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Symmetry, Representations, and Invariants

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Appendix F
Decomposition of Tensor Spaces for $\mathbf{O(V)}$ and $\mathbf{Sp(V)}$

Abstract In this appendix we complete the results of Chapter 10 concerning the decomposition of tensor space under the action of the orthogonal group or the symplectic group. Following ideas of Weyl [7], we decompose the full tensor space into spaces of *partially harmonic tensors*. This is the full tensor generalization of the decompositions of the algebras of symmetric tensors and skew-symmetric tensors obtained in Chapter 5.

F.1 Partially harmonic tensors

Let V be a finite-dimensional complex vector space, and let $G \subset \mathbf{GL}(V)$ be the group preserving the nondegenerate symmetric or skew-symmetric bilinear form ω on V. We consider the problem of decomposing the tensor space $\bigotimes^k V$ under the joint action of G and the centralizer algebra

$$\mathcal{B}_k(V,\omega) = \mathrm{End}_G(\textstyle\bigotimes^k V)$$

(following the notation of Section 10.1.1).

For $r = 0, 1, \ldots, [k/2]$ let $\mathcal{B}_{k,r}(V,\omega)$ be the subspace of $\mathcal{B}_k(V,\omega)$ spanned by the operators involving r or more contractions (these operators correspond to Brauer diagrams with r or more bars). From the relations in Section 10.1.2 we see that $\mathcal{B}_{k,r}(V,\omega)$ is a 2-sided ideal in $\mathcal{B}_k(V,\omega)$. Thus we have a chain of ideals

$$\mathcal{B}_k(V,\omega) = \mathcal{B}_{k,0}(V,\omega) \supset \mathcal{B}_{k,1}(V,\omega) \supset \cdots \supset \mathcal{B}_{k,[k/2]}(V,\omega)\ .$$

We set $\mathcal{T}_r^{\otimes k} = \mathcal{B}_{k,r}(V,\omega)(\bigotimes^k V)$ (we will not indicate the dependence of these spaces on the pair (V,ω), which will remain fixed throughout this section). Clearly, $\mathcal{T}_r^{\otimes k} \supset \mathcal{T}_{r+1}^{\otimes k}$, and we have a filtration

$$\textstyle\bigotimes^k V = \mathcal{T}_0^{\otimes k} \supset \mathcal{T}_1^{\otimes k} \supset \cdots \supset \mathcal{T}_{[k/2]}^{\otimes k} \tag{F.1}$$

by $\mathcal{A}[G] \otimes \mathcal{B}_k(V,\omega)$ submodules. We define

$$\mathcal{HT}_r^{\otimes k} = \{u \in \mathcal{T}_r^{\otimes k} : z \cdot u = 0 \quad \text{for all } z \in \mathcal{B}_{k,r+1}(V,\omega)\} ,$$

and we call $\mathcal{HT}_r^{\otimes k}$ the space of *partially harmonic tensors of valence* r. When $r = 0$, then $\mathcal{T}_0^{\otimes k} = \bigotimes^k V$ and

$$\mathcal{HT}_0^{\otimes k} = \mathcal{H}(\bigotimes^k V, \omega)$$

is the space of ω-harmonic k-tensors. The spaces $\mathcal{HT}_r^{\otimes k}$ are obviously invariant under G and $\mathcal{B}_k(V,\omega)$. In fact, the quotient algebra $\mathcal{B}_k(V,\omega)/\mathcal{B}_{k,r+1}(V,\omega)$ acts on $\mathcal{HT}_r^{\otimes k}$.

Theorem F.1.1. *The space $\bigotimes^k V$ is the direct sum of the partially harmonic tensors of valences* $0, 1, \ldots, [k/2]$:

$$\bigotimes{}^k V = \bigoplus_{r=0}^{[k/2]} \mathcal{HT}_r^{\otimes k} . \tag{F.2}$$

We shall prove this theorem in the next section. Now we consider the structure of $\mathcal{HT}_r^{\otimes k}$ as a G-module. First we need some notation. For $r = 1, 2, \ldots, [k/2]$ we denote by $\mathcal{M}(k,r)$ the set of all *matchings* of r pairs of numbers from the set $\{1, 2, \ldots, k\}$. An element $\gamma \in \mathcal{M}(k,r)$ is a set

$$\gamma = \{\{m_1, n_1\}, \{m_2, n_2\}, \ldots, \{m_r, n_r\}\}$$

of r unordered pairs of integers $\{m_i, n_i\}$ such that

(1) $1 \le m_i \le k$ and $1 \le n_i \le k$;
(2) the set $[[\gamma]] = \{m_1, n_1, \ldots, m_r, n_r\}$ has $2r$ distinct elements.

For example, $\mathcal{M}(4,2)$ consists of the three matchings

$$\{\{1,2\},\{3,4\}\}, \quad \{\{1,3\},\{2,4\}\}, \quad \{\{1,4\},\{2,3\}\} .$$

The action of $\mathfrak{S}_k$ as permutations of $\{1, \ldots, k\}$ induces an action of $\mathfrak{S}_k$ on $\mathcal{M}(k,r)$ for each r. This action is transitive, and the stabilizer of a point in $\mathcal{M}(k,r)$ is isomorphic to the subgroup $\mathfrak{B}_r$ of $\mathfrak{S}_{2r}$, as in Section 10.1.1.

Let $\mathcal{M}(k,0)$ be the empty set, and write

$$\mathcal{M}(k) = \bigcup_{r=0}^{[k/2]} \mathcal{M}(k,r) .$$

Let $\gamma \in \mathcal{M}(k,r)$. There is a unique labeling for the pairs $\{m_i, n_i\}$ in γ so that

$$m_1 < \cdots < m_r, \qquad m_i < n_i \quad \text{for } i = 1, \ldots, r .$$

We write $\gamma = (\mathbf{m}, \mathbf{n})$ when this labeling is understood, where $\mathbf{m} = (m_1, \ldots, m_r)$ and $\mathbf{n} = (n_1, \ldots, n_r)$. We use the notation

$$[[\gamma]]^c = \{1,2,\dots,k\} \setminus [[\gamma]]$$

for the set of numbers not matched by γ.

The normalized r-bar Brauer diagrams with $2k$ dots from Section 10.1.1 are parameterized by the matchings $\gamma \in \mathcal{M}(k,r)$, where a pair of integers $\{m_i, n_i\}$ in γ corresponds to a bar joining dots numbered $2m_i+1$ and $2n_i+1$ in the top row and a bar joining dots numbered $2m_i$ and $2n_i$ in the bottom row of the diagram. For $\gamma = (\mathbf{m}, \mathbf{n}) \in \mathcal{M}(k,r)$ define

$$P_\gamma = (\dim V)^{-r} \tau_{m_1 n_1} \cdots \tau_{m_r n_r} \in \mathcal{B}_k,$$

where the operators τ_{ij} are as in Section 10.1.2. Since the pairs of indices $\{m_i, n_i\}$ are all distinct, the operators $\tau_{m_i n_i}$ mutually commute for $i = 1, \dots, r$; therefore, the ordering in the product is irrelevant and the map $\gamma \mapsto P_\gamma$ is well defined (see Lemma 10.1.2). For $r = 0$ we set $P_\emptyset = I$ (the identity operator on $\bigotimes^k V$). Since $\tau_{ij}^2 = (\dim V)\tau_{ji}$ by Lemma 10.1.5, we have $P_\gamma^2 = P_\gamma$. Hence P_γ is a projection operator. For $s \in \mathfrak{S}_k$ we have

$$\sigma_k(s) P_\gamma \sigma_k(s)^{-1} = P_{s\cdot\gamma}$$

from Lemma 10.1.5.

