Integrable Hamiltonian Systems, Commuting Flows on Flag Manifolds, and the QR Algorithm *

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Themes

(A) Matrix factorization algorithms $\leftrightarrow$ geometry of Lie groups and symmetric spaces:

1. Gaussian factorization with pivots $\leftrightarrow$ cell decomposition of Flag Manifolds
2. QR factorization $\leftrightarrow$ Horospherical coordinates on symmetric space
3. Singular value decomposition $\leftrightarrow$ Polar coordinates on symmetric space

(B) Matrix Factorizations integrate some Hamiltonian dynamical systems on Lie groups.

(C) Geometry of flag manifolds determines the asymptotic behaviour of flows in **(B)** and explains the QR matrix diagonalization algorithm.

Hamiltonian Systems on Lie Groups

- S = Lie group, $\mathbf{s}$ = Lie algebra of S.
- Cotangent bundle $T^*(S) \cong S \times \mathbf{s}^*$
  Smooth left S-invariant functions on $T^*(S) \leftrightarrow C^\infty(\mathbf{s}^*)$
- Lie-Poisson bracket for $f, g \in C^\infty(\mathbf{s}^*)$:

$$\{f, g\}(\xi) = \langle \xi, \, [df_\xi, \, dg_\xi] \rangle, \quad \text{for } \xi \in \mathbf{s}^*$$

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- Hamiltonian vector field H^f on $\mathbf{s}^*$ defined by f**:**

  As a derivation of $C^\infty(\mathbf{s}^*)$
  $$g \mapsto H^f(g) = \{f, g\}$$
  As a field of tangent vectors,
  $$H^f_\xi = df_\xi \cdot \xi \quad (\text{coadjoint action of } \mathbf{s} \text{ on } \mathbf{s}^*)$$
  Consequence: Integral curves for H^f lie in S orbits on $\mathbf{s}^*$.

Geodesic Flow

Take positive definite inner product on $\mathbf{s}^*$ ($\leftrightarrow$ left-invariant Riemannian metric on S). Set $F(\xi) = \frac{1}{2}||\xi||^2$. Then
$$H^F_\xi = \xi^\flat \cdot \xi$$
($\xi \mapsto \xi^\flat \in \mathbf{s}$ by inner product duality).
Let $\mathcal{O} \subset \mathbf{s}^*$ be coadjoint S orbit, $\dim \mathcal{O} = 2r$. Set $F_1 = F$.

Definition: The orbit $\mathcal{O}$ is *completely integrable for the geodesic flow* if $\exists\ F_2, \ldots, F_r$ in $C^\infty(\mathbf{s}^*)$ with
(1) $\{F_i,\, F_j\} = 0$ for $i, j = 1, \ldots, r$
(2) $dF_1 \wedge \ldots \wedge dF_r \neq 0$ generically on $\mathcal{O}$

Main Examples

$G = \mathrm{SL}(n, \mathbf{R})$ – (or any connected real reductive Lie group)
$K = \mathrm{SO}(n)$ – maximal compact subgroup of G.
S = upper triangular $n \times n$ real matrices, positive diagonal, det = 1
$S = AN$:
  A = positive diagonal, det = 1
  N = unipotent upper triangular
P = positive definite $n \times n$ real symmetric matrices, det = 1

Factorizations of G:

$$G = \begin{cases} K \cdot S & \text{(QR)} \\ K \cdot A^+ \cdot K & \text{(Singular value)} \\ K \cdot P & \text{(Polar)} \end{cases}$$

(A^+ = diagonal, positive entries in decreasing order)

Lie algebra decompositions:
$\mathbf{k}$ = skew-symmetric $n \times n$ real matrices
$\mathbf{p}$ = symmetric $n \times n$ real matrices, trace zero
$\mathbf{a}$ = diagonal matrices in $\mathbf{p}$

$\mathbf{n}$ = upper triangular matrices, diagonal zero

(1) $\quad \mathbf{g} = \mathbf{k} + \mathbf{s} \quad$ Projections: $\begin{cases} \pi_{\mathbf{k}} : \mathbf{g} \to \mathbf{k} \\ \pi_{\mathbf{s}} : \mathbf{g} \to \mathbf{s} \end{cases}$

