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Symmetry, Representations, and Invariants

Graduate Texts in Mathematics 255

Springer

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This material is a web-based supplement to the book published by Springer 2009.

Appendix G
Group Representations and the Platonic Solids

Abstract In this appendix we shall find all the irreducible representations of the symmetry groups of the Platonic solids, by a mixture of geometric methods and algebraic methods similar to those used in Chapters 5 for representations of the classical groups. We shall also see how these representations occur naturally in the harmonic analysis of functions on the Platonic solids. This is a discrete analogue of the decomposition of functions on the 2-sphere under the action of the orthogonal group, and it will serve to illustrate the methods of Fourier analysis on groups in relatively simple but beautiful examples.

As has been known from antiquity (Weyl [5], Sternberg [3, §1.8]), there are three types of three-dimensional Platonic solids:

1. the tetrahedron, with rotational symmetry group the alternating group $\mathfrak{A}_4$,
2. the octahedron (and its dual cube), with rotational symmetry group $\mathfrak{S}_4$,
3. the icosahedron (and its dual dodecahedron), with rotational symmetry group the alternating group $\mathfrak{A}_5$.

In considering each of type of solid, we urge the reader to draw pictures, or even better to construct a 3-dimensional model, in order to verify the properties of the symmetries of the solid. As Felix Klein writes, "we are treating of concrete matters, which may easily be conceived with the assistance of the suggested aids, but which may occasionally offer difficulties if these are neglected" ([2, §1.4]).

G.1 Symmetries of the tetrahedron

We begin with the simplest Platonic solid, the regular tetrahedron $\mathcal{T}$. The full group of symmetries of $\mathcal{T}$ is $\mathfrak{S}_4$, acting by permuting the 4 vertices, and the subgroup of orientation-preserving symmetries is the alternating group $\mathfrak{A}_4$.

The action of $\mathfrak{S}_4$ on $\mathcal{T}$ is linear and arises from the standard three-dimensional representation (ρ, V) of $\mathfrak{S}_4$ of Section 4.4.3. To see this explicitly, we take the orthogonal basis

$$\mathbf{e}_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \qquad \mathbf{e}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \qquad \mathbf{e}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$$

for V. Let $\mathfrak{T}$ be the regular tetrahedron in the three-dimensional Euclidean space $\mathbb{E}^3 = \mathbb{R}\mathbf{e}_1 \oplus \mathbb{R}\mathbf{e}_2 \oplus \mathbb{R}\mathbf{e}_3$ whose edges have midpoints $\pm\mathbf{e}_i$, for $i = 1,2,3$. Enumerate the vertices so that $\mathbf{e}_1$ is the midpoint of the edge joining v_2 and v_3, $\mathbf{e}_2$ is the midpoint of the edge joining v_2 and v_4, and $\mathbf{e}_3$ is the midpoint of the edge joining v_1 and v_2. Each permutation of the vertices of $\mathfrak{T}$ is given by an orthogonal transformation of $\mathbb{E}^3$ which preserves the sets of edges and faces of $\mathfrak{T}$. The even permutations act by transformations with determinant 1.

Let $C \subset \mathfrak{A}_4$ be the cyclic subgroup of order 3 generated by $s = (123)$, and let $H \subset \mathfrak{A}_4$ be the commutative subgroup of order 4 consisting of 1 and the 3 elements of order 2 in $\mathfrak{A}_4$:

$$\begin{aligned} h_1 &= (14)(23) \\ h_2 &= (13)(24) \\ h_3 &= (12)(34) \end{aligned}$$

Then H is a normal subgroup of $\mathfrak{A}_4$, and $\mathfrak{A}_4 = C \cdot H$ (semidirect product). The element h_i acts by 180° rotation around the $\mathbf{e}_i$ axis. Let F be the face of $\mathfrak{T}$ with vertices $\{v_1, v_2, v_3\}$. The element s acts by 120° rotation around the axis joining the midpoint of F and the opposite vertex v_4.

Each vector $\mathbf{e}_i$ is an eigenvector for h_j with eigenvalue 1 $(i = j)$ or -1 $(i \neq j)$. The set of eigenvalues associated with $\mathbf{e}_i$ defines a character $h \mapsto \lambda_i(h)$ of the group H, where $\lambda_i(h_j) = -\delta_{ij}$, and we have

$$\rho(h)\mathbf{e}_i = \lambda_i(h)\mathbf{e}_i\,, \quad \text{for } h \in H\ .$$

We call $\{\lambda_1, \lambda_2, \lambda_3\}$ the *weights* of H on V, and the subspace $\mathbb{C}\mathbf{e}_i$ the *weight space* for the weight λ_i. Notice that each weight space is one-dimensional, and V is the direct sum of the weight spaces.

In any representation (π, W) of $\mathfrak{A}_4$, the space W will decompose as the direct sum of weight spaces W_λ, where λ runs over the characters of H and

$$W_\lambda = \{w \in W \,:\, \pi(h)w = \lambda(h)w\}\ .$$

We say that λ is a *weight* of $\pi|_H$ if $W_\lambda \neq 0$. The operator $\pi(s)$ permutes the spaces W_λ: if w is a vector of weight λ, then

$$\pi(h)\pi(s)w = \pi(hs)w = \pi(s)\pi(s^{-1}hs)w = \lambda(s^{-1}hs)\pi(s)w$$

for $h \in H$. Thus $\pi(s)w$ is a weight vector with weight $s \cdot \lambda$, where $s \cdot \lambda(h) = \lambda(s^{-1}hs)$. For the standard representation V this is obvious: $\rho(s)$ cyclically permutes $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$.