For $\gamma \in \mathcal{M}(k)$ we define

$$\mathcal{T}_\gamma^{\otimes k} = P_\gamma(\textstyle\bigotimes^k V)\ .$$

Then $\mathcal{T}_\gamma^{\otimes k}$ is a G-invariant subspace of $\bigotimes^k V$. In particular, $\mathcal{T}_\emptyset^{\otimes k} = \bigotimes^k V$. Define the subspace of *partially harmonic k-tensors of type* γ:

$$\mathcal{H}\mathcal{T}_\gamma^{\otimes k} = \{u \in \mathcal{T}_\gamma^{\otimes k} : C_{ij}u = 0 \text{ for all } i \neq j \text{ with } i,j \in [[\gamma]]^c\}\ .$$

Here C_{ij} is the ij-contraction operator. In particular, $\mathcal{H}\mathcal{T}_\emptyset^{\otimes k} = \mathcal{H}(\bigotimes^k V, \omega)$. The space $\mathcal{H}\mathcal{T}_\gamma^{\otimes k}$ is invariant under G.

We have shown in Sections 10.2.3, 10.2.4, and 10.2.5 that for each G-admissible partition λ of $k-2r$ and Young tableau A of shape λ, there is an associated irreducible G-module

$$U(\lambda) = \mathbf{s}(A)\mathcal{H}(\textstyle\bigotimes^{k-2r} V, \omega)\ ,$$

where $\mathbf{s}(A)$ is the Young symmetrizer for A (here we fix some choice of tableau A for each λ). Let G^λ be the irreducible representation of $\mathfrak{S}_{k-2r}$ corresponding to λ by Schur duality.

Theorem F.1.2. *For every matching $\gamma \in \mathcal{M}(k,r)$, the G-module $\mathcal{H}\mathcal{T}_\gamma^{\otimes k}$ is isomorphic to $\mathcal{H}(\bigotimes^{k-2r} V, \omega)$. Furthermore,*

$$\mathcal{H}\mathcal{T}_r^{\otimes k} = \sum_{\gamma \in \mathcal{M}(k,r)} \mathcal{H}\mathcal{T}_\gamma^{\otimes k}\ . \tag{F.3}$$

Hence there is a decomposition

$$\mathcal{H}\mathcal{T}_r^{\otimes k} \cong \bigoplus_{\lambda} m(r,\lambda) U(\lambda)$$

as a G-module, where λ ranges over all the G-admissible partitions of $k-2r$ and the multiplicities $m(r,\lambda)$ satisfy

$$1 \leq m(r,\lambda) \leq \dim(G^{\lambda}) |\mathcal{M}(k,r)| \ . \tag{F.4}$$

This theorem, which we will prove in the next section, does not assert that the sum (F.3) is direct, in general. However, when n is sufficiently large (relative to k), we will prove that the sum is direct and hence the upper bound in (F.4) becomes an equality. Together with Theorem F.1.1 this gives the complete decomposition of $\bigotimes^k V$ as a G-module when the rank of G is sufficiently large relative to k (we call this the *stable range*).

Theorem F.1.3. *Let $n = \dim V$. Let $r \geq 0$ and assume $2k \leq n + 3r$. Then*

$$\mathcal{H}\mathcal{T}_r^{\otimes k} = \bigoplus_{\gamma \in \mathcal{M}(k,r)} \mathcal{H}\mathcal{T}_{\gamma}^{\otimes k} \ .$$

Hence if $2k \leq n+3$, then

$$\bigotimes^k V \cong \bigoplus_{r=0}^{[k/2]} m(r) \mathcal{H}(\bigotimes^{k-2r} V, \omega)$$

as a G-module, where the multiplicities are $m(0) = 1$ and

$$m(r) = |\mathcal{M}(k,r)| = k(k-1)\cdots(k-2r+1)/(2^r r!) \quad \textit{for } r \geq 1 \ .$$

We shall prove this theorem in Section F.3. It is the tensor generalization of the harmonic decompositions in Chapter 5.

Examples

1. Suppose $k = 2m$ is even and take $r = m$. Since $\mathcal{H}(\bigotimes^0 V, \omega)$ is the trivial G-module and the spaces $\mathcal{H}(\bigotimes^p V, \omega)$ have no G-invariants for $p > 0$, we see from Theorem F.1.2 that

$$\mathcal{H}\mathcal{T}_m^{\otimes(2m)} = \left(\bigotimes^{2m} V\right)^G, \qquad \dim \mathcal{H}\mathcal{T}_m^{\otimes(2m)} \leq |\mathcal{M}(2m,m)| = \tfrac{(2m)!}{2^m (m!)^2} \ .$$

Furthermore, when $m \leq n$ then Theorem F.1.3 applies and the inequality above for the dimension of the space of G-invariant $2m$-tensors becomes an equality.

2. In the range $2k > n+3$ the multiplicities of the harmonic $k-2r$ tensors in the decomposition of $\bigotimes^k V$ can be strictly less than $|\mathcal{M}(k,r)|$. For example, let $G = \mathbf{Sp}(1,\mathbb{C}) = \mathbf{SL}(2,\mathbb{C})$ and let V_k be the $(k+1)$-dimensional irreducible G-module (see Section 2.3.2). Then $\mathcal{H}(\bigotimes^k(\mathbb{C}^2, \Omega) \cong V_k$, and from the Clebsch–Gordan formula (see Exercises 7.1.4) we calculate that

$$\bigotimes^3 \mathbb{C}^2 \cong V_3 \oplus 2V_1 \,, \qquad \bigotimes^4 \mathbb{C}^2 \cong V_4 \oplus 3V_2 \oplus V_0 \,. \tag{F.5}$$

This shows that the multiplicities given in Theorem F.1.3 are not always attained outside the stable range, since $|\mathcal{M}(3,1)| = 3$ whereas the actual multiplicity of $\mathcal{H}(\mathbb{C}^2) = V_1$ in $\bigotimes^3 \mathbb{C}^2$ is 2. If Theorem F.1.3 were valid in this case, the multiplicities of $V_2 = \mathcal{H}(\bigotimes^2 \mathbb{C}^2)$ and of $V_0 = \mathcal{H}(\bigotimes^0 \mathbb{C}^2)$ in $\bigotimes^4 \mathbb{C}^2$ would be $|\mathcal{M}(4,1)| = 6$ and $|\mathcal{M}(4,2)| = 3$, respectively. But the actual multiplicities are 3 and 2. Thus the right inequality in (F.4) is strict in these cases. (Here the G-admissible partitions have one part, so λ corresponds to the trivial representation of the symmetric group).