In terms of triangular decomposition $x = x_- + x_0 + x_+$,

$$\pi_{\mathbf{k}}(x) = x_- - x_-^t, \qquad \pi_{\mathbf{s}}(x) = x_0 + x_+ + x_-^t$$

(2) $\quad \mathbf{g} = \mathbf{k} + \mathbf{p} \quad$ ($\mathbf{k} \perp \mathbf{p}$ relative to trace form)

Decompositions **(1)** and **(2)** give linear isomorphism $\psi : \mathbf{s}^* \to \mathbf{p}$ (trace form on $\mathbf{g}$).
Norm on $\mathbf{s}^*$: $\quad ||\xi||^2 = \mathrm{tr}(\psi(\xi)^2)$.
Have linear isomorphism

$$f \;\mapsto\; \psi^*(f), \qquad C^\infty(\mathbf{p}) \;\to\; C^\infty(\mathbf{s}^*).$$

Main Idea: Take Hamiltonians on $\mathbf{s}^*$ corresponding to K-invariant functions on $\mathbf{p}$. These give commuting vector fields on $\mathbf{s}^*$:

Involution Theorem
If $f, g \in C^\infty(\mathbf{p})^K$, then $\{\psi^*(f), \psi^*(g)\} = 0$.

Generators for K-invariant polynomials on $\mathbf{p}$:

$$f_j(x) = \frac{1}{j}\mathrm{tr}(x^j), \qquad j = 2, \ldots, n$$

(number of generators = $\dim A = r$). Set $F_j = \psi^*(f_j)$.

Corollary
Suppose $\mathcal{O} \subset \mathbf{s}^*$ is an S orbit so that
(1) $\quad \dim \mathcal{O} = 2r$
(2) $\quad dF_1 \wedge \ldots \wedge dF_r \neq 0 \quad$ (generically) on $\mathcal{O}$.
Then $\mathcal{O}$ is completely integrable for the geodesic flow.

Problems:

- Find the orbits $\mathcal{O}$ satisfying **(1)** and **(2)**
- Integrate the flows generated by the mutually commuting vector fields $H^{F_1}, \ldots, H^{F_r}$.

Lax Equations

Let $f \in C^\infty(\mathbf{p})^K$.

$$\nabla f = \text{gradient relative to } K\text{-invariant inner product on } \mathbf{p}.$$

Then the vector field $H^{\psi^*(f)}$ on $\mathbf{s}^*$ corresponds to the vector field

$$x \mapsto [x, \pi_{\mathbf{k}}(\nabla f(x))]$$

on $\mathbf{p}$. In particular, the generator for the geodesic flow on $\mathbf{s}^*$ corresponds to the vector field

$$x \mapsto [x, \pi_{\mathbf{k}}(x)]$$

on $\mathbf{p}$.

The **Lax equation** for the geodesic flow is the nonlinear differential equation

$$x'(t) = [x(t), \pi_{\mathbf{k}}(x(t))]$$

($x(t)$ a real symmetric matrix)

Solution to Lax Equation by QR Factorization

For $g \in \mathrm{GL}(n, \mathbf{R})$, can factor $g = Q(g)R(g)$ uniquely (Gram-Schmidt orthogonalization):

- $Q(g)$ is an orthogonal matrix (product of reflections)
- $R(g)$ is upper triangular, positive diagonal
- $g \mapsto Q(g)$ and $g \mapsto R(g)$ are real analytic maps

Lemma 1
Let Y_0 be $n \times n$ real matrix. Set

$$s(t) = R(\exp tY_0), \qquad u(t) = Q(\exp tY_0)$$

for $t \in \mathbf{R}$. Then

$$\begin{aligned} u(t)^{-1}u'(t) &= \pi_{\mathbf{k}}(u(t)^{-1}Y_0u(t)) \\ s'(t)s(t)^{-1} &= \pi_{\mathbf{s}}(u(t)^{-1}Y_0u(t)) \end{aligned}$$

Proof. Differentiate both sides of identity $u(t)s(t) = \exp tY_0$ to get

$$u'(t)s(t) + u(t)s'(t) = Y_0 \exp tY_0 = Y_0u(t)s(t).$$

Hence

$$\underbrace{u(t)^{-1}u'(t)}_{\in \mathbf{k}} = -\underbrace{s'(t)s(t)^{-1}}_{\in \mathbf{s}} + u(t)^{-1}Y_0u(t) \qquad (*)$$