Proposition G.1.1. *The set $\widehat{A}$ of irreducible representations of $\mathfrak{A}_4$ consists of the three-dimensional tetrahedral representation ρ and the three one-dimensional representations $(123) \mapsto \omega^m$ $(m = 0,1,2)$ that are trivial on the normal subgroup H (where $\omega = e^{2\pi i/3}$ is a primitive cube root of unity).*

Proof. The tetrahedral representation ρ is irreducible by Corollary 4.4.5. There are four conjugacy classes in $\mathfrak{A}_4$: $\{1\}$, $\{(12)(34)\}$, $\{(123)\}$, $\{(213)\}$, hence there are four irreducible representations by (4.44). ☐

We have thus obtained the following character table for $\mathfrak{A}_4$:

Table G.1 Character table of $\mathfrak{A}_4$.

conjugacy class	1	(123)	(132)	$(12)(34)$
# elements	1	4	4	3
χ_1	1	1	1	1
χ_ω	1	ω	ω^2	1
χ_{ω^2}	1	ω^2	ω	1
χ_ρ	3	0	0	-1

G.2 Symmetries of the octahedron

We continue our study of the groups and representations associated with the Platonic solids. Let (ρ, V) be the standard three-dimensional representations of $\mathfrak{S}_4$, and let the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ for V be chosen as in Section G.1. Let $\mathcal{O}$ be the octahedron whose vertices are $\pm\mathbf{e}_i$, $i = 1,2,3$. The character table for $\mathfrak{S}_4$ was obtained in Exercises 9.1.4 using the Frobenius character formula. We shall now show that all the irreducible representations of $\mathfrak{S}_4$ of dimension greater than one have natural geometric descriptions in terms of $\mathcal{O}$.

There are two inequivalent actions of $\mathfrak{S}_4$ on $\mathcal{O}$. In the representation ρ, the action of $\mathfrak{S}_4$ permutes the vertices of $\mathcal{O}$ but does not preserve orientation. However, if we form the representation $\rho' = \mathrm{sgn} \otimes \rho$ (and identify $\mathbb{C} \otimes V$ with V as usual), then $\rho'(\mathfrak{S}_4)$ does preserve orientation.

Example. The following assertions are easily seen using a model of the octahedron: The transposition (12) acts via ρ' by 180° rotation about the midpoint of the edge joining $\mathbf{e}_1$ and $-\mathbf{e}_2$. The element (123) acts via ρ' as a rotation by 120° about an axis joining the midpoint of opposite triangular faces, as in the standard representation. The element (1234) acts via ρ' as rotation by 90° about a diagonal axis of $\mathcal{O}$, and its square $(13)(24)$ is rotation by 180° about the same axis.

We can obtain yet another representation of $\mathfrak{S}_4$ from $\mathcal{O}$. For $g \in \mathfrak{S}_4$ the rotation $\rho'(g)$ permutes the set of three axes through opposite vertices of $\mathcal{O}$. If we number the six vertices as 1, 1′, 2, 2′, 3, 3′, where x and x' are opposite vertices, then $\rho'(g)$ permutes the three (unordered) pairs $\{x, x'\}$. Thus we obtain a homomorphism from $\mathfrak{S}_4$ to $\mathfrak{S}_3$. Composing this homomorphism with the standard two-dimensional representation of $\mathfrak{S}_3$, we obtain a two-dimensional representation, call it δ, of $\mathfrak{S}_4$. We shall call δ the *vertex representation* of $\mathfrak{S}_4$.

We now have five inequivalent irreducible representations of $\mathfrak{S}_4$: 1, sgn, ρ, ρ', and δ. Likewise, since there are five partitions of 4, there are five conjugacy classes in $\mathfrak{S}_4$, with representative elements 1, (12) , (123), $(12)(34)$, and (1234). Hence we have found the complete set of irreducible representations of $\mathfrak{S}_4$. As a check, we calculate the sum of the squares of the dimensions of these representations:

$$1+1+3^2+3^2+2^2 = 24 = |\mathfrak{S}_4| .$$

To calculate the characters of these representations, we use formula (4.59) for ρ; multiplying by sgn then furnishes the character of ρ'). We can calculate the character of δ using the fixed-point formula as follows:

1. The elements in the class (12) fix one pair of opposite vertices.
2. The elements in the class (123) fix no pair of opposite vertices.
3. The elements in the class $(12)(34)$ fix all three pairs of opposite vertices.
4. The elements in the class (1234) fix one pair of opposite vertices.

Hence from (4.59) we obtain the character values in the following table:

Table G.2 Character table of $\mathfrak{S}_4$.

conj. class:	$C(1^4)$	$C(1^2 2^1)$	$C(1^1 3^1)$	$C(2^2)$	$C(4^1)$
# elements:	1	6	8	3	6
χ_1	1	1	1	1	1
χ_ρ	3	1	0	−1	−1
χ_δ	2	0	−1	2	0
$\chi_{\rho'}$	3	−1	0	−1	1
χ_{sgn}	1	−1	1	1	−1

(Compare with Table 9.2 to match these representations with partitions of 4 according to Schur–Weyl duality.)

We can find a distinguished basis in each irreducible representation of $\mathfrak{S}_4$ that diagonalizes the action of a maximal commutative subgroup, just as we did for $\mathfrak{A}_4$. We fix a commutative subgroup H generated by an element h of order 4, say $h = (1234)$. This is a maximal commutative subgroup of $\mathfrak{S}_4$ and it is cyclic of order 4. There are 4 characters of H, defined by

$$h \mapsto \mathrm{i}^p \quad \text{for } p = 0,1,2,3 \quad (\mathrm{i} = \sqrt{-1}) .$$

In any representation (π, W) of $\mathfrak{S}_4$ the linear transformation $\pi(h)$ is diagonalizable and satisfies $\pi(h)^4 = I$. Hence the space W will decompose as the direct sum of the weight spaces W_p, where

$$W_p = \{w \in W \,:\, \pi(h)w = \mathrm{i}^P w\} \,.$$

Proposition G.2.1. *Let (π, W) be an irreducible representation of $\mathfrak{S}_4$. Assume that* $\dim W > 1$.

1. *Let $s = (24)$. Then $\pi(s)W_1 = W_{-1}$ and $\pi(s)W_2 = W_2$. In particular,* $\dim W_1 = \dim W_{-1}$.
2. *When π is the vertex representation δ, then* $\dim W_p = 1$ *for* $p = 0$ *and* $p = 2$.
3. *When π is the orientation-preserving octahedral representation ρ', then* $\dim W_p = 1$ *for* $p = 0$, 1, *and* 3.
4. *When π is the orientation-reversing octahedral representation ρ, then* $\dim W_p = 1$ *for* $p = 1$, 2, *and* 3.

Proof. If $s = (24)$, then $shs^{-1} = h^{-1}$. This implies assertion (1).

For (2), note that the transformation $\rho'(h)$ is rotation by 90° in $\mathbb{E}^3$, so its eigenvalues are 1 and $\pm$i.

For (3), we have $\rho(h) = -\rho'(h)$, so by (2) the eigenvalues of $\rho(h)$ are -1 and $\pm$i.

