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Symmetry, Representations, and Invariants

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Appendix E
Cohomology and Character Formulas

Abstract In this appendix we give another algebraic proof of the Weyl character formula, using methods that have many other applications in Lie theory. We begin by setting up the machinery of Lie algebra cohomology (without assuming any previous background in homological algebra). We define the cohomology spaces for a Lie algebra representation in terms of a cochain complex and differential, with the cohomology in degree zero being the subspace of invariants. We show that the short left-exact sequence of invariants associated to a submodule and quotient module extends to a long exact sequence in cohomology. Then we determine the cohomology of the universal enveloping algebra of a Lie algebra. For a semisimple Lie algebra $\mathfrak{g}$ the cohomology spaces associated with the nilradial $\mathfrak{n}^+$ of a Borel subalgebra of $\mathfrak{g}$ are particularly important (we already saw this in the classification of irreducible representations using the $\mathfrak{n}^+$-invariant vectors in Chapter 3). These cohomology spaces are completely described by a theorem of Kostant, which we prove using an identity for the Casimir operator due to Casselman and Osborne. From Kostant's theorem we obtain the Weyl character formula via the Euler-Poincaré principle.

E.1 Lie algebra cohomology

E.1.1 Cochain complex

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$, and let (ρ, V) be a representation of $\mathfrak{g}$. Here we do *not* assume that $\mathfrak{g}$ or V is finite dimensional. For each integer $p = 0, 1, \ldots$ consider the space $C^p(\mathfrak{g}, V)$ of all p-multilinear maps

$$\omega : \underbrace{\mathfrak{g} \times \cdots \times \mathfrak{g}}_{p} \longrightarrow V$$

that are alternating in their arguments. We call such a map ω a *p-cochain*. For example, a 0-cochain is a constant map from $\mathfrak{g}$ to V, which we identify with its value as an element of V. A 1-cochain is a linear map from $\mathfrak{g}$ to V. We set

$$C^\bullet(\mathfrak{g},V) = \bigoplus_{p\geq 0} C^p(\mathfrak{g},V) .$$

We make $\bigwedge^p \mathfrak{g}$ into a $\mathfrak{g}$-module by restricting the representation $\mathrm{ad}^{\otimes p}$ of $\mathfrak{g}$ on $\bigotimes^p \mathfrak{g}$. Since $C^p(\mathfrak{g},V) \cong \mathrm{Hom}(\bigwedge^p \mathfrak{g},V)$ (see Section B.2.4), the space of p-cochains is then a $\mathfrak{g}$-module in a canonical way. For $X \in \mathfrak{g}$ we denote the action of X by $\theta(X)$. Thus for $\omega \in C^p(\mathfrak{g},V)$ we have

$$\begin{aligned}(\theta(X)\omega)(X_1,\ldots,X_p) &= \rho(X)\omega(X_1,\ldots,X_p) \\ &\quad - \sum_{j=1}^{p} \omega(X_1,\ldots,[X,X_j],\ldots,X_p) .\end{aligned} \tag{E.1}$$

We call $\theta(X)\omega$ the *Lie derivative* of ω relative to X.

The Lie derivative on 0-cochains coincides with the given representation ρ on V. If ω is a 1-cochain then

$$\theta(X)\omega(Y) = \rho(X)\omega(Y) - \omega([X,Y]) .$$

In particular, $\theta(\mathfrak{g})\omega = 0$ precisely when ω is an intertwining map between the adjoint representation of $\mathfrak{g}$ and the representation ρ.

We also have the *interior product* operator

$$\iota(X) : C^p(\mathfrak{g},V) \longrightarrow C^{p-1}(\mathfrak{g},V)$$

defined by evaluation on the first argument:

$$(\iota(X)\omega)(X_1,\ldots,X_{p-1}) = \omega(X,X_1,\ldots,X_{p-1})$$

(where $\iota(X)v = 0$ for $v \in V$).

Lemma E.1.1. *For $X,Y \in \mathfrak{g}$ the operators $\theta(X)$ and $\iota(Y)$ satisfy the commutation relations*

$$[\theta(X),\iota(Y)] = \iota([X,Y]) . \tag{E.2}$$

Proof. Both sides of (E.2) annihilate 0-cochains. If ω is a $(p+1)$-cochain, then from (E.1) we calculate that

$$\begin{aligned}(\iota(Y)\theta(X)\omega)(X_1,\ldots,X_p) &= (\theta(X)\omega)(Y,X_1,\ldots,X_p) \\ &= \rho(X)\omega(Y,X_1,\ldots,X_p) - \omega([X,Y],X_1,\ldots,X_p) \\ &\quad - \sum_{j=1}^{p} \omega(Y,X_1,\ldots,[X,X_j],\ldots,X_p) \\ &= -(\iota([X,Y])\omega)(X_1,\ldots,X_p) \\ &\quad + (\theta(X)\iota(Y)\omega)(X_1,\ldots,X_p) .\end{aligned}$$

This proves (E.2). □

Note that if ω is a p-cochain, then

$$\omega(X_1,\dots,X_p) = \iota(X_p)\cdots\iota(X_1)\omega\,.$$

From this identity we see that a linear operator on $C^\bullet(\mathfrak{g},V)$ is completely determined by its action on $C^0(\mathfrak{g},V) = V$ and its commutation relations with the interior product operators. We shall use this principle several times.

Let $\omega \in C^p(\mathfrak{g},V)$ and let $X_1,\dots,X_{p+1}$ be in $\mathfrak{g}$. Define

$$\begin{aligned} \mathrm{d}\omega(X_1,\dots,X_{p+1}) \;=\; & \sum_{j=1}^{p+1}(-1)^{j+1}\rho(X_j)\,\omega(X_1,\dots,\widehat{X_j},\dots,X_{p+1}) \\ & + \sum_{1\le r<s\le p+1}(-1)^{r+s}\,\omega([X_r,X_s],X_1,\dots,\widehat{X_r},\dots,\widehat{X_s},\dots,X_{p+1}) \end{aligned}$$

for $\omega \in C^p(\mathfrak{g},V)$. Here $\widehat{X_j}$ means to omit the argument X_j. The right-hand side in this formula obviously changes sign when X_j and X_{j+1} are interchanged, by the skew symmetry of the bracket and the factor $(-1)^{j+1}$ in the first sum. Hence $\mathrm{d}\omega$ is a $(p+1)$-cochain. We call $\mathrm{d}\omega$ the *coboundary* of ω and the operator

$$\mathrm{d} : C^p(\mathfrak{g},V) \longrightarrow C^{p+1}(\mathfrak{g},V)$$

the *coboundary operator*. For example, when $p=0$ then $\omega \in V$ and

$$\mathrm{d}\omega(X) = \rho(X)\omega \quad \text{for } X \in \mathfrak{g}$$

(the second sum in the definition is over an empty range of indices in this case). When $p=1$ then $\omega \in \mathrm{Hom}(\mathfrak{g},V)$ and

$$\mathrm{d}\omega(X,Y) = \rho(X)\omega(Y) - \rho(Y)\omega(X) - \omega([X,Y]) \quad \text{for } X,Y \in \mathfrak{g}\,.$$

Lemma E.1.2. *The operator* d *satisfies*

$$\mathrm{d}\circ\iota(X) + \iota(X)\circ\mathrm{d} = \theta(X) \quad \textit{for all } X \in \mathfrak{g}\,. \tag{E.3}$$

It is uniquely defined by this relation and its action on V.

Proof. Let $\omega \in C^p(\mathfrak{g},V)$ and $X_0,X_1,\dots,X_p \in \mathfrak{g}$. Then from the definitions of the operators $\iota(X_0)$ and d we have

$$\begin{aligned} \mathrm{d}(\iota(X_0)\omega)(X_1,\dots,X_p) \;=\; & \sum_{j=1}^{p}(-1)^{j+1}\rho(X_j)\,\omega(X_0,X_1,\dots,\widehat{X_j},\dots,X_p) \\ & + \sum_{1\le r<s\le p}(-1)^{r+s}\,\omega(X_0,[X_r,X_s],X_1,\dots,\widehat{X_r},\dots,\widehat{X_s},\dots,X_p)\,. \end{aligned}$$

Applying these operators in the opposite order yields

$$
\begin{aligned}
(\iota(X_0)\mathrm{d}\omega)(X_1,\ldots,X_p) &= \sum_{j=0}^{p}(-1)^j \rho(X_j)\,\omega(X_0,\ldots,\widehat{X}_j,\ldots,X_p) \\
&\quad + \sum_{0\le r<s\le p}(-1)^{r+s}\,\omega([X_r,X_s],X_0,\ldots,\widehat{X}_r,\ldots,\widehat{X}_s,\ldots,X_p)\,.
\end{aligned}
$$

Set $\eta = \mathrm{d}\,\iota(X_0)\omega + \iota(X_0)\,\mathrm{d}\omega$. Noting the ranges of summation in each of the equations above and using the skew symmetry of ω, we find that

$$
\begin{aligned}
\eta(X_1,\ldots,X_p) &= \rho(X_0)\omega(X_1,\ldots,X_p) + \sum_{1\le s\le p}(-1)^s\,\omega([X_0,X_s],X_1,\ldots,\widehat{X}_s,\ldots,X_p) \\
&= (\theta(X_0)\omega)(X_1,\ldots,X_p)\,.
\end{aligned}
$$

Thus d satisfies (E.3).

It is clear that (E.3) uniquely determines d on $C^p(\mathfrak{g},V)$ in terms of the action of d on $C^{p-1}(\mathfrak{g},V)$. Indeed, if $\omega \in C^p(\mathfrak{g},V)$ then

$$
\begin{aligned}
\mathrm{d}\omega(X_0,\ldots,X_p) &= (\iota(X_0)\mathrm{d}\omega)(X_1,\ldots,X_p) \\
&= (\theta(X_0)\omega)(X_1,\ldots,X_p) - (\mathrm{d}\,\iota(X_0)\omega)(X_1,\ldots,X_p)\,.
\end{aligned}
$$

$\square$

Lemma E.1.3. *The Lie derivative commutes with the coboundary operator:*

$$
\mathrm{d}\,\theta(X) = \theta(X)\,\mathrm{d} \quad \textit{for all } X \in \mathfrak{g}\,. \tag{E.4}
$$

Proof. Consider the operator $T = \mathrm{d}\,\theta(X) - \theta(X)\,\mathrm{d}$. If $v \in V$ is a 0-cochain, then

$$
(\mathrm{d}\theta(X)v)(Y) = (\mathrm{d}(\rho(X)v))(Y) = \rho(Y)\rho(X)v
$$

and

$$
(\theta(X)\mathrm{d}v)(Y) = \rho(X)\mathrm{d}v(Y) - \mathrm{d}v([X,Y]) = \rho(X)\rho(Y)v - \rho([X,Y])v\,.
$$

Hence T annihilates 0-cochains, since ρ is a Lie algebra homomorphism. If we can show that T anticommutes with all interior product operators, it will follow that $T = 0$.

Let $Y \in \mathfrak{g}$. Then

$$
\begin{aligned}
\iota(Y)\mathrm{d}\theta(X) &= \theta(Y)\theta(X) - \mathrm{d}\,\iota(Y)\theta(X) \\
&= \theta(Y)\theta(X) - \mathrm{d}\theta(X)\iota(Y) + \mathrm{d}\,\iota([X,Y])\,,
\end{aligned}
$$

where we have used (E.2) and (E.3) to interchange the order of the operators. Similarly, we have

$$
\begin{aligned}
\iota(Y)\theta(X)\mathrm{d} &= \theta(X)\iota(Y)\mathrm{d} - \iota([X,Y])\mathrm{d} \\
&= \theta(X)\theta(Y) - \theta(X)\mathrm{d}\,\iota(Y) - \iota([X,Y])\mathrm{d}\,.
\end{aligned}
$$

Subtracting these two equations and using the property that θ is a Lie algebra homomorphism, we find that $\iota(Y)T = -T\iota(Y)$. Hence $T = 0$. ◻

Lemma E.1.4. *The coboundary operator* d *satisfies* $\mathrm{d}^2 = 0$.

Proof. From Lemma E.1.3 and the defining equation (E.3) for d we have

$$\begin{aligned}\mathrm{d}^2\,\iota(X) + \mathrm{d}\,\iota(X)\mathrm{d} &= \mathrm{d}\,\theta(X) = \theta(X)\mathrm{d} \\ &= \iota(X)\mathrm{d}^2 + \mathrm{d}\,\iota(X)\mathrm{d}\end{aligned}$$

for all $X \in \mathfrak{g}$. Hence d^2 commutes with $\iota(X)$ for all X. Consequently, we only need to check that d^2 annihilates 0-cochains.

If $v \in V$ then

$$\begin{aligned}(\mathrm{d}^2 v)(X,Y) &= \rho(X)\mathrm{d}v(Y) - \rho(Y)\mathrm{d}v(X) - \mathrm{d}v([X,Y]) \\ &= \rho(X)\rho(Y)v - \rho(Y)\rho(X)v - \rho([X,Y])v\,.\end{aligned}$$

Because ρ is a representation, the right-hand side of this equation vanishes. Thus $\mathrm{d}^2 = 0$. ◻

Assume now that $\mathfrak{g}$ is a finite-dimensional Lie algebra over $\mathbb{C}$ and that (ρ, V) is a $\mathfrak{g}$-module (we allow $\dim V = \infty$). In this case there is a natural identification

$$C^p(\mathfrak{g}, V) = \left(\textstyle\bigwedge^p \mathfrak{g}^*\right) \otimes V$$

as a $\mathfrak{g}$-module (see Sections B.2.2 and B.2.4). In order to make some explicit calculations of cohomology spaces for $\mathfrak{g}$, we derive a formula for the operator d that reflects this tensor product structure of the cochain spaces.