F.2 Proof of partial harmonic decomposition

We turn to the proofs of Theorem F.1.1 and Theorem F.1.2. Fix a basis $\{e_j : j = 1,\dots,n\}$ for V so that

$$\omega(e_i, e_j) = \delta_{n+1,i+j} \text{ for } i \le j \,.$$

We identify V with $\mathbb{C}^n$ via this basis. Then the dual basis is $e^i = e_{n+1-i}$ for $1 \le i \le [n/2]$ and $e^i = \varepsilon e_{n+1-i}$ for $[n/2] < i \le n$. Let $u \mapsto u^*$ be the conjugate-linear map on $\mathbb{C}^n$ such that $(ce_i)^* = c^* e^i$ for $c \in \mathbb{C}$ (where c^* is the complex conjugate of c). Clearly,

$$(u^*)^* = \varepsilon u \,, \tag{F.6}$$

$$\omega(u^*, v^*) = \omega(u, v)^* \tag{F.7}$$

for $u, v \in \mathbb{C}^n$. We extend $*$ to a conjugate-linear map on $\bigotimes^k \mathbb{C}^n$ by

$$(u_1 \otimes \cdots \otimes u_k)^* = u_1^* \otimes \cdots \otimes u_k^* \,.$$

Then (F.7) is also valid for $u, v \in \bigotimes^k \mathbb{C}^n$. It is easy to verify that the operators P_γ commute with $*$, so the spaces $\mathcal{HT}_\gamma^{\otimes k}$ are $*$-invariant.

Define $\langle u|v\rangle = \omega(u, v^*)$. This is a Hermitian form on $\bigotimes^k \mathbb{C}^n$ that we claim is positive definite. Indeed, if $u = \sum_{|I|=k} c_I e_I$, then

$$\omega(u, u^*) = \sum_I c_I c_I^* \, \omega(e_I, e^I) = \sum_I |c_I|^2 \,.$$

It follows that if $L \subset M \subset \bigotimes^k \mathbb{C}^n$ are any linear subspaces such that $L^* = L$ and $M^* = M$, then there is a decomposition $M = L \oplus N$, where

$$N = \{u \in M : \omega(u, v) = 0 \text{ for all } v \in L\} \,.$$

Indeed, $\omega(u, v) = \varepsilon^k \langle u|v^*\rangle$, so N is also the orthogonal complement of L in M relative to the Hermitian inner product $\langle\cdot|\cdot\rangle$ on M. For $*$-invariant subspaces we can use the term *orthogonal* without confusion between the bilinear form ω and the associated Hermitian form. We write $L \perp N$ to denote orthogonality relative to the Hermitian form.

Given a matching $\gamma = (\mathbf{m}, \mathbf{n}) \in \mathcal{M}(k,r)$, we define the *r-fold contraction operator* $C_\gamma : \bigotimes^k \mathbb{C}^n \longrightarrow \bigotimes^{k-2r} \mathbb{C}^n$ by

$$C_\gamma(v_1 \otimes \cdots \otimes v_k) = \Big\{ \prod_{i=1}^{r} \omega(v_{m_i}, v_{n_i}) \Big\} v_{j_1} \otimes \cdots \otimes v_{j_{k-2r}} ,$$

where $\{j_1, \ldots, j_{k-2r}\}$ is the set $[[\gamma]]^c$ enumerated in increasing order. For example, if $\gamma = \{\{1,2\}, \{3,5\}\} \in \mathcal{M}(6,2)$, then

$$C_\gamma(v_1 \otimes \cdots \otimes v_6) = \omega(v_1, v_2)\omega(v_3, v_5) v_4 \otimes v_6 .$$

When $\gamma = \{i, j\}$, then C_γ is the previously defined contraction operator C_{ij}. Define the *r-fold expansion operator* $D_\gamma : \bigotimes^{k-2r} \mathbb{C}^n \longrightarrow \bigotimes^k \mathbb{C}^n$ by

$$D_\gamma(v_1 \otimes \cdots \otimes v_{k-2r}) = \sum_{p_1, \ldots, p_r} u(p_1, \ldots, p_k) ,$$

where $u(p_1, \ldots, p_k)$ is the decomposable $2r$-tensor with e_{p_i} in position m_i and e^{p_i} in position n_i, for $i = 1, \ldots, r$, while the other positions in $u(p_1, \ldots, p_k)$ are filled in order by $v_1, \ldots, v_{k-2r}$. For example, if $\gamma = \{\{1,2\}, \{3,5\}\} \in \mathcal{M}(6,2)$ then

$$D_\gamma(v_1 \otimes v_2) = \sum_{p_1, p_2} e_{p_1} \otimes e^{p_1} \otimes e_{p_2} \otimes v_1 \otimes e^{p_2} \otimes v_2 .$$

When $\gamma = \{i, j\}$ then D_γ is the previously defined expansion operator D_{ij}. When $\gamma = \{\{1,2\}, \ldots, \{2r-1, 2r\}\}$ then $D_\gamma u = \theta_r \otimes u$ for $u \in \bigotimes^{k-2r} \mathbb{C}^n$ (tensor multiplication by the G-invariant tensor θ_r).

The contraction and expansion operators obviously intertwine the actions of G on $\bigotimes^k \mathbb{C}^n$ and $\bigotimes^{k-2} \mathbb{C}^n$. They give a factorization

$$P_\gamma = n^{-r} D_\gamma C_\gamma . \tag{F.8}$$

Let $E_\gamma : \bigotimes^{k-2r} \mathbb{C}^n \longrightarrow \bigotimes^k \mathbb{C}^n$ be the operator that inserts e_1 in positions $m_1, \ldots, m_r$ and inserts e^1 in positions $n_1, \ldots, n_r$. A simple calculation shows that

$$D_\gamma = n^r P_\gamma E_\gamma . \tag{F.9}$$

Just as in the case $r = 1$, the product of the contraction and expansion operators in the opposite order is a multiple of the identity operator:

$$C_\gamma D_\gamma = n^r I . \tag{F.10}$$

(This follows by the same calculation as (10.13).) In particular, from (F.8) and (F.10) we have $C_\gamma P_\gamma = C_\gamma$. Hence

$$\mathrm{Ker}(P_\gamma) = \mathrm{Ker}(C_\gamma)\ . \tag{F.11}$$

Furthermore, from (F.8) and (F.9) we see that

$$\mathcal{T}_\gamma^{\otimes k} = \mathrm{Range}(P_\gamma) = \mathrm{Range}(D_\gamma)\ . \tag{F.12}$$

It is straightforward to verify that

$$\langle C_\gamma u \,|\, v\rangle = \langle u \,|\, D_\gamma v\rangle \tag{F.13}$$

for $u \in \bigotimes^k \mathbb{C}^n$ and $v \in \bigotimes^{k-2r} \mathbb{C}^n$. This implies that the projection operator P_γ is self-adjoint, relative to the Hermitian inner product.

