)} \to V^{\otimes k}$ by

$$D_{ij}(v_1 \otimes \cdots \otimes v_{k-2}) = \sum_{p=1}^{n} v_1 \otimes \cdots \otimes \underbrace{f_p}_{i\text{th}} \otimes \cdots \otimes \underbrace{f^p}_{j\text{th}} \otimes \cdots \otimes v_{k-2}.$$

These operators intertwine the action of $G$ and are mutually adjoint, relative to the invariant form $\omega$ on $V^{\otimes k}$:

$$\omega(C_{ij}u,\, w) = \omega(u,\, D_{ij}w) \qquad \text{for } u \in V^{\otimes k},\, w \in V^{\otimes(k-2)}. \tag{13.4}$$

Set $\tau_{ij} = D_{ij}C_{ij} \in \mathrm{End}_G(V^{\otimes k})$. If $u = v_1 \otimes \cdots \otimes v_k$ with $v_i \in V$, then

$$\tau_{ij}(u) = \omega(v_i, v_j) \sum_{p=1}^{n} v_1 \otimes \cdots \otimes \underbrace{f_p}_{i\text{th}} \otimes \cdots \otimes \underbrace{f^p}_{j\text{th}} \otimes \cdots \otimes v_k\,. \tag{13.5}$$

The contraction and expansion operators satisfy the symmetry properties

$$C_{ij} = \varepsilon C_{ji}, \quad D_{ij} = \varepsilon D_{ji}\,, \tag{13.6}$$

since $\sum_p f_p \otimes f^p = \varepsilon \sum_p f^p \otimes f_p$. Hence $\tau_{ij} = \tau_{ji}$, so the operator $\tau_{ij}$ only depends on the set $\{i, j\}$.

Let $Z_{k,r} \subset X_k$ be the set of normalized $r$-bar Brauer diagrams, and set

$$Z_k = \bigcup_{r=0}^{[k/2]} Z_{k,r}\,.$$

**Lemma 13.2.** *Suppose that $z \in Z_{k,r}$ is a normalized $r$-bar Brauer diagram with bars $\{i_1, j_1\}, \ldots, \{i_r, j_r\}$. Then*

$$\tau_{i_p j_p}\tau_{i_q j_q} = \tau_{i_q j_q}\tau_{i_p j_p} \quad \textit{for } p \neq q. \tag{13.7}$$

*Thus the operator $\tau_z = \tau_{i_1 j_1} \cdots \tau_{i_r j_r}$ only depends on $z$ and not on the enumeration of the bars in $z$. Furthermore, $\tau_z = T(\lambda_z)$.*

**Proposition 13.3.** *The algebra $\mathrm{End}_G(V^{\otimes k})$ is spanned by the set of operators $\sigma_k(s)\tau_z$ with $s \in \mathfrak{S}_k$ and $z \in Z_k$. Furthermore, if $\dim V \geq 2k$ then this set of operators is linearly independent.*

13.3. **Relations in the Centralizer Algebra.** We next determine the algebraic relations among the operators in Proposition 13.3.

**Lemma 13.4.** *Let $n = \dim V$ and set $\varepsilon = 1$ if $\omega$ is symmetric and $\varepsilon = -1$ if $\omega$ is skew. The operators $\tau_{ij}$ (where $1 \leq i, j \leq k$ and $i \neq j$) satisfy the following relations, where $(il)$ denotes the transposition of $i$ and $l$:*

(1) $\tau_{ij} = \tau_{ji}$ *and* $\tau_{ij}^2 = n\,\tau_{ij}$

(2) $\tau_{ij}\tau_{lm} = \tau_{lm}\tau_{ij}$ *for distinct* $i, j, l, m$

(3) $\tau_{ij}\tau_{jl} = \sigma_k(il)\tau_{jl}$ *for distinct* $i, j, l$

(4) $\sigma_k(s)\tau_{ij}\sigma_k(s)^{-1} = \tau_{s(i),s(j)}$ *for all* $s \in \mathfrak{S}_k$

(5) $\sigma_k(ij)\tau_{ij} = \varepsilon\tau_{ij}$

Define the *Brauer Algebra* $\mathcal{B}_k(\varepsilon, n)$ with parameters $k, \varepsilon, n$ to be the associative algebra generated by $\mathfrak{S}_k$ and elements $\{\tau_{ij} : 1 \leq i < j \leq k\}$ subject to the relations (1)–(5) in Lemma 13.4; here $n$ can be any complex number and $\varepsilon = \pm 1$. From these relations it is clear that $\mathcal{B}_k(\varepsilon, n)$ is finite-dimensional. If $\mathcal{T}$ is the subalgebra generated by $\{\tau_{ij}\}$, then $\mathcal{T}$ is an ideal and we have the decomposition

$$\mathcal{B}_k(\varepsilon, n) = \mathbb{C}[\mathfrak{S}_k] \oplus \mathcal{T}. \tag{13.8}$$

