Restricted Roots and Weyl Dimension Formula for Spherical Varieties

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G reductive algebraic group over $\mathbb{C}$
$H \subset G$ reductive subgroup ($X = G/H$ affine variety)
$B = TU \subset G$ Borel subgroup (T max. torus, U unipotent)
E_λ irreducible G module with highest weight λ ($\dim E_\lambda^U = 1$)
Definition: (G, H) is a spherical pair if (3 equivalent conditions):

- B has an open orbit on X (BgH open in G for some $g \in G$)
- $\dim E_\lambda^H \leq 1$ for all highest weights λ
- $\mathbb{C}[X] = \bigoplus_{\lambda \in \Gamma(X)} E_\lambda$ (multiplicity-free decomposition)
  where $\Gamma(X)$ = semigroup of H-spherical highest weights

Examples

- Involution θ of G, $H = G^\theta$ (symmetric subgroup),
  X = symmetric space
- Krämer's list: G simple, H nonsymmetric subgroup
  X = weakly symmetric space (Akhiezer-Vinberg)

Assume G simple. Let $\Phi \subset \mathfrak{t}^*$ be the roots of T on $\mathfrak{g}$
$\Phi^+ =$ positive roots (relative to B) $\quad \Delta \subset \Phi^+$ simple roots
$\varpi_1, \ldots, \varpi_\ell$ fundamental highest weights (dual to simple coroots)
The highest weights are $\lambda = n_1\varpi_1 + \cdots + n_\ell\varpi_\ell, \;\; n_i \in \mathbb{N}$
Define $\mathrm{Supp}(\lambda) = \{\varpi_i \,:\, n_i \neq 0\}$

Definition (G, H) is an excellent spherical pair if

- $\Gamma(X)$ is generated by highest weights $\mu_1, \ldots, \mu_r$ $\;(r = \mathrm{rank}\, X)$
- $\mathrm{Supp}(\mu_i) \cap \mathrm{Supp}(\mu_j) = \emptyset$ for $i \neq j$

Examples of Excellent Pairs

- $H = G^\theta$ (symmetric subgroup) — Cartan–Helgason theorem
- 10 of the 12 pairs on Krämer's list (case-by-case verification)
  Rank-one: $(\mathbf{Spin}_7, \mathbf{G}_2)$, $(\mathbf{G}_2, \mathbf{SL}_3)$
  Rank $r > 1$: $(\mathbf{SL}_{p+q}, \mathbf{SL}_p \times \mathbf{SL}_q)$ $(p \neq q)$, $(\mathbf{SL}_{2n+1}, \mathbf{Sp}_{2n})$,
  $(\mathbf{Spin}_{4p+2}, \mathbf{SL}_{2p})$, $(\mathbf{Spin}_{2\ell+1}, \mathbf{GL}_\ell)$, $(\mathbf{Spin}_9, \mathbf{Spin}_7)$,
  $(\mathbf{Spin}_8, \mathbf{G}_2)$, $(\mathbf{Sp}_{2\ell}, \mathbb{C}^\times \times \mathbf{Sp}_{2\ell-2})$, $(\mathbf{E}_6, \mathbf{Spin}_{10})$

Parabolic Subgroup for Excellent Spherical Pair

Assume G simple, simply connected, (G, H) excellent spherical pair
$P \supset B$ – parabolic subgroup for (G, H) (2 equivalent conditions):

- (geometry) $P =$ stabilizer of open B orbit in X
- (harmonic analysis) $P =$ stabilizer of E_λ^U for all $\lambda \in \Gamma(X)$

Structure of P:

- $\mathrm{Lie}(P) = \mathfrak{m} + \mathfrak{a} + \mathfrak{n}, \quad \mathfrak{a} =$ Cartan subspace for X
- Levi subalgebra $\mathfrak{m} + \mathfrak{a}$, nilradical $\mathfrak{n} \subset \mathrm{Lie}(U)$
- $\mathfrak{a} \subset \mathrm{Lie}(T)$, $\dim \mathfrak{a} = r$, $[\mathfrak{a}, \mathfrak{m}] = 0$
- $\mathfrak{m} \cdot E_\lambda^U = 0$ for all $\lambda \in \Gamma(X)$