To prove (4), label the axis of $\mathcal{O}$ fixed by h as 3. Then $h \mapsto (12)$ under the homomorphism $\mathfrak{S}_4 \longrightarrow \mathfrak{S}_3$. Hence $\delta(h)^2 = 1$. From the character table we see that $\delta(h)$ has trace 0, and so its eigenvalues are ± 1. $\square$

Corollary G.2.2. *Let $H \subset \mathfrak{S}_4$ be a cyclic subgroup of order four. If (π, W) is an irreducible representation of $\mathfrak{S}_4$, then the space W^H of H-fixed vectors has dimension* 0 *or* 1. *This space is of dimension 1 for the orientation-preserving octahedral representation ρ', the vertex representation δ, and the trivial representation.*

G.3 Symmetries of the icosahedron

We complete our study of the groups and representations associated with the Platonic solids by examining the icosahedron $\mathcal{I}$ and its group $\mathfrak{A}_5$ of rotational symmetries (see Klein [2], Grove and Benson [1], and Sternberg [3, §1.10]).

We begin by finding the irreducible representations of $\mathfrak{A}_5$. Let ρ be the standard four-dimensional representation of $\mathfrak{S}_5$. We know from Corollary 4.4.5 that the restriction of ρ to $\mathfrak{A}_5$ is irreducible. We can calculate its character from (4.59):

conjugacy class:	1	(12)(34)	(123)	(12345)	(13524)
χ_ρ :	4	0	1	-1	-1

We next obtain the other three irreducible representations of $\mathfrak{A}_5$ from its action on the icosahedron. Let $\varphi = (1+\sqrt{5})/2$ be the golden mean. Take the three points $v_1 = (\varphi, 1, 0)$, $v_2 = (0, \varphi, 1)$, $v_3 = (1, 0, \varphi)$ in $\mathbb{E}^3$, and let v_4, …, v_{12} be obtained by all changes of signs of the coordinates of these three points. Then $\{v_1, \ldots, v_{12}\}$

are the vertices of an icosahedron with edge length 2 (see Grove and Benson [1, Exercise 2.23]). The elements of $\mathfrak{A}_5$ give symmetries of $\mathcal{I}$ of the following three types:

1. Rotation by 180° about an axis through the midpoints of opposite edges of $\mathcal{I}$. The elements of $\mathfrak{A}_5$ giving this symmetry have order two and comprise the conjugacy class of $(12)(34)$, which has $\frac{1}{2}\binom{5}{2}\binom{3}{2} = 15$ elements (geometric argument: $\mathcal{I}$ has 30 edges, so there are 15 such symmetry axes).
2. Rotation by 120° about an axis through the midpoints of opposite faces of $\mathcal{I}$. These elements have order three and comprise the conjugacy class of (123) in $\mathfrak{A}_5$, which has $2\binom{5}{3} = 20$ elements (geometric argument: $\mathcal{I}$ has 20 faces, so there are 10 such symmetry axes, and for each axis there are two distinct 120° rotations).
3. Rotation by 72° or 144° about an axis through opposite vertices of $\mathcal{I}$. These elements have order five and comprise two conjugacy classes in $\mathfrak{A}_5$, that of $h = (12345)$ and $h^2 = (13524)$. Each class has $(1/2)4! = 12$ elements (geometric argument: $\mathcal{I}$ has 12 vertices, so there are 6 such symmetry axes. For each axis there are two distinct 72° rotations and two distinct 144° rotations).

Let α denote the three-dimensional representation of $\mathfrak{A}_5$ as rotations of the icosahedron in which the five-cycle (12345) acts by a 72° rotation (extend α to a representation on the complexification $\mathbb{C}^3$ of $\mathbb{E}^3$ by making $\alpha(g)$ complex linear for $g \in \mathfrak{A}_5$). Note that a rotation through an angle θ in $\mathbb{E}^3$ is represented by the matrix

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

in a suitable basis, and hence has trace $1 + 2\cos\theta$. Using the description just given of the action of each conjugacy class in $\mathfrak{A}_5$ as rotations in $\mathbb{E}^3$, and recalling that $1 + 2\cos(2\pi/5) = \varphi$, we can immediately write down the character of α:

conjugacy class:	1	(12)(34)	(123)	(12345)	(13524)
χ_α :	3	-1	0	φ	$-\varphi^{-1}$

It is easy to verify algebraically that α is irreducible. In fact, from the character values we see that the restriction of α to $\mathfrak{A}_4$ is the standard three-dimensional representation of $\mathfrak{A}_4$ as symmetries of the tetrahedron, which we already know to be irreducible. To obtain a geometric interpretation of this property, we note that there are five mutually conjugate copies of $\mathfrak{A}_4$ in $\mathfrak{A}_5$; each copy of $\mathfrak{A}_4$ leaves invariant one of the five inscribed tetrahedra with vertices at the center of faces of $\mathcal{I}$. Furthermore, these five tetrahedra are permuted cyclically by the element $(12345) \in \mathfrak{A}_5$. This is most easily visualized by coloring the faces of $\mathcal{I}$ with five different colors in such a way that five faces around each vertex are all of different colors. Then the centers of the four faces of a give color are the vertices of a tetrahedron. The action of $\mathfrak{A}_5$ on $\mathcal{I}$

permutes the faces of $\mathfrak{I}$ and gives a permutation of the five colors which corresponds to the embedding of $\mathfrak{A}_5$ in $\mathfrak{S}_5$.

