Restricted Roots and Weyl Dimension Formula for Spherical Varieties

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CUNY Representation Theory Seminar

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- Krämer's list: G simple, H nonsymmetric subgroup $X =$ weakly symmetric space (Akhiezer-Vinberg)

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The highest weights are $\lambda = n_1\varpi_1 + \cdots + n_\ell\varpi_\ell, \;\; n_i \in \mathbb{N}$
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  Rank $r > 1$: $(\mathbf{SL}_{p+q}, \mathbf{SL}_p \times \mathbf{SL}_q)$ $(p \neq q)$, $(\mathbf{SL}_{2n+1}, \mathbf{Sp}_{2n})$, $(\mathbf{Spin}_{4p+2}, \mathbf{SL}_{2p})$, $(\mathbf{Spin}_{2\ell+1}, \mathbf{GL}_\ell)$, $(\mathbf{Spin}_9, \mathbf{Spin}_7)$, $(\mathbf{Spin}_8, \mathbf{G}_2)$, $(\mathbf{Sp}_{2\ell}, \mathbb{C}^\times \times \mathbf{Sp}_{2\ell-2})$, $(\mathbf{E}_6, \mathbf{Spin}_{10})$

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Structure of P:

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Let $\Sigma^+ =$ all positive restricted roots ξ, $\quad \mathfrak{n}_\xi = \bigoplus_{\alpha \in \Phi^+(\xi)} \mathfrak{g}_\alpha$
Then $\quad \mathfrak{n} = \bigoplus_{\xi \in \Sigma^+} \mathfrak{n}_\xi \qquad \dim \mathfrak{n}_\xi = m_\xi =$ multiplicity of ξ

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$d(\lambda) = \dim E_\lambda$ (Weyl dimension formula)

$\lambda^* =$ highest weight of dual representation $(E_\lambda)^*$ $(\lambda^* \in \Gamma(X))$

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4. Calculate horospherical function $f_\lambda(gH) = \langle \mathbf{e}_\lambda , g \cdot \mathbf{e}_{\lambda^*}^H \rangle$ (MN-invariant function on X for horospherical Cauchy-Radon transform)

Solution to problems 1, 2:

Take product over each restricted root nest so Weyl's formula is

$$d(\lambda) = \prod_{\xi \in \Sigma^+} d_\xi(\lambda) \quad \text{where} \quad d_\xi(\lambda) = \prod_{\alpha \in \Phi^+(\xi)} \frac{\langle \lambda + \rho_{\mathfrak{g}} \mid \alpha \rangle}{\langle \rho_{\mathfrak{g}} \mid \alpha \rangle}$$

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shift vectors for $\mathfrak{m}$ and $\mathfrak{n}$: $\quad 2\rho_{\mathfrak{m}} = \sum_{\alpha \in \Psi^+} \alpha, \quad 2\delta = \sum_{\xi \in \Sigma^+} m_\xi \, \xi$

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Result: (*i*) & (*ii*) $\Longrightarrow$ $d_\xi(\lambda) = \prod_{\alpha \in \Phi^+(\xi)} \frac{\langle \lambda + \delta \mid \xi \rangle + \langle \rho_\mathfrak{m} \mid \alpha \rangle}{\langle \delta \mid \xi \rangle + \langle \rho_\mathfrak{m} \mid \alpha \rangle}$ $(\star)$

Combine $(\star)$ and root nest information to obtain explicit dimension formula:

Take char. poly. for $\begin{bmatrix} 1/2 & 0 \\ 0 & -1/2 \end{bmatrix}$ in $(2t+1)$-dim. rep. of $\mathfrak{sl}_2$:

$$\varphi(x\,;\,t) = (x-t)(x-t+1)\cdots(x+t-1)(x+t) \quad (t \in \tfrac{1}{2}\mathbb{N})$$

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Let $\Sigma_0^+ =$ indivisible pos. restricted roots ($c\xi \notin \Sigma_0^+$ for $0 < c < 1$)
For $\xi \in \Sigma_0^+$ we use $\Phi(x, y, t)$ to define Weyl dimension function

$$W(x, y; m_\xi, m_{2\xi}, m_{3\xi})$$

(restricted root multiplicities as parameters)

Combine $(\star)$ and root nest information to obtain explicit dimension formula:

Take char. poly. for $\begin{bmatrix} 1/2 & 0 \\ 0 & -1/2 \end{bmatrix}$ in $(2t+1)$-dim. rep. of $\mathfrak{sl}_2$:

$$\varphi(x\,;\,t) = (x-t)(x-t+1)\cdots(x+t-1)(x+t) \quad (t \in \tfrac{1}{2}\mathbb{N})$$

Normalize with shift: $\Phi(x, y\,;\,t) = \varphi(x+y\,;\,t)/\varphi(y\,;\,t)$

For $t = 0$ write $\Phi(x, y) = \Phi(x, y\,;\,0) = (x+y)/y$

Let Σ_0^+ = indivisible pos. restricted roots ($c\xi \notin \Sigma_0^+$ for $0 < c < 1$)
For $\xi \in \Sigma_0^+$ we use $\Phi(x, y, t)$ to define Weyl dimension function

$$W(x, y; m_\xi, m_{2\xi}, m_{3\xi})$$

(restricted root multiplicities as parameters)

Special Cases:

$$W(x, y; m_\xi) = W(x, y; m_\xi, 0, 0)$$
$$W(x, y; m_\xi, m_{2\xi}) = W(x, y; m_\xi, m_{2\xi}, 0)$$

Let $m = m_\xi$ (ξ indivisible positive restricted root)

