Journal of Lie Theory
Volume **23** (2013) 257–311

Restricted Roots and Restricted Form of the Weyl Dimension Formula for Spherical Varieties

Simon Gindikin and Roe Goodman

Communicated by E. B. Vinberg

Abstract. We study in this paper the restricted roots for a class of spherical homogeneous spaces of semisimple groups which includes simply connected symmetric spaces. For these spaces we give a detailed description (case by case) of the set of roots of the group associated with each restricted root of the space (the *nest* of the restricted root). As an application, we obtain a refinement of the Weyl dimension formula in the case of spherical representations, expressing the dimension as a product over the set of indivisible positive restricted roots.
Mathematics Subject Classification 2000: Primary: 14M27; Secondary: 17B10, 20G20, 22E46.
Key Words and Phrases: Spherical variety, symmetric space, restricted root system, Weyl dimension formula.

1. Introduction

In this paper we consider restricted roots for a class of affine spherical homogeneous spaces $X = G/H$, where G is a semisimple group, H is a reductive subgroup, and a Borel subgroup of G has an open orbit on X. All groups are complex linear algebraic groups and all topological notions refer to the Zariski topology. Following [Av2], we assume that X is "excellent" (see Section 3 for the definition; this class of spherical homogeneous spaces was introduced in [VG]). All simply-connected symmetric spaces and all rank-one spaces have this property. When G is simple and H is not the fixed points of an involution of G, we obtain from Krämer's tables [Kr] a relatively short list of excellent affine spherical homogeneous spaces. These spaces exhibit several new phenomena. For example, by contrast with the case of a symmetric space, the restricted roots are not a root system in the usual sense. We want to understand how these restricted roots behave in some problems associated with such spaces. We concentrate in this paper on constructing a restricted version of Weyl's dimension formula, and we obtain a refined version of the Plancherel formula.

We recall that Weyl's dimension formula is the product over a system of positive roots of the group G. It is natural to expect that for spherical

ISSN 0949–5932 / $2.50

representations on X the dimension can be expressed as a product over a system of positive restricted roots of X. However, as far as we know, an explicit dimension formula of this type has not appeared in print for a general symmetric space. The $\mathbf{c}$-function of Harish-Chandra for a non-compact Riemannian symmetric space has such a product formula, and as a result the Plancherel density for such a space also has a product formula. For a symmetric space Vretare [Vr] and Helgason [He2, Ch. III §9.4] showed that Weyl's dimension formula can be obtained as a regularization of $\mathbf{c}$-functions. As a consequence we know that for a symmetric space a version of Weyl's formula for the dimension of a spherical representation as a product over restricted roots exists, but the computation of specific factors corresponding to individual restricted roots is a substantial work.

There is another approach to this problem which we follow in this paper. To each restricted root of X we associate a set of roots of G that we call the "nest" of the restricted root. Then in Weyl's formula we combine the factors from the same nest. We know that in the case of a symmetric space it is convenient to consider together all scalar multiples of a restricted root, say α and 2α, each with a multiplicity. We can associate with this system an "atomic" symmetric space of rank one (which has these roots and multiplicities). In this paper we show that the spherical dimension function (in the symmetric case) is a product of some explicit combinatorial functions, corresponding to the "atomic" symmetric spaces of rank one. These functions for rank one are explicit but not simple. This is similar to the situation for the rank-one factors in the Plancherel formula for a non-compact Riemannian symmetric space, which are quite complicated in contrast to the factors occurring in the $\mathbf{c}$-function [Gi3].

We obtain similar product formulas for non-symmetric excellent affine spherical homogeneous spaces. The focus here is again on the detailed study of the nests of restricted roots (and, as a result, of atomic spaces of rank one). This takes up the major part of the paper. We believe the result can be useful in other problems, such as the horospherical Cauchy transform [Gi1], [Gi2] (cf. [Go]), and that our dimension formula gives a hint as to how a product formula for the $\mathbf{c}$-function for an excellent homogeneous spherical space might look.