Let $\Psi =$ roots of $\mathfrak{m}$ (generated by simple roots $\perp \Gamma(X)$)
Definition: $\xi \in \mathfrak{a}^*$ is a restricted root if $\xi = \alpha|_{\mathfrak{a}} \neq 0$ $(\alpha \in \Phi^+)$
root nest $\quad \Phi^+(\xi) = \{\alpha \in \Phi^+ \setminus \Psi : \alpha|_{\mathfrak{a}} = \xi\}$
Let $\Sigma^+ =$ all positive restricted roots ξ, $\quad \mathfrak{n}_\xi = \bigoplus_{\alpha \in \Phi^+(\xi)} \mathfrak{g}_\alpha$
Then $\quad \mathfrak{n} = \bigoplus_{\xi \in \Sigma^+} \mathfrak{n}_\xi \qquad \dim \mathfrak{n}_\xi = m_\xi =$ multiplicity of ξ

For $\lambda \in \Gamma(X)$:

$d(\lambda) = \dim E_\lambda$ (Weyl dimension formula)

$\lambda^* =$ highest weight of dual representation $(E_\lambda)^*$ $(\lambda^* \in \Gamma(X))$

$\mathbf{e}_\lambda =$ U-fixed vector in E_λ, $\mathbf{e}_\lambda^H =$ H-fixed vector in E_λ

Problems:

1. Determine restricted roots Σ and root nests (when X nonsymmetric then Σ is not a root system)
2. Express $d(\lambda)$ in terms of λ and restricted root data (gives explicit Plancherel formula for compact real form of X)
3. Calculate spherical function $\varphi_\lambda(gH) = \langle \mathbf{e}_\lambda^H \,, \, g \cdot \mathbf{e}_{\lambda^*}^H \rangle$ (H-invariant function on X for spherical Fourier transform)
4. Calculate horospherical function $f_\lambda(gH) = \langle \mathbf{e}_\lambda \,, \, g \cdot \mathbf{e}_{\lambda^*}^H \rangle$ (MN-invariant function on X for horospherical Cauchy-Radon transform)

Solution to problems 1, 2:
Take product over each restricted root nest so Weyl's formula is

$$d(\lambda) = \prod_{\xi \in \Sigma^+} d_\xi(\lambda) \quad \text{where} \quad d_\xi(\lambda) = \prod_{\alpha \in \Phi^+(\xi)} \frac{\langle \lambda + \rho_{\mathfrak{g}} \mid \alpha \rangle}{\langle \rho_{\mathfrak{g}} \mid \alpha \rangle}$$

Here $\langle \cdot \mid \cdot \rangle$ = normalized Killing form with shift $2\rho_{\mathfrak{g}} = \sum_{\alpha \in \Phi^+} \alpha$

New Problem: Express $d_\xi(\lambda)$ in terms of restricted root nest data

Identify $\mathfrak{t} = \mathfrak{t}^*$, $\quad \mathfrak{a} = \mathfrak{a}^* \subset \mathfrak{t}$, $\quad \Psi^+$ = positive roots of $\mathfrak{m}'$

shift vectors for $\mathfrak{m}$ and $\mathfrak{n}$: $\quad 2\rho_{\mathfrak{m}} = \sum_{\alpha \in \Psi^+} \alpha$, $\quad 2\delta = \sum_{\xi \in \Sigma^+} m_\xi \, \xi$

(i) If $\alpha \in \Phi^+(\xi)$ then $\langle \lambda \mid \alpha \rangle = \langle \lambda \mid \xi \rangle$, $\langle \delta \mid \alpha \rangle = \langle \delta \mid \xi \rangle$ (easy)

(ii) $\rho_{\mathfrak{g}} = \rho_{\mathfrak{m}} + \delta$ (use classification and diagram symmetries)