Split

$$u(t)^{-1}Y_0u(t) = \pi_{\mathbf{k}}(u(t)^{-1}Y_0u(t)) + \pi_{\mathbf{s}}(u(t)^{-1}Y_0u(t))$$

and equate $\mathbf{k}$ and $\mathbf{s}$ components in $(*)$. □

Integration Theorem
Let $f \in C^\infty(\mathbf{p})^K$ and fix $X_0 \in \mathbf{p}$. Set

$$Y_0 = \nabla f(X_0), \qquad u(t) = Q(\exp tY_0).$$

Then the integral curve through X_0 corresponding to the flow generated by the Hamiltonian $H^{\psi^*(f)}$ on $\mathbf{s}^*$ is

$$t \;\mapsto\; u(t)^{-1}X_0u(t).$$

Proof. Set $X(t) = u(t)^{-1}X_0u(t)$. Then

$$X'(t) = -u(t)^{-1}u'(t)u(t)^{-1}X_0u(t) + u(t)^{-1}X_0u'(t)$$

Set $Y(t) = u(t)^{-1}Y_0u(t)$. Then Lemma 1 gives

$$X'(t) = -\pi_{\mathbf{k}}(Y(t))X(t) + X(t)\pi_{\mathbf{k}}(Y(t)) = [X(t),\, \pi_{\mathbf{k}}(Y(t))].$$

But f is K-invariant, so

$$Y(t) = u(t)^{-1}\nabla f(X_0)u(t) = \nabla f(X(t)).$$

Hence $X(t)$ satisfies the Lax equation. $\square$

Commuting Flows on p

Set

$$u(\mathbf{t}) = Q\left(\exp \sum_j t_j \nabla f_j(X_0)\right), \qquad \mathbf{t} = [t_1, \dots t_r] \in \mathbf{R}^r$$

(where $\{f_1, \dots, f_r\}$ are generators for the K invariant polynomials on $\mathbf{p}$) Then get r-parameter commuting flow

$$\mathbf{t} \mapsto \mathbf{t} \cdot X_0 = u(\mathbf{t})^{-1}X_0u(\mathbf{t})$$

on $\mathbf{p}$ generated by the Poisson-commuting Hamiltonians $H^{F_1}, \dots, H^{F_r}$ on $\mathbf{s}^*$.

Geometric Description of Flows

- M = centralizer of A in K (= diagonal $\{\pm 1\}$ for $K = \mathrm{SO}(n)$)
- $W = \mathrm{Norm}_K(A)/M$ – Weyl group of G/K (symmetric group S_n)
- $\mathbf{a}^+$ = real diagonal matrices, entries decreasing, trace zero

Diagonalization of real symmetric matrix gives polar coordinates for $\mathbf{p}$:

$$(K/M) \times \mathbf{a}^+ \longrightarrow \mathbf{p}, \qquad (kM, H) \mapsto kHk^{-1} = \mathrm{Ad}(k)H$$

Flag Manifold $= G/B$, where $B = MAN$ (upper triangular matrices)
QR factorization gives a K equivariant diffeomorphism

$$G/B \quad \longrightarrow \quad K/M, \qquad gB \mapsto Q(g)M$$

Transfer left action of A on G/B to left action on K/M by

$$a \cdot kM = Q(ak)M$$

Theorem (Geometric Linearization of Flows)
Let $X_0 = \mathrm{Ad}(k_0)H_0 \in \mathbf{p}$, where $H_0 \in \mathbf{a}^+$. Set

$$a(\mathbf{t}) = \exp\left(\sum t_j \nabla f_j(H_0)\right), \qquad \mathbf{t} \in \mathbf{R}^r$$

Then the r-parameter flow $\mathbf{t} \mapsto \mathbf{t} \cdot X_0$ through X_0 is given by

$$\mathbf{t} \cdot X_0 = \mathrm{Ad}(a(\mathbf{t}) \cdot k_0^{-1})^{-1} H_0$$

Hence the flow is the (nonlinear) action of A on the flag manifold composed with the linear action $\mathrm{Ad}(K) : \mathbf{a} \to \mathbf{p}$.