There are several new interesting facts that emerge from our investigations. Some of the atomic spaces of rank one are symmetric spaces of rank one, but there are two nonsymmetric spherical rank-one spaces (one of them having restricted roots α, 2α, 3α). Furthermore, there are some "virtual" rank-one spaces: they are not realized as spherical spaces of rank one, but participate as "atomic" spaces in certain excellent affine spherical homogeneous spaces of higher rank.

Here is a brief description of the organization of the paper. The main results concerning dimension formulas are stated in Section 2. Some general results concerning excellent affine spherical spaces and associated parabolic subgroups are established in Sections 3 and 4. With these structural properties of excellent affine spherical pairs established, we turn to the detailed consideration of restricted roots and the dimension formula in Section 5. We introduce a principal $\mathfrak{sl}_2$ subalgebra that plays a key role in determining the shifts in the dimension factors. In the following sections we then work out all the rank-one cases in detail, followed by the higher-rank non-symmetric excellent affine spherical homogeneous spaces, and conclude with the higher-rank symmetric spaces.

Some Notational Conventions.

1. $\mathbb{Z}_+$ is the set of nonnegative integers and $\mathbb{C}^\times$ is the multiplicative group of the field $\mathbb{C}$ of complex numbers.

2. Denote the $n\times n$ matrix $\mathbf{x}$ with diagonal entries $x_i \in \mathbb{C}$ and other entries zero by $\mathrm{diag}[x_1,\dots,x_n]$. Let ε_i be the ith coordinate function on the diagonal matrices, so that $\varepsilon_i(\mathbf{x}) = x_i$. If $\mathbf{x} = \mathrm{diag}[x_1,\dots,x_n]$ and $\mathbf{y} = \mathrm{diag}[y_1,\dots,y_n]$ then $\langle \mathbf{x} \mid \mathbf{y}\rangle = \mathrm{tr}(\mathbf{xy}) = x_1y_1 + \cdots + x_ny_n$.

3. For $\mathbf{x} = [x_1,\dots,x_n]$ with $x_i \in \mathbb{C}$ let $\check{\mathbf{x}} = [x_n,\dots,x_1]$.

4. If V is a complex vector space with dual space $V^* = \mathrm{Hom}(V,\mathbb{C})$, then $\langle\cdot\,,\,\cdot\rangle$ denotes the tautological duality pairing of V^* with V.

5. Lie algebras of algebraic groups are denoted by the corresponding German lower case letters. For an algebraic group L let $\mathbb{C}[L]$ be the algebra of regular functions on L. Let $\mathfrak{X}(L) = \mathrm{Hom}(L,\mathbb{C}^\times)$ be the character group of L (written additively); the value of $\lambda \in \mathfrak{X}(L)$ on $y \in L$ will be denoted by y^λ. If V is an L module, then V^L denotes the subspace of L-fixed vectors.

6. Let G be a semisimple simply-connected algebraic group over $\mathbb{C}$. Fix a choice of Borel subgroup $B \subset G$ and a choice of maximal torus $T \subset B$. Let U be the unipotent radical of B. Then $B = TU$ and $(tu)^\lambda = t^\lambda$ for $t \in T$, $u \in U$, and $\lambda \in \mathfrak{X}(B)$, so we may identify $\mathfrak{X}(B)$ with $\mathfrak{X}(T)$.

7. The set of dominant weights of B is denoted by $\mathfrak{X}_+(B)$. Let $\varpi_1,\dots,\varpi_\ell$ be the fundamental dominant weights, where $\ell = \mathrm{rank}(G)$. Let $\lambda = k_1\varpi_1 + \cdots + k_\ell\varpi_\ell$ with $k_i \in \mathbb{Z}_+$ be a dominant weight. The *support* of λ is the set $\mathrm{Supp}\,\lambda = \{\varpi_i \,:\, k_i > 0\}$.