We can now prove Theorem F.1.1. From the relations in Section 10.1.2 we see that $\mathcal{B}_k^r$ is spanned by the operators $P_\gamma \sigma_k(s)$ with $\gamma \in \mathcal{M}(k,r)$ and $s \in \mathfrak{S}_k$. Thus

$$\mathcal{T}_r^{\otimes k} = \sum_{\gamma \in \mathcal{M}(k,r)} \mathcal{T}_\gamma^{\otimes k}\ .$$

Likewise, we have

$$\mathcal{H}\mathcal{T}_r^{\otimes k} = \{u \in \mathcal{T}_r^{\otimes k} : P_\gamma u = 0 \text{ for all } \gamma \in \mathcal{M}(k, r+1)\}\ .$$

Thus a k-tensor is in $\mathcal{H}\mathcal{T}_r^{\otimes k}$ if it is in the span of the r-fold expansion operators and is annihilated by all $(r+1)$-fold contraction operators. Note that $\mathcal{H}\mathcal{T}_{[k/2]}^{\otimes k} = \mathcal{T}_{[k/2]}^{\otimes k}$ since $\mathcal{M}(k,r)$ is empty for $r > [k/2]$. Clearly, $\mathcal{T}_r^{\otimes k}$ and $\mathcal{H}\mathcal{T}_r^{\otimes k}$ are $*$ invariant. We see from (F.12) and (F.13) that $\langle u|w\rangle = 0$ for all $w \in \mathcal{T}_{r+1}^{\otimes k}$ if and only if $C_\gamma u = 0$ for all $\gamma \in \mathcal{M}(k, r+1)$. This implies that there is an orthogonal decomposition

$$\mathcal{T}_r^{\otimes k} = \mathcal{H}\mathcal{T}_r^{\otimes k} \oplus \mathcal{T}_{r+1}^{\otimes k} \tag{F.14}$$

for $r = 0, 1, \ldots, [k/2]-1$. This together with the filtration (F.1) then give the decomposition

$$\bigotimes^k \mathbb{C}^n = \mathcal{H}\mathcal{T}_0^{\otimes k} \oplus \mathcal{H}\mathcal{T}_1^{\otimes k} \oplus \cdots \oplus \mathcal{H}\mathcal{T}_{[k/2]}^{\otimes k}$$

as claimed in Theorem F.1.1

We now turn to the proof of Theorem F.1.2. For $r \geq 0$ set

$$\widetilde{\mathcal{T}}_r^{\otimes k} = \sum_{\gamma \in \mathcal{M}(k,r)} \mathcal{H}\mathcal{T}_\gamma^{\otimes k}\ .$$

Then $\widetilde{\mathcal{T}}_r^{\otimes k} \subset \mathcal{T}_r^{\otimes k}$. We first prove that for $r = 0, 1, \ldots, [k/2]$, one has

$$\mathcal{T}_r^{\otimes k} = \widetilde{\mathcal{T}}_r^{\otimes k} + \mathcal{T}_{r+1}^{\otimes k}\ . \tag{F.15}$$

Let $\gamma \in \mathcal{M}(k,r)$. There is an orthogonal decomposition

$$\mathcal{T}_\gamma^{\otimes k} = \mathcal{H}\mathcal{T}_\gamma^{\otimes k} \oplus \mathcal{N}_\gamma^{\otimes k}\ . \tag{F.16}$$

We claim that

$$\mathcal{N}_\gamma^{\otimes k} \subset \mathcal{T}_{r+1}^{\otimes k} \,. \tag{F.17}$$

Indeed, for every $i, j \in [[\gamma]]^c$ with $i \neq j$, the self-adjoint operator P_{ij} leaves the space $\mathcal{T}_\gamma^{\otimes k}$ invariant, since it commutes with P_γ. By definition $\mathcal{H}\mathcal{T}_\gamma^{\otimes k}$ is the intersection of the kernels of all such operators P_{ij} on $\mathcal{T}_\gamma^{\otimes k}$. Hence the orthogonal space $\mathcal{N}_\gamma^{\otimes k}$ is the span of the ranges of P_{ij} on $\mathcal{T}_\gamma^{\otimes k}$, for all such pairs i, j. Since

$$P_{ij}\mathcal{T}_\gamma^{\otimes k} \subset \mathcal{T}_{r+1}^{\otimes k} \,,$$

this proves (F.17). Summing (F.16) over $\gamma \in \mathcal{M}(k,r)$ and using (F.17), we obtain (F.15).

We now show that the sum (F.15) is direct by a combinatorial argument using the relations in the algebra $\mathcal{B}_k$.

Let $\gamma \in \mathcal{M}(k,r)$ and $\mu \in \mathcal{M}(k,r+1)$ be matchings with r and $r+1$ pairs, respectively. If $\mu' \in \mathcal{M}(k,s+1)$ for some $s \leq r$, we say that μ' is a *submatching* of μ and write $\mu' \subset \mu$ if every pair $\{p,q\} \in \mu'$ is also in μ. If μ' is a submatching of μ, we say that μ is *γ-linked with ends x,y* if

$$\mu' = \{\{x,m_1\},\{n_1,m_2\},\ldots,\{n_{s-1},m_s\},\{n_s,y\}\} \,, \tag{F.18}$$

where the pairs $\{m_i,n_i\} \in \gamma$ for $i = 1,\ldots,s$ (here we do not assume that m_i is enumerated in increasing order or that $m_i < n_i$). Thus there are s pairs from γ that "link" the $s+1$ pairs in μ'.

Example
Let $k = 6$ and $r = 2$. Consider the pairs γ and μ and the elements $x,y \in [[\mu]]$ shown in Figure F.1.

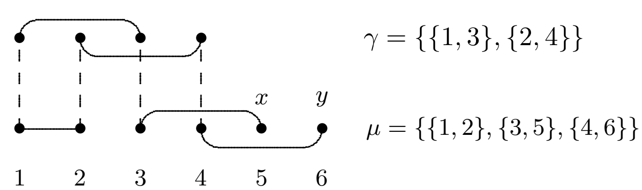

Fig. F.1 γ-linked submatching.

Then $\mu' = \{\{5,3\},\{1,2\}\}$ is a γ-linked submatching of μ with ends $x = 5$ and 2 (we have indicated the linking by the vertical dashed lines). In this example the *maximal* γ-linked submatching of μ having 5 as an end is μ itself, and the other end of this maximal submatching is $y = 6$. Note that starting with $x \in [[\gamma]]^c$ and taking the maximum γ-linked submatching of μ, we ended at $y \in [[\gamma]]^c$. We will show in the scholium below that this is a general phenomenon.