For $V = \mathbb{C}$ with trivial $\mathfrak{g}$ action we have $C^p(\mathfrak{g}, \mathbb{C}) = \bigwedge^p \mathfrak{g}^*$. We write d_0 for the coboundary operator and $\theta_0(X)$ for the Lie derivative in this case. For $\xi \in \mathfrak{g}^*$, define the *exterior product* operator $\varepsilon(\xi) : \bigwedge^p \mathfrak{g}^* \longrightarrow \bigwedge^{p+1} \mathfrak{g}^*$ as in Section 6.1.2:

$$\varepsilon(\xi)\omega(X_0, \ldots, X_p) = \sum_{j=0}^{p} (-1)^j \langle \xi, X_j \rangle \omega(X_0, \ldots, \widehat{X_j}, \ldots, X_p)\,.$$

Recall that the interior and exterior product operators satisfy the relations

$$\begin{aligned}\iota(X)\iota(Y) + \iota(Y)\iota(X) &= 0, \qquad \varepsilon(\xi)\varepsilon(\eta) + \varepsilon(\eta)\varepsilon(\xi) = 0 \\ \iota(X)\varepsilon(\xi) + \varepsilon(\xi)\iota(X) &= \langle \xi, X \rangle\end{aligned} \tag{E.5}$$

for $X, Y \in \mathfrak{g}$ and $\xi, \eta \in \mathfrak{g}^*$.

Fix a basis $\{X_i\}$ for $\mathfrak{g}$, and let $\{\xi_i\}$ be the dual basis for $\mathfrak{g}^*$. We have the following explicit formulas for the Lie derivative and the coboundary operators in terms of interior and exterior products.

Lemma E.1.5. *For* $X \in \mathfrak{g}$,

$$\theta_0(X) = \sum_i \varepsilon(\xi_i)\,\iota([X_i, X])\;, \tag{E.6}$$

$$2\mathrm{d}_0 = \sum_i \varepsilon(\xi_i)\,\theta_0(X_i)\;. \tag{E.7}$$

Furthermore, if (ρ, V) *is any* $\mathfrak{g}$*-module and* d *is the coboundary operator on* $(\bigwedge^\bullet \mathfrak{g}^*) \otimes V$*, then*

$$\mathrm{d} = \mathrm{d}_0 \otimes 1 + \sum_i \varepsilon(\xi_i) \otimes \rho(X_i)\;. \tag{E.8}$$

Proof. Let $T(X)$ be the operator on the right side of (E.6). Then the operators $\theta_0(X)$ and $T(X)$ annihilate constants, so to show they are equal, we only need to check that they have the same commutation relations with the operators $\iota(Y)$, $Y \in \mathfrak{g}$. We have

$$\begin{aligned} T(X)\iota(Y) &= -\sum_i \varepsilon(\xi_i)\,\iota(Y)\,\iota([X_i, X]) \\ &= \iota(Y)T(X) - \sum_i \langle \xi_i, Y\rangle\,\iota([X_i, X]) \\ &= \iota(Y)T(X) - \iota([Y, X]) \end{aligned}$$

from (E.5) and the expansion of Y in terms of the basis. Now use (E.2) to conclude that $T(X) = \theta_0(X)$.

We use a similar argument to prove (E.7). Call the right-hand side of this equation D_0, and note that D_0 and d_0 annihilate constants. Let $Y \in \mathfrak{g}$. Then

$$\begin{aligned} D_0\iota(Y) &= \sum_i \varepsilon(\xi_i)\,(\iota(Y)\theta_0(X_i) - \iota([Y, X_i])) \\ &= -\iota(Y)D_0 + \sum_i \langle \xi_i, Y\rangle \theta_0(X_i) + \theta_0(Y) \\ &= -\iota(Y)D_0 + 2\theta_0(Y) \end{aligned}$$

by (E.6). Thus $2\mathrm{d}_0 - D_0$ commutes with all $\iota(Y)$ and annihilates constants. Hence $2\mathrm{d}_0 = D_0$.

Finally, let D denote the right-hand side of (E.8). Since d_0 annihilates 0-cochains, we have $(Dv)(X) = \rho(X)v = (\mathrm{d}v)(X)$ for all $v \in V$. Moreover, from (E.5) we calculate that

$$\begin{aligned} D\iota(Y) &= -\iota(Y)D + \theta_0(Y) \otimes 1 + \sum_i \langle \xi_i, Y\rangle 1 \otimes \rho(X_i) \\ &= -\iota(Y)D + \theta_0(Y) \otimes 1 + 1 \otimes \rho(Y)\;. \end{aligned}$$

But $\theta(Y) = \theta_0(Y) \otimes 1 + 1 \otimes \rho(Y)$. Thus $D\iota(Y) = -\iota(Y)D + \theta(Y)$. Hence $D = \mathrm{d}$ by Lemma E.1.2. $\square$

E.1.2 Cohomology spaces

We continue the notation of the previous section. Define the space of *p-cocycles*

$$Z^p(\mathfrak{g},V) = \mathrm{Ker}\left(\mathrm{d} : C^p(\mathfrak{g},V) \longrightarrow C^{p+1}(\mathfrak{g},V)\right)$$

and the space of *p-coboundaries*

$$B^p(\mathfrak{g},V) = \mathrm{d}\left(C^{p-1}(\mathfrak{g},V)\right) .$$

Since $\mathrm{d}^2 = 0$, we have $B^p(\mathfrak{g},V) \subset Z^p(\mathfrak{g},V)$. By Lemma E.1.3 each of these spaces is a $\mathfrak{g}$-submodule of $C^p(\mathfrak{g},V)$. The quotient space

$$H^p(\mathfrak{g},V) = Z^p(\mathfrak{g},V)/B^p(\mathfrak{g},V)$$

is the pth *cohomology space* of $\mathfrak{g}$ with coefficients in V. For $p = 0$ we have $H^0(\mathfrak{g},V) = V^{\mathfrak{g}}$, the subspace of $\mathfrak{g}$-invariant vectors in V. If $\omega \in Z^p(\mathfrak{g},V)$, we write $[\omega] = \omega + B^p(\mathfrak{g},V)$ for the cohomology class of ω in $H^p(\mathfrak{g},V)$. Define

$$H^\bullet(\mathfrak{g},V) = \bigoplus_{p\geq 0} H^p(\mathfrak{g},V)$$

as a graded vector space. If $\omega \in Z^p(\mathfrak{g},V)$ and $X \in \mathfrak{g}$, then from (E.3) we have

$$\theta(X)\omega = \iota(X)\mathrm{d}\omega + \mathrm{d}\iota(X)\omega = \mathrm{d}\iota(X)\omega .$$

Hence $\theta(X) : Z^p(\mathfrak{g},V) \longrightarrow B^p(\mathfrak{g},V)$, and so $\mathfrak{g}$ acts by zero on $H^p(\mathfrak{g},V)$. When $V = \mathbb{C}$ with trivial $\mathfrak{g}$ action, we write $H^p(\mathfrak{g},\mathbb{C}) = H^p(\mathfrak{g})$.

Consider a Lie algebra $\mathfrak{b}$ and an ideal $\mathfrak{n} \subset \mathfrak{b}$. Let (ρ,V) be a $\mathfrak{b}$-module. We view V as a $\mathfrak{n}$-module by restriction. If $X \in \mathfrak{b}$ and $\omega \in C^p(\mathfrak{n},V)$ then

$$\rho(X)\omega(X_1,\ldots,X_p) - \sum_{j=1}^{p} \omega(X_1,\ldots,[X,X_j],\ldots,X_p) \tag{E.9}$$

is a well-defined vector in V for any $X_1,\ldots,X_p \in \mathfrak{n}$, since $[\mathfrak{b},\mathfrak{n}] \subset \mathfrak{n}$. We define $\theta(X)\omega \in C^p(\mathfrak{n},V)$ to be the p-cochain whose value at $(X_1,\ldots,X_p)$ is given by (E.9). This is consistent with the earlier definition of the Lie derivative when $X \in \mathfrak{n}$, and $X \mapsto \theta(X)$ is just the natural action of $\mathfrak{b}$ on $\mathrm{Hom}(\bigwedge^p \mathfrak{n},V)$.

Lemma E.1.6. *For $X,Y \in \mathfrak{b}$,*

$$\theta([X,Y]) = \theta(X)\theta(Y) - \theta(Y)\theta(X) . \tag{E.10}$$

Thus $(C^\bullet(\mathfrak{n},V),\theta)$ is a $\mathfrak{b}$-module.

Proof. This is clear from the identification $C^p(\mathfrak{n},V) = \mathrm{Hom}(\bigwedge^p \mathfrak{n},V)$. □

Proposition E.1.7. *Let $\mathfrak{b}$ be a Lie algebra and let (ρ,V) be a $\mathfrak{b}$-module. Suppose $\mathfrak{n}\subset\mathfrak{b}$ is an ideal. The spaces $Z^p(\mathfrak{n},V)$ and $B^p(\mathfrak{n},V)$ are invariant under $\theta(\mathfrak{b})$, and $\theta(\mathfrak{n})Z^p(\mathfrak{n},V)\subset B^p(\mathfrak{n},V)$. Hence the $\mathfrak{n}$-cohomology spaces $H^p(\mathfrak{n},V)$ have natural structures as $\mathfrak{b}/\mathfrak{n}$-modules.*

Proof. It is obvious that the proof of Lemma E.1.3 remains valid when $X\in\mathfrak{b}$, since (E.2) holds for $X\in\mathfrak{b}$ and $Y\in\mathfrak{n}$. Hence $\theta(X)$ commutes with d and preserves $Z^p(\mathfrak{n},V)$ and $B^p(\mathfrak{n},V)$. We already observed that $\mathfrak{n}$ acts trivially on $H^p(\mathfrak{n},V)$. Thus the action of $\mathfrak{b}$ lifts to an action of $\mathfrak{b}/\mathfrak{n}$. □

E.1.3 Cohomology exact sequences

Suppose U and V are $\mathfrak{g}$-modules. Given $\alpha\in\mathrm{Hom}_{\mathfrak{g}}(U,V)$, we extend α to a linear map from $C^p(\mathfrak{g},U)$ to $C^p(\mathfrak{g},V)$ for all p by letting it act on the values of the cochains:

$$(\alpha(\omega))(X_1,\dots,X_p)=\alpha(\omega(X_1,\dots,X_p))$$

for $\omega\in C^p(\mathfrak{g},V)$. From the formula for the cochain differential d we see that $\alpha\mathrm{d}=\mathrm{d}\alpha$, since α is a $\mathfrak{g}$-intertwining map. It follows that α leaves invariant the subspaces of p-cocycles and coboundaries and induces maps

$$\alpha^{(p)}:H^p(\mathfrak{g},U)\longrightarrow H^p(\mathfrak{g},V)$$

for $p=0,1,\dots$. We write $\alpha^{\bullet}$ to denote this family of maps. Note that $\alpha^{(0)}=\alpha$, under the canonical identifications of $H^0(\mathfrak{g},U)$ and $H^0(\mathfrak{g},V)$ with U and V, respectively.

Let W be another $\mathfrak{g}$-module, and suppose $\beta\in\mathrm{Hom}_{\mathfrak{g}}(V,W)$. Then $\beta\alpha$ is in $\mathrm{Hom}_{\mathfrak{g}}(U,W)$ and it is clear from the definitions that

$$(\beta\alpha)^{(p)}=\beta^{(p)}\circ\alpha^{(p)}:H^p(\mathfrak{g},U)\longrightarrow H^p(\mathfrak{g},W)\ .$$

Suppose U is a $\mathfrak{g}$-submodule of V and W is the quotient module. Then we have an exact sequence of $\mathfrak{g}$-maps

$$0\longrightarrow U\xrightarrow{\ \alpha\ }V\xrightarrow{\ \beta\ }W\longrightarrow 0\ , \tag{E.11}$$

where α is the injection map and β is the natural projection onto the quotient. When we pass to the subspaces of $\mathfrak{g}$-invariant vectors, we have a *left exact* sequence

$$0\longrightarrow U^{\mathfrak{g}}\xrightarrow{\ \alpha\ }V^{\mathfrak{g}}\xrightarrow{\ \beta\ }W^{\mathfrak{g}}\ . \tag{E.12}$$

The map β in (E.12) is not necessarily surjective. For example, suppose that $\mathfrak{g}$ is one dimensional, with basis X acting on $V=\mathbb{C}^2$ by $Xe_1=0$, $Xe_2=e_1$. Take $U=\mathbb{C}\cdot e_1$. Then $V^{\mathfrak{g}}=\mathbb{C}\cdot e_1$ maps to 0 in $W=V/U$, but $W^{\mathfrak{g}}=\mathbb{C}\cdot e_2+U$ is nonzero. We can measure the lack of surjectivity of β by constructing a *connecting map* that embeds the left exact sequence (E.12) into a long exact sequence of cohomology spaces, as follows:

Lemma E.1.8. *For* $p = 0,1,\dots$ *there is a map* $\delta^{(p)} : H^p(\mathfrak{g},W) \longrightarrow H^{p+1}(\mathfrak{g},U)$ *such that the sequence*

$$\begin{aligned} &0 \longrightarrow U^{\mathfrak{g}} \xrightarrow{\alpha} V^{\mathfrak{g}} \xrightarrow{\beta} W^{\mathfrak{g}} \xrightarrow{\delta^{(0)}} H^1(\mathfrak{g},U) \longrightarrow \cdots \\ &\longrightarrow H^p(\mathfrak{g},U) \xrightarrow{\alpha^{(p)}} H^p(\mathfrak{g},V) \xrightarrow{\beta^{(p)}} H^p(\mathfrak{g},W) \xrightarrow{\delta^{(p)}} H^{p+1}(\mathfrak{g},U) \longrightarrow \cdots \end{aligned} \tag{E.13}$$

is exact.