There is another inequivalent irreducible representation of $\mathfrak{A}_5$ as rotational symmetries of $\mathfrak{I}$ that is obtained by twisting α using the outer automorphism τ of $\mathfrak{A}_5$ given by $\tau(g) = \sigma g \sigma^{-1}$ with $\sigma = (1452)$ (note that $\mathrm{sgn}(\sigma) = -1$, so $\sigma \notin \mathfrak{A}_5$). We define a representation β of $\mathfrak{A}_5$ by

$$\beta(g) = \alpha(\tau(g)) \quad \text{for } g \in \mathfrak{A}_5.$$

For the element $h = (12345)$ we have $\tau(h) = h^2$. Hence τ interchanges the two conjugacy classes of elements of order 5, whereas it fixes the other conjugacy classes in $\mathfrak{A}_5$. Thus we obtain the character of β from that of α by interchanging the values on the two conjugacy classes of elements of order 5:

conjugacy class:	1	(12)(34)	(123)	(12345)	(13524)
χ_β :	3	-1	0	$-\varphi^{-1}$	φ

We can obtain the remaining irreducible representation by considering the action of $\mathfrak{A}_5$ as permutations of the set of six axes through opposite vertices of $\mathfrak{I}$, by analogy with the case of the octahedron. If we number the twelve vertices as $1, 1', \ldots, 6, 6'$, where x and x' are opposite vertices, then $\alpha(g)$ permutes the six (unordered) pairs $\{x, x'\}$. Thus we obtain a homomorphism $\nu : \mathfrak{A}_5 \longrightarrow \mathfrak{S}_6$. Composing ν with the standard representation of $\mathfrak{S}_6$, we obtain a five-dimensional representation γ that we shall call the *vertex representation* of $\mathfrak{A}_5$.

We can calculate the character of γ from (4.59), as follows: the elements of order 2 fix 2 pairs of opposite vertices; the elements of order 3 fix no pair of opposite vertices; the elements of order 5 fix one pair of opposite vertices. Hence the character values are

conjugacy class:	1	(12)(34)	(123)	(12345)	(13524)
χ_γ :	5	1	-1	0	0

To show that γ is irreducible, we calculate

$$(\chi_\gamma, \chi_\gamma) = \frac{1}{60}\left\{1 \cdot 5^2 + 15 \cdot 1^2 + 20 \cdot (-1)^2\right\} = 1\ .$$

We now have found five inequivalent irreducible representations of $\mathfrak{A}_5$ of dimensions 1, 3, 3, 4, and 5, respectively. Since $\mathfrak{A}_5$ has five conjugacy classes, this is the complete set. As a check, we note that $|\mathfrak{A}_5| = 60 = 1^2 + 3^2 + 3^2 + 4^2 + 5^2$. We summarize these character calculations in Table G.3.

We can find a distinguished basis for each irreducible representation of $\mathfrak{A}_5$ that diagonalizes the action of a commutative subgroup of maximal order, just as we did for the representations of $\mathfrak{A}_4$ and $\mathfrak{S}_4$. We fix a commutative subgroup H generated by an element h of order 5, say $h = (12345)$. This is a maximal commutative subgroup of $\mathfrak{A}_5$ and it is cyclic of order 5. There are 5 characters of H, defined by

Table G.3 Character table of $\mathfrak{A}_5$.

conj. class:	1	(12)(34)	(123)	(12345)	(13424)
# elements:	1	15	20	12	12
χ_1	1	1	1	1	1
χ_α	3	-1	0	φ	$-\varphi^{-1}$
χ_β	3	-1	0	$-\varphi^{-1}$	φ
χ_ρ	4	0	1	-1	-1
χ_γ	5	1	-1	0	0

$$h \mapsto \varepsilon^p \quad \text{for } p = 0, \pm 1, \pm 2 \quad (\varepsilon = e^{2\pi i/5}) .$$

In any representation (π, W) of $\mathfrak{A}_5$ the linear transformation $\pi(h)$ is diagonalizable and satisfies $\pi(h)^5 = I$. Hence the space W will decompose as the direct sum of the weight spaces W_p, where

$$W_p = \{w \in W \,:\, \pi(h)w = \varepsilon^P w\} \quad \text{for } p = 0, \pm 1, \pm 2 .$$

Proposition G.3.1. *Let (π, W) be a irreducible representation of $\mathfrak{A}_5$. Assume that* $\dim W > 1$.

1. *Let $t = (25)(34)$. Then $\pi(t)W_1 = W_{-1}$ and $\pi(t)W_2 = W_{-2}$. In particular,* $\dim W_1 = \dim W_{-1}$ *and* $\dim W_2 = \dim W_{-2}$. *Furthermore, either $W_1 \neq 0$ or $W_2 \neq 0$.*
2. *When π is the vertex representation γ, then* $\dim W_p = 1$ *for $p = 0, \pm 1, \pm 2$.*
3. *When π is the standard representation ρ, then* $\dim W_p = 1$ *for $p = \pm 1, \pm 2$.*
4. *When π is the icosahedral representation α, then* $\dim W_p = 1$ *for $p = 0, \pm 1$.*
5. *When π is the twisted icosahedral representation β, then* $\dim W_p = 1$ *for $p = 0, \pm 2$.*

Proof. Note that $t^{-1}ht = h^{-1}$, and hence $t^{-1}h^2t = h^{-2}$. This implies the first assertion in (1). Suppose, for the sake of reaching a contradiction, that $W_1 = W_2 = 0$. Then from what was just proved, we conclude that H acts by the identity on W. But the conjugates of H generate $\mathfrak{A}_5$, since $\mathfrak{A}_5$ is a simple group. Hence $\dim W = 1$, since π is irreducible, which is a contraction.

To prove (2), recall that γ is the restriction to $\nu(\mathfrak{A}_5)$ of the standard representation, call it ψ, of $\mathfrak{S}_6$. If we label the axis of $\mathfrak{I}$ fixed by h as $(6, 6')$, then $\nu(h)$ is the permutation (12345) of $\mathfrak{S}_6$. Let $s = (2453) \in \mathfrak{S}_6$. Then $s^{-1}hs = h^2$, and hence $s^{-1}h^{-1}s = h^{-2}$. This implies that

$$\psi(s)W_1 = W_2 \ , \quad \psi(s^2)W_1 = W_{-1} \ , \quad \psi(s^3)W_1 = W_{-2} \ . \tag{G.1}$$

Since $\dim W = 5$ and $s^2 = (25)(34)$, it follows from (G.1) and part (1) that $\dim W_k = 1$ for $k = 0, \pm 1, \pm 2$.

For assertion (3), recall that ρ is the restriction to $\mathfrak{A}_5$ of the standard representation, call it σ, of $\mathfrak{S}_5$. Let $s = (2453) \in \mathfrak{S}_5$. Then by the same argument as in (2) we see that

$$\sigma(s)W_1 = W_2 \ , \quad \sigma(s^2)W_1 = W_{-1} \ , \quad \sigma(s^3)W_1 = W_{-2} \ . \tag{G.2}$$

Since $\dim W = 4$ and $s^2 = (25)(34)$, it follows from (G.2) and part (1) that $\dim W_k = 1$ for $k = \pm 1, \pm 2$.