The notion of a γ-linked submatching is important because of the following identity in the algebra $\mathcal{B}_k$:

Lemma F.2.1. *Let $\gamma \in \mathcal{M}(k,r)$ and $\mu \in \mathcal{M}(k,r+1)$. Suppose μ' is a γ-linked submatching of μ with ends x,y. If $x,y \in [[\gamma]]^c$ then there exists $t \in \mathfrak{S}_k$ so that*

$$\tau_{\mu'}\tau_\gamma = \sigma_k(t)\tau_{xy}\tau_\gamma \,. \tag{F.19}$$

Proof. Let μ' be given by (F.18). We argue by induction on s. If $s = 0$ then $\mu' = \{x,y\}$ and (F.19) holds with $t = 1$. If $s > 0$ we have the identity

$$\tau_{yn_s}\tau_{m_sn_s} = \sigma_k(ym_s)\tau_{m_sn_s}$$

by Lemma 10.1.5 (2), where (ym_s) is the transposition $y \leftrightarrow m_s$. Hence

$$\tau_{n_{s-1}m_s}\tau_{yn_s}\tau_{m_sn_s} = \sigma_k(ym_s)\tau_{n_{s-1}y}\tau_{m_sn_s} \,. \tag{F.20}$$

The operator $\sigma_k(ym_s)$ commutes with the remaining factors in $\tau_{\mu'}$, namely $\tau_{n_im_{i+1}}$ (for $1 \le i \le s-2$) and τ_{xm_1}, and so it may be moved to the left. From (F.20) we thus have

$$\tau_{\mu'}\tau_\gamma = \sigma_k(ym_s)\tau_{\mu''}\tau_\gamma \,,$$

where μ'' is obtained from μ' by omitting $\{m_s,n_s\}$. The lemma now follows by induction on s. $\square$

Scholium F.2.2. *For every $\gamma \in \mathcal{M}(k,r)$ and $\mu \in \mathcal{M}(k,r+1)$ there exists a γ-linked submatching $\mu' \subset \mu$ whose ends are in $[[\gamma]]^c$.*

We defer the combinatorial proof of this scholium to the end of the section. Let us first see how it can be combined with Lemma F.2.1 to obtain the first part of Theorem F.1.2.

Let $\gamma \in \mathcal{M}(k,r)$ and $u \in \mathcal{H}\mathcal{T}_\gamma^{\otimes k}$. We claim that $\tau_\mu u = 0$ for all $\mu \in \mathcal{M}(k,r+1)$. Indeed, by Scholium F.2.2 there exists an γ-linked submatching $\mu' \subset \mu$ with ends $x,y \in [[\gamma]]^c$. Since $\tau_\mu = \tau_{\mu''}\tau_{\mu'}$, where $\mu'' \subset \mu$ is the complement to μ', it will suffice to show that $\tau_{\mu'}u = 0$. Since $u = \tau_\gamma z$ for some $z \in \bigotimes^k \mathbb{C}^n$, we have

$$\tau_{\mu'}u = \sigma_k(t)\tau_{xy}\tau_\gamma z = \sigma_k(\gamma)\tau_{xy}u$$

from (F.19), where $x,y \in [[\gamma]]^c$. But $\tau_{xy}u = 0$ since u is partially harmonic of type γ.

We have thus shown that

$$\mathcal{H}\mathcal{T}_\gamma^{\otimes k} \subset \mathcal{H}\mathcal{T}_r^{\otimes k}$$

for every $\gamma \in \mathcal{M}(k,r)$. Hence $\widetilde{\mathcal{T}}_r^{\otimes k} \subset \mathcal{H}\mathcal{T}_r^{\otimes k}$, so by (F.14) and (F.15) we conclude that $\mathcal{H}\mathcal{T}_r^{\otimes k} = \widetilde{\mathcal{T}}_r^{\otimes k}$ and $\mathcal{T}_r^{\otimes k} = \mathcal{H}\mathcal{T}_r^{\otimes k} \oplus \mathcal{T}_{r+1}^{\otimes k}$. This completes the proof of the first part of Theorem F.1.2. Now we turn to the structure of the individual spaces $\mathcal{H}\mathcal{T}_\gamma^{\otimes k}$.

Lemma F.2.3. *Let $\gamma \in \mathcal{M}(k,r)$.*

1. *$D_\gamma : \bigotimes^{k-2r} \mathbb{C}^n \longrightarrow \mathcal{T}_\gamma^{\otimes k}$ is an isomorphism of G-modules.*
2. *$D_\gamma\big(\mathcal{H}(\bigotimes^{k-2r} \mathbb{C}^n)\big) = \mathcal{H}\mathcal{T}_\gamma^{\otimes k}$.*

Proof. We know that D_γ intertwines the G actions on $\bigotimes^{k-2r} \mathbb{C}^n$ and $\bigotimes^{k} \mathbb{C}^n$. From (F.12) and (F.10) we see that

$$\mathcal{T}_\gamma^{\otimes k} = D_\gamma \left(\bigotimes\nolimits^{k-2r} \mathbb{C}^n\right)$$

and $n^{-r} C_\gamma : \mathcal{T}_\gamma^{\otimes k} \longrightarrow \bigotimes^{k-2r} \mathbb{C}^n$ is the inverse to D_γ. This proves statement (1).

For statement (2), let $i, j \in [[\gamma]]^c$. We claim that there exists i', j' with $1 \le i' < j' \le k-2r$ and $\gamma' \in \mathcal{M}(k,r)$ such that $(i', j') \in [[\gamma']]^c$ and

$$C_{ij} D_\gamma u = D_{\gamma'} C_{i'j'} u \tag{F.21}$$

for all $u \in \bigotimes^{k-2r} \mathbb{C}^n$. For example, if $\gamma = \{\{1,2\},\ldots,\{2r-1,2r\}\}$ and $i > 2r$, $j > 2r$, then

$$C_{ij} D_\gamma u = C_{ij}(\theta_r \otimes u) = D_\gamma C_{i-2r,j-2r} u$$

for $u \in \bigotimes^{k-2r} \mathbb{C}^n$, so we can take $i' = i-2r$, $j' = j-2r$, and $\gamma' = \gamma$ in this case. In general, if we define

$$s(i) = \mathrm{Card}\{p : m_p < i\} + \mathrm{Card}\{p : n_p < i\} ,$$

then (F.21) holds with $i' = i - s(i)$, $j' = j - s(j)$, and a suitable $\gamma' = (\mathbf{m}', \mathbf{n}')$. Here m'_p is either m_p, $m_p - 1$, or $m_p - 2$ (depending on the relation between p and i, j), and likewise for n'_p. The map $i \mapsto i - s(i)$ is a monotone bijection from $[[\gamma]]^c$ onto $\{1,\ldots,k-2r\}$. It now follows from (F.21) and the injectivity of $D_{\gamma'}$ that $u \in \mathcal{H}(\bigotimes^{k-2r} \mathbb{C}^n)$ if and only if $D_\gamma u \in \mathcal{H}\mathcal{T}_\gamma^{\otimes k}$. This proves statement (2). $\square$

Now that we have established Lemma F.2.3, it only remains to prove the last part of Theorem F.1.2. Let λ be a G-admissible partition of $k-2r$ and let A be a tableau of shape λ. Consider the G-module

$$E = \bigoplus_{\gamma \in \mathcal{M}(k,r)} \mathcal{H}\mathcal{T}_\gamma^{\otimes k} .$$