From Proposition 13.3 and Lemma 13.4 we see that there is a surjective algebra homomorphism

$$\mathcal{B}_k(\varepsilon, n) \longrightarrow \mathrm{End}_G(V^{\otimes k}) \qquad (n = \dim V)$$

with $\varepsilon = \pm 1$ determined as in Lemma 13.4 (5). The two algebras are isomorphic if $n \geq 2k$. In any case, the centralizer algebra $\mathrm{End}_G(V^{\otimes k})$ is the quotient of the associated Brauer algebra by a two-sided ideal, so the representations of the centralizer algebra can be viewed as representations of the Brauer algebra.[1]

We can describe the multiplication in $\mathcal{B}_k(\varepsilon, n)$ and the relations in Lemma 13.4 in terms of concatenation of Brauer diagrams. Let $s_r \in \mathfrak{S}_k$ be the transposition $r \leftrightarrow r + 1$. It corresponds to the Brauer

[1]See [9] and [10] for recent work on the representation theory of the Brauer algebra and citations of earlier work.

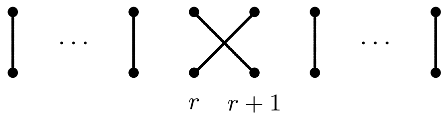


FIGURE 6. Brauer Diagram for $s_r$.

diagram shown in Figure 6. Let $z_r = \tau_{r,r+1}$ be the operator corresponding to the normalized Brauer diagram with a single $(r, r+1)$ bar, as in Figure 7. Since $\mathfrak{S}_k$ is generated by $s_1, \ldots, s_{k-1}$, we see from Proposition 13.3 and property (3) in Lemma 13.4 that the algebra $\mathcal{B}_k(\varepsilon n)$ is generated by the operators $s_1, \ldots, s_{k-1}$ and $z_1, \ldots, z_{k-1}$. If $x, y$ are Brauer diagrams, then their product $xy$ in the Brauer algebra is obtained by placing the $x$ above $y$ and joining the lower row of dots in $x$ to the upper row of dots in $y$. When $x, y$ correspond to elements of $\mathfrak{S}_k$ (no bars) this procedure obviously gives the multiplication in $\mathfrak{S}_k$. When $x$ or $y$ have bars, we remove the closed loops from the concatenated graph using relation (1) in Lemma 13.4.

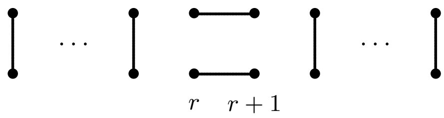


FIGURE 7. Brauer Diagram for $z_r$.

The general recipe for transforming the concatenated Brauer diagrams of $x$ and $y$ into a scalar multiple of the Brauer diagram for $xy$ is as follows:

(1)**:** Delete each closed loop in the concatenated diagram and multiply by a scalar factor of $n^r$ if there are $r$ such loops.

(2)**:** Multiply by a factor of $\varepsilon$ for every path in the concatenated diagram that begins and ends on the top row of $x$ or on bottom row of $y$.

For example, if $x = \sigma(236)\tau_{35}\tau_{46}$ and $y = \sigma(46)\tau_{12}\tau_{34}\tau_{56}$, then $xy$ is obtained as shown in Figure 8 (see [16, §10.1.2 and Exercises 10.1.3] for further details and examples).

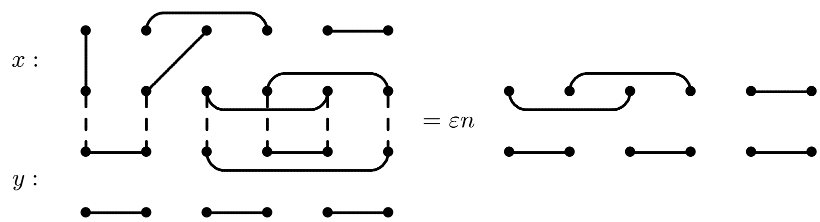


FIGURE 8. The Relation $(\sigma(236)\tau_{35}\tau_{46}) \cdot (\sigma(46)\tau_{12}\tau_{34}\tau_{56}) = \varepsilon n\, \sigma(23)\, \tau_{12}\tau_{34}\tau_{56}$.