Proof. We first observe that (E.11) gives rise to the exact sequences of $\mathfrak{g}$-modules

$$0 \longrightarrow C^p(\mathfrak{g},U) \xrightarrow{\alpha} C^p(\mathfrak{g},V) \xrightarrow{\beta} C^p(\mathfrak{g},W) \longrightarrow 0\ , \tag{E.14}$$

since the maps in (E.14) only act on the values of the cochains. Thus since β commutes with d we have

$$B^p(\mathfrak{g},W) = \beta B^p(\mathfrak{g},V)\ . \tag{E.15}$$

Given $c \in H^p(\mathfrak{g},W)$, we define $\delta^{(p)}c$ as follows: Choose $\zeta \in Z^p(\mathfrak{g},W)$ with $[\zeta] = c$. By (E.14) there is a cochain $\omega \in C^p(\mathfrak{g},V)$ such that $\beta(\omega) = \zeta$. Then $\mathrm{d}\omega \in B^{p+1}(\mathfrak{g},V)$ satisfies

$$\beta(\mathrm{d}\omega) = \mathrm{d}(\beta(\omega)) = \mathrm{d}\zeta = 0\ .$$

Hence by (E.14) there exists $\mu \in C^{p+1}(\mathfrak{g},U)$ such that $\alpha(\mu) = \mathrm{d}\omega$. Furthermore, $\mathrm{d}\mu = 0$, since $\alpha(\mathrm{d}\mu) = \mathrm{d}^2\omega = 0$. Define $\delta^{(p)}c = [\mu] \in H^{p+1}(\mathfrak{g},U)$.

It is easy to check that $\delta^{(p)}c$ depends only on the cohomology class c and not on the particular choice of ω. Indeed, any other choice, say ω', must satisfy

$$\beta(\omega') = \zeta - \beta(\mathrm{d}\xi)$$

for some $\xi \in C^p(\mathfrak{g},V)$, by (E.15). Hence $\beta(\omega - \omega' - \mathrm{d}\xi) = 0$, so by (E.14) there exists $\gamma \in C^p(\mathfrak{g},U)$ with $\omega - \omega' - \mathrm{d}\xi = \alpha(\gamma)$. Thus the cocycle μ' such that $\alpha(\mu') = \mathrm{d}\omega'$ satisfies

$$\alpha(\mu - \mu') = \mathrm{d}(\alpha(\gamma) + \mathrm{d}\xi) = \alpha(\mathrm{d}\gamma)\ .$$

We conclude from (E.14) again that $[\mu] = [\mu']$, as claimed. It follows that $\delta^{(p)}$ is a well-defined linear map.

It follows directly from the definition of $\delta^{(p)}$ that

$$\Im\beta^{(p)} \subset \mathrm{Ker}\,\delta^{(p)}, \quad \Im\delta^{(p)} \subset \mathrm{Ker}\,\alpha^{(p+1)}\ . \tag{E.16}$$

For the opposite inclusions, suppose that $\delta^{(p)}c = 0$. Then, in the notation above, $\mu = \mathrm{d}\nu$ for some $\nu \in C^p(\mathfrak{g},U)$. Hence $\mathrm{d}(\omega - \alpha(\nu)) = 0$, so that $\omega' = \omega - \alpha(\nu)$ is a cocycle. But by (E.14) we have $\beta(\omega') = \zeta$. Hence $[\zeta] \in \Im(\beta^{(p)})$.

Finally, we check that $\mathrm{Ker}(\alpha^{(p+1)}) \subset \mathfrak{I}\delta^{(p)}$. Let $\mu \in Z^{p+1}(\mathfrak{g},U)$ satisfy $\alpha(\mu) = \mathrm{d}\omega$ for some $\omega \in C^p(\mathfrak{g},V)$. Set $\zeta = \beta(\omega)$. Then $\mathrm{d}\zeta = \beta(\mathrm{d}\omega) = \beta\alpha(\mu) = 0$ by (E.14), so ζ is a cocycle, and by definition $\delta^{(p)}([\zeta]) = [\mu]$. This completes the proof of exactness of (E.13). $\square$

Suppose $A \subset B$ and $U \subset V$ are two pairs of $\mathfrak{g}$-modules and submodules, with quotient modules $C = B/A$ and $W = V/U$. If $f \in \mathrm{Hom}_{\mathfrak{g}}(V,B)$ satisfies $f(U) \subset A$, then f induces maps $\underline{f} \in \mathrm{Hom}_{\mathfrak{g}}(U,A)$ and $\bar{f} \in \mathrm{Hom}_{\mathfrak{g}}(W,C)$ by restriction and passage to the quotient, respectively. This situation can be described by the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U & \longrightarrow & V & \longrightarrow & W & \longrightarrow & 0 \\ & & \downarrow \underline{f} & & \downarrow f & & \downarrow \bar{f} & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

where the horizontal arrows denote inclusion and quotient maps.

Lemma E.1.9. *Let the maps $\delta^{(p)}$ be defined as in Lemma E.1.8. Then for $p = 0, 1, \ldots$ the diagram*

$$\begin{array}{ccc} H^p(\mathfrak{g},W) & \xrightarrow{\delta^{(p)}} & H^{p+1}(\mathfrak{g},U) \\ \Big\downarrow \bar{f}^{(p)} & & \Big\downarrow \underline{f}^{(p+1)} \\ H^p(\mathfrak{g},C) & \xrightarrow{\delta^{(p)}} & H^{p+1}(\mathfrak{g},A) \end{array}$$

is commutative.

Proof. Let $\beta : V \longrightarrow W$ and $b : B \longrightarrow C$ be the quotient maps. Given $\zeta \in Z^p(\mathfrak{g},W)$, take $\omega \in C^p(\mathfrak{g},V)$ with $\beta(\omega) = \zeta$. Then $\delta^{(p)}[\zeta] = [\mathrm{d}\omega]$, and hence $\underline{f}^{(p+1)}\delta^{(p)}[\zeta] = [f\mathrm{d}\omega]$. However,

$$\bar{f}^{(p)}[\zeta] = [f\beta(\omega)] = [bf(\omega)] ,$$

so $\delta^{(p)}\bar{f}^{(p)}[\zeta] = [\mathrm{d}f(\omega)]$. Since f commutes with d, we conclude that

$$\underline{f}^{(p+1)}\delta^{(p)} = \delta^{(p)}\bar{f}^{(p)} ,$$

as claimed. $\square$

Suppose $\mathfrak{n}$ is an ideal in a Lie algebra $\mathfrak{b}$, and assume that (E.11) is an exact sequence of $\mathfrak{b}$-modules, which we view as an exact sequence of $\mathfrak{n}$-modules by restriction. By Proposition E.1.7 the cohomology spaces in (E.13) are modules for $\mathfrak{b}/\mathfrak{n}$. Clearly $\alpha^{(p)}$ and $\beta^{(p)}$ intertwine the action of $\mathfrak{b}/\mathfrak{n}$.

Proposition E.1.10. *Replace $\mathfrak{g}$ by $\mathfrak{b}$ in* (E.13). *Then the connecting maps $\delta^{(p)}$ intertwine the action of $\mathfrak{b}$ on the cohomology spaces. Hence* (E.13) *is an exact sequence of $\mathfrak{b}/\mathfrak{n}$-modules.*

Proof. Let $X \in \mathfrak{b}$ and $\zeta \in C^p(\mathfrak{n},W)$. Choose $\omega \in C^p(\mathfrak{n},V)$ such that $\beta(\omega) = \zeta$ and choose $\mu \in C^{p+1}(\mathfrak{n},U)$ such that $\alpha(\mu) = \mathrm{d}\omega$. Then

$$\alpha(\theta(X)\mu) = \mathrm{d}\theta(X)\omega , \quad \beta(\theta(X)\omega) = \theta(X)\zeta .$$

It follows from the definition of the connecting homomorphism that

$$\delta^{(p)}[\theta(X)\zeta] = [\theta(X)\mu] = \theta(X)[\mu] = \theta(X)\delta^{(p)}[\zeta]$$

as claimed. $\square$

E.1.4 The Koszul complex

Now consider any finite-dimensional vector space $\mathfrak{g}$ viewed as a Lie algebra with $[X,Y] = 0$ for all $X, Y \in \mathfrak{g}$. The symmetric tensor algebra $S(\mathfrak{g})$ is a $\mathfrak{g}$-module relative to the action μ, where

$$\mu(X) : S^k(\mathfrak{g}) \longrightarrow S^{k+1}(\mathfrak{g})$$

is multiplication by X and $S^k(\mathfrak{g})$ is the space of homogeneous symmetric tensors of degree k (see Section B.2.3). For $\xi \in \mathfrak{g}^*$ define $\partial(\xi)$ as the derivation on $S(\mathfrak{g})$ such that $\partial(\xi)X = \langle \xi, X \rangle$ for $X \in \mathfrak{g}$. Then $\partial(\xi) : S^k(\mathfrak{g}) \longrightarrow S^{k-1}(\mathfrak{g})$. Recall that there is a canonical identification $S(\mathfrak{g}) = \mathcal{P}(\mathfrak{g}^*)$, the polynomial functions on $\mathfrak{g}^*$, with $\partial(\xi)$ becoming the directional derivative in the direction ξ.

The operators $\mu(X)$ and $\partial(\xi)$ satisfy the *canonical commutation relations*

$$[\mu(X), \mu(Y)] = 0, \quad [\partial(\xi), \partial(\eta)] = 0, \quad [\partial(\xi), \mu(X)] = \langle \xi, X \rangle \tag{E.17}$$

for $X, Y \in \mathfrak{g}$, $\xi, \eta \in \mathfrak{g}^*$.

Define an operator E on $S(\mathfrak{g})$ by $E = \sum_i \mu(X_i)\partial(\xi_i)$, where $\{X_i\}$ is a basis for $\mathfrak{g}$ and $\{\xi_i\}$ is the dual basis for $\mathfrak{g}^*$ (E is the *Euler operator*; see Section 5.6.1). From (E.17) we calculate that

$$[E, \mu(X)] = \mu(X), \quad [E, \partial(\xi)] = -\partial(\xi)\,. \tag{E.18}$$

Since E annihilates constants, it follows by induction on k that $Eu = ku$ if $u \in S^k(\mathfrak{g})$. Similarly, the operator $\widehat{E}$ on $\bigwedge \mathfrak{g}^*$ defined by $\widehat{E} = \sum_i \varepsilon(\xi_i)\iota(X_i)$ satisfies $\widehat{E}\omega = p\omega$ if $\omega \in \bigwedge^p \mathfrak{g}^*$. Furthermore,

$$[\widehat{E}, \varepsilon(\xi)] = \varepsilon(\xi), \quad [\widehat{E}, \iota(X)] = -\iota(X) \tag{E.19}$$

(see Lemma 5.5.1).

Proposition E.1.11. *Let $\mathfrak{g}$ be an abelian Lie algebra of dimension n. Then for $p \neq n$ the cohomology space $H^p(\mathfrak{g}, S(\mathfrak{g})) = 0$, while $H^n(\mathfrak{g}, S(\mathfrak{g})) = [\bigwedge^n \mathfrak{g}^* \otimes 1]$ is one dimensional.*

Proof. Since $\mathfrak{g}$ is abelian, the Lie derivative operator $\theta_0(X) = 0$ on $\bigwedge \mathfrak{g}^*$ for all $X \in \mathfrak{g}$. Hence $\mathrm{d}_0 = 0$ and by (E.8) the differential on the complex $(\bigwedge \mathfrak{g}^*) \otimes S(\mathfrak{g})$ is

$$\mathrm{d} = \sum_i \varepsilon(\xi_i) \otimes \mu(X_i)\,. \tag{E.20}$$

Note that $\mathrm{d} : (\bigwedge^p \mathfrak{g}^*) \otimes S^k(\mathfrak{g}) \longrightarrow \left(\bigwedge^{p+1} \mathfrak{g}^*\right) \otimes S^{k+1}(\mathfrak{g})$. This differential complex is called the *Koszul complex*.

We define the *codifferential*

$$\widehat{\mathrm{d}} = \sum_i \iota(X_i) \otimes \partial(\xi_i) . \tag{E.21}$$

Since $\widehat{\mathrm{d}} : (\bigwedge^p \mathfrak{g}^*) \otimes S^k(\mathfrak{g}) \longrightarrow \left(\bigwedge^{p-1} \mathfrak{g}^*\right) \otimes S^{k-1}(\mathfrak{g})$, the operator $\mathrm{d}\widehat{\mathrm{d}} + \widehat{\mathrm{d}}\mathrm{d}$ preserves degree in both factors. From (E.20) and (E.21) we calculate the anticommutator

$$\{\mathrm{d}, \widehat{\mathrm{d}}\} = \sum_{i,j} \varepsilon(\xi_i)\iota(X_j) \otimes \mu(X_i)\partial(\xi_j) + \iota(X_j)\varepsilon(\xi_i) \otimes \partial(\xi_j)\mu(X_i) .$$

Adding and subtracting $\iota(X_j)\varepsilon(\xi_i) \otimes \mu(X_i)\partial(\xi_j)$, we can write $\{\mathrm{d}, \widehat{\mathrm{d}}\}$ in terms of commutators and anticommutators as

$$\{\mathrm{d}, \widehat{\mathrm{d}}\} = \sum_{i,j} \{\varepsilon(\xi_i), \iota(X_j)\} \otimes \mu(X_i)\partial(\xi_j) + \iota(X_j)\varepsilon(\xi_i) \otimes [\partial(\xi_j), \mu(X_i)] .$$

From the anticommutation and commutation relations (E.5) and (E.17), this simplifies to

$$\{\mathrm{d}, \widehat{\mathrm{d}}\} = I \otimes E + \sum_i \iota(X_i)\varepsilon(\xi_i) \otimes I .$$

Finally, using the anticommutation relations to reverse the order of multiplication in the sum, we obtain

$$\mathrm{d}\widehat{\mathrm{d}} + \widehat{\mathrm{d}}\mathrm{d} = I \otimes E + nI - \widehat{E} \otimes I , \tag{E.22}$$

where $n = \dim \mathfrak{g}$.