To prove (4) and (5), note that $\alpha(h)$ acts on $\mathbb{E}^3$ as a rotation by $2\pi/5$, and so its eigenvalues are 1, ε, and ε^{-1}. Likewise, $\beta(h)$ acts on $\mathbb{E}^3$ as a rotation by $4\pi/5$, and so its eigenvalues are 1, ε^2, and ε^{-2}. □

Corollary G.3.2. *Let $H \subset \mathfrak{A}_5$ be a cyclic subgroup of order five. If (π, W) is an irreducible representation of $\mathfrak{S}_4$, then the space W^H of H-fixed vectors has dimension* 0 *or* 1*. This space is of dimension 1 for the two icosahedral representations α and β, the vertex representation γ, and the trivial representation.*

We shall now show how the representations in Corollary G.3.2 occur naturally in connection with harmonic analysis of functions on the icosahedron. Let V be the set of vertices of $\mathfrak{I}$, and let π be the permutation representation of $G = \mathfrak{A}_5$ on the twelve-dimensional space $\mathbb{C}[V]$ of complex-valued functions on V:

$$\pi(g)f(v) = f(g^{-1}v) \quad \text{for } g \in G,\ v \in V, \text{ and } f \in \mathbb{C}[V]\ .$$

We may assume that H is generated by the element $h = (12345)$, which acts on $\mathfrak{I}$ by 72° rotation about an axis through some vertex v_0 and its opposite vertex. Thus H is the subgroup of G that fixes v_0, and we can identify V with the coset space G/H by the map $g \mapsto \alpha(g)v_0$. With this identification π becomes the induced representation $\mathrm{Ind}_H^G(1)$.

From the Frobenius reciprocity formula (4.53) and Corollary G.3.2 we obtain the decomposition

$$\pi = 1 \oplus \alpha \oplus \beta \oplus \gamma\ .$$

In particular, π is *multiplicity free*: each irreducible representation of G occurs at most once. From this decomposition we can obtain another model for the vertex representation γ. The projection P_γ onto the γ-isotypic component of π is given by

$$\begin{aligned} P_\gamma &= \frac{\dim\gamma}{|G|} \sum_{g \in G} \overline{\chi_\gamma(g)}\pi(g) \\ &= \frac{5}{60}\Big\{5I + \sum_{g \in C(2^2)} \pi(g) - \sum_{g \in C(3^1)} \pi(g)\Big\}\ , \end{aligned}$$

where $C(3^1)$ denotes the conjugacy class of (123) and $C(2^2)$ denotes the conjugacy class of $(12)(34)$.

To obtain a nonzero function in the range of P_γ, we fix the vertex v_0 as above. Define $f_0(v_0) = 1$ and $f(v) = 0$ for $v \neq v_0$. Then

$$\pi(g)f_0(v) = \begin{cases} 1 & \text{if } gv_0 = v\ , \\ 0 & \text{otherwise}\ . \end{cases}$$

Since no element in the class $C(3^1)$ or $C(2^2)$ fixes v_0, we see from the projection formula that $P_\gamma f_0(v_0) = 5/12$. If the vertex v_1 is separated from v_0 by one or two edges, then two elements in the class $C(3^1)$ carry v_0 to v_1, whereas one element in the class $C(2^2)$ does this. Thus

$$P_\gamma f_0(v_1) = \frac{5}{60}\Big(1\cdot 1 - 2\cdot 1\Big) = -\frac{1}{12}\ .$$

Finally, if v_0' is the antipodal vertex to v_0, the five elements in the class $C(2^2)$ carry v_0 to v_0', whereas no elements in the class $C(3^1)$ do this. The projection formula implies that $P_\gamma f_0(v_0') = 5/12$.

Define the function $\varphi_\gamma = (12/5)P_\gamma f_0$. The calculations just made show that φ_γ takes the value 1 at v_0 and v_0', and takes the value $-1/5$ at the other ten vertices of $\mathcal{I}$. In particular,

$$\pi(h)\varphi_\gamma = \varphi_\gamma\ ,$$

and so φ_γ is the unique normalized H-fixed function in the γ-isotypic subspace of $\mathbb{C}[V]$. We call φ_γ the *spherical function of type* γ for the pair (G,H). The translates of φ_γ under G span a five-dimensional subspace of $\mathbb{C}[V]$ on which G acts by γ (see Terras [4] for more examples of spherical functions on finite groups.)

G.4 Irreducible representations of $\mathfrak{S}_5$

Now that we have constructed all the irreducible representations of $\mathfrak{A}_5$, it is easy to obtain the irreducible representations of $\mathfrak{S}_5$, since $\mathfrak{A}_5$ is a normal subgroup of index two (just as in Section 5.5.5 for the special and full orthogonal groups).

In $\mathfrak{S}_5$ there are seven conjugacy classes, corresponding to the seven partitions of 5, so we must find seven irreducible representations[1]. We already have the trivial representation, the determinant representation sgn, the standard four-dimensional representation ρ and its associated twist $\rho' = \mathrm{sgn}\otimes\rho$, so we need three more representations. We will construct a five-dimensional irreducible representation by extending the vertex representation of $\mathfrak{A}_5$ to $\mathfrak{S}_5$. Tensoring this representation with sgn will furnish another inequivalent representation. Finally, we will construct a six-dimensional representation from the pair of three-dimensional representations of $\mathfrak{A}_5$. We now proceed to carry out the details of these constructions.