13.4. **Harmonic Tensors.** Let $k \geq 2$. A tensor $u \in V^{\otimes k}$ is called $\omega$-*harmonic*[m] if it is annihilated by all the contraction operators $C_{ij}$. Denote by

$$\mathcal{H}(V^{\otimes k}, \omega) = \textstyle\bigcap_{1 \leq i < j \leq k} \operatorname{Ker}(C_{ij})$$

the space of all $\omega$-harmonic $k$-tensors. We will simply call these tensors *harmonic* and write $\mathcal{H}(V^{\otimes k}, \omega) = \mathcal{H}(V^{\otimes k})$ when $\omega$ is clear from the context.

**Example.** Assume $\omega$ is symmetric and let $v \in V$. Then $C_{ij} v^{\otimes k} = \omega(v, v) v^{\otimes(k-2)}$. Thus the symmetric tensor $v^{\otimes k}$ is harmonic if and only if $v$ is an *isotropic vector* for $\omega$. This is the same as the polynomial function $\xi \mapsto \langle \xi, x \rangle^k$ on $V^*$ being harmonic relative to the Laplace operator defined by $\omega$. On the other hand, every skew-symmetric tensor is harmonic when $\omega$ is symmetric.

[m]Weyl uses the term *traceless*; we prefer the term harmonic because when $\omega$ is symmetric the contraction operators $C_{ij}$ act as Laplacians on the symmetric tensors.

**Theorem 13.5** (Harmonic Tensor Duality). *The space $\mathcal{H}(V^{\otimes k})$ is invariant under $\mathfrak{S}_k \times G$ and decomposes as*

$$\mathcal{H}(V^{\otimes k}) \cong \bigoplus_{\lambda \in \Lambda} E^\lambda \otimes U^\lambda \tag{13.9}$$

*Here $\Lambda \subset \mathrm{Par}(k)$, $E^\lambda$ is the irreducible $\mathfrak{S}_k$-module corresponding to the partition $\lambda$ by Schur-Weyl duality, and $U^\lambda$ is an irreducible $G$-module. Furthermore, the modules $U^\lambda$ are all distinct.*

13.5. **Decomposition of Harmonic Tensors for** $\mathrm{Sp}(V)$**.** We now determine the set $\Lambda$ of partitions of $k$ occurring in Theorem 13.5 and the corresponding irreducible representations $U^\lambda$ when $G$ is the symplectic group.[n] We take $V = \mathbb{C}^n$ with $n = 2l$ and the bilinear form

$$\omega(x, y) = \sum_{i=1}^{l} x_i y_{n+1-i} - y_i x_{n+1-i}$$

(so the standard basis vectors $e_i$ are $\omega$-isotropic and $e_1$ is paired with $e_n$, $e_2$ is paired with $e_{n-1}$, and so forth). Let $G = \mathrm{Sp}(V, \omega)$ and let $H$ be the diagonal matrices in $G$. Then $H$ is a maximal torus whose elements are of the form

$$h = \mathrm{diag}[x_1, \ldots, x_l, x_l^{-1}, \ldots, x_1^{-1}], \quad x_i \in \mathbb{C}^\times \tag{13.10}$$

Following the notation in Lecture 10, we let $D_n$ be the diagonal matrices, $B_n$ the upper-triangular matrices, and $N_n$ the upper-triangular unipotent matrices in $\mathrm{GL}(V)$. With our choice of $\omega$ the group $B = G \cap B_n$ is a Borel subgroup of $G$ and $N = G \cap N_n$ is its unipotent radical. The weight lattice of $H$ is identified with $\mathbb{Z}^l$, where $\lambda = [m_1, \ldots, m_l]$ gives the character

$$h \mapsto x_1^{m_1} \cdots x_l^{m_l}$$

when $h$ is given by (13.10). The Weyl group $W$ of $G$ acts by all permutations and sign changes of the coordinations of $\lambda$. The set of $B$-dominant weights is thus identified with $\mathbb{N}^l_{++}$ (see [16, §2.5]).

For $\lambda \in \mathbb{N}^l_{++}$ let $(\pi^\lambda, U^\lambda)$ be the irreducible representation of $G$ with highest weight $\lambda$. If $|\lambda| = k$ then we view $\lambda$ as a partition of $k$ with at most $l$ parts. Let $E^\lambda = (V^{\otimes k})^{N_n}(\lambda)$ be the corresponding irreducible representation of $\mathfrak{S}_k$ on the space of $\mathrm{GL}(n, \mathbb{C})$ highest weight vectors of weight $\lambda$, as in Theorem 3.8.