8. For each $\lambda \in \mathfrak{X}_+(B)$ there is an irreducible finite-dimensional rational G-module E_λ with highest weight λ. The action of $g \in G$ on $\mathbf{x} \in E_\lambda$ is denoted by $g\cdot\mathbf{x}$. Write λ^* for the highest weight of the dual representation $(E_\lambda)^*$. Fix a highest weight vector $\mathbf{e}_\lambda \in E_\lambda$; thus $b\cdot\mathbf{e}_\lambda = b^\lambda\,\mathbf{e}_\lambda$ for $b \in B$.

2. Restricted Weyl Dimension Formula

To state our theorems concerning dimension formulas, we introduce the following functions. For $t = m/2$ with m a nonnegative integer let $\varphi(x\,;\,t)$ be the monic polynomial of degree $2t+1$ in x whose zeros are at $t,\ t-1,\ \dots,\ -t+1,\ -t$. Thus $\varphi(x\,;\,0) = x$ and

$$\varphi(x\,;\,t) = (x-t)(x-t+1)\cdots(x+t-1)(x+t) \tag{1}$$

when $t > 0$. This polynomial arises naturally as the characteristic polynomial for the matrix $\begin{bmatrix} \frac12 & 0 \\ 0 & -\frac12 \end{bmatrix} \in \mathfrak{sl}(2,\mathbb{C})$ in the irreducible representation of dimension $2t+1$. We extend $\varphi(x\,;\,t)$ to be a meromorphic function of x and t by setting $\varphi(x\,;\,t) = \Gamma(x+t+1)/\Gamma(x-t)$. Define

$$\Phi(x,y\,;\,t) = \frac{\varphi(x+y\,;\,t)}{\varphi(y\,;\,t)}\,. \tag{2}$$

This is a meromorphic function of x, y, t that is normalized to satisfy $\Phi(0,y\,;\,t)=1$. We write $\Phi(x,y)=\Phi(x,y\,;\,0)=(x+y)/y$.

The *regular* Weyl dimension functions are defined as follows; here m (the *multiplicity parameter*) is a positive integer subject to the additional conditions indicated.

$$W(x,y\,;\,m)=\begin{cases}\Phi(x,y) & \text{if } m=1\,,\\ \Phi(x,y)^2 & \text{if } m=2\,,\\ \Phi(x,y\,;\,1) & \text{if } m=3\,,\\ \Phi(x,y)\,\Phi\big(x,y\,;\,\tfrac{1}{2}m-1\big) & \text{if } m\geq 4\,,\end{cases}\tag{3}$$

when there is a single multiplicity parameter, and

$$\begin{cases} W(x,y\,;\,m,1)=\Phi(x,y)\,\left\{\Phi\big(x,y\,;\,\tfrac{1}{4}m-\tfrac{1}{2}\big)\right\}^2 \quad \text{if } m\geq 2 \text{ is even}\,,\\ W(x,y\,;\,m,3)=\dfrac{\Phi\big(x,y\,;\,\tfrac{1}{4}m-\tfrac{1}{2}\big)\,\Phi\big(x,y\,;\,\tfrac{1}{4}m+\tfrac{1}{2}\big)}{\Phi\big(x,y\,;\,\tfrac{1}{2}\big)}\,\Phi(2x,2y\,;\,1)\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } m\geq 2 \text{ is even}\,,\\ W(x,y\,;\,8,7)=\Phi(x,y)\,\Phi\big(2x,2y\,;\,\tfrac{3}{2}\big)\,\Phi\big(2x,2y\,;\,\tfrac{9}{2}\big)\,,\end{cases}\tag{4}$$