Proof. If $f \in \mathcal{P}(\mathbf{p})^K$ then $\nabla f(X_0) = \mathrm{Ad}(k_0)(\nabla f(H_0))$. Apply Integration Theorem. □

Corollary
(1) The geodesic flow on $\mathbf{s}^*$ corresponds to the flow of the one-parameter group $a_t = \exp tH_0$ on the flag manifold composed with the linear action of K on $\mathbf{a}$:

$$t \mapsto \mathrm{Ad}(a_t \cdot k_0^{-1})^{-1} H_0 \qquad (t \in \mathbf{R}) \qquad\qquad (**)$$

(2) The behaviour of the geodesic flow as $t \to \infty$ is determined by the orbits of $\{a_t\}$ on the flag manifold.

Geodesic Flow and the QR Iteration

QR Iteration (Francis, 1961): Given (real) matrix B_0, define:

$$Q_1 = Q(B_0), \qquad R_1 = R(B_0), \qquad B_1 = R_1 Q_1$$

(NOTE: Reversed Order of Multiplication)
Continue the iteration:

$$Q_{n+1} = Q(B_n), \quad R_{n+1} = R(B_n), \quad B_{n+1} = R_{n+1} Q_{n+1}$$

Lemma 2 (proof by induction)
(1) $(B_0)^n = (Q_1 \cdots Q_n)(R_n \cdots R_1)$
(2) $B_n = (Q_1 \cdots Q_n)^{-1} B_0 (Q_1 \cdots Q_n)$
Hence B_n is orthogonally equivalent to B_0.

QR Iteration Theorem
Suppose B_0 is symmetric, positive definite.
(1) B_n converges to diagonal matrix B_∞ as $n \to \infty$ (exponential rate)
(2) If B_0 is generic, then the diagonal entries in B_∞ are in decreasing order.

Numerical Example of QR Algorithm (MATLAB)

Generate random 4×4 matrix A to get positive-definite matrix

$$B_0 = I + A^t * A = \begin{bmatrix} 3.4003 & 1.3765 & 1.9490 & 1.3069 \\ 1.3765 & 1.8868 & 1.0608 & 0.5705 \\ 1.9490 & 1.0608 & 2.9702 & 1.1196 \\ 1.3069 & 0.5705 & 1.1196 & 2.0520 \end{bmatrix}$$

Iterate QR algorithm, starting with B_0, and $U_0 = I$:

$$B_{\text{old}} = Q * R \qquad B_{\text{new}} = R * Q \qquad U_{\text{new}} = U_{\text{old}} * Q$$

10 iterations: (zero entries omitted)

$$B_{10} = \begin{bmatrix} 6.6118 & & & \\ & 1.4237 & -0.0039 & -0.0265 \\ & -0.0039 & 1.2687 & -0.0267 \\ & -0.0265 & -0.0267 & 1.0051 \end{bmatrix} \qquad U_{10} = \begin{bmatrix} 0.6460 & 0.1704 & -0.4402 & -0.5998 \\ 0.3593 & 0.6377 & -0.1357 & 0.6677 \\ 0.5637 & -0.1166 & 0.8173 & -0.0258 \\ 0.3685 & -0.7421 & -0.3461 & 0.4401 \end{bmatrix}$$

20 iterations (zero entries omitted):

$$B_{20} = \begin{bmatrix} 6.6118 & & & \\ & 1.4254 & -0.0004 & -0.0008 \\ & -0.0004 & 1.2713 & -0.0025 \\ & -0.0008 & -0.0025 & 1.0008 \end{bmatrix} \qquad U_{20} = \begin{bmatrix} 0.6460 & 0.2114 & -0.3826 & -0.6258 \\ 0.3593 & 0.5977 & -0.1923 & 0.6904 \\ 0.5637 & -0.1243 & 0.8155 & 0.0414 \\ 0.3685 & -0.7632 & -0.3894 & 0.3606 \end{bmatrix}$$