Result: (*i*) & (*ii*) $\implies$

$$d_\xi(\lambda) = \prod_{\alpha \in \Phi^+(\xi)} \frac{\langle \lambda + \delta \mid \xi \rangle + \langle \rho_{\mathfrak{m}} \mid \alpha \rangle}{\langle \delta \mid \xi \rangle + \langle \rho_{\mathfrak{m}} \mid \alpha \rangle} \quad (\star)$$

Combine $(\star)$ and root nest information to obtain explicit dimension formula:

Take char. poly. for $\begin{bmatrix} 1/2 & 0 \\ 0 & -1/2 \end{bmatrix}$ in $(2t+1)$-dim. rep. of $\mathfrak{sl}_2$:

$$\varphi(x\,;\,t) = (x-t)(x-t+1)\cdots(x+t-1)(x+t) \quad (t \in \tfrac{1}{2}\mathbb{N})$$

Normalize with shift: $\Phi(x, y\,;\,t) = \varphi(x+y\,;\,t)/\varphi(y\,;\,t)$

For $t = 0$ write $\Phi(x, y) = \Phi(x, y\,;\,0) = (x+y)/y$

Let $\Sigma_0^+ =$ indivisible pos. restricted roots ($c\xi \notin \Sigma_0^+$ for $0 < c < 1$)
For $\xi \in \Sigma_0^+$ we use $\Phi(x, y, t)$ to define Weyl dimension function

$$W(x, y; m_\xi, m_{2\xi}, m_{3\xi})$$

(restricted root multiplicities as parameters)

Special Cases:

$$W(x, y; m_\xi) = W(x, y; m_\xi, 0, 0)$$
$$W(x, y; m_\xi, m_{2\xi}) = W(x, y; m_\xi, m_{2\xi}, 0)$$

Formulas for Weyl Dimension Functions

Let $m = m_\xi$ (ξ indivisible positive restricted root)

- Define $W(x, y, m)$ when $m_{2\xi} = 0$ using $\Phi(x, y; t)$
- When $m_{2\xi} \neq 0$ and X symmetric space, then $m \geq 2$ is even and $m_{2\xi} = 1, 3$, or 7 ($X = F_4/\mathbf{Spin}_9$). Use $\Phi(x, y; t)$ to define $W(x, y\,;\, m, 1)$, $W(x, y\,;\, m, 3)$, $W(x, y\,;\, 8, 7)$
- When X is non-symmetric, rank one and $m_{2\xi} \neq 0$ have $(m_\xi, m_{2\xi}, m_{3\xi}) = (3, 3, 0)$ or $(2, 1, 2)$. Use $\Phi(x, y; t)$ to define $W(x, y\,;\, 3, 3)$ and $W(x, y\,;\, 2, 1, 2)$

Theorem

(1) Assume that X is a symmetric space. Then

$$d(\lambda) = \prod_{\xi \in \Sigma_0^+} W(\langle \lambda \mid \xi \rangle, \langle \delta \mid \xi \rangle \,;\, m_\xi \,, m_{2\xi} \,)$$

(2) Assume $\operatorname{rank} X = 1$ *($\Sigma_0^+ = \{\xi\}$), not symmetric (2 examples). Then*

$$d(\lambda) = W(\langle \lambda \mid \xi \rangle, \langle \delta \mid \xi \rangle \,;\, m_\xi \,, m_{2\xi} \,, m_{3\xi})$$

Remarks

- Dimension determined by root multiplicities and $\langle \lambda \mid \xi \rangle$, $\langle \delta \mid \xi \rangle$
- Formulas use specific normalized Killing form
- If $m_\xi = 1$ for all $\xi \in \Sigma_0^+$, then $m_{2\xi} = 0$ and get Weyl formula

X nonsymmetric, rank > 1 (8 cases) In dimension formula have

- singular Weyl dimension functions (don't occur in rank one): $W_{\mathrm{sing}}(x, y\,;\, m)$, $W_{\mathrm{sing}}(x, y\,;\, m, 1)$ (with m even)
- two types of positive restricted roots: regular and singular
$$\Sigma_0^+ = \Sigma_{\mathrm{reg}}^+ \cup \Sigma_{\mathrm{sing}}^+$$