From the results of Sections 10.2.3, 10.2.4, and 10.2.5 we know that $U(\lambda)$ occurs in E with multiplicity $\dim(G^\lambda)|\mathcal{M}(k,r)|$. By (F.3) we have a surjective G-module map $E \longrightarrow \mathcal{H}\mathcal{T}_r^{\otimes k}$ given by addition. The multiplicity bound (F.4) is then a consequence of the following general result:

Proposition F.2.4. *Let $\mathcal{A}$ be an associative algebra, and let U and V be completely-reducible $\mathcal{A}$-modules. Suppose $\xi \in \widehat{\mathcal{A}}$.*

1. *If $T : U \longrightarrow V$ is a surjective $\mathcal{A}$-module map, then $m_U(\xi) \ge m_V(\xi)$.*

2. If $T : U \longrightarrow V$ is an injective $\mathcal{A}$-module map, then $m_U(\xi) \leq m_V(\xi)$.

Proof. (1): Let E_ξ be an irreducible $\mathcal{A}$-module of type ξ. Let $S \in \mathrm{Hom}_{\mathcal{A}}(V, E_\xi)$. Then $ST \in \mathrm{Hom}_{\mathcal{A}}(U, E_\xi)$. Since T is surjective, the map $S \mapsto ST$ is an injection from $\mathrm{Hom}_{\mathcal{A}}(V, E_\xi)$ into $\mathrm{Hom}_{\mathcal{A}}(U, E_\xi)$. Thus

$$m_U(\xi) = \dim \mathrm{Hom}_{\mathcal{A}}(U, E_\xi) \geq \dim \mathrm{Hom}_{\mathcal{A}}(V, E_\xi) = m_V(\xi) .$$

(2): Let $S \in \mathrm{Hom}_{\mathcal{A}}(E_\xi, U)$. Then $TS \in \mathrm{Hom}_{\mathcal{A}}(E_\xi, V)$. Since T is injective, the map $S \mapsto TS$ is an injection from $\mathrm{Hom}_{\mathcal{A}}(E_\xi, U)$ into $\mathrm{Hom}_{\mathcal{A}}(E_\xi, V)$. Thus $m_U(\xi) \leq m_V(\xi)$ in this case. $\square$

Proof of Scholium F.2.2 Let $\mathcal{E} = [[\mu]] \setminus [[\gamma]]$. Then $\mathrm{Card}(\mathcal{E}) \geq 2$. Take $x \in \mathcal{E}$. The collection of all γ-linked submatchings of μ with x as one end is totally ordered under inclusion, so there is a maximum such submatching. Let y be the other end in this maximum submatching and define $\varphi(x) = y$. Letting x vary, we obtain a map

$$\varphi : \mathcal{E} \longrightarrow [[\mu]] .$$

The map φ is injective. Indeed, since $x \notin [[\gamma]]$, the maximum γ-linked submatching starting at x is also the maximal γ-linked submatching starting at $\varphi(x)$, and x and $\varphi(x)$ are the ends of this submatching. Thus $\varphi(x)$ uniquely determines x. We may enumerate $\gamma = \{\{m_1, n_1\}, \ldots, \{m_r, n_r\}\}$ so that

(1) If $[[\mu]] \cap \{m_i, n_i\} \neq \emptyset$ then $n_i \in [[\mu]]$.

Suppose the Scholium is false. Then for each $x \in \mathcal{E}$ we have $\varphi(x) \in [[\gamma]]$. We may choose the enumeration of γ so that (1) holds and also

(2) For all $x \in \mathcal{E}$ one has $\varphi(x) = n_i$ for some i.

Since φ is injective, there is a unique $y \in [[\gamma]]$ so that the pair $\{\varphi(x), y\} \in \gamma$. Set $\psi(x) = y$. Then $\psi : \mathcal{E} \longrightarrow [[\gamma]]$ is also injective. By the maximality condition defining $\varphi(x)$, we must have $\psi(x) \notin [[\mu]]$, since otherwise the γ-linked submatching starting at x and ending at $\varphi(x)$ could be extended using the link $\{\psi(x), \varphi(x)\} \in \gamma$.

We write $\{m_1, \ldots, m_r\} \cap [[\mu]] = \mathbf{m} \cap [[\mu]]$ and $\{n_1, \ldots, n_r\} \cap [[\mu]] = \mathbf{n} \cap [[\mu]]$. Define a map $f : \mathcal{E} \cup \big(\mathbf{m} \cap [[\mu]]\big) \longrightarrow \mathbf{n} \cap [[\mu]]$ by

$$f(x) = \begin{cases} \varphi(x) & \text{for } x \in \mathcal{E} \\ n_i & \text{for } x = m_i \in \mathbf{m} \cap [[\mu]]. \end{cases}$$

Note that if $m_i \in \mathbf{m} \cap [[\mu]]$ then $n_i \in [[\mu]]$ by property (1) above. Furthermore, since $\psi(x) \notin [[\mu]]$ for $x \in \mathcal{E}$, we cannot have $\varphi(x) = n_i$ for any $m_i \in \mathbf{m} \cap [[\mu]]$, by property (2) above. Since φ is injective, it follows that f is injective. Thus

$$\mathrm{Card}(\mathcal{E}) + \mathrm{Card}(\mathbf{m} \cap [[\mu]]) \leq \mathrm{Card}(\mathbf{n} \cap [[\mu]]) .$$

This gives the inequalities

$$\begin{aligned}\mathrm{Card}([[\mu]]) &= \mathrm{Card}(\mathcal{E}) + \mathrm{Card}(\mathbf{m}\cap[[\mu]]) + \mathrm{Card}(\mathbf{n}\cap[[\mu]]) \\ &\le 2\,\mathrm{Card}(\mathbf{n}\cap[[\mu]]) \le 2r\,.\end{aligned}$$

But $\mathrm{Card}([[\mu]]) = 2r+2$, so we have reached a contradiction. $\square$

F.3 Decomposition in the stable range

We now turn to the proof of Theorem F.1.3. We continue with the notation of Section F.2. Let $\lambda \in \mathrm{Par}(k-2r)$ for some r with $0 \le r \le [k/2]$. Recall from Theorem 9.3.12 that a basis for the space $\widetilde{W}^{k-2r}(\lambda)$ of $\mathbf{GL}(n,\mathbb{C})$ highest-weight tensors consists of the tensors $\mathbf{s}(A)e_A$, where $A \in \mathrm{STab}(\lambda)$ and $\mathbf{s}(A)$ is the Young symmetrizer for A.

Lemma F.3.1. *Assume that λ has d parts and $2d+r \le n$. Then the set*

$$\{D_\gamma \mathbf{s}(A)e_A \,:\, \gamma \in \mathcal{M}(k,r),\, A \in \mathrm{STab}(\lambda)\}$$

is linearly independent.