**Theorem 13.6.** *Let $\lambda \in \mathrm{Par}(k, n)$. Then $E^\lambda \subset \mathcal{H}(V^{\otimes k})$ if and only if $\lambda$ has at most $l$ parts. Furthermore, the space of $\omega$-harmonic $k$-tensors has isotypic decomposition*

$$\mathcal{H}(V^{\otimes k}) \cong \bigoplus_{\lambda \in \mathrm{Par}(k,l)} E^\lambda \otimes U^\lambda. \tag{13.11}$$

*under $\mathfrak{S}_k \times \mathrm{Sp}(V, \omega)$. Thus all the irreducible representations of $\mathrm{Sp}(V, \omega)$ occur in the decomposition of the harmonic $k$-tensors, for $k = 1, 2, \ldots$.*

The general form of decomposition (13.11) follows from Theorem 13.5. To determine the spectrum $\Lambda$ of $\mathrm{Sp}(V, \omega)$ on the harmonic tensors, we will compare the spaces of $B$ eigenvectors in $V^{\otimes k}$ with the spaces $E^\lambda$. If $\mu = [m_1, \ldots, m_n] \in \mathbb{Z}^n$ is a weight of $D_n$, then we denote by $\bar{\mu}$ the restriction of $\mu$ to $H$. From (13.10)

$$\bar{\mu} = [m_1 - m_n, m_2 - m_{n-1}, \ldots, m_l - m_{l+1}]. \tag{13.12}$$

Hence if $\mu \in \mathbb{N}^n_{++}$ is a $B_n$-dominant weight, then $\bar{\mu}$ is a $B$-dominant weight. We introduce the notation

$$W^k(\lambda) = (V^{\otimes k})^N(\lambda), \quad \text{for } \lambda \in \mathbb{N}^l_{++}.$$

Since $N \subset N_n$, we have $E^\mu \subset W^k(\bar{\mu})$.

**Proposition 13.7.** *There are the following dichotomies:*

(1) *Assume $\lambda \in \mathbb{N}^l_{++}$. Then either $W^k(\lambda) \cap \mathcal{H}(V^{\otimes k}) = 0$ or else $W^k(\lambda) \subset \mathcal{H}(V^{\otimes k})$.*

(2) *Assume $\mu \in \mathrm{Par}(k, n)$. Then either $E^\mu \cap \mathcal{H}(V^{\otimes k}) = 0$ or else $E^\mu = W^k(\bar{\mu})$.*

[n]Similar methods work for the orthogonal groups but the details are considerable more intricate–see [16, §10.2] for a full treatment.

*Proof.* (1): By Theorems 3.7 and 13.1 we know that $W^k(\lambda)$ is an irreducible module for $\mathcal{B}_k(\varepsilon, n)$. Since $W^k(\lambda) \cap \mathcal{H}(V^{\otimes k})$ is a $\mathcal{B}_k(\varepsilon, n)$-invariant subspace of $W^k(\lambda)$, it must be 0 or $W^k(\lambda)$.

(2): Assume $E^\mu \cap \mathcal{H}(V^{\otimes k}) \neq 0$. Since $E^\mu \subset W^k(\bar{\mu})$, it follows by (1) that $W^k(\bar{\mu}) \subset \mathcal{H}(V^{\otimes k})$. Furthermore, $W^k(\bar{\mu})$ is irreducible under $\mathfrak{S}_k$. Indeed, it is irreducible under $\mathcal{B}_k(\varepsilon, n)$ by (1), and on the harmonic tensors the action of $\mathcal{B}_k(\varepsilon, n)$ is the same as the action of $\mathfrak{S}_k$. By Theorem 3.8 it follows that $W^k(\bar{\mu}) = E^\mu$. □

**Proposition 13.8.** *Let $\mu \in \mathrm{Par}(k, n)$. Then $E^\mu \subset \mathcal{H}(V^{\otimes k})$ if and only if $\mu$ has at most $l$ parts.*

*Proof.* Let $\mu = [m_1, \ldots, m_n]$. Define $u_\mu = u_1^{\otimes c_1} \otimes \cdots \otimes u_n^{\otimes c_n}$ as in Theorem 3.8, where $u_p = e_1 \wedge \cdots \wedge e_p$ and $c_j = m_j - m_{j+1}$ (with $m_{n+1} = 0$). Then $u_\mu \in E^\mu$ and the depth of $\mu$ is the largest integer $d$ such that $c_d \neq 0$.

We first verify that $u_p$ is harmonic if and only if $p \leq l$. To see this, take any pair $i, j$ with $1 \leq i < j \leq p$. If $p \leq l$, then $C_{ij} u_p = 0$ since $\omega(e_i, e_j) = 0$. Conversely, if $p > l$ then

$$C_{l,l+1} u_p = e_1 \wedge \ldots \wedge e_{l-1} \wedge e_{l+2} \wedge \ldots \wedge e_p \neq 0,$$

since $\omega(e_l, e_{l+1}) = 1$. So $u_p$ is not harmonic in this case.