Theorem

Assume G is simple, simply connected, and (G, H) is an excellent spherical pair with H connected, not symmetric. Then

$$d(\lambda) = \prod_{\xi \in \Sigma_{\mathrm{reg}}^+} W\big(\langle \lambda \mid \xi \rangle, \langle \delta \mid \xi \rangle\,;\, m_\xi\,, m_{2\xi}\,, m_{3\xi}\,\big)$$
$$\times \prod_{\xi \in \Sigma_{\mathrm{sing}}^+} W_{\mathrm{sing}}\big(\langle \lambda \mid \xi \rangle, \langle \delta \mid \xi \rangle\,;\, m_\xi\,, m_{2\xi}\,\big)\,.$$

Remark: Regular and singular roots can have same multiplicities

Proof of Theorems: Calculate shifts $\langle \rho_{\mathfrak{m}} \mid \alpha \rangle$ in dimension factors

$$d_\xi(\lambda) = \prod_{\alpha \in \Phi^+(\xi)} \frac{\langle \lambda \mid \xi \rangle + \langle \delta \mid \xi \rangle + \langle \rho_{\mathfrak{m}} \mid \alpha \rangle}{\langle \delta \mid \xi \rangle + \langle \rho_{\mathfrak{m}} \mid \alpha \rangle} \qquad (\star)$$

Difficulties:

- No rank-one reduction (even in symmetric case)
- Want $d_\xi(\lambda)$ in terms of $\langle \lambda \mid \xi \rangle$, $\langle \delta \mid \xi \rangle$, and root multiplicities

Method: Use principal TDS $\mathfrak{s} \subset \mathfrak{m}'$ with diagonal element $h^0_{\mathfrak{m}}$

- $\mathfrak{n}_\xi$ is an $\mathfrak{s}$ module
- Lowest weight spaces in $\mathfrak{n}_\xi$ for $\mathrm{ad}(h^0_{\mathfrak{m}}) \overset{\mathrm{def}}{\longleftrightarrow}$ basic roots in root nest for ξ
- If $\mathfrak{m}$ is simply-laced, then $h^0_{\mathfrak{m}} = 2\rho_{\mathfrak{m}}$ and shifts $\langle \rho_{\mathfrak{m}} \mid \alpha \rangle \longleftrightarrow$ eigenvalues of $\frac{1}{2}\,\mathrm{ad}(h^0_{\mathfrak{m}})$ on $\mathfrak{n}_\xi$ (determined by basic roots)
- If $\mathfrak{m}$ not simply-laced, then $h^0_{\mathfrak{m}} = 2\rho_{\mathfrak{m}^\vee}$ ($\mathfrak{m}^\vee$ = dual algebra), so have additional shift $\langle \rho_{\mathfrak{m}} - \rho_{\mathfrak{m}^\vee} \mid \alpha \rangle$

rank = 3, $\{\mu_1, \mu_2, \mu_3\}$ = highest weights for the 8-dim. reps.
$\mathfrak{m} = \mathfrak{sl}_2$, $\{\xi_1, \xi_2, \xi_3\}$ = orthonormal basis for $\mathfrak{a}$

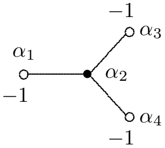
Root Data:

r/s	restricted root ξ	mult.	$\langle \delta \mid \xi \rangle$	# basic roots β	$\langle h^0_{\mathfrak{m}} \mid \beta \rangle$
(s)	$\overline{\alpha_1} = \xi_1 - \xi_2$ $\overline{\alpha_3} = \xi_2 - \xi_3$ $\overline{\alpha_4} = \xi_2 + \xi_3$	2	3/2	1	-1
(r)	$\xi_1 - \xi_3$ $\xi_1 + \xi_3$ $2\xi_2$	1	3	1	0
(s)	$\xi_1 + \xi_2$	2	9/2	1	-1