Let $\omega \in Z^p(\mathfrak{g}, S(\mathfrak{g}))$. We can decompose ω into homogeneous components as

$$\omega = \sum \omega_k ,$$

where $\omega_k \in C^p(\mathfrak{g}, S^k(\mathfrak{g}))$. Since $\mathrm{d}\omega_k \in C^{p+1}(\mathfrak{g}, S^{k+1}(\mathfrak{g}))$, we must have $\mathrm{d}\omega_k = 0$ for all k. Thus the space of p-cocycles is the direct sum

$$Z^p(\mathfrak{g}, S(\mathfrak{g})) = \bigoplus_{k \geq 0} Z^p(\mathfrak{g}, S^k(\mathfrak{g})) .$$

From (E.22) we see that the operator $\mathrm{d}\widehat{\mathrm{d}}$ acts by the scalar $k+n-p$ on $Z^p(\mathfrak{g}, S^k(\mathfrak{g}))$. Hence $\omega_k = (k+n-p)^{-1}\mathrm{d}\widehat{\mathrm{d}}\omega_k$ provided $k+n-p \neq 0$. Thus we have

$$Z^p(\mathfrak{g}, S^k(\mathfrak{g})) = B^p(\mathfrak{g}, S^{k-1}(\mathfrak{g})) \text{ if } p < n \text{ or if } p = n \text{ and } k > 0 . \tag{E.23}$$

In particular,

$$Z^p(\mathfrak{g}, S^0(\mathfrak{g})) = 0 \text{ if } p < n, \text{ and } Z^n(\mathfrak{g}, S(\mathfrak{g})) = C^n(\mathfrak{g}, \mathbb{C}) . \tag{E.24}$$

We conclude that $H^p(\mathfrak{g}, S(\mathfrak{g})) = 0$ for $p < n$.

Since d increases degree, we have $B^n(\mathfrak{g},\mathbb{C})=0$. Hence from (E.23) we see that

$$H^n(\mathfrak{g},S(\mathfrak{g}))=[\bigwedge^n \mathfrak{g}^*\otimes 1]$$

is one dimensional. $\square$

E.1.5 Cohomology of enveloping algebras

Let ρ be the representation of a finite-dimensional Lie algebra $\mathfrak{g}$ on its universal enveloping algebra $U(\mathfrak{g})$ given by left multiplication: $\rho(X)u = Xu$ for $X\in\mathfrak{g}$ and $u\in U(\mathfrak{g})$. We shall calculate the $\mathfrak{g}$ cohomology of $(\rho,U(\mathfrak{g}))$.

Let $\{U_r(\mathfrak{g}) : r=0,1,\ldots\}$ be the *standard filtration* of $U(\mathfrak{g})$ (see Section C.2.3). By Theorem C.2.4 the associated graded algebra

$$\mathrm{Gr}(U(\mathfrak{g}))=\bigoplus_{r\geq 0}(U_r(\mathfrak{g})/U_{r-1}(\mathfrak{g})) \tag{E.25}$$

is canonically isomorphic to the graded algebra $S(\mathfrak{g})$. Furthermore, if $\pi_r : U_r(\mathfrak{g}) \longrightarrow S^r(\mathfrak{g})$ denotes the associated projection, then

$$\pi_{r+1}(\rho(X)u)=\mu(X)\pi_r(u) \tag{E.26}$$

for $X\in\mathfrak{g}$ and $u\in U_r(\mathfrak{g})$.

Suppose (σ,V) is any $\mathfrak{g}$-module. We form the representation $\gamma=\rho\otimes\sigma$ of $\mathfrak{g}$ on $U(\mathfrak{g})\otimes V$:

$$\gamma(X)(u\otimes v)=(Xu)\otimes v+u\otimes(\sigma(X)v)$$

for $X\in\mathfrak{g}$. By the universal property of $U(\mathfrak{g})$ the map γ extends to a representation of $U(\mathfrak{g})$ on $U(\mathfrak{g})\otimes V$. For example, for $X,Y\in\mathfrak{g}$ we have

$$\begin{aligned}\gamma(XY)(u\otimes v) &= \gamma(X)(Yu\otimes v+u\otimes\sigma(Y)v)\\ &= XYu\otimes v+Yu\otimes\sigma(X)v+Xu\otimes\sigma(Y)v+u\otimes\sigma(XY)v\,.\end{aligned}$$

Let σ_0 be the trivial representation of $\mathfrak{g}$ on V, and let γ_0 be the associated representation of $\mathfrak{g}$ on $U(\mathfrak{g})\otimes V$:

$$\gamma_0(X)(u\otimes v)=(Xu)\otimes v\,.$$

Let $T : U(\mathfrak{g})\otimes V\longrightarrow U(\mathfrak{g})\otimes V$ be the unique linear map such that

$$T(u\otimes v)=\gamma(u)(1\otimes v)\quad \text{for } u\in U(\mathfrak{g}) \text{ and } v\in V\,.$$

Lemma E.1.12. *The map T is a linear automorphism of the vector space $U(\mathfrak{g})\otimes V$, and*

$$\gamma(X)T=T\gamma_0(X)\quad \textit{for } X\in\mathfrak{g}\,.$$

Proof. We first verify that T intertwines the actions γ and γ_0. Let $X \in \mathfrak{g}$ and $u \in U(\mathfrak{g}), v \in V$. Then

$$\begin{aligned} T(\gamma_0(X)(u \otimes v)) &= T(Xu \otimes v) = \gamma(Xu)(1 \otimes v) \\ &= \gamma(X)\gamma(u)(1 \otimes v) = \gamma(X)T(u \otimes v) \ . \end{aligned}$$

Now we prove that T is bijective. Set $W = U(\mathfrak{g}) \otimes V$ and define an increasing filtration on W by $W_r = U_r(\mathfrak{g}) \otimes V$. We have $W_0 = 1 \otimes V$ and $T(1 \otimes v) = 1 \otimes v$ for $v \in V$. Thus T acts as the identity on W_0. For $u \in U_r(\mathfrak{g})$ and $v \in V$ we claim that

$$T(u \otimes v) = u \otimes v + \sum_j u_j \otimes v_j \ , \tag{E.27}$$

where $u_j \in U_{r-1}(\mathfrak{g}), v_j \in V$. Indeed, we just verified this when $r = 0$. If we assume equation (E.27) holds for r, then for $X \in \mathfrak{g}$ we have

$$\begin{aligned} T(Xu \otimes v) &= \gamma(X)T(u \otimes v) \\ &= Xu \otimes v + u \otimes \sigma(X)v + \sum_j (Xu_j \otimes v_j + u_j \otimes \sigma(X)v_j) \ , \end{aligned}$$

and hence the equation holds for $r+1$. It follows that $T : W_r \longrightarrow W_r$ and T induces the identity map on W_r/W_{r-1}. This shows that T is an isomorphism. $\square$

Theorem E.1.13. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over $\mathbb{C}$, and let $U(\mathfrak{g})$ have the $\mathfrak{g}$-module structure given by left multiplication. Then* $\dim H^p(\mathfrak{g}, U(\mathfrak{g})) = 1$ *if $p = \dim \mathfrak{g}$, and $H^p(\mathfrak{g}, U(\mathfrak{g})) = 0$ otherwise.*

Proof. Let $u \in U_r(\mathfrak{g})$ and $\omega \in \bigwedge^p \mathfrak{g}^*$. By formulas (E.7) and (E.8) for the differential, we have

$$\mathrm{d}(\omega \otimes u) = \mathrm{d}_0\omega \otimes u + \sum_i \varepsilon(\xi_i)\omega \otimes X_i u \ . \tag{E.28}$$

From (E.28) we see that

$$\mathrm{d} : (\textstyle\bigwedge^p \mathfrak{g}^*) \otimes U_r(\mathfrak{g}) \longrightarrow (\textstyle\bigwedge^{p+1} \mathfrak{g}^*) \otimes U_{r+1}(\mathfrak{g}) \ .$$

So if we project $\mathrm{d}(\omega \otimes u)$ onto $(\bigwedge^{p+1} \mathfrak{g}^*) \otimes S^{r+1}(\mathfrak{g})$ by the operator $\bar{\pi}_{r+1} = 1 \otimes \pi_{r+1}$, then the image only depends on $\pi_r(u)$:

$$\bar{\pi}_{r+1}\mathrm{d}(\omega \otimes (u+v)) = \bar{\pi}_{r+1}\mathrm{d}(\omega \otimes u)$$

for all $v \in U_{r-1}(\mathfrak{g})$. Thus we can define an operator

$$\bar{\mathrm{d}} : (\textstyle\bigwedge^p \mathfrak{g}^*) \otimes S^r(\mathfrak{g}) \longrightarrow (\textstyle\bigwedge^{p+1} \mathfrak{g}^*) \otimes S^{r+1}(\mathfrak{g})$$

by setting

$$\bar{\mathrm{d}}\pi_r(f) = \bar{\pi}_{r+1}(\mathrm{d}f) \tag{E.29}$$

for $f \in (\bigwedge^p \mathfrak{g}^*) \otimes U_r(\mathfrak{g})$. From (E.28) and (E.26) we see that

$$\bar{\mathrm{d}}(\omega \otimes \pi_r(u)) = \sum_i \varepsilon(\xi_i)\omega \otimes \mu(X_i)\pi_r(u) .$$

Thus $\bar{\mathrm{d}}$ is just the differential on the Koszul complex $(\bigwedge^\bullet \mathfrak{g}^*) \otimes S(\mathfrak{g})$ for the vector space $\mathfrak{g}$ and does not depend on the Lie bracket.

Suppose that $p < \dim \mathfrak{g}$ and $f \in Z^p(\mathfrak{g}, U(\mathfrak{g}))$. Then $f \in (\bigwedge^p \mathfrak{g}^*) \otimes U_r(\mathfrak{g})$ for some integer $r \geq 0$. If $r = 0$ then (E.24) shows that $\bar{\pi}_0(f) = 0$ and hence $f = 0$. If $r > 0$ then from (E.29) we have $\bar{\mathrm{d}}\bar{\pi}_r(f) = 0$; therefore, by the surjectivity of $\bar{\pi}_{r-1}$ and (E.23) there is a $p-1$ cochain $\eta \in (\bigwedge^{p-1} \mathfrak{g}^*) \otimes U_{r-1}(\mathfrak{g})$ such that

$$\bar{\pi}_r(f) = \bar{\mathrm{d}}\bar{\pi}_{r-1}(\eta) .$$

Set $f_1 = f - \mathrm{d}\eta$. Then f_1 is still a p-cocycle, and $f_1 \in (\bigwedge^p \mathfrak{g}^*) \otimes U_{r-1}(\mathfrak{g})$, since $\bar{\pi}_r(f - \mathrm{d}\eta) = \bar{\pi}_r(f) - \bar{\mathrm{d}}\bar{\pi}_{r-1}(\eta) = 0$. By induction on r we may assume that f_1 is a coboundary and hence so is f.

Now consider the case $p = \dim \mathfrak{g}$. Fix $0 \neq \omega \in \bigwedge^n \mathfrak{g}$. We first prove that for any $\mathfrak{g}$-module V,

$$H^n(\mathfrak{g}, V) = \omega \otimes V / \theta(\mathfrak{g})(\omega \otimes V) \quad (n = \dim \mathfrak{g}) . \tag{E.30}$$

Indeed, since $C^{n+1}(\mathfrak{g}, V) = 0$, every n-cochain is a cocyle. Moreover, $C^n(\mathfrak{g}, V) = \omega \otimes V$. If $\{X_1, \ldots, X_n\}$ is a basis for $\mathfrak{g}$ then $\{\iota(X_1)\omega, \ldots, \iota(X_n)\omega\}$ is a basis for $\bigwedge^{n-1} \mathfrak{g}^*$. Hence

$$C^{n-1}(\mathfrak{g}, V) = \bigoplus_{j=1}^n (\iota(X_j)\omega) \otimes V .$$

But $\mathrm{d}(\iota(X)(\omega \otimes v)) = \theta(X)(\omega \otimes v)$ by (E.3), and so we have

$$B^n(\mathfrak{g}, V) = \theta(\mathfrak{g})(\omega \otimes V) .$$

This proves (E.30).

Now consider the $\mathfrak{g}$-module $V = (\bigwedge^n \mathfrak{g}^*) \otimes U(\mathfrak{g})$. From Lemma E.1.12 this module is equivalent to the module with the same underlying vector space but trivial $\mathfrak{g}$ action on $\bigwedge^n \mathfrak{g}^*$ and left multiplication action on $U(\mathfrak{g})$. Since $\dim U(\mathfrak{g})/(\mathfrak{g} \cdot U(\mathfrak{g})) = 1$, it follows that $\mathfrak{g} \cdot ((\bigwedge^n \mathfrak{g}^*) \otimes U(\mathfrak{g}))$ is of codimension 1 when $n = \dim \mathfrak{g}$. Hence from (E.30) we conclude that $\dim H^n(\mathfrak{g}, U(\mathfrak{g})) = 1$. $\square$

E.1.6 Exercises

In all these exercises, $\mathfrak{g}$ is a Lie algebra and V is a $\mathfrak{g}$-module.