$$\begin{cases} W(x,y\,;\,3,3)=\Phi(x,y\,;\,1)\,\Phi(2x,2y\,;\,1)\,,\\ W(x,y\,;\,2,1,2)=\Phi\big(x,y\,;\,\tfrac{1}{2}\big)\,\Phi(2x,2y)\,\Phi\big(3x,3y\,;\,\tfrac{1}{2}\big)\,,\end{cases}\tag{5}$$

when there are two or three multiplicity parameters. These functions of x, y occur in the dimension formulas for rank-one affine spherical spaces, with the first parameter the multiplicity of the indivisible restricted root ξ. The second and third parameters are the multiplicity of 2ξ and 3ξ (when these multiplicities are nonzero). The original Weyl dimension formula is expressed in terms of the function $W(x,y\,;\,1)=\Phi(x,y)$, whereas the functions in (5) occur for non-symmetric spherical spaces of rank one.

The *singular* Weyl dimension functions are defined as follows.

$$\begin{cases} W_{\mathrm{sing}}(x,y\,;\,m)=\Phi\big(x,y\,;\,\tfrac{1}{2}m-\tfrac{1}{2}\big)\,,\\ W_{\mathrm{sing}}(x,y\,;\,m,1)=\Phi\big(x,y\,;\,\tfrac{1}{2}m\big) \quad \text{if } m \text{ is even}\,.\end{cases}\tag{6}$$

These functions only occur in the dimension formulas for some excellent non-symmetric spherical homogeneous spaces of rank greater than one.

When a multiplicity parameter is zero, we omit it from the notation; thus we write

$$\begin{aligned} W(x,y\,;\,m,0)&=W(x,y\,;\,m)\,,\\ W(x,y\,;\,m,n,0)&=W(x,y\,;\,m,n)\quad (n=1,3,7)\,,\\ W_{\mathrm{sing}}(x,y\,;\,m,0)&=W_{\mathrm{sing}}(x,y\,;\,m)\,.\end{aligned}$$

With the indicated restrictions on m all these dimension functions are polynomials in x and rational functions of y. They are normalized to take the value 1 when $x=0$.

Assume that G/K is an irreducible simply-connected symmetric space. Fix a Cartan subspace $\mathfrak{a} \subset \mathfrak{p}$, where $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is the Cartan decomposition corresponding to the involution. Then $\mathfrak{a} \subset \mathfrak{t}$ where $\mathfrak{t}$ is the Lie algebra of a maximal torus of $\mathfrak{g}$, and the roots of $\mathfrak{t}$ on $\mathfrak{g}$ can be restricted to $\mathfrak{a}$. Fix a set of positive restricted roots $\Sigma^+ \subset \mathfrak{a}^*$ and let Σ_0^+ be the indivisible positive roots. For $\xi \in \Sigma_0^+$ let m_ξ and $m_{2\xi}$ be the associated root multiplicities, and let

$$\delta = \frac{1}{2} \sum_{\xi \in \Sigma_0^+} (m_\xi + 2m_{2\xi})\xi \, .$$

Let $\langle \lambda \mid \xi \rangle$ be the bilinear form on $\mathfrak{a}^*$ obtained by duality from the restriction to $\mathfrak{a}$ of a positive multiple of the Killing form of $\mathfrak{g}$ (the appropriate normalization of the form is described in Section 5).

Theorem 2.1. *The finite-dimensional irreducible K-spherical representation of G with highest weight λ has dimension*

$$d(\lambda) = \prod_{\xi \in \Sigma_0^+} W(\langle \lambda \mid \xi \rangle, \langle \delta \mid \xi \rangle \, ; \, m_\xi \, , m_{2\xi} \,) \, . \tag{7}$$