Let (γ, W) be the five-dimensional vertex representation of $\mathfrak{A}_5$. We can extend it to a representation of $\mathfrak{S}_5$ as follows: Let r be the transposition (12) in $\mathfrak{S}_5$. Define $\theta(g) = rgr$ for $g \in \mathfrak{A}_5$. Then θ is an outer automorphism of $\mathfrak{A}_5$ that fixes the character of γ:

$$\chi_\gamma(rgr) = \chi_\gamma(g) \quad \text{for } g \in \mathfrak{A}_5\ .$$

This follows from Table G.3, since χ_γ takes the same value on the two conjugacy classes of five-cycles that are interchanged by θ. Hence the representation $g \mapsto \gamma(rgr)$ of $\mathfrak{A}_5$ is equivalent to γ, and so there exists $R \in \mathbf{GL}(W)$ such that

[1] An explicit matching between partitions of 5 and irreducible representations of $\mathfrak{S}_5$ is furnished by Schur–Weyl duality in Section 9.1.1

$$\gamma(rgr) = R\gamma(g)R^{-1} \quad \text{for } g \in \mathfrak{A}_5 \ . \tag{G.3}$$

Since $r^2 = 1$, we have $\gamma(g) = \gamma(r^2gr^2) = R^2\gamma(g)R^{-2}$ for all $g \in \mathfrak{A}_5$. Thus by Schur's lemma $R^2 = \lambda I$ for some $\lambda \in \mathbb{C}^\times$. Replacing R by $(1/\sqrt{\lambda})R$, we may assume that $R = R^{-1}$ in (G.3). The subgroup $\mathfrak{A}_5$ is of index two in $\mathfrak{S}_5$, and $\mathfrak{S}_5 = \mathfrak{A}_5 \cup r\mathfrak{A}_5$. We define $\tilde{\gamma} : \mathfrak{S}_5 \longrightarrow \mathbf{GL}(W)$ by

$$\tilde{\gamma}(g) = \gamma(g) \ , \quad \tilde{\gamma}(rg) = R\gamma(g) \ , \quad \text{for } g \in \mathfrak{A}_5 \ .$$

From (G.3) and the property $R^2 = I$ it is easy to check that $\tilde{\gamma}$ is a representation of $\mathfrak{S}_5$. Indeed, if $g, g' \in \mathfrak{A}_5$, then

$$\tilde{\gamma}(g)\tilde{\gamma}(rg') = \gamma(g)R\gamma(g') = R\gamma(rgr)\gamma(g') = \gamma(rgrg') = \tilde{\gamma}(grg') \ ,$$
$$\tilde{\gamma}(rg)\tilde{\gamma}(rg') = R\gamma(g)R\gamma(g') = \gamma(rgr)\gamma(g') = \gamma(rgrg') = \tilde{\gamma}(rgrg') \ .$$

Of course $\tilde{\gamma}$ is irreducible, since its restriction to $\mathfrak{A}_5$ is irreducible.

We now know part of the character table for $\mathfrak{S}_5$:

conj. class:	1	(12)	(12)(34)	(123)	(1234)	(12)(345)	(12345)
# elements:	1	10	15	20	30	20	24
χ_1	1	1	1	1	1	1	1
χ_{sgn}	1	-1	1	1	-1	-1	1
χ_ρ	4	2	0	1	0	-1	-1
$\chi_{\rho'}$	4	-2	0	1	0	1	-1
$\chi_{\tilde{\gamma}}$	5	a	1	-1	b	c	0

Here we have used (4.59) to calculate χ_ρ (which then yields $\chi_{\rho'}$), together with the character values of χ_γ that we already know for elements of $\mathfrak{A}_5$. The unknown character values a, b, and c are related by $a = c = -b$, since $\chi_{\tilde{\gamma}}$ is orthogonal to χ_1 and χ_ρ. Since $\tilde{\gamma}$ is irreducible, we also know that $\langle \chi_{\tilde{\gamma}}, \chi_{\tilde{\gamma}} \rangle = 1$ by Corollary 4.3.11, which forces $a = \pm 1$. In the definition of the operator R, a choice of square root was made. The opposite choice would replace R by $-R$, and hence a by $-a$. So we may assume that $a = 1$, and hence $b = -1$ and $c = 1$. The other choice of $a = -1$ occurs for the representation $\tilde{\gamma}' = \mathrm{sgn} \otimes \tilde{\gamma}$. From the character values we see that $\tilde{\gamma}'$ is not equivalent to $\tilde{\gamma}$. We have thus obtained two irreducible 5-dimensional representations of $\mathfrak{S}_5$.

We construct the final irreducible representation of $\mathfrak{S}_5$ by combining the two icosahedral representations (α, U) and (β, U) of $\mathfrak{A}_5$ as follows: Let θ be the outer automorphism of $\mathfrak{A}_5$ as before (conjugation by $r = (12)$). Define

$$\alpha^\theta(g) = \alpha(rgr) \quad \text{for } g \in \mathfrak{A}_5 \ .$$

Then $\alpha^\theta \cong \beta$ (the other icosahedral representation), since θ interchanges the two conjugacy classes of five-cycles in $\mathfrak{A}_5$. Set $W = U \oplus U$ and let $T \in \mathbf{GL}(W)$ be the transformation that interchanges the two summands:

$$T(u \oplus u') = u' \oplus u \, .$$

Define $\pi : \mathfrak{S}_5 \longrightarrow \mathbf{GL}(W)$ by

$$\pi(g) = \alpha(g) \oplus \alpha^\theta(g) \, , \quad \pi(rg) = T\pi(g) \quad \text{for } g \in \mathfrak{A}_5 \, ,$$

Since $T^2 = I$ and $T\pi(g)T = \pi(rgr)$ for $g \in \mathfrak{A}_5$, it follows that π is a representation of $\mathfrak{S}_5$ (by the same calculation as for $\tilde{\gamma}$).

To prove that π is irreducible, suppose $X \in \mathrm{End}_G(W)$. Write X in block matrix form as

$$X = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

with $A, B, C, D \in \mathrm{End}(U)$. Then the equation $X\pi(g) = \pi(g)X$ for $g \in \mathfrak{A}_5$ implies that

$$\begin{aligned} A\alpha(g) &= \alpha(g)A \, , & D\alpha^\theta(g) &= \alpha^\theta(g)D \, , \\ B\alpha^\theta(g) &= \alpha(g)B \, , & C\alpha(g) &= \alpha^\theta(g)C \, . \end{aligned}$$

Since α and α^θ are inequivalent irreducible representations, it follows from Schur's lemma that $B = C = 0$, $A = aI$, and $B = bI$ for some scalars a, b. But we also have $XT = TX$, so $a = b$ and hence $X = aI$. This proves that π is irreducible, since $\mathfrak{S}_5$ is a reductive group.

For $g \in \mathfrak{A}_5$ we have $\chi_\pi(g) = \chi_\alpha(g) + \chi_\beta(g)$ and $\chi_\pi(rg) = 0$ (this is clear from the block matrix form of $\pi(g)$ and $\pi(rg)$. From Table G.3 we then obtain the character values for π as follows:

conjugacy class:	1	(12)	(12)(34)	(123)	(1234)	(12)(345)	(12345)
χ_π	6	0	-2	0	0	0	1

As a check that our seven representations have the correct dimensions, we calculate

$$1^2 + 1^2 + 4^2 + 4^2 + 5^2 + 5^2 + 6^2 = 120 = |\mathfrak{S}_5| \, .$$

Here is the complete character table; compare with Table 9.3 to obtain the Schur–Weyl correspondence between these representations and the partitions of 5.