1. Show that if $\omega \in C^2(\mathfrak{g}, V)$ then $\mathrm{d}\omega$ is the cyclic sum

$$\begin{aligned} \mathrm{d}\omega(X,Y,Z) = {} & X \cdot \omega(Y,Z) + Y \cdot \omega(Z,X) + Z \cdot \omega(X,Y) \\ & - \omega([X,Y],Z) - \omega([Y,Z],X) - \omega([Z,X],Y). \end{aligned}$$

2. Show that $H^0(\mathfrak{g},V) = V^{\mathfrak{g}}$ is the space of $\mathfrak{g}$-invariant vectors in V. (HINT: Note that $B^0(\mathfrak{g},V) = 0$, so $H^0(\mathfrak{g},V) = Z^0(\mathfrak{g},V)$.)
3. Let $\mathfrak{b}$ be a Lie algebra, and let $\mathfrak{a}$ be a $\mathfrak{b}$-module. Given $\omega \in C^2(\mathfrak{b},\mathfrak{a})$, let $\mathfrak{g}_\omega$ be the vector space $\mathfrak{a} \oplus \mathfrak{b}$ with skew-symmetric multiplication

$$[(X,Y),\,(X',Y')]_\omega = (Y \cdot X' - Y' \cdot X + \omega(Y,Y'),\,[Y,Y'])$$

for $X,X' \in \mathfrak{a}$ and $Y,Y' \in \mathfrak{b}$.
(a) Prove that $\mathfrak{g}_\omega$ is a Lie algebra with this multiplication (i.e. $[\cdot,\cdot]_\omega$ satisfies the Jacobi identity) if and only if $\mathrm{d}\omega = 0$.
(b) Assume $\omega, \omega' \in Z^2(\mathfrak{b},\mathfrak{a})$. Show that $\mathfrak{g}_{\omega'}$ is isomorphic to $\mathfrak{g}_\omega$ if and only if $\omega' = \omega + \mathrm{d}\phi$ for some linear map $\phi : \mathfrak{b} \longrightarrow \mathfrak{a}$. In particular, prove that $\mathfrak{g}_\omega$ is isomorphic to $\mathfrak{g}_0$ if and only if $[\omega] = 0$ in $H^2(\mathfrak{b},\mathfrak{a})$. (The Lie algebra $\mathfrak{g}_0$ is called the *semidirect product* of $\mathfrak{a}$ and $\mathfrak{b}$, relative to the given $\mathfrak{b}$-module structure on $\mathfrak{a}$.)
4. Suppose $\mathfrak{a}$ is an abelian ideal in $\mathfrak{g}$. Let $\mathfrak{b} = \mathfrak{g}/\mathfrak{a}$ and let $\pi : \mathfrak{g} \longrightarrow \mathfrak{b}$ be the quotient map. (The Lie algebra $\mathfrak{g}$ is called an *abelian extension* of $\mathfrak{b}$ by $\mathfrak{a}$.)
(a) Show that $\mathfrak{a}$ is a $\mathfrak{b}$-module relative to the action $Y \cdot Z = [X,Z]$, where $Y \in \mathfrak{b}$, $Z \in \mathfrak{a}$, and $\pi(X) = Y$.
(b) Let $f : \mathfrak{b} \longrightarrow \mathfrak{g}$ be a linear map such that $\pi \circ f = I$. Set

$$\omega_f(X,Y) = [f(X), f(Y)] - f([X,Y])$$

for $X,Y \in \mathfrak{b}$. Prove that $\omega_f \in Z^2(\mathfrak{b},\mathfrak{a})$ and the cohomology class $[\omega_f] \in H^2(\mathfrak{b},\mathfrak{a})$ does not depend on the choice of f.
(c) Fix any f as in (b), and set $\omega = \omega_f$. Prove that $\mathfrak{g}$ is isomorphic to the Lie algebra $\mathfrak{g}_\omega$ constructed in the previous exercise. (This shows that the abelian extensions of $\mathfrak{b}$ by a vector space $\mathfrak{a}$ are classified by the $\mathfrak{b}$-module structure of $\mathfrak{a}$ and the space $H^2(\mathfrak{b},\mathfrak{a})$.)
5. Let $X \in \mathfrak{g}$, $\lambda \in \mathfrak{g}^*$, and $\omega \in C^p(\mathfrak{g},V)$. Show that

$$\mathrm{d}(\lambda \wedge \omega) = \mathrm{d}\lambda \wedge \omega - \lambda \wedge \mathrm{d}\omega\,, \quad \theta(X)(\lambda \wedge \omega) = (\theta(X)\lambda) \wedge \omega + \lambda \wedge (\theta(X)\omega)\,.$$

6. Suppose $\mathfrak{g}$ is finite dimensional. Show that the natural map $(\bigwedge^p \mathfrak{g}^*) \otimes V \longrightarrow C^p(\mathfrak{g},V)$ is a $\mathfrak{g}$-module isomorphism, relative to the natural action of $\mathfrak{g}$ on these spaces. (HINT: Express the value of a p-cochain in terms of a basis for $\mathfrak{g}$ and a dual basis for $\mathfrak{g}^*$.)
7. Suppose W is a *trivial* $\mathfrak{g}$-module. Prove that $H^p(\mathfrak{g}, V \otimes W) = H^p(\mathfrak{g},V) \otimes W$.
8. Suppose $\dim \mathfrak{g} < \infty$. Pick a basis $\{c_j\}$ for $H^2(\mathfrak{g})$ and representative cocycles $z_j \in c_j$. Define a skew-symmetric mapping $\Omega : \mathfrak{g} \times \mathfrak{g} \longrightarrow H^2(\mathfrak{g})$ by

$$\Omega(X,Y) = \sum_j z_j(X,Y) c_j\,.$$

Prove that the multiplication $[X \oplus \alpha,\, Y \oplus \beta)] = [X,Y] \oplus \Omega(X,Y)$ defines a Lie algebra structure on the vector space $\mathfrak{b} = \mathfrak{g} \oplus H^2(\mathfrak{g})$. ($\mathfrak{b}$ is called a *central extension* of $\mathfrak{g}$ by $H^2(\mathfrak{g})$, since $H^2(\mathfrak{g})$ is in the center of $\mathfrak{b}$.)

9. (*Continuation of previous exercise*) Define a 1-cochain α_i on $\mathfrak{b}$ by $\alpha_i(X \oplus c_j) = -\delta_{ij}$. Show that $z_i = \mathrm{d}\alpha_i|_{\mathfrak{g}}$.
10. Verify that the sequences (E.14) are exact. (HINT: Cochains are determined by their values on a basis for $\mathfrak{g}$.)
11. Assume $\dim \mathfrak{g} < \infty$ and $\dim V < \infty$. Let U be a submodule of V and set $W = V/U$.
(a) Show that the inclusion $U \subset V$ and quotient map $\pi : V \longrightarrow W$ give rise to an exact sequence

$$0 \longrightarrow \mathrm{Hom}(W,U) \longrightarrow \mathrm{Hom}(W,V) \longrightarrow \mathrm{Hom}(W,W) \longrightarrow 0$$

of $\mathfrak{g}$-modules.
(b) Let $\delta^{(0)}$ be the connecting homomorphism

$$\mathrm{Hom}_{\mathfrak{g}}(W,V) \longrightarrow \mathrm{Hom}_{\mathfrak{g}}(W,W) \xrightarrow{\delta^{(0)}} H^1(\mathfrak{g}, \mathrm{Hom}(W,U)) \,,$$

and let $I_W \in \mathrm{Hom}_{\mathfrak{g}}(W,W)$ be the identity map. Set $\gamma = \delta^{(0)}(I_W)$. Prove that $\gamma = 0$ if and only if there is a $\mathfrak{g}$-invariant subspace $U' \subset V$ such that $V = U + U'$ and $U \cap U' = 0$.
(c) Suppose $\mathfrak{g}$ is abelian. Prove that there exists a $\mathfrak{g}$-module V such that $H^1(\mathfrak{g}, V)$ is nonzero. (HINT: Find a representation of $\mathfrak{g}$ that is not completely reducible.)
12. Let M be an open subset in $\mathbb{R}^n$, and let $\mathfrak{g}$ be the Lie algebra of C^∞ vector fields on M. The space $V = C^\infty(M)$ is a $\mathfrak{g}$-module, relative to the natural action of vector fields on functions. The space $C^p(\mathfrak{g}, V)$ in this case consists of the smooth differential forms of degree i on M, and the operator d is the exterior derivative. Let $x_1, \ldots, x_n$ be the coordinate functions on M. Show that

$$2\mathrm{d} = \sum_{j=1}^{n} \varepsilon(\mathrm{d}x_j)\theta\left(\frac{\partial}{\partial x_j}\right).$$

E.2 Cohomology approach to Weyl character formula

E.2.1 Casimir identity on cohomology

Let $\mathfrak{g}$ be a semi-simple Lie algebra. Fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and a system $\Phi^+ \subset \Phi(\mathfrak{g}, \mathfrak{h})$ of positive roots. Let the nilpotent subalgebras $\mathfrak{n}^\pm$ and the associated *triangular decomposition*

$$\mathfrak{g} = \mathfrak{n}^- + \mathfrak{h} + \mathfrak{n}^+ \tag{E.31}$$

be as in Corollary 2.5.25. We set $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}^+$ and $\bar{\mathfrak{b}} = \mathfrak{h} + \mathfrak{n}^-$.

Let B be the Killing form on $\mathfrak{g}$. Let X_i be a basis for $\mathfrak{g}$ and let X^i be the B-dual basis. Define

$$C = \sum_i X_i X^i \in U(\mathfrak{g}) \,.$$

We call C the (universal) *Casimir operator* associated with the form B. If ρ is a representation of $\mathfrak{g}$ then $\rho(C)$ is the operator denoted by C_ρ in Section 3.3.2. The operator C does not depend on the choice of basis and dual basis, and it is in the center of $U(\mathfrak{g})$ by Lemma 3.3.7.

Since B is an invariant form, then for all $\alpha, \beta \in \Phi$ we have $B(\mathfrak{h}, \mathfrak{g}_\alpha) = 0$ and $B(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ if $\alpha \neq -\beta$ (see Theorem 3.2.13). Hence we can choose $X_\alpha \in \mathfrak{g}_\alpha$ and $X_{-\alpha} \in \mathfrak{g}_{-\alpha}$ so that $B(X_\alpha, X_{-\alpha}) = 1$, and a B-orthonormal basis $H_1, \ldots, H_l$ for $\mathfrak{h}$. In terms of this basis, the Casimir operator is

$$C = \sum_i H_i^2 + \sum_{\alpha \in \Phi^+} (X_\alpha X_{-\alpha} + X_{-\alpha} X_\alpha) .$$

For $\alpha \in \mathfrak{h}^*$ we denote by $H_\alpha \in \mathfrak{h}$ the element such that

$$\alpha(H) = B(H_\alpha, H), \quad \text{for } H \in \mathfrak{h} .$$

Since $[X_\alpha, X_{-\alpha}] = H_\alpha$, we may write C in two different *polarized* forms as

$$C = \sum_i H_i^2 - 2H_\rho + 2 \sum_{\alpha \in \Phi^+} X_\alpha X_{-\alpha} , \tag{E.32}$$

$$C = \sum_i H_i^2 + 2H_\rho + 2 \sum_{\alpha \in \Phi^+} X_{-\alpha} X_\alpha , \tag{E.33}$$

where $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$.

Let (σ, V) be a $\mathfrak{g}$-module. By restriction, V is a $\mathfrak{b}$-module; therefore, by Proposition E.1.7 the cohomology spaces $H^\bullet(\mathfrak{n}^+, V)$ carry a natural structure as $\mathfrak{h}$-modules via the Lie derivative θ (note that $\mathfrak{b}/\mathfrak{n}^+ \cong \mathfrak{h}$). Thus for $u \in U(\mathfrak{h})$ we have an operator

$$\theta^{(p)}(u) : H^p(\mathfrak{n}^+, V) \to H^p(\mathfrak{n}^+, V)$$

that acts on cocycles both internally via the adjoint action of $\mathfrak{h}$ on $\mathfrak{n}^+$ and externally via the representation σ on V.

In contrast, the Casimir operator $\sigma(C)$ commutes with the action of $\mathfrak{g}$ on V (see Lemma 3.3.7). In particular, $\sigma(C) \in \mathrm{Hom}_{\mathfrak{n}^+}(V, V)$. Hence there are operators

$$\sigma(C)^{(p)} : H^p(\mathfrak{n}^+, V) \to H^p(\mathfrak{n}^+, V)$$

for $p = 0, 1, \ldots, \dim \mathfrak{n}$ that act only on the values of representative cocycles. For example, when $p = 0$, then $H^0(\mathfrak{n}^+, V) = V^{\mathfrak{n}^+}$ is the space of $\mathfrak{n}^+$-invariants. Set

$$\Gamma = \sum_i H_i^2 + 2H_\rho .$$

Since $\sigma(X_\alpha)v = 0$ for $v \in V^{\mathfrak{n}^+}$ and $\alpha \in \Phi^+$, we see from (E.33) that C acts on $V^{\mathfrak{n}^+}$ by $\sigma(\Gamma)$. This is the same as the operator $\theta^{(0)}(\Gamma)$, so we have shown that

$$\sigma(C)^{(0)} = \theta^{(0)}(\Gamma) .$$

We now prove that this relation between the two actions of the Casimir operator is true on all the $\mathfrak{n}^+$ cohomology spaces.

Theorem E.2.1 (Casselman-Osborne). *For any* $\mathfrak{g}$*-module* (σ, V)*, the transformations* $\sigma(C)^{(p)}$ *and* $\theta^{(p)}(\Gamma)$ *have the same action on* $H^p(\mathfrak{n}^+, V)$*.*

Proof. We know the theorem is true for $p = 0$, but this does not give a good starting point for an inductive proof. Instead, we shall use downward induction on p, staring with the highest possible nonvanishing cohomology dimension $p = r = \dim \mathfrak{n}^+$.

Fix $0 \neq \omega \in \bigwedge^r(\mathfrak{n}^+)^*$. Since $\mathrm{ad}(X)^*\omega = \mathrm{tr}(\mathrm{ad}_{\mathfrak{n}^+} X)\omega = 0$ for $X \in \mathfrak{n}^+$, we find by (E.30) that

$$H^r(\mathfrak{n}^+, V) = (\omega \otimes V) \,/\, \big(\omega \otimes \mathfrak{n}^+ \cdot V\big) \ . \tag{E.34}$$

From (E.32) we have

$$\sigma(C)v \equiv \sigma\Big(\sum_i H_i^2 - 2H_\rho\Big)v \qquad (\mathrm{mod}\ \mathfrak{n}^+ \cdot V)$$

for $v \in V$. Since $\sigma(C)^{(r)}$ acts only on the values of cochains, it follows from (E.34) that the induced action on cohomology classes is

$$\sigma(C)^{(r)}[\omega \otimes v] = [\omega \otimes \sigma(C)v] = \big[\omega \otimes \sigma\big(\textstyle\sum_i H_i^2 - 2H_\rho\big)v\big] \ .$$

However, $H \in \mathfrak{h}$ acts by $-2\rho(H)$ on $\bigwedge^r(\mathfrak{n}^+)^*$, so

$$\theta^{(r)}(H) = 1 \otimes \sigma(H - 2\rho(H)) \ .$$

Thus

$$\begin{aligned} \theta^{(r)}(\Gamma) &= 1 \otimes \sigma\Big(\sum_i (H_i - 2\rho(H_i))^2 + 2H_\rho - 4\rho(H_\rho)\Big) \\ &= 1 \otimes \sigma\Big(\sum_i H_i^2 - 4\rho(H_i)H_i + 4\rho(H_i)^2\Big) \\ &\qquad + 1 \otimes \sigma(2H_\rho - 4\rho(H_\rho)) \ . \end{aligned}$$

Now $\sum_i \rho(H_i)H_i = H_\rho$ and $\sum_i \rho(H_i)^2 = \rho(H_\rho)$, so the formula above simplifies to

$$\theta^{(r)}(\Gamma) = 1 \otimes \sigma\Big(\sum_i H_i^2 - 2\rho(H_i)H_i\Big) = 1 \otimes \sigma\Big(\sum_i H_i^2 - 2H_\rho\Big) \ .$$

This proves the theorem when $p = r$.