30 iterations (zero entries omitted):

$$B_{30} = \begin{bmatrix} 6.6118 & & & \\ & 1.4254 & -0.0001 & \\ & -0.0001 & 1.2714 & -0.0002 \\ & & -0.0002 & 1.0007 \end{bmatrix} \qquad U_{30} = \begin{bmatrix} 0.6460 & 0.2132 & -0.3769 & -0.6286 \\ 0.3593 & 0.5969 & -0.1970 & 0.6898 \\ 0.5637 & -0.1258 & 0.8149 & 0.0480 \\ 0.3685 & -0.7632 & -0.3937 & 0.3560 \end{bmatrix}$$

39 iterations (zero entries omitted):

$$B_{39} = \begin{bmatrix} 6.6118 & & & \\ & 1.4254 & & \\ & & 1.2714 & \\ & & & 1.0007 \end{bmatrix} \qquad U_{39} = \begin{bmatrix} -0.6460 & -0.2135 & 0.3763 & -0.6288 \\ -0.3593 & -0.5969 & 0.1972 & 0.6897 \\ -0.5637 & 0.1262 & -0.8148 & 0.0486 \\ -0.3685 & 0.7630 & 0.3944 & 0.3556 \end{bmatrix}$$

Note convergence of B to diagonal matrix and convergence of U in flag manifold (can multiply columns by arbitrary choice of ± 1)
Compare with original matrix:

$$U_{39} * B_{39} * U_{39}^t = \begin{bmatrix} 3.4003 & 1.3765 & 1.9490 & 1.3069 \\ 1.3765 & 1.8868 & 1.0608 & 0.5705 \\ 1.9490 & 1.0608 & 2.9702 & 1.1196 \\ 1.3069 & 0.5705 & 1.1196 & 2.0520 \end{bmatrix} = B_0$$

(agreement to 4 decimal places)

Key Observation:
QR iteration is discrete-time version of exponentiated geodesic flow.

- Take $X_0 \in \mathbf{p}$. Geodesic flow is $X(t) = U_t^{-1} X_0 U_t$, with $U_t = Q(\exp tX_0)$.
- Set $T_t = \exp X(t) = U_t^{-1} T_0 U_t$. Apply QR iteration with $B_0 = T_0$. Then by **(1)** of Lemma 2,

  $$\exp(nX_0) = (B_0)^n = (Q_1 \cdots Q_n)(R_n \cdots R_1)$$

  Hence $Q_1 \cdots Q_n = U_n$, so by **(2)** of Lemma 2,

  $$B_n = U_n^{-1} B_0 U_n = T_n$$

Flag Manifold Fixed Point Theorem
Let $X_0 \in \mathbf{p}$. Then $\lim_{t\to+\infty} X(t) = H_0$ is diagonal. For generic X_0, the entries in H_0 are in decreasing order.

Proof. Use formula $(**)$ for geodesic flow and cell decomposition of flag manifold (Gaussian factorization with pivoting):

$$G/B = \bigcup_{w \in W} N^- wB$$

- N^- = lower triangular unipotent matrices, W = Weyl group
- cosets wB are the fixed points for A action
- If $H_0 \in \mathbf{a}^+$ then eigenvalues of $\mathrm{ad}H_0$ on $\mathrm{Lie}(N^-)$ are negative:

  $$\lim_{t\to+\infty} \mathrm{Ad}(\exp tH_0)N^- = \{1\}$$

- If X_0 generic, then $k_0 M$ in open cell $N^- B$ and $w = 1$.

□

Further Results:

- Restrict X_0 to $\psi(\mathcal{O})$, with $\mathcal{O} \subset \mathbf{s}^*$ a coadjoint S orbit.

  Example: (Toda lattice) $\psi(\mathcal{O})$ = symmetric tridiagonal matrices

- Determine *scattering transformation:* There exists $w \in W$ so that

$$X(+\infty) = w \cdot X(-\infty)$$

  Generic case: w = longest element of W ($1 \leftrightarrow n$, $2 \leftrightarrow n-1, \ldots$ when $G = \mathrm{SL}(n, \mathbf{R})$, $K = \mathrm{SO}(n, \mathbf{R})$)

- Determine all the integrable S orbits in $\mathbf{s}^*$ (*minimal orbits)*

  Many Known Examples: Toda lattice orbit, ...(see G-W paper II). Complete classification ??

Some References:

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