Marked Satake Diagram:

(α_2 = root of $\mathfrak{m}$)

labels $\langle h^0_{\mathfrak{m}} \mid \alpha_i \rangle$ on simple roots



$d(\lambda)$ formula

(G, H) excellent $\Longrightarrow \mathbb{C}[G]^{MN} = \mathbb{C}[X_0]$
$X_0 =$ affine contraction of $X = G/H$ ($\dim X_0 = \dim X$)
$X_0 \supset G/MN$ quasi-affine, complement in X_0 has codim ≥ 2
$\mathbb{C}[X] \cong \mathbb{C}[X_0]$ as G-modules (not as algebras)

For further investigation:

- Properties of spherical and horospherical functions on X
- Horospherical Cauchy-Radon transform $\mathbb{C}[X] \to \mathbb{C}[X_0]$
  For inversion formula $d(\lambda)$ becomes differential operator $W(D)$ (right $\mathfrak{a}$ action on fibers of $G/MN \longrightarrow G/P$)

References:

- Gindikin–Goodman: Journal of Lie Theory **23** (2013) 257-311 (abridged version in arXiv:1209.3002)
- Avdeev: *Excellent Affine Spherical Homogeneous Spaces of Semisimple Algebraic Groups*, Trans. Moscow Math. Soc. **2010**, 209-240

Appendix: Formulas for Weyl Dimension Functions

- $$W(x,y\,;\,m) = \begin{cases} \Phi(x,y) & \text{if } m=1 \\ \Phi(x,y)^2 & \text{if } m=2 \\ \Phi(x,y\,;\,1) & \text{if } m=3 \\ \Phi(x,y)\,\Phi\big(x,y\,;\,\tfrac{1}{2}m-1\big) & \text{if } m \geq 4 \end{cases}$$

- $$W(x,y\,;\,m,1) = \Phi(x,y)\,\big\{\Phi\big(x,y\,;\,\tfrac{1}{4}m-\tfrac{1}{2}\big)\big\}^2$$

  $$W(x,y\,;\,m,3) = \frac{\Phi\big(x,y\,;\,\tfrac{1}{4}m-\tfrac{1}{2}\big)\,\Phi\big(x,y\,;\,\tfrac{1}{4}m+\tfrac{1}{2}\big)}{\Phi\big(x,y\,;\,\tfrac{1}{2}\big)}\,\Phi(2x,2y\,;\,1)$$

  $$W(x,y\,;8,7) = \Phi(x,y)\,\Phi\big(2x,2y\,;\,\tfrac{3}{2}\big)\,\Phi\big(2x,2y\,;\,\tfrac{9}{2}\big)$$

- $$\begin{cases} W(x,y\,;3,3) & = \Phi(x,y\,;1)\,\Phi(2x,2y\,;1) \\ W(x,y\,;2,1,2) & = \Phi\big(x,y\,;\,\tfrac{1}{2}\big)\,\Phi(2x,2y)\,\Phi\big(3x,3y\,;\,\tfrac{1}{2}\big) \end{cases}$$

- $$\begin{cases} W_{\text{sing}}(x,y\,;\,m) & = \Phi\big(x,y\,;\,\tfrac{1}{2}m-\tfrac{1}{2}\big) \\ W_{\text{sing}}(x,y\,;\,m,1) & = \Phi\big(x,y\,;\,\tfrac{1}{2}m\big) \quad (m \text{ even}) \end{cases}$$

$\mathbf{Spin}_8\,/G_2$ Dimension Formula

If $\lambda = k_1\mu_1 + k_2\mu_2 + k_3\mu_3$ then

$$d(\lambda) = c_1 \prod_{i=1}^{3} \binom{k_i + 2}{2} \prod_{1 \le i < j \le 3} (k_i + k_j + 3) \prod_{j=1}^{2} (k_1 + k_2 + k_3 + j + 3)$$

where $c_1 = 1/(3^3 \cdot 4 \cdot 5)$. Note symmetry in $k_1,\ k_2,\ k_3$ (triality)