Take $p < r$ and assume by induction that the theorem is true in degrees greater than p for every $\mathfrak{g}$-module. Given a $\mathfrak{g}$-module V, denote by $\widetilde{V}$ the vector space V with $\mathfrak{g}$ acting by 0 and set $F = U(\mathfrak{g}) \otimes \widetilde{V}$. We make F into a $\mathfrak{g}$-module by the left multiplication action of $\mathfrak{g}$ on $U(\mathfrak{g})$. Let m be the multiplication map

$$m : U(\mathfrak{g}) \otimes \widetilde{V} \longrightarrow V, \quad m(u \otimes v) = \sigma(u)v \ .$$

Then the map m intertwines the $\mathfrak{g}$ actions and is obviously surjective. Denoting its kernel by E, we thus obtain a short exact sequence

$$0 \longrightarrow E \longrightarrow F \xrightarrow{m} V \longrightarrow 0 \tag{E.35}$$

of $\mathfrak{g}$-modules.

To carry out the induction step we shall use the cohomology long exact sequence associated with (E.35). We observe that the $\mathfrak{n}^+$-module structure of F is just $U(\mathfrak{n}^+)\otimes W$ for a suitable trivial $\mathfrak{n}^+$-module W, with $\mathfrak{n}^+$ acting by left multiplication on the first tensor factor. Indeed, by the triangular decomposition (E.31) and the Poincaré–Birkoff–Witt theorem (Theorem C.2.2) we have $U(\mathfrak{g}) \cong U(\mathfrak{n}^+)\otimes U(\bar{\mathfrak{b}})$ as a module for $\mathfrak{n}^+$ under left multiplication. Define the vector space $W = U(\bar{\mathfrak{b}})\otimes \widetilde{V}$, and let $\mathfrak{n}^+$ act by zero on it. Then $F \cong U(\mathfrak{n}^+)\otimes W$ as a $\mathfrak{n}^+$-module. Hence

$$H^p(\mathfrak{n}^+, F) = H^p(\mathfrak{n}^+, U(\mathfrak{n}^+))\otimes W = 0 \tag{E.36}$$

by Theorem E.1.13. It follows from Lemma E.1.8 that

$$0 \longrightarrow H^p(\mathfrak{n}^+, V) \xrightarrow{\delta^{(p)}} H^{p+1}(\mathfrak{n}^+, E) \tag{E.37}$$

is exact.

Consider now the action of the Casimir operator on (E.35). Since the horizontal arrows are $\mathfrak{g}$ intertwining maps, we have the commutative diagram

$$\begin{array}{ccccccc}
0 \longrightarrow & E & \longrightarrow & F & \longrightarrow & V & \longrightarrow 0 \\
& \downarrow C & & \downarrow C & & \downarrow C & \\
0 \longrightarrow & E & \longrightarrow & F & \longrightarrow & V & \longrightarrow 0
\end{array}$$

Let S denote the operator $\sigma(C)^{(p)} - \theta^{(p)}(\Gamma)$ on $H^p(\mathfrak{n}^+, V)$, and let T denote the operator $\sigma(C)^{(p+1)} - \theta^{(p+1)}(\Gamma)$ on $H^{p+1}(\mathfrak{n}^+, E)$. By Lemma E.1.9 and Proposition E.1.10 these operators fit into the commutative diagram

$$\begin{array}{ccc}
H^p(\mathfrak{n}^+, V) & \xrightarrow{\delta^{(p)}} & H^{p+1}(\mathfrak{n}^+, E) \\
S\downarrow & & \downarrow T \\
H^p(\mathfrak{n}^+, V) & \xrightarrow{\delta^{(p)}} & H^{p+1}(\mathfrak{n}^+, E)
\end{array}$$

By (E.37) we know that $\delta^{(p)}$ is injective. The induction hypothesis is that $T = 0$. Hence $S = 0$ also, which completes the induction. $\square$

Corollary E.2.2. *Suppose $\sigma(C)$ acts by a scalar κ on V. Then*

$$\theta^{(p)}\Big(\sum_i H_i^2 + 2H_\rho\Big) = \kappa I \tag{E.38}$$

on $H^p(\mathfrak{n}^+, V)$.

Proof. Since $\sigma(C)^{(p)}$ just acts on the values of the cochains, it acts by the same scalar κ on all the cohomology spaces. Thus (E.38) follows from theorem E.2.1.

□

E.2.2 Weyl group and sets of positive roots

Let W be the Weyl group of $\mathfrak{g}$. Let $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ be an enumeration of the simple roots in Φ^+ and write $s_i = s_{\alpha_i}$ for the reflection through the simple root α_i. We recall the following facts about Δ and W; see Sections 2.4.3 and 3.1.2 and equation (3.8) for details.

(R1) Every $\beta \in \Phi^+$ has a unique expression $\beta = \sum n_i \alpha_i$ with $n_i \in \mathbb{N}$.
(R2) $W \cdot \Delta = \Phi$ and $W \cdot \Phi = \Phi$.
(R3) W is generated by the simple reflections $\{s_1, \ldots, s_l\}$.
(R4) $s_i \alpha_i = -\alpha_i$ and s_i permutes the set $\Phi^+ \setminus \{\alpha_i\}$.
(R5) If $\alpha, \beta \in \Phi^+$ and $\alpha + \beta \in \Phi$, then $\alpha + \beta \in \Phi^+$.

If $s \in W$ and $s \neq 1$, then by (R3) there are indices $i_1, \ldots, i_k$ so that $s = s_{i_1} \cdots s_{i_k}$. If k is minimal, this is called a *reduced expression* for s. We set $l(1) = 0$ and for $s \neq 1$ in W we define the *length* of s, relative to the choice Φ^+ of positive roots, to be

$$l(s) = \min\{k : s = s_{i_1} \cdots s_{i_k}\} .$$

This nonnegative function on W satisfies $l(s) = l(s^{-1})$. To each $s \in W$, we also associate the subset

$$Q(s) = \{\alpha \in \Phi^+ : s\alpha \in -\Phi^+\}$$

of positive roots. For example, if $s = 1$ then $Q(s) = \emptyset$. If $s = s_i$ is a simple reflection, then $Q(s) = \{\alpha_i\}$ by (R4).

Lemma E.2.3. *Let $Q \subset \Phi^+$. Then $Q = Q(s)$ for some $s \in W$ if and only if Q satisfies the following conditions:*

1. If $\alpha, \beta \in Q$ and $\alpha + \beta \in \Phi$, then $\alpha + \beta \in Q$.
2. If $\alpha \in Q$ and $\alpha = \gamma + \delta$, where $\gamma, \delta \in \Phi^+$, then either $\gamma \in Q$ or $\delta \in Q$.

Proof. Suppose $Q = Q(s)$. Given $\alpha, \beta \in Q(s)$ such that $\alpha + \beta \in \Phi$, we have $s \cdot (\alpha + \beta) \in \Phi$ by (R2). But $-s \cdot \alpha$ and $-s \cdot \beta$ are both in Φ^+, and $\Phi = -\Phi$. Hence $-s \cdot \alpha - s \cdot \beta \in \Phi^+$ by (R5), which verifies property (1). For property (2), suppose $\gamma, \delta, \gamma + \delta \in \Phi^+$ and $s\gamma, s\delta \in \Phi^+$. Since $s \cdot (\gamma + \delta) \in \Phi$, it follows again by (R5) that $s \cdot (\gamma + \delta) \in \Phi^+$. Hence $\gamma + \delta \notin Q(s)$.

Conversely, suppose Q satisfies conditions (1) and (2). We shall prove that $Q = Q(s)$ for some $s \in W$ by induction on $|Q|$. When $Q = \emptyset$, then $Q = Q(1)$ and (1) and (2) are vacuous. If $|Q| > 0$, then Q must contain some simple root α_i by (2). Set $Q' = s_i \cdot Q \setminus \{-\alpha_i\}$. Then $Q' \subset \Phi^+ \setminus \{\alpha_i\}$ and $|Q'| = |Q| - 1$ by (R4). We claim that Q' satisfies (1) and (2).

(1): Suppose $\gamma', \delta' \in Q'$ such that $\gamma' + \delta' \in \Phi^+$. Then $\gamma' = s_i \cdot \gamma$, $\delta' = s_i \cdot \delta$ with $\gamma, \delta \in Q \setminus \{\alpha_i\}$. Since $\gamma' + \delta'$ is not a simple root, we have $\gamma + \delta = s_i \cdot (\gamma' + \delta') \in \Phi^+ \setminus \{\alpha_i\}$ by (R4). Hence $\gamma + \delta \in Q \setminus \{\alpha_i\}$ by (1). But $\gamma' + \delta' = s_i \cdot (\gamma + \delta)$, which implies that $\gamma' + \delta' \in Q'$.

(2): Suppose $\beta' = s_i\beta \in Q'$ can be decomposed as $\beta' = \delta' + \gamma'$, where $\delta', \gamma' \in \Phi^+$. Set $\gamma = s_i \cdot \gamma'$ and $\delta = s_i \cdot \delta'$. Since $\beta = \gamma + \delta$ is a positive root, at least one of γ, δ is positive. Thus we only need to consider the following two cases:

Case 1: Suppose $\gamma \in -\Phi^+$ and $\delta \in \Phi^+$. Then $\gamma = -\alpha_i$ by (R4), since $s_i\gamma \in \Phi^+$. In this case we can write $\delta = \beta + \alpha_i$, which implies that $\delta \in Q$ by (1) and hence $\delta' \in Q'$.

Case 2: Suppose $\gamma, \delta \in \Phi^+$. Then neither one can be α_i, since $\delta', \gamma' \in \Phi^+$. Since $\beta \in Q$, it follows by condition (2) that either γ or δ must be in $Q \setminus \{\alpha_i\}$. Hence either γ' or δ' is in Q'.

Now we apply the induction hypothesis to represent $Q' = Q(s')$ for some $s' \in W$. We thus have

$$Q = \{\alpha_i\} \cup s_i \cdot Q(s') \ .$$

Set $s = s's_i$. We shall show that $Q = Q(s)$. We first observe that $s' \cdot \alpha_i$ is a positive root, since $\alpha_i \notin Q' = Q(s')$. Hence $s \cdot \alpha_i = -s' \cdot \alpha_i$ is negative, and so we have $\alpha_i \in Q(s)$. Suppose $\beta \in \Phi^+ \setminus \{\alpha_i\}$ and $s \cdot \beta$ is negative. Since $s_i \cdot \beta$ is a positive root by (R4), we have $s_i \cdot \beta \in Q(s')$. Hence $\beta = s_i \cdot (s_i \cdot \beta) \in Q$. Conversely, if $\beta \in Q \setminus \{\alpha_i\}$, then $s_i \cdot \beta \in Q(s')$ and consequently $s \cdot \beta$ is negative. Thus $\beta \in Q(s)$. This completes the proof that $Q = Q(s)$. $\square$

Corollary E.2.4. *If $s \in W$ and $\alpha \in \Delta$, then*

$$Q(ss_\alpha) = \begin{cases} s_\alpha \cdot Q(s) \setminus \{-\alpha\} & \text{if } \alpha \in Q(s) \ , \\ s_\alpha \cdot Q(s) \cup \{\alpha\} & \text{if } \alpha \notin Q(s) \ . \end{cases}$$

Proof. If $\alpha \in Q(s)$, one has $ss_\alpha(\alpha) = -s \cdot \alpha \in \Phi^+$, and hence $\alpha \notin Q(ss_\alpha)$. Now use the argument at the end of the previous proof, with $\alpha_i = \alpha$ and $s' = ss_\alpha$, to conclude that $Q(s)$ is the disjoint union of $s_\alpha \cdot Q(s')$ and $\{\alpha\}$. Since $s_\alpha = s_\alpha^{-1}$, this gives the first formula.

Now suppose $\alpha \notin Q(s)$. Then $ss_\alpha(\alpha) \in -\Phi^+$, so $\alpha \in Q(ss_\alpha)$. Replacing s by ss_α and s' by s in the previous argument, we find now that $Q(ss_\alpha)$ is the disjoint union of $s_\alpha \cdot Q(s)$ and $\{\alpha\}$, and we obtain the second formula. $\square$

Lemma E.2.5. *If $s \in W$ then $l(s) = |Q(s)|$.*

Proof. Let α be a simple root. From Corollary E.2.4 we have

$$|Q(ss_\alpha)| = \begin{cases} |Q(s)| - 1 & \text{if } \alpha \in Q(s) \ , \\ |Q(s)| + 1 & \text{if } \alpha \notin Q(s) \ . \end{cases} \tag{E.39}$$

We shall prove that whenever $s = s_{i_1} \cdots s_{i_k}$ is a reduced expression, then the last root $\alpha_{i_k} \in Q(s)$. Assume for the sake of contradiction that $s \cdot \alpha_{i_k} \in \Phi^+$. Set $w_{k-1} = I$ and $w_j = s_{i_{j+1}} \cdots s_{i_{k-1}}$ for $0 \le j < k-1$. Then $w_0 \cdot \alpha_{i_k} = -s \cdot \alpha_{i_k} \in -\Phi^+$, while

$w_{k-1} \cdot \alpha_{i_k} = \alpha_{i_k} \in \Phi^+$. Thus there exists an integer $j \geq 1$ such that $w_{j-1} \cdot \alpha_{i_k} \in -\Phi^+$ and $w_j \cdot \alpha_{i_k} \in \Phi^+$. Set $\beta = w_j \cdot \alpha_{i_k}$. Since $s_{i_j} \cdot \beta = w_{j-1} \cdot \alpha_{i_k} \in -\Phi^+$, it follows from property (R4) above that $\beta = \alpha_{i_j}$. Now for any $w \in W$ and $\alpha \in \Phi$, one has

$$w s_\alpha w^{-1} = s_{w\cdot\alpha} \ .$$

Hence we obtain the relation $w_j s_{i_k} w_j^{-1} = s_{i_j}$, which can be written as

$$s_{i_{j+1}} \cdots s_{i_k} = s_{i_j} \cdots s_{i_{k-1}} \ .$$

Substituting this relation in the expression for s we find that

$$s = s_{i_1} \cdots s_{i_{j-1}} s_{i_{j+1}} \cdots s_{i_{k-1}} \ .$$

This is a product of $k-2$ simple reflections, which contradicts the assumption that $l(s) = k$.