Thus to finish the proof of the proposition, we may assume that $\mu$ has at most $l$ parts. Let $1 \leq p \leq q \leq l$. We now show that

$$C_{ij}(u_p \otimes u_q) = 0 \quad \text{for all } 1 \leq i \leq p < j \leq p + q.$$

Set $v = u_p \otimes u_q$. In terms of the basis $\{e_I\}$, $v$ is obtained by a double alternation:

$$v = \frac{1}{p!q!} \sum_{s \in \mathfrak{S}_p} \sum_{t \in \mathfrak{S}_q} \mathrm{sgn}(s)\,\mathrm{sgn}(t)\, e_{s(1)} \otimes \cdots \otimes e_{s(p)} \otimes e_{t(1)} \otimes \cdots \otimes e_{t(q)}.$$

The contraction operator $C_{ij}$ removes $e_{s(i)}$ and $e_{t(j-p)}$ from each term of the sum and multiplies the resulting $(p+q-2)$-tensor by $\omega(e_{s(i)}, e_{t(j-p)})$. But $s(i) + t(j-p) \leq p + q \leq n$, while $\omega(e_a, e_b) = 0$ unless $a + b = n + 1$. Hence $C_{i,j}(v) = 0$. □

We now complete the proof of Theorem 13.6. By Propositions 13.7 and 13.8, it will suffice to prove the following:

(∗) *If $\lambda \in \mathbb{N}^l_{++}$ is such that $0 \neq W^k(\lambda) \subset \mathcal{H}(V^{\otimes k})$, then $|\lambda| = k$.*

To establish (∗), take a nonzero tensor $u \in W^k(\lambda)$ and decompose $u$ under the action of $D_n$ as

$$u = \sum_\mu u_\mu, \quad \text{where } \mu \in \mathrm{Par}(k, n) \text{ and } u_\mu \in V^{\otimes k}(\mu).$$

Fix some $\mu = [m_1, \ldots, m_n]$ such that $u_\mu \neq 0$. Then $\bar{\mu} = \lambda$ and from (13.12) we see that

$$|\lambda| = \sum_{i=1}^{l} (m_i - m_{n+1-i}) = k - 2r, \quad \text{where } \ r = \sum_{i=l+1}^{n} m_i\,.$$

Thus $\lambda \in \mathrm{Par}(k - 2r, l)$ so from Theorem 3.8 we know that $0 \neq E^\lambda \subset V^{\otimes(k-2r)}$. Suppose for the sake of contradiction that $r > 0$. Since the expansion operator $D_{12}$ is injective and intertwines the action of $G$ on tensors, we have

$$0 \neq (D_{12})^r E^\lambda \subset W^k(\lambda).$$

Since $C_{12} D_{12} = nI$, this implies that

$$C_{12}(D_{12})^r E^\lambda = (D_{12})^{r-1} E^\lambda \neq 0.$$

But we have assumed that $W^k(\lambda)$ in contained in the harmonic tensors, a contradiction. Thus $r = 0$ and (∗) is proved. □

*Examples.* **1.** Assume $k \leq l$ and take $\lambda = [1, \ldots, 1] \in \mathrm{Par}(k)$. Then $E^\lambda$ is the sgn representation of $\mathfrak{S}_k$ and $U^\lambda$ is the $k$th *fundamental representation* of $\mathrm{Sp}(V, \omega)$. From Theorem 13.6 we know that $U^\lambda$ is the sgn-isotypic component for $\mathfrak{S}_k$ in $\mathcal{H}(V^{\otimes k})$. Hence

$$U^\lambda = \mathcal{H}(V^{\otimes k}) \cap \bigwedge^k V = \mathcal{H}_{\mathrm{skew}}(V^{\otimes k})$$

is the space of *harmonic skew $k$-tensors.*

**2.** Let $k = 2$ and assume $l \geq 2$. Then the two partitions of 2 give the trivial and sgn representations of $\mathfrak{S}_k$, respectively. Because the form $\omega$ is skew-symmetric, every symmetric tensor is harmonic. Hence by Theorem 13.6 we have

$$\mathcal{H}(V^{\otimes 2}) = S^2(V) \oplus \mathcal{H}_{\mathrm{skew}}(V).$$

The summands are the irreducible representations of $G$ with highest weights $2\varepsilon_1$ and $\varepsilon_1 + \varepsilon_2$.

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