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- When X is non-symmetric, rank one and $m_{2\xi} \neq 0$ have $(m_\xi, m_{2\xi}, m_{3\xi}) = (3, 3, 0)$ or $(2, 1, 2)$. Use $\Phi(x, y; t)$ to define $W(x, y\,;\, 3, 3)$ and $W(x, y\,;\, 2, 1, 2)$

Theorem

(1) Assume that X is a symmetric space. Then

$$d(\lambda) = \prod_{\xi \in \Sigma_0^+} W(\langle \lambda \mid \xi \rangle, \langle \delta \mid \xi \rangle \, ; \, m_\xi \, , m_{2\xi} \,)$$

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X nonsymmetric, rank > 1 (8 cases) In dimension formula have

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Remark: Regular and singular roots can have same multiplicities

Proof of Theorems: Calculate shifts $\langle \rho_{\mathfrak{m}} \mid \alpha \rangle$ in dimension factors

$$d_\xi(\lambda) = \prod_{\alpha \in \Phi^+(\xi)} \frac{\langle \lambda \mid \xi \rangle + \langle \delta \mid \xi \rangle + \langle \rho_{\mathfrak{m}} \mid \alpha \rangle}{\langle \delta \mid \xi \rangle + \langle \rho_{\mathfrak{m}} \mid \alpha \rangle} \qquad (\star)$$

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- If $\mathfrak{m}$ not simply-laced, then $h^0_{\mathfrak{m}} = 2\rho_{\mathfrak{m}^\vee}$ ($\mathfrak{m}^\vee$ = dual algebra), so have additional shift $\langle \rho_{\mathfrak{m}} - \rho_{\mathfrak{m}^\vee} \mid \alpha \rangle$

Example: $\mathbf{Spin}_8 / G_2$

rank $= 3$, $\{\mu_1, \mu_2, \mu_3\}$ = highest weights for the 8-dim. reps.
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r/s	restricted root ξ	mult.	$\langle \delta \mid \xi \rangle$	# basic roots β	$\langle h^0_{\mathfrak{m}} \mid \beta \rangle$
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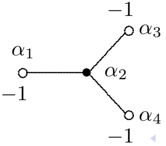
Marked Satake Diagram:

(α_2 = root of $\mathfrak{m}$)

labels $\langle h^0_{\mathfrak{m}} \mid \alpha_i \rangle$ on simple roots



$d(\lambda)$ formula

(G, H) excellent $\Longrightarrow \mathbb{C}[G]^{MN} = \mathbb{C}[X_0]$
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- Properties of spherical and horospherical functions on X

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References:

- Gindikin–Goodman: Journal of Lie Theory **23** (2013) 257-311 (abridged version in arXiv:1209.3002)
- Avdeev: *Excellent Affine Spherical Homogeneous Spaces of Semisimple Algebraic Groups*, Trans. Moscow Math. Soc. **2010**, 209-240

- $$W(x, y\,;\, m) = \begin{cases} \Phi(x, y) & \text{if } m = 1 \\ \Phi(x, y)^2 & \text{if } m = 2 \\ \Phi(x, y\,;\, 1) & \text{if } m = 3 \\ \Phi(x, y)\,\Phi\big(x, y\,;\, \tfrac{1}{2}m - 1\big) & \text{if } m \geq 4 \end{cases}$$

Appendix: Formulas for Weyl Dimension Functions

- $$W(x, y\,;\, m, 1) = \Phi(x, y)\ \left\{\Phi\left(x, y\,;\, \tfrac{1}{4}m - \tfrac{1}{2}\right)\right\}^2$$

  $$W(x, y\,;\, m, 3) = \frac{\Phi\left(x, y\,;\, \tfrac{1}{4}m - \tfrac{1}{2}\right)\Phi\left(x, y\,;\, \tfrac{1}{4}m + \tfrac{1}{2}\right)}{\Phi\left(x, y\,;\, \tfrac{1}{2}\right)}\,\Phi(2x, 2y\,;\, 1)$$

  $$W(x, y\,;\, 8, 7) = \Phi(x, y)\,\Phi\left(2x, 2y\,;\, \tfrac{3}{2}\right)\Phi\left(2x, 2y\,;\, \tfrac{9}{2}\right)$$

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- $$\begin{cases} W(x, y\,;\,3, 3) & = \Phi(x, y\,;\,1)\,\Phi(2x, 2y\,;\,1) \\ W(x, y\,;\,2, 1, 2) & = \Phi\big(x, y\,;\,\tfrac{1}{2}\big)\,\Phi(2x, 2y)\,\Phi\big(3x, 3y\,;\,\tfrac{1}{2}\big) \end{cases}$$

Appendix: Formulas for Weyl Dimension Functions

- $$\begin{cases} W_{\mathrm{sing}}(x, y\,;\ m) & = \Phi\big(x, y\,;\ \tfrac{1}{2}m - \tfrac{1}{2}\big) \\ W_{\mathrm{sing}}(x, y\,;\ m, 1) & = \Phi\big(x, y\,;\ \tfrac{1}{2}m\big) \quad (m \text{ even}) \end{cases}$$

If $\lambda = k_1\mu_1 + k_2\mu_2 + k_3\mu_3$ then

$$d(\lambda) = c_1 \prod_{i=1}^{3} \binom{k_i+2}{2} \prod_{1\le i<j\le 3} (k_i+k_j+3) \prod_{j=1}^{2} (k_1+k_2+k_3+j+3)$$

where $c_1 = 1/(3^3 \cdot 4 \cdot 5)$. Note symmetry in $k_1,\ k_2,\ k_3$ (triality)