Let $s = s_{i_1} \cdots s_{i_k}$ be a reduced expression and set $s' = s s_{i_k}$. Since $s_{i_1} \cdots s_{i_{k-1}}$ is a reduced expression for s', we have $l(s) = l(s') + 1$. By induction we may assume that $l(s') = |Q(s')| = k-1$. By the result just proved and (E.39) we have $|Q(s')| = |Q(s)| - 1$; hence $|Q(s)| = k$. □

Lemma E.2.6. *For $Q \subset \Phi^+$ set $\langle Q \rangle = \sum_{\alpha \in Q} \alpha$. Let $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$.*

1. If $s \in W$ then $s \cdot \rho - \rho = s \cdot \langle Q(s) \rangle$. In particular, if $s \cdot \rho = \rho$ then $s = 1$.
2. If $s \in W$ and $Q \subset \Phi^+$ then there exists $Q' \subset \Phi^+$ such that $s \cdot (\rho - \langle Q \rangle) = \rho - \langle Q' \rangle$.

Proof. If $\beta \in \Phi^+$, then either $s^{-1}\beta = \alpha \in \Phi^+$, in which case $\alpha \in \Phi^+ \setminus Q(s)$, or else $s^{-1} \cdot \beta = -\alpha \in -\Phi^+$, with $\alpha \in Q(s)$. Thus

$$\Phi^+ = \left\{ s \cdot (\Phi^+ \setminus Q(s)) \right\} \cup \{ -s \cdot Q(s) \}$$

is a disjoint union. It follows that

$$\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+ \setminus Q(s)} s \cdot \alpha - \frac{1}{2} \sum_{\alpha \in Q(s)} s \cdot \alpha \ .$$

It is obvious, however, that

$$s \cdot \rho = \frac{1}{2} \sum_{\alpha \in \Phi^+ \setminus Q(s)} s \cdot \alpha + \frac{1}{2} \sum_{\alpha \in Q(s)} s \cdot \alpha \ .$$

Subtracting these two expressions, we obtain the first statement of (1). This implies the second statement of (1) since $\langle Q \rangle = 0$ if and only if Q is empty.

To prove (2), we note that since W is generated by simple reflections, it is enough to consider the case $s = s_i$. Set $Q' = s \cdot (Q \setminus \{\alpha_i\})$. From (R4) we have $s \cdot \rho = \rho - \alpha_i$ and $Q' \subset \Phi^+$. Hence if $\alpha_i \in Q$ then

$$s \cdot (\rho - \langle Q \rangle) = \rho - \alpha_i - \langle Q' \rangle + \alpha_i = \rho - \langle Q' \rangle \ .$$

If $\alpha_i \notin Q$ then $s \cdot (\rho - \langle Q \rangle) = \rho - \langle Q'' \rangle$, where $Q'' = s \cdot Q \cup \{\alpha_i\} \subset \Phi^+$. □

E.2.3 Expansion of an invariant

We next consider the W-invariant analogue of the Weyl denominator. Set

$$\xi = e^{\rho} \prod_{\alpha \in \Phi^+} (1 + e^{-\alpha}) \, .$$

Writing ξ as $\prod_{\alpha \in \Phi^+} (e^{\alpha/2} + e^{-\alpha/2})$, we see that ξ is invariant under W, since a simple reflection permutes the factors. Expanding the product yields the following sum version of the formula:

$$\xi = \sum_{Q \subset \Phi^+} e^{\rho - \langle Q \rangle} \, . \tag{E.40}$$

Here the sum is over all subsets of Φ^+, including the empty set.

Lemma E.2.7. *If $Q \subset \Phi^+$ and $\langle Q \rangle = \langle Q(s) \rangle$ for some $s \in W$, then $Q = Q(s)$. Furthermore, s is uniquely determined by $\langle Q(s) \rangle$.*

Proof. Since $s^{-1} \cdot \xi = \xi$, we have

$$\sum_{Q' \subset \Phi^+} e^{s^{-1} \cdot (\rho - \langle Q' \rangle)} = \sum_{Q \subset \Phi^+} e^{\rho - \langle Q \rangle} \, .$$

Since the coefficient of $e^{s^{-1} \cdot \rho}$ on the left side is 1, there exists a *unique* subset $Q \subset \Phi^+$ such that $\rho - \langle Q \rangle = s^{-1} \cdot \rho$. But by Lemma E.2.6, the subset $Q(s)$ has this property. Hence $Q = Q(s)$. □

E.2.4 Kostant's lemma

We now come to the key result about subsets of positive roots and weights of finite-dimensional $\mathfrak{g}$ modules. Let (α, β) be the symmetric bilinear form on $\mathfrak{h}^*$ as in Section 2.5.3.

Lemma E.2.8 (Kostant). *Let V be the finite-dimensional irreducible $\mathfrak{g}$-module with highest weight λ. Suppose μ is a weight of V and Q is a subset of Φ^+ such that*

$$(\mu + \rho - \langle Q \rangle, \mu + \rho - \langle Q \rangle) = (\lambda + \rho, \lambda + \rho) \, .$$

Then there exists a unique element $s \in W$ such that $\mu = s \cdot \lambda$ and $Q = Q(s^{-1})$. Furthermore, $\mu + \rho - \langle Q \rangle = s \cdot (\lambda + \rho)$.

Proof. By Proposition 3.1.20 there exists $s \in W$ such that $s^{-1} \cdot (\mu + \rho - \langle Q \rangle) \in P_{++}(\mathfrak{g})$. However, $s^{-1} \cdot \mu$ is a weight of V, so by Corollary 3.2.3 we have $s^{-1} \cdot \mu = \lambda - \beta$ with β a sum of positive roots (with possible repetitions). Also, by Lemma E.2.6 there exists $Q' \subset \Phi^+$ such that $s^{-1} \cdot (\rho - \langle Q \rangle) = \rho - \langle Q' \rangle$.

Set $\eta = s^{-1} \cdot (\mu + \rho - \langle Q \rangle)$. By the observations above, we can write

$$\eta = \lambda + \rho - \beta - \langle Q' \rangle \,.$$

Moreover, we have $(\lambda + \rho, \lambda + \rho) = (\eta, \eta)$ by assumption and the orthogonality of the W action. Substituting the expression above for η in the second factor of the inner product, we obtain

$$\begin{aligned}(\lambda+\rho,\lambda+\rho) &= (\eta,\lambda+\rho-\beta-\langle Q'\rangle) = (\eta,\lambda+\rho) - (\eta,\beta+\langle Q'\rangle), \\ &\leq (\eta,\lambda+\rho)\,,\end{aligned}$$

since $(\eta, \alpha) \geq 0$ for all $\alpha \in \Phi^+$ (recall that $\eta \in P_{++}(\mathfrak{g})$). Substituting once again the expression for η, we obtain

$$\begin{aligned}(\lambda+\rho,\lambda+\rho) &\leq (\lambda+\rho-\beta-\langle Q'\rangle,\lambda+\rho) = (\lambda+\rho,\lambda+\rho) - (\beta+\langle Q'\rangle,\lambda+\rho) \\ &\leq (\lambda+\rho,\lambda+\rho)\,,\end{aligned}$$

since $(\lambda + \rho, \alpha) \geq 0$ for all $\alpha \in \Phi^+$. We conclude that equality holds throughout, and hence

$$(\beta + \langle Q' \rangle, \lambda + \rho) = 0 \,. \tag{E.41}$$

Since $\lambda + \rho$ is dominant regular, whereas β and $\langle Q' \rangle$ are sums of positive roots, (E.41) implies that $\beta = 0$ and $Q' = \emptyset$. Thus $\mu = s \cdot \lambda$ and $\rho - \langle Q \rangle = s \cdot \rho$. In particular, $\mu + \rho - \langle Q \rangle = s \cdot (\lambda + \rho)$. Furthermore, by Lemma E.2.6 we have $\rho - s \cdot \rho = \langle Q(s^{-1}) \rangle$. Hence $\langle Q \rangle = \langle Q(s^{-1}) \rangle$, and so by Lemma E.2.7 it follows that $Q = Q(s^{-1})$ and that s is uniquely determined. □

E.2.5 Kostant's theorem

We now turn to the calculation of $H^\bullet(\mathfrak{n}^+, V)$, where (σ, V) is the irreducible finite-dimensional representation of $\mathfrak{g}$ with highest weight λ. We begin by determining the weight spaces for the cochain complex.

The weights of $\mathfrak{h}$ on $(\mathfrak{n}^+)^*$ are $-\alpha$, where $\alpha \in \Phi^+$, and each has multiplicity one. We choose nonzero elements $\omega_{-\alpha} \in (\mathfrak{n}^+)^*(-\alpha)$. Then $\omega_{-\alpha}(\mathfrak{g}_\beta) = 0$ for all roots $\beta \neq \alpha$. For each subset $Q \subset \Phi^+$ with $|Q| = p$, fix an enumeration $Q = \{\beta_1, \ldots, \beta_p\}$ and set

$$\omega_{-Q} = \omega_{-\beta_1} \wedge \cdots \wedge \omega_{-\beta_p} \in \textstyle\bigwedge^p (\mathfrak{n}^+)^* \,.$$

Different enumerations of Q or choices of functionals $\omega_{-\alpha}$ change ω_{-Q} by a nonzero scalar multiple. Hence the one-dimensional space $\mathbb{C}\omega_{-Q}$ is uniquely determined by Q. If $v_\mu \in V(\mu)$, then $\mathfrak{h}$ acts on $\omega_{-Q} \otimes v_\mu$ by the weight $\mu - \langle Q \rangle$. So the space of p-cochains $C^p(\mathfrak{n}^+, V)$ decomposes as an $\mathfrak{h}$-module into weight spaces

$$C^p(\mathfrak{n}^+, V) = \bigoplus_{\substack{Q \subset \Phi^+,\, |Q|=p \\ \mu \in \mathcal{X}(V)}} \omega_{-Q} \otimes V(\mu) \,, \tag{E.42}$$

where $\mathcal{X}(V)$ is the set of weights of V.

We now determine the $\mathfrak{n}^+$ cohomology of V. Fix a non-zero vector $v_{s\cdot\lambda}$ in the one-dimensional space $V(s\cdot\lambda)$ for each $s \in W$.

Theorem E.2.9 (Kostant). *If V is an irreducible $\mathfrak{g}$-module with highest weight λ, then for every $s \in W$ the cochain $\omega_{-Q(s^{-1})} \otimes v_{s\cdot\lambda}$ is a cocycle whose cohomology class is nonzero. Furthermore,*

$$H^p(\mathfrak{n}^+, V) = \bigoplus_{s \in W,\, l(s)=p} \mathbb{C}[\omega_{-Q(s^{-1})} \otimes v_{s\cdot\lambda}] \,. \tag{E.43}$$

Proof. Recall from Proposition E.1.7 that $H^p(\mathfrak{n}^+, V)$ is an $\mathfrak{h}$-module with action θ. Since the differential d commutes with $\theta(\mathfrak{h})$, each weight space $H^p(\mathfrak{n}^+, V)(\xi)$, for $\xi \in \mathfrak{h}^*$, is the pth cohomology space of the differential complex $\{C^\bullet(\mathfrak{n}^+, V)(\xi), \mathrm{d}\}$. By (E.42) we see that the weights of $\mathfrak{h}$ on $H^p(\mathfrak{n}^+, V)$ are contained in the set

$$\{\mu - \langle Q \rangle : \mu \in \mathcal{X}(V),\ Q \subset \Phi^+,\ |Q| = p\} \,.$$

The problem is to determine the pairs (μ, Q) that actually occur and their multiplicities.

Take $s \in W$ with length p and define $\gamma(s) = \omega_{-Q(s^{-1})} \otimes v_{s\cdot\lambda}$. We begin by showing that $\gamma(s)$ is a p-cocycle whose cohomology class is nonzero. We observe that $\gamma(s)$ has weight $s\cdot\lambda - \langle Q(s^{-1}) \rangle = s\cdot(\lambda+\rho) - \rho$. We claim that

$$C^p(\mathfrak{n}^+, V)(s\cdot(\lambda+\rho)-\rho) = \begin{cases} \mathbb{C}\gamma(s) & \text{if } p = l(s)\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{E.44}$$

To prove this, suppose $C^p(\mathfrak{n}^+, V)(s\cdot(\lambda+\rho)-\rho) \neq 0$. Then from (E.42) we see that there exists a subset $Q \subset \Phi^+$ with $|Q| = p$ and a weight $\mu \in \mathcal{X}(V)$ such that $\mu - \langle Q \rangle = s\cdot(\lambda+\rho) - \rho$. Thus

$$(\mu + \rho - \langle Q \rangle, \mu + \rho - \langle Q \rangle) = (\lambda+\rho, \lambda+\rho) \,.$$

Hence by Lemma E.2.8 there exists $w \in W$ such that $Q = Q(w^{-1})$ and $\mu = w\cdot\lambda$. But $w\cdot(\lambda+\rho) = s\cdot(\lambda+\rho)$, so we conclude that $w = s$ and $Q = Q(s^{-1})$. Since $\dim V(s\cdot\lambda) = 1$, this proves (E.44).

Now set $p = l(s)$. From (E.44) we have $\mathrm{d}\gamma(s) \in C^{p+1}(\mathfrak{n}^+, V)(s\cdot(\lambda+\rho)-\rho) = 0$, so $\gamma(s)$ is a cocycle. Likewise, we have $C^{p-1}(\mathfrak{n}^+, V)(s\cdot(\lambda+\rho)-\rho) = 0$, so $\gamma(s)$ is not a coboundary. Since the weight $s\cdot(\lambda+\rho)-\rho$ uniquely determines s, we have $[\gamma(s)] \neq [\gamma(s')]$ if $s \neq s'$.