Table G.4 Character table of $\mathfrak{S}_5$.

conj. class:	1	(12)	(12)(34)	(123)	(1234)	(12)(345)	(12345)
# elements:	1	10	15	20	30	20	24
χ_1	1	1	1	1	1	1	1
χ_{sgn}	1	−1	1	1	−1	−1	1
χ_ρ	4	2	0	1	0	−1	−1
$\chi_{\rho'}$	4	−2	0	1	0	1	−1
$\chi_{\tilde{\gamma}}$	5	1	1	−1	−1	1	0
$\chi_{\tilde{\gamma}\otimes\mathrm{sgn}}$	5	−1	1	−1	1	−1	0
χ_π	6	0	−2	0	0	0	1

There is another way to obtain the representation π. Take the standard representation (ρ, V) and form the representation $\bigwedge^2 V$. This space has dimension $\binom{4}{2} = 6$. We will show that it is irreducible by calculating its character and using the following general result:

Scholium G.4.1. *Let (μ, E) be a finite-dimensional representation of a group G with character χ_μ. Then the character of the representation $(\mu \wedge \mu, \bigwedge^2 E)$ is given by*

$$\chi_{\mu\wedge\mu}(g) = \frac{1}{2}\Big\{\chi_\mu(g)^2 - \chi_\mu(g^2)\Big\} . \tag{G.4}$$

Proof. Let $g \in G$. We choose an ordered basis $\{\mathbf{e}_i\}$ for E such that the matrix $[g_{ij}]$ for $\mu(g)$ is upper triangular. The wedge products $\mathbf{e}_i \wedge \mathbf{e}_j$ with $1 \le i < j \le \dim E$ then give a basis for $\bigwedge^2 E$ that we order lexicographically: $(i, j) < (k,l)$ if either $i < k$ or else $i = k$ and $j < l$. Since

$$\begin{aligned}(\mu \wedge \mu)(g)(\mathbf{e}_j \wedge \mathbf{e}_l) &= \sum_{i\le j}\sum_{k\le l} g_{ij}g_{kl}\, \mathbf{e}_i \wedge \mathbf{e}_k \\ &= g_{jj}g_{ll}\mathbf{e}_j \wedge \mathbf{e}_l + \sum_{i<j}\sum_{k\le l} g_{ij}g_{kl}\, \mathbf{e}_i \wedge \mathbf{e}_k \\ &\qquad + \sum_{i\le j}\sum_{k<l} g_{ij}g_{kl}\, \mathbf{e}_i \wedge \mathbf{e}_k ,\end{aligned}$$

we see that the matrix for $(\mu \wedge \mu)(g)$ is also upper triangular relative to this ordered basis. Furthermore, the diagonal entries are $g_{ii}g_{jj}$ for all pairs $i < j$. Hence

$$\chi_{\mu\wedge\mu}(g) = \sum_{i<j} g_{ii}g_{jj} = \frac{1}{2}\Big(\sum_i g_{ii}\Big) - \frac{1}{2}\sum_i g_{ii}^2 ,$$

which proves (G.4). □

We return to the case $G = \mathfrak{S}_5$ and the standard representation (ρ, V). From (G.4) and Table G.4 we calculate the character of $\rho \wedge \rho$ as follows:

conjugacy class:	1	(12)	(12)(34)	(123)	(1234)	(12)(345)	(12345)
$\chi_\rho(g)^2$	16	4	0	0	0	0	1
$\chi_\rho(g^2)$	4	4	4	0	0	0	-1
$\chi_{\rho\wedge\rho}(g)$	6	0	-2	0	0	0	1

This shows that $(\rho \wedge \rho, \bigwedge^2 V) \cong (\pi, W)$. In particular, $\bigwedge^2 V$ is irreducible for $\mathfrak{S}_5$ (see Exercises 9.2.4 for more about the wedge products of standard representations of $\mathfrak{S}_n$).

G.5 Exercises

In exercises 1–3, G denotes $\mathfrak{A}_4$ acting as orientation-preserving isometries of the regular tetrahedron $\mathcal{T}$.

1. Let H be the group $\{1, h_1, h_2, h_3\}$ as in Section G.1.
   (a) Show that H is maximal abelian in G.
   (b) Let C be the cyclic subgroup generated by (123) (this is the subgroup fixing a face of $\mathcal{T}$). Show that $C \cong \mathrm{Norm}_G(H)/H$
2. Let E be the set of edges of $\mathcal{T}$. G acts transitively as permutations of E, and $E \cong G/L$, where L is the subgroup of order two fixing an edge. Let $\pi = \mathrm{Ind}_L^G(1)$ be the permutation representation of G on the six-dimensional space of all functions on E.
   (a) Use the fixed-point formula (4.57) to calculate the character of π.
   (b) Obtain the isotypic decomposition of π.
   (c) For each irreducible representation W of G, find the dimension of the space W^L of L-fixed vectors and the decomposition of W under L. (HINT: Use the characters of L and the restriction to L of the character of W.)
3. Let F be the set of faces of $\mathcal{T}$. Then G acts transitively as permutations of F, and $F \cong G/C$, where C is the cyclic subgroup of order 3 fixing a face. Let $\pi = \mathrm{Ind}_C^G(1)$ be the permutation representation of G on the four-dimensional space of all functions on F.
   (a) Use the fixed-point formula (4.57) to calculate the character of π.
   (b) Obtain the isotypic decomposition of π.
   (c) For each irreducible representation W of G, find the dimension of the space W^C of C-fixed vectors and the decomposition of W under C. (HINT: Use the characters of C and the restriction to C of the character of W.)

In exercises 4–7, $\mathcal{O}$ is the regular octahedron and G is $\mathfrak{S}_4$, acting as orientation-preserving isometries of $\mathcal{O}$.