For rank-one symmetric spaces we prove Theorem 2.1 in Section 6 using classification and the results from Section 5. Helgason's formula for $d(\lambda)$ in terms of a regularization of ratios of $\mathbf{c}$-functions [He2, Ch. III §9.4] suggests that the result should then hold in higher rank. However, in the rank-one case the multiplicity m_ξ determines the value of $\langle \delta \mid \xi \rangle$; this is not true in rank greater than one, and it seems necessary to do a case-by-case argument for higher rank to obtain the explicit factors corresponding to each positive restricted root. We carry this out in Section 8 using techniques similar to those for rank-one symmetric spaces. By Weyl group symmetry of the restricted root system, however, we only need to consider the simple restricted roots in most cases. The necessary information about the restricted roots is summarized in a *marked Satake diagram* (the Satake diagram as in [He1, Ch. X, Table VI] with additional labels on certain vertices) and a table of root data in each case.

Remark 2.2. If $\xi \in \Sigma_0^+$ then by Cartan's classification of symmetric spaces the only possible values for $m_{2\xi}$ are 0, 1, 3, and 7, and $m_{3\xi} = 0$. Furthermore, when $m_{2\xi} \neq 0$ then m_ξ is even. Thus all the dimension functions in formula (7) are defined in (3) and (4).

For an irreducible simply-connected excellent spherical space that is not symmetric there is an analogue of the subspace $\mathfrak{a}$ that was introduced in [Br], and there is a corresponding set Σ of restricted roots (although this set is not a root system). As a consequence of our dimension formulas we can separate the indivisible positive roots Σ_0^+ into *regular* and *singular* roots:

$$\Sigma_0^+ = \Sigma_{\mathrm{reg}}^+ \cup \Sigma_{\mathrm{sing}}^+ \, .$$

By definition, an indivisible positive restricted root is called *regular* if its dimension function occurs in a rank-one affine spherical space; these functions are given in

(3), (4), and (5). Otherwise, the root is called *singular*. The dimension functions for singular roots are given in (6). With this terminology established, we can state our second main result.

Theorem 2.3. *Assume that G is simple and simply-connected, H is reductive and connected, and G/H is an excellent spherical space that is not symmetric. The finite-dimensional irreducible H-spherical representation of G with highest weight λ has dimension*

$$\begin{aligned} d(\lambda) = \prod_{\xi \in \Sigma^+_{\mathrm{reg}}} & W\big(\langle \lambda \mid \xi \rangle, \langle \delta \mid \xi \rangle\, ;\, m_\xi\,, m_{2\xi}\,, m_{3\xi}\,\big) \\ & \times \prod_{\xi \in \Sigma^+_{\mathrm{sing}}} W_{\mathrm{sing}}\big(\langle \lambda \mid \xi \rangle, \langle \delta \mid \xi \rangle\, ;\, m_\xi\,, m_{2\xi}\,\big)\,. \end{aligned} \tag{8}$$

(If $\operatorname{rank} G/H = 1$ *then* Σ^+_{sing} *is empty.)*

We prove Theorem 2.3 in Section 7 using methods similar to those for symmetric spaces. Determination of the root nests and dimension factors requires more calculation in this case because there is no Weyl group action on the restricted roots. The necessary information about the restricted roots in each case is summarized in a marked Satake diagram (as in the symmetric case) and a table of root data.

Remark 2.4. For a singular root ξ the multiplicity of 2ξ turns out to be either zero or one, and the multiplicity of 3ξ is zero. Thus the dimension functions in (8) are all defined in (3), (4), (5), and (6), with the convention that

$$W\big(\langle \lambda \mid \xi \rangle, \langle \delta \mid \xi \rangle\, ;\, m_\xi\,, m_{2\xi}\,, 0\,\big) = W\big(\langle \lambda \mid \xi \rangle, \langle \delta \mid \xi \rangle\, ;\, m_\xi\,, m_{2\xi}\,\big)\,.$$

Remark 2.5. For non-symmetric excellent affine spherical homogeneous spaces of rank greater than one, the calculations in Sections 7 show that regular and singular restricted roots can have the same multiplicities. Hence the dimension functions for these spaces are not completely determined just by the restricted roots and their multiplicities, unlike the case of rank one spaces or higher rank symmetric spaces.