It remains to show that the classes $[\gamma(s)]$, for $s \in W$ of length p, give all of $H^p(\mathfrak{n}^+, V)$. For this we will use the Casselman–Osborne formula for the action of the Casimir operator C on the cohomology spaces, together with Kostant's lemma.

Since V is irreducible, $C = \kappa I$ on V for some scalar κ by Schur's lemma. We calculate κ by applying C in the polarized form (E.33) to the highest-weight vector v_λ. This gives

$$\kappa v_\lambda = \sum_i \sigma(H_i^2) v_\lambda + 2\sigma(H_\rho) v_\lambda = \Big(\sum_i \lambda(H_i)^2 + 2\lambda(H_\rho)\Big) v_\lambda \ .$$

Hence $\kappa = \sum_i \lambda(H_i)^2 + 2\lambda(H_\rho) = (\lambda+\rho, \lambda+\rho) - (\rho, \rho)$.

Suppose $\xi \in \mathfrak{h}^*$ and $H^p(\mathfrak{n}^+, V)(\xi) \neq 0$. Then there is a subset $Q \subset \Phi^+$ such that $|Q| = p$ and $\mu \in \mathfrak{h}^*$ such that $\xi = \mu - \langle Q \rangle$. By Corollary E.2.2 we have the identity

$$\theta^{(p)}\Big(\sum_i H_i^2 + 2H_\rho\Big) = (\lambda+\rho, \lambda+\rho) - (\rho, \rho) \tag{E.45}$$

on $H^p(\mathfrak{n}^+, V)$. But we also know that the operator $\sum_i H_i^2 + 2H_\rho$ acts by the scalar $(\mu + \rho - \langle Q \rangle, \mu + \rho - \langle Q \rangle) - (\rho, \rho)$ on $H^p(\mathfrak{n}^+, V)(\xi)$. Hence

$$(\mu + \rho - \langle Q \rangle, \mu + \rho - \langle Q \rangle) = (\lambda+\rho, \lambda+\rho) \ .$$

Lemma E.2.8 asserts that there exists a unique $s \in W$ such that $\mu = s \cdot \lambda$ and $Q = Q(s^{-1})$. Furthermore, $\xi = s \cdot (\lambda+\rho) - \rho$. Hence $H^p(\mathfrak{n}^+, V)(\xi) = \mathbb{C}[\gamma(s)]$ by (E.44). This completes the proof that the classes on the right side of (E.43) give all the cohomology. $\square$

Corollary E.2.10. *As an $\mathfrak{h}$-module,*

$$H^p(\mathfrak{n}^+, V) = \bigoplus_{s \in W,\, l(s) = p} \mathbb{C}_{s\cdot(\lambda+\rho)-\rho} \ ,$$

where $\mathbb{C}_\mu$ denotes the one-dimensional $\mathfrak{h}$-module with weight μ. In particular $H^p(\mathfrak{n}^+, V)$ is multiplicity free.

E.2.6 Cohomology proof of Weyl character formula

Let $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}^+$ be the Borel subalgebra of the semisimple Lie algebra $\mathfrak{g}$ corresponding to the choice Φ^+ of positive roots. Let L be a finite dimensional $\mathfrak{b}$-module, which we assume to be the direct sum of $\mathfrak{h}$ weight spaces $L(\mu)$ with $\mu \in \mathfrak{h}^*$. We define the *formal character* of L to be

$$\mathrm{ch}\, L = \sum_{\mu \in \mathfrak{h}^*} \dim L(\mu)\, e^\mu$$

as an element of the group algebra $\mathcal{A}[\mathfrak{h}^*]$.

We now take V to be the irreducible $\mathfrak{g}$-module with highest weight λ and let $L = H^p(\mathfrak{n}^+, V)$, viewed as a $\mathfrak{h}$-module. Then by Corollary E.2.10

$$\mathrm{ch}\, H^p(\mathfrak{n}^+, V) = \sum_{s\in W,\, l(s)=p} e^{s\cdot(\lambda+\rho)-\rho} \ . \tag{E.46}$$

We define the *Euler characteristic*

$$\chi(V) = \sum_{p=0}^{\dim \mathfrak{n}} (-1)^p \ \mathrm{ch}\, H^p(\mathfrak{n}^+, V) \ .$$

From (E.46) we see that

$$\chi(V) = \sum_{s\in W} \mathrm{sgn}(s)\, e^{s\cdot(\lambda+\rho)-\rho} \ . \tag{E.47}$$

Although the Euler characteristic of V carries less information than the cohomology of V, it can be calculated directly from the cochain complex without using the coboundary operator, as follows:

Lemma E.2.11 (Euler–Poincaré Principle). *The Euler characteristic is the alternating sum*

$$\chi(V) = \sum_{p=0}^{\dim \mathfrak{n}} (-1)^p \ \mathrm{ch}\, C^p(\mathfrak{n}^+, V) \ . \tag{E.48}$$

Proof. We already observed in the proof of Theorem E.2.9 that the $\mathfrak{h}$-weight spaces in the $\mathfrak{n}^+$-cohomology are $H^p(\mathfrak{n}^+, V)(\xi) = Z^p(\mathfrak{n}^+, V)(\xi)/B^p(\mathfrak{n}^+, V)(\xi)$. Hence

$$\dim H^p(\mathfrak{n}^+, V)(\xi) = \dim Z^p(\mathfrak{n}^+, V)(\xi) - \dim B^p(\mathfrak{n}^+, V)(\xi) \ .$$

Furthermore, we have the short exact sequence

$$0 \longrightarrow Z^p(\mathfrak{n}^+, V)(\xi) \longrightarrow C^p(\mathfrak{n}^+, V)(\xi) \xrightarrow{\ \mathrm{d}\ } B^{p+1}(\mathfrak{n}^+, V)(\xi) \longrightarrow 0 \ ,$$

which gives the relation

$$\dim C^p(\mathfrak{n}^+, V)(\xi) = \dim Z^p(\mathfrak{n}^+, V)(\xi) + \dim B^{p+1}(\mathfrak{n}^+, V)(\xi) \ .$$

Because of the dimension shift $p \to p+1$ in this relation, it is clear that

$$\sum_p (-1)^p \dim C^p(\mathfrak{n}^+, V)(\xi) = \sum_p (-1)^p \dim H^p(\mathfrak{n}^+, V)(\xi)$$

for every $\xi \in \mathfrak{h}^*$, which yields (E.48). $\square$

We can now prove the Weyl character formula as a consequence of Kostant's Theorem and the Euler–Poincaré principle. By equation (E.42) we have

$$\operatorname{ch} C^p(\mathfrak{n}^+, V) = \sum_{Q \subset \Phi^+,\, |Q|=p} e^{-\langle Q \rangle} \operatorname{ch} V \ .$$

Taking the alternating sum over p, we obtain

$$\sum_p (-1)^p \operatorname{ch} C^p(\mathfrak{n}^+, V) = \operatorname{ch} V \sum_{Q \subset \Phi^+} (-1)^{|Q|} e^{-\langle Q \rangle} = \operatorname{ch} V \prod_{\alpha \in \Phi^+} (1 - e^{-\alpha}) \ .$$

But by the Euler–Poincaré principle, this is the Euler characteristic (E.47) of V. Multiplying by e^{ρ}, we obtain the Weyl character formula (7.2). □

E.2.7 Exercises

In the following exercises $\mathfrak{g}$ *is a semisimple Lie algebra.*

1. Let $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ with Weyl group $W = \mathfrak{S}_n$ and take the positive roots $\Phi^+ = \{\varepsilon_i - \varepsilon_j : 1 \le i < j \le n\}$. For $s \in \mathfrak{S}_n$ show that the length of s is the number of *inversions* in s (the number of pairs (i, j) such that $s(i) > j$).
2. Let $\{X_j\}$, $\{X^j\}$ be a pair of dual bases for $\mathfrak{g}$ relative to the Killing form B. Let (σ, V) be a $\mathfrak{g}$-module (not necessarily finite dimensional). Define a linear map $\Gamma : C^p(\mathfrak{g}, V) \longrightarrow C^{p-1}(\mathfrak{g}, V)$ by $\Gamma = \sum_j \iota(X_j) \otimes \sigma(X^j)$, and let $C \in U(\mathfrak{g})$ be the Casimir operator defined by B.
(a) Prove that $[\Gamma, \theta(X)] = 0$ for all $X \in \mathfrak{g}$. (HINT: See the proof of Lemma 3.3.7.)
(b) Prove that $\Gamma \mathrm{d} v = \sigma(C) v$ for all $v \in V$.
(c) Prove that $\mathrm{d}\Gamma + \Gamma\mathrm{d}$ commutes with $\iota(X)$ for all $X \in \mathfrak{g}$. (HINT: First show that $\iota(X)$ anticommutes with Γ. Then use (a) and Equation (E.3).)
(d) Prove that $\mathrm{d}\Gamma + \Gamma\mathrm{d} = 1 \otimes \sigma(C)$ on $C^p(\mathfrak{g}, V)$ for all p. (HINT: Use (b), (c), and induction on p.)
(e) Suppose $\sigma(C)$ is invertible on V. Prove that $H^p(\mathfrak{g}, V) = 0$ for all p. (*Hint:* Use (d).)
(f) Suppose V is irreducible with highest weight $\lambda \neq 0$. Prove that $H^p(\mathfrak{g}, V) = 0$ for all p. (HINT: $\sigma(C)$ is the scalar $\kappa = (\lambda, \lambda) + 2(\lambda, \rho)$. Use the dominance of λ to show that $\kappa > 0$ and then apply (e).)
3. Let (π, V) be a finite-dimensional representation of $\mathfrak{g}$. Use the complete reducibility of $\mathfrak{g}$ to write $V = V^{\mathfrak{g}} \oplus V_{\mathfrak{g}}$, where $V_{\mathfrak{g}}$ is the sum of all the nonzero irreducible $\mathfrak{g}$-modules in V. Show that $V_{\mathfrak{g}}$ consists of all vectors of the form $\sum_i \pi(X_i) v_i$, where $X_i \in \mathfrak{g}$ and $v_i \in V$. (HINT: Let C be the Casimir operator for $\mathfrak{g}$. Then C acts by the nonzero scalar $(\lambda + 2\rho, \lambda)$ in the irreducible $\mathfrak{g}$-module with highest weight $\lambda \neq 0$. Hence the operator $\Gamma = C|_{V_{\mathfrak{g}}}$ is invertible. Write $v \in V_{\mathfrak{g}}$ as $\pi(C)\Gamma^{-1} v$.)
4. This exercise gives the cohomology of $\mathfrak{g}$ with trivial coefficients as the $\mathfrak{g}$-invariant exterior forms.
(a) Show that $(\bigwedge^p \mathfrak{g}^*)^{\mathfrak{g}} \subset Z^p(\mathfrak{g})$. (HINT: Use (E.7).)

(b) Use the previous exercise to decompose $Z^p(\mathfrak{g}) = (\bigwedge^p \mathfrak{g}^*)^{\mathfrak{g}} \oplus V_p$, where V_p consists of the p-cocycles that can be written as $\sum_i \theta(X_i)\alpha_i$ for some $\alpha_i \in Z^p(\mathfrak{g}^*)$.
(c) Show that $V_p \subset B^p(\mathfrak{g})$. (HINT: If $\alpha \in Z^p(\mathfrak{g})$ and $X \in \mathfrak{g}$ then $\theta(X)\alpha = d\iota(X)\alpha$.)
(d) Show that $V_p = B^p(\mathfrak{g})$, and hence $H^p(\mathfrak{g}) \cong (\bigwedge^p \mathfrak{g}^*)^{\mathfrak{g}}$. (HINT: If $\beta = d\alpha$, decompose $\alpha = \alpha_0 + \alpha_1$, where α_0 is $\mathfrak{g}$-invariant and $\alpha_1 \in V_{p-1}$. Then $\beta = d\alpha_1$ by part (a). Now use the commutativity of $\theta(X)$ and d to show that $\beta \in V_p$.)
(e) Suppose $\mathfrak{g}$ is a simple Lie algebra. Show that $H^1(\mathfrak{g}) = H^2(\mathfrak{g}) = 0$. (HINT: Use part (d), the fact that $\mathfrak{g}^* \cong \mathfrak{g}$ is an irreducible $\mathfrak{g}$-module, and Theorem 3.2.14.)
(f) Suppose $\mathfrak{g}$ is a simple Lie algebra. Define $\omega(X,Y,Z) = B(X,[Y,Z])$ for $X,Y,Z \in \mathfrak{g}$, where B is the Killing form on $\mathfrak{g}$. Show that $\omega \in \left(\bigwedge^3 \mathfrak{g}^*\right)^{\mathfrak{g}}$ and that $[\omega]$ is a basis for $H^3(\mathfrak{g})$. (HINT: Let $\beta \in \left(\bigwedge^3 \mathfrak{g}^*\right)^{\mathfrak{g}}$ and $X \in \mathfrak{g}$. Then $\iota(X)\beta \in Z^2(\mathfrak{g})$ since $\theta(X)\beta = 0$. Since $H^1(\mathfrak{g}) = 0$, there is $\alpha_X \in \mathfrak{g}^*$ so that $\iota(X)\beta = d\alpha_X$. Show that $\theta(Y)\alpha_X = \alpha_{[Y,X]}$ and that the bilinear form $\langle X, \alpha_Y \rangle$ is $\mathfrak{g}$ invariant. Now use the argument of (e) to conclude that this form is proportional to B.)

E.3 Notes

Section E.1.2. The cohomology of Lie algebras was introduced by Chevalley–Eilenberg [2] as a general setting for the results of Whitehead [6], [7]. For a more detailed introduction to Lie algebra cohomology and its applications to representation theory, see Knapp [3].

Section E.2.1. Theorem E.2.1 is in Casselman–Osborne [1]; the proof here is from Vogan [5].

Section E.2.5. Theorem E.2.9 was first proved by Kostant [4]. The proof in the text takes advantage of Theorem E.2.1.

References for Appendix E

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