4. Let H be a cyclic group of order 4 in G.
   (a) Show that H is maximal abelian in G.
   (b) Find $\mathrm{Norm}_G(H)/H$.
5. Let V be the set of vertices of O, and let π be the permutation representation of G on the 6-dimensional space $\mathbb{C}[V]$ of complex-valued functions on V:

$$\pi(g)f(v) = f(g^{-1}v) .$$

   Let H be the cyclic group generated by (1234), and let v_0 be a vertex of V fixed by H. Identify V with G/H by the map $g \mapsto \rho'(g)v_0$.
   (a) Calculate the character of π.
   (b) Find the decomposition of the representation π by calculating the inner products of χ_π with the irreducible characters of G.
   (c) Find the decomposition of π by Frobenius reciprocity.

(d) Give an explicit description of the three-dimensional subspace of $\mathbb{C}[V]$ which transforms by ρ' under the action of G.
(e) Show that the projection P_δ onto the δ-isotypic component in any representation π is given by

$$P_\delta = \frac{\dim \delta}{|G|} \sum_{g\in G} \bar{\chi}_\delta(g)\pi(g) = \frac{2}{24} \left\{ 2I - \sum_{g\in 3^1} \pi(g) + 2 \sum_{g\in 2^2} \pi(g) \right\} ,$$

where 3^1 is the conjugacy class of (123) and 2^2 is the conjugacy class of $(12)(34)$.
(f) Define $f_0(v_0) = 1$ and $f_0(v) = 0$ for $v \neq v_0$. Let $\varphi_\delta = 3P_\delta f_0$. Show that φ_δ takes the value 1 at v_0 and the opposite vertex, and the value $-1/2$ at the other four vertices. In particular, $\pi(h)\varphi_\delta = \varphi_\delta$ for $h \in H$. (We call φ_δ the *spherical* function of type δ relative to the subgroup H of G.)

6. Let E be the set of edges of $\mathcal{O}$. G acts transitively as permutations of E, and $E \cong G/L$, where L is the subgroup of order 2 fixing an edge. Let $\rho = \mathrm{Ind}_L^G(1)$ be the permutation representation of G on the eight-dimensional space of all functions on E.
(a) Use the fixed-point formula (4.57) to calculate the character of ρ.
(b) Obtain the isotypic decomposition of ρ.
(c) For each irreducible representation W of G, find the dimension of the space W^L of L-fixed vectors and the decomposition of W under L.
7. Let F be the set of faces of $\mathcal{O}$. G acts transitively as permutations of F, and $F \cong G/C$, where C is the cyclic subgroup of order 3 fixing a face. Let $\rho = \mathrm{Ind}_C^G(1)$ be the permutation representation of G on the eight-dimensional space of all functions on F.
(a) Use the fixed-point formula (4.57) to calculate the character of ρ.
(b) Obtain the isotypic decomposition of ρ.
(c) For each irreducible representation W of G, find the dimension of the space W^C of C-fixed vectors and the decomposition of W under C.

In exercises 8–10, G denotes $\mathfrak{A}_5$ acting as orientation-preserving isometries of the regular icosahedron $\mathcal{I}$.

8. Let V be the set of vertices of $\mathcal{I}$, and let π be the permutation representation of $G = \mathfrak{A}_5$ on the 12-dimensional space $\mathbb{C}[V]$ of complex-valued functions on V.
(a) Calculate the character of π using the character formula (4.55) for an induced representation.
(b) Find the decomposition of the representation π by calculating the inner products of χ_π with the irreducible characters.
9. Let E be the set of edges of I. G acts transitively as permutations of E, and $E \cong G/L$, where L is the subgroup of order 2 fixing an edge. Let $\pi = \mathrm{Ind}_L^G(1)$ be the permutation representation of G on the 30-dimensional space of all functions on E.
(a) Use the fixed-point formula (4.57) to calculate the character of π.

(b) Obtain the isotypic decomposition of π.
(c) For each irreducible representation W of G, find the dimension of the space W^L of L-fixed vectors and the decomposition of W under L.

10. Let F be the set of faces of $\mathcal{I}$. G acts transitively as permutations of E, and $E \cong G/C$, where C is the cyclic subgroup of order 3 fixing a face. Let $\rho = \mathrm{Ind}_C^G(1)$ be the permutation representation of G on the 20-dimensional space of all functions on F.
(a) Use the fixed-point formula (4.57) to calculate the character of ρ.
(b) Obtain the isotypic decomposition of ρ.
(c) For each irreducible representation W of G, find the dimension of the space W^C of C-fixed vectors and the decomposition of W under C.

The following exercises consider the representations of $\mathfrak{S}_5$.

11. (a) Embed $\mathfrak{S}_5 \subset \mathfrak{S}_6$ as the subgroup fixing 6. Let π be the standard 5-dimensional representation of $\mathfrak{S}_6$. Show that $\mathrm{Res}_{\mathfrak{S}_5}^{\mathfrak{S}_6}(\pi)$ is *not* irreducible by calculating its character, and find its isotypic decomposition. (HINT: Use Corollary 4.3.11; check your answer using Corollary 9.2.7.)
(b) To obtain the representation γ of $\mathfrak{A}_5$, we found a homomorphism $\mathfrak{A}_5 \longrightarrow \mathfrak{S}_6$ via the permutation action on six symmetry axes of the icosahedron, and then composed this with the standard representation of $\mathfrak{S}_6$. Use the result in (a) to show that this homomorphism does *not* extend to a homomorphism $\mathfrak{S}_5 \longrightarrow \mathfrak{S}_6$.
12. Let (μ, E) be a representation of a group G. Show that the character of the symmetric square $(\mu^{\mathrm{sym2}}, S^2(E))$ is given by

$$\chi_{\mu^{\mathrm{sym2}}}(g) = \frac{1}{2}\left\{\chi_\mu(g)^2 + \chi_\mu(g^2)\right\} .$$

Find the relation between this formula and the decomposition

$$\bigotimes^2 E = S^2(E) \oplus \bigwedge^2 E .$$

13. Let (ρ, V) be the standard representation of $\mathfrak{S}_5$. Since $\rho^{\wedge 2}$ is equivalent to the six-dimensional representation π of $\mathfrak{S}_5$, we know from the construction of π that $\rho^{\wedge 2}|_{\mathfrak{A}_5}$ is the sum of the two three-dimensional icosahedral representations. Use the character of α to find the three-dimensional subspace of $\bigwedge^2 V$ that transforms by α under the action of $\mathfrak{A}_5$.

References for Appendix G

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