3. Spherical Pairs

Let G be a simply-connected semi-simple complex algebraic group and H an algebraic subgroup of G. The pair (G, H) (and by extension the homogeneous space G/H and the subgroup $H \subset G$) is called *spherical* if B has an open orbit on the variety G/H. The existence of such an orbit implies that $\dim E^H_\lambda \le 1$ for all $\lambda \in \mathfrak{X}_+(B)$. If G/H is quasi-affine (such a subgroup H is called *observable*), then the converse is true [VK, Theorem 1]. Since all Borel subgroups of G are conjugate, the notion of spherical pair does not depend on the choice of B.

Assume that (G, H) is a spherical pair. If $\lambda \in \mathfrak{X}_+(B)$ and $E^H_{\lambda^*} \neq 0$, then λ will be called an *H-spherical* highest weight and E_λ an *H-spherical* representation (thus $\mathbb{C}[G/H]$ contains E_λ as a submodule in this case). Following [Av1] we let $\Gamma(G/H)$ denote the set of H-spherical highest weights for G. Then $\Gamma(G/H)$ is a subsemigroup of $\mathfrak{X}_+(B)$. If H is reductive, then for $\lambda \in \mathfrak{X}_+(B)$ we have $E_\lambda^H \neq 0$ if and only if $E_{\lambda^*}^H \neq 0$. Hence $\Gamma(G/H)$ is invariant under the map $\lambda \mapsto \lambda^*$ in this case.

The following class of spherical pairs was introduced in [VG] (cf. [Av2]).

Definition 3.1. The spherical pair (G, H) is *excellent* if G/H is quasi-affine and $\Gamma(G/H)$ is generated by $\mu_1, \ldots, \mu_r$ with $\operatorname{Supp}\mu_i \cap \operatorname{Supp}\mu_j = \emptyset$ for $i \neq j$.

When (G, H) is an excellent spherical pair, then the support condition implies that $\{\mu_1, \ldots, \mu_r\}$ is linearly independent and $\Gamma(G/H)$ is a free semigroup. For example, from [He2, Ch. II, Prop. 4.23] one knows that (G, H) is excellent when G is simply-connected and H is any symmetric subgroup of G (the fixed-point group of an involutive automorphism of G). Here G/H is affine because H is reductive.

4. Parabolic Subgroups for Excellent Affine Spherical Pairs

For the rest of the paper we assume that (G, H) is an excellent spherical pair with G simply connected and simple, H connected and reductive (the list of such pairs with H not a symmetric subgroup of G is given in Sections 6 and 7). Fix a Borel subgroup B in G. Let $\mu_1, \ldots, \mu_r$ satisfy the conditions of Definition 3.1. The integer r is the *spherical rank* of the pair (G, H).

For a vector space V let $\mathbb{P}(V)$ be the associated projective space, and denote the canonical map from $V \setminus \{0\}$ to $\mathbb{P}(V)$ by $\mathbf{x} \mapsto [\mathbf{x}] = \mathbb{C}^\times \cdot \mathbf{x}$. Define

$$P = \{g \in G \;:\; [g \cdot \mathbf{e}_{\mu_i}] = [\mathbf{e}_{\mu_i}] \;\text{ for } i = 1, \ldots, r\}\,.$$

Then P is a parabolic subgroup of G since it contains B.

We can describe the structure of P as follows (see, e.g., [Hu, §30.2] and [VP]). Let Φ be the roots of T on $\mathfrak{g}$ and let Φ^+ be the positive roots determined by the Borel subgroup B. Let Δ be the simple roots in Φ^+. For $\alpha \in \Phi^+$ let $h_\alpha \in \mathfrak{t}$ be the coroot to α. There is a unique regular homomorphism $\psi_\alpha : \mathbf{SL}(2, \mathbb{C}) \to G$ whose differential $d\psi_\alpha : \mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{g}$ satisfies

$$d\psi_\alpha \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = h_\alpha\,, \quad d\psi_\alpha \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in \mathfrak{g}_\alpha\,, \quad d\psi_\alpha \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \in \mathfrak{g}_{-\alpha}\,.$$

Write $G^{(\alpha)}$ for the image of ψ_α; this is a closed subgroup of G.