Proof. In the present situation, where we are considering tensors of ranks $k, k-2, k-4, \ldots$ simultaneously, it will be convenient to have a more flexible definition of Young symmetrizer. For any set $\mathcal{L}$ of positive integers with $|\mathcal{L}| = |\lambda| = k-2r$ we define an *$\mathcal{L}$-tableau of shape* λ to be a Young tableau of shape λ with each element of $\mathcal{L}$ occurring in exactly one box of λ. We call such a tableau *standard* if the sequence of numbers in any row (or column) is monotonically increasing. We denote by $\mathrm{Tab}(\lambda,\mathcal{L})$ the set of all $\mathcal{L}$-tableau of shape λ. We write $\mathrm{STab}(\lambda,\mathcal{L})$ for the set of all standard $\mathcal{L}$-tableau. The row group, column group, and Young symmetrizer associated with an $\mathcal{L}$-tableau are defined in the obvious way as before.

Now take $\gamma = (\mathbf{m},\mathbf{n}) \in \mathcal{M}(k,r)$ and $\mathcal{L} = [[\gamma]]^c$. For $A \in \mathrm{Tab}(\lambda)$ let $A_\gamma \in \mathrm{Tab}(\lambda,\mathcal{L})$ be the tableau obtained from A by replacing the integer j by the jth integer in $\mathcal{L}$ (enumerated in increasing order), for $j = 1,2,\ldots,k-2r$. It is clear that the map $A \mapsto A_\gamma$ is a bijection between $\mathrm{STab}(\lambda)$ and $\mathrm{STab}(\lambda,\mathcal{L})$. Furthermore, since the expansion operator D_γ only acts on the tensor positions in $[[\gamma]]$ while the Young symmetrizer $\mathbf{s}(A_\gamma)$ acts on the tensor positions in $[[\gamma]]^c$, we have

$$D_\gamma \mathbf{s}(A) = \mathbf{s}(A_\gamma) D_\gamma\,. \tag{F.22}$$

Let $D_n \subset \mathbf{GL}(n,\mathbb{C})$ be the n-torus of diagonal matrices. Then the torus $T = \times^k D_n$ acts on $\bigotimes^k \mathbb{C}^n$ with one-dimensional weight spaces spanned by the decomposable tensors $e_{p_1}\otimes\cdots\otimes e_{p_k}$. Since λ has d parts, the tensor e_A only involves $e_1,\ldots,e_d$ (see the definition for e_A from Section 9.3.1). Let $\gamma = (\mathbf{m},\mathbf{n})$. Define the subspace $K_\gamma \subset \bigotimes^k \mathbb{C}^n$ to be the span of the decomposable tensors $e_{p_1}\otimes\cdots\otimes e_{p_k}$, where e_{d+i} occurs in tensor position m_i, e^{d+i} occurs in tensor position n_i, for $i = 1,\ldots, r$, and all the other tensor positions are filled with $e_1, \ldots, e_d$ (with repetitions and in any order). Since we are assuming $2d+r \le n$ we have $K_\gamma \ne \{0\}$. Furthermore, since $e^{d+i} = \varepsilon e_{n+1-i}$, we have

$$e^{d+i} \neq e_j \qquad \text{for all } i = 1, \ldots, r \quad \text{and } j = 1, \ldots, d \,. \tag{F.23}$$

Clearly, K_γ is invariant under T. Let $A \in \mathrm{Tab}(\lambda)$. Since the row and column groups of A_γ only operate on the tensor positions in $[[\gamma]]^c$, we have

($\star$) The space K_γ is invariant under $\mathbf{s}(A_\gamma)$ for all $A \in \mathrm{Tab}(\lambda)$.

When we express the operator D_γ using the basis $\{e_i\}$ and ω-dual basis $\{e^i\}$, we see that $D_\gamma e_A$ is a sum of decomposable tensors, and from (F.23) exactly one of them is in K_γ. Call this term $u_{\gamma,A}$. Then $\mathbf{s}(A_\gamma)u_{\gamma,A} \in K_\gamma$ by ($\star$). Also $\mathbf{s}(A_\gamma)u_{\gamma,A} \neq 0$ by (F.23) and the same argument that gives ($\star$).

If $\gamma = (\mathbf{m}, \mathbf{n})$ and $u \in K_\gamma$ is a decomposable tensor, then the vectors $e_1, \ldots, e_d$ do not occur in positions $m_1, \ldots, m_r, n_1, \ldots, n_r$ in u by (F.23). Since distinct matchings γ determine distinct positions, it follows that the weights of T on K_γ uniquely determine γ. This implies that the spaces K_γ, for $\gamma \in \mathcal{M}(k,r)$, are linearly independent. Since T is reductive, there is a T-invariant subspace $F \subset \bigotimes^k \mathbb{C}^n$ so that

$$\bigotimes^k \mathbb{C}^n = F \oplus \bigoplus_{\gamma \in \mathcal{M}(k,r)} K_\gamma \,. \tag{F.24}$$

Now suppose there is a linear relation

$$\sum_{\gamma \in \mathcal{M}(k,r)} \sum_{A \in \mathrm{STab}(\lambda)} a_{\gamma,A}\, D_\gamma \mathbf{s}(A) e_A = 0 \,,$$

where $a_{\gamma,A} \in \mathbb{C}$. Then by (F.22) we can write this relation as

$$\sum_{\gamma \in \mathcal{M}(k,r)} \sum_{A \in \mathrm{STab}(\lambda)} a_{\gamma,A}\, \mathbf{s}(A_\gamma) D_\gamma e_A = 0 \,.$$

We have $\mathbf{s}(A_\gamma)u_{\gamma,A} \in K_\gamma$, while all the other decomposable tensors in $\mathbf{s}(A_\gamma)D_\gamma e_A$ are in the subspace F. Hence by (F.24)

$$\sum_{A \in \mathrm{STab}(\lambda)} a_{\gamma,A}\, \mathbf{s}(A_\gamma) u_{\gamma,A} = 0 \tag{F.25}$$

for each $\gamma \in \mathcal{M}(k,r)$.

Assume now for the sake of contradiction that some coefficient $a_{\gamma,A} \neq 0$. Let A' be the smallest standard tableau (relative to the lexicographic order as in Section 9.3.3) for which this is true. Since normalized Young symmetrizers are idempotent and $\mathbf{s}(A'_\gamma)\mathbf{s}(A) = 0$ for all $A > A'$ by Lemma 9.3.13, we can apply $\mathbf{s}(A'_\gamma)$ to the left side of (F.25) to obtain

$$a_{\gamma,A'}\mathbf{s}(A_\gamma)u_{\gamma,A} = 0 \,.$$

Hence $a_{\gamma,A'} = 0$, which is a contradiction. $\square$

We now prove Theorem F.1.3. Let $0 < r \leq [k/2]$ and assume $2k \leq n + 3r$. By Theorem F.1.2, it is enough to show that for each G-admissible partition λ of $k - 2r$, the set of irreducible G-modules

$$\{D_\gamma \mathbb{C}[G]\mathbf{s}(A)e_A : \gamma \in \mathcal{M}(k,r), A \in \mathrm{STab}(\lambda)\} \qquad \text{(F.26)}$$

are linearly independent. When $G = \mathrm{O}(2l,\mathbb{C})$ and $\lambda \in A_0(k-2r,2l)$, these modules are of the form

$$\big(D_\gamma U(\mathfrak{g})\mathbf{s}(A)e_A\big) \oplus \big(D_\gamma U(\mathfrak{g})\mathbf{s}(A)g_0 \cdot e_A\big)\,,$$

where $g_0 \in \mathbf{O}(2l,\mathbb{C})$ is as in Section 10.2.5. In all other cases these modules are of the form $D_\gamma U(\mathfrak{g})\mathbf{s}(A)e_A$. Now $\mathbf{s}(A)e_A$ is a $\mathfrak{b}$-extreme tensor of weight λ, whereas $\mathbf{s}(A)g_0 \cdot e_A$ is $\mathfrak{b}$-extreme of weight $\mathrm{Ad}^*(\xi)\lambda$. Thus the linear independence of the modules (F.26) is a consequence of Lemma F.3.1 and the following general result:

Scholium F.3.2. *Suppose* $\{u_1,\ldots,u_k\} \subset V^{\mathfrak{n}^+}(\mu)$ *is a linearly independent set of highest-weight vectors. Let* $U_1, \ldots, U_k$ *be, respectively, the cyclic* $\mathfrak{g}$*-modules generated by* $u_1,\ldots,u_k$*. Then* $U_1,\ldots,U_k$ *are linearly independent.*

Proof. We use the notation in the proof of Theorem 4.2.12. We have $k \le d(\mu)$ by (4.26). Hence there is a linear transformation $T_0 \in \mathrm{End}(V^{\mathfrak{n}^+}(\mu))$ so that

$$T_0(u_i) = v_{\mu,i} \qquad \text{for } i = 1,\ldots,k\,.$$

Set $T = \varphi^{-1}(T_0) \in \mathrm{End}_{\mathfrak{g}}(V)$. Then $T(U_i) = V_{\mu,i}$, so (4.25) implies that $U_1,\ldots,U_k$ are linearly independent. $\square$

F.4 Exercises

1. Verify the formula for $m(r)$ in Theorem F.1.1.
2. Use the Clebsch–Gordan formula (see Exercises 7.1.4) to prove (F.5).
3. Let $G = \mathbf{O}(\mathbb{C}^3,B)$. For $k = 0,2,4,\ldots$ and $\varepsilon = \pm$, let $V_{k,\varepsilon}$ be the $(k+1)$-dimensional irreducible representation of G with highest weight $(k/2)\varepsilon_1$ in which $-I$ acts by ε. Write $\mathcal{H}^{\otimes k} = \mathcal{H}(\bigotimes^k \mathbb{C}^3,B)$.
   (a) Show that $\mathcal{H}^{\otimes 3} \cong V_{6,-} \oplus 2V_{4,-} \oplus V_{0,-}$ and $\mathcal{H}^{\otimes k} \cong V_{2k,\varepsilon} \oplus (k-1)V_{2k-2,\varepsilon}$ for $k \ne 3$ with $\varepsilon = (-1)^k$. (Hint: Determine the G-admissible partitions; the associated representations of the symmetric groups are either the trivial, the sign, or the standard.)
   (b) Theorem F.1.3 asserts that $\bigotimes^3 \mathbb{C}^3 \cong \mathcal{H}^{\otimes 3} \oplus 3\mathcal{H}^1$ as a G-module; give an alternate proof of this using (a) and the Clebsch–Gordan formula (use the covering homomorphism $\mathbf{SL}(2,\mathbb{C}) \longrightarrow \mathbf{SO}(3,\mathbb{C})$).
   (c) Use the same method as in (b) to show that $\bigotimes^4 \mathbb{C}^3 \cong \mathcal{H}^{\otimes 4} \oplus 6\mathcal{H}^{\otimes 2} \oplus 3\mathcal{H}^0$ as a G-module. Note that the stable multiplicity of $\mathcal{H}^{\otimes 2}$ in $\bigotimes^4 V$ would be $|\mathcal{M}(4,1)| = 6$ and the stable multiplicity of $\mathcal{H}^0$ in $\bigotimes^4 V$ would be $|\mathcal{M}(4,2)| = 3$. Thus Theorem F.1.3 holds in this case even though the inequality $2k \le n+3$ is violated (here $k = 4$, $n = 3$).
4. (Same notation as in previous problem) By Theorems F.1.1 and F.1.2 one has $\bigotimes^5 \mathbb{C}^3 \cong \mathcal{H}^{\otimes 5} \oplus \mathcal{H}\mathcal{T}_1^{\otimes 5} \oplus \mathcal{H}\mathcal{T}_2^{\otimes 5}$. Note that the inequality $2k \le n+3$ is violated (here $k = 5$ and $n = 3$), so Theorem F.1.3 is not applicable.

(a) Use the method and results of the previous problem to show that $\mathcal{HT}_1^{\otimes 5} \cong 10V_{6,-} \oplus 15V_{4,-} \oplus 6V_{0,-}$ and $\mathcal{HT}_2^{\otimes 5} \cong 15V_{2,-}$ in this case.
(b) Show that the upper bound in (F.4) for the multiplicity of the representation $V_{4,-}$ in $\mathcal{HT}_1^{\otimes 5}$ is 20, whereas from (a) the actual multiplicity is 15. (HINT: The partition λ of 3 associated with $V_{4,-}$ is $3 = 2+1$, and $\dim G^{\lambda} = 2$.)

5. Let γ and $\gamma' \in \mathrm{Par}_2(k)$. Suppose there is a pair $\{i, j\} \in \gamma$ so that $\{i, j\} \notin \gamma'$, or vice versa. Prove that $\mathcal{HT}_{\gamma}^{\otimes k}$ is orthogonal to $\mathcal{HT}_{\gamma'}^{\otimes k}$ relative to the form ω.

F.5 Notes

The decomposition of tensor space into a sum of harmonic tensors of "valences" $k, k-2, \ldots$ is given (in broad outline) in Weyl [7, Ch. V, §6 and §7]; see also Brown [2]. Our presentation follows the presentation in Benkart, Britten, and Lemire [1] with some changes in terminology, notation, and details of proof. See also Gavarini and Papi [4], who obtain this decomposition using the representations of the Brauer algebra.

When G is the symplectic group a combinatorial formula for the multiplicities $m(r, \lambda)$ in Theorem F.1.2 was obtained by Sundaram [6]. The condition $2k \le n+3$ in Theorem F.1.3 can possibly be weakened, in light of the results on semisimplicity of the Brauer algebra in Brown [3] and the determination of the structure of the algebra in Brown [2] (see the exercises in Section F.1). See Hanlon [5] for the "stable limit" of the decomposition of the tensor algebra over $\mathfrak{g}$ as a module under $\mathrm{Ad}(G)$, with G a classical group.

References for Appendix F

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