Define

$$\Delta_0 = \{\alpha \in \Delta \;:\; \langle \mu_i\,,\, h_\alpha \rangle = 0 \;\text{ for } i = 1, \ldots, r\;\}\,.$$

Thus Δ_0 consists of the simple roots α such that h_α acts by zero on $\mathbf{e}_\lambda$ for all $\lambda \in \Gamma(G/H)$. By the representation theory of $\mathbf{SL}_2$, one knows that $\alpha \in \Delta_0$ if and only if $G^{(\alpha)}$ fixes $\mathbf{e}_\lambda$ for all $\lambda \in \Gamma(G/H)$.

Viewing the elements of Δ_0 as characters of T, we define

$$C = \Big(\bigcap_{\alpha \in \Delta_0} \mathrm{Ker}(\alpha) \Big)^{\circ} ,$$

where K° denotes the identity component of an algebraic group K. Then C is a subtorus of T, and elements of C commute with $G^{(\alpha)}$ for all $\alpha \in \Delta_0$. The Lie algebra of C is

$$\mathfrak{c} = \{ x \in \mathfrak{t} \,:\, \langle \alpha\,, x \rangle = 0 \text{ for all } \alpha \in \Delta_0 \ \} .$$

Define

$$L = \{ g \in G \,:\, gc = cg \ \text{ for all } c \in C \} . \tag{9}$$

Then L contains the subgroups $G^{(\alpha)}$ for all $\alpha \in \Delta_0$. Since C is a torus, one knows that L is a connected reductive group containing T, and that $P = LN$ (Levi decomposition), where N is the unipotent radical of P. Furthermore, C is the identity component of the center of L. The Lie algebras of L and N are

$$\mathfrak{l} = \mathfrak{c} + \sum_{\alpha \in \Delta_0} \mathbb{C} h_\alpha + \sum_{\beta \in \Psi} \mathfrak{g}_\beta \,, \tag{10}$$

$$\mathfrak{n} = \sum_{\alpha \in \Phi^+ \setminus \Psi} \mathfrak{g}_\alpha \,, \tag{11}$$

where $\Psi = \big(\mathrm{Span}\,\Delta_0\big) \cap \Phi$ is the root system with simple roots Δ_0.

Define M to be the subgroup of L that fixes all the highest weight vectors $\mathbf{e}_{\mu_i}$ for $i = 1, \ldots, r$. Then M is reductive (but not necessarily connected). Let M' be the commutator subgroup of M and let $C_0 = M \cap C$. The Lie algebras are

$$\mathfrak{c}_0 = \{ Y \in \mathfrak{c} \,:\, \langle \mu_i\,, Y \rangle = 0 \ \text{ for } i = 1, \ldots, r \} \,, \tag{12}$$

$$\mathfrak{m}' = \sum_{\alpha \in \Delta_0} \mathbb{C} h_\alpha + \sum_{\beta \in \Psi} \mathfrak{g}_\beta \,, \tag{13}$$

$$\mathfrak{m} = \mathfrak{c}_0 + \mathfrak{m}' . \tag{14}$$

Following [Br], we let $\mathfrak{a}$ be the orthogonal complement to $\mathfrak{c}_0$ in $\mathfrak{c}$ relative to the Killing form on $\mathfrak{t}$. Since the Killing form is positive definite on the real span $\mathfrak{t}_{\mathbb{R}}$ of the coroots, and since $\mathfrak{c}$ and $\mathfrak{c}_0$ are complexifications of real subspaces of $\mathfrak{t}_{\mathbb{R}}$, we have $\mathfrak{c} = \mathfrak{a} \oplus \mathfrak{c}_0$. Thus $\mathfrak{l} = \mathfrak{a} \oplus \mathfrak{m}$ as a Lie algebra and $\mathfrak{l}' = \mathfrak{m}'$. Furthermore, $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ as a vector space and

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n} \quad \text{with} \quad \mathfrak{n}^- = \sum_{\beta \in \Phi^+ \setminus \Psi} \mathfrak{g}_{-\beta} \,. \tag{15}$$

From (15) it follows that

$$\mathfrak{l} = \{ X \in \mathfrak{g} \,:\, [X, \mathfrak{a}] = 0 \} \,. \tag{16}$$

Lemma 4.1. *Let* $d = |\bigcup_{i=1}^{r} \operatorname{Supp} \mu_i|$. *Then* $\dim \mathfrak{c} = d$ *and* $\dim \mathfrak{c}_0 = d - r$. *Hence* $\dim \mathfrak{a} = r$. *In particular, if* $|\operatorname{Supp} \mu_i| = 1$ *for all* i, *then* $\mathfrak{a} = \mathfrak{c}$.

Proof. For $\alpha \in \Delta$ write ϖ_α for the corresponding fundamental weight. Then $\varpi_\alpha \in \operatorname{Supp} \mu_i$ if and only if $\langle \mu_i, h_\alpha \rangle \neq 0$. Thus if we use the Killing form to identify $\mathfrak{t}$ with $\mathfrak{t}^*$, then μ_i is orthogonal to Δ_0 since h_α and α are proportional. By the support condition in Definition 3.1, for each $\alpha \in \Delta$ there exists at most one index i such that $\varpi_\alpha \in \operatorname{Supp} \mu_i$. Hence $|\Delta_0| = \ell - d$ (where $\ell = \dim \mathfrak{t}$ is the rank of $\mathfrak{g}$). This shows that $\dim \mathfrak{c} = d$ since Δ_0 is linearly independent. Furthermore, we see that the set $\Delta_0 \cup \{\mu_1, \ldots, \mu_r\}$ is linearly independent.

Let $h \in \mathfrak{t}$. Then $h \in \mathfrak{c}_0$ if and only if $\langle \alpha, h \rangle = 0$ for all $\alpha \in \Delta_0$ and $\langle \mu_i, h \rangle = 0$ for $i = 1, \ldots, r$. Hence by the linear independence of $\Delta_0 \cup \{\mu_1, \ldots, \mu_r\}$ we conclude that $\dim \mathfrak{c}_0 = \ell - (\ell - d + r) = d - r$. ∎

5. Restricted Roots and Dimension Factors

The spherical subgroup H defines a partition of the root system of $\mathfrak{g}$ as $\Phi = \Psi \cup (\Phi \setminus \Psi)$, where we recall that Ψ is the root system of $\mathfrak{m}$ (these are the roots whose restriction to $\mathfrak{a}$ is zero) and $\Delta_0 = \Psi \cap \Delta$ is a set of simple roots for Ψ. Define the set of *restricted roots* Σ to be the restrictions of the roots in $\Phi \setminus \Psi$ to $\mathfrak{a}$.[1] For $\lambda \in \mathfrak{t}^*$ we write $\overline{\lambda}$ for the restriction of λ to $\mathfrak{a}$.

Let Σ^+ be the set of restrictions to $\mathfrak{a}$ of the roots in $\Phi^+ \setminus \Psi$. For $\xi \in \Sigma^+$ define

$$\Phi^+(\xi) = \{\alpha \in \Phi^+ \setminus \Psi \,:\, \overline{\alpha} = \xi\}. \tag{17}$$

We call $\Phi^+(\xi)$ the *nest* of roots for ξ. We define

$$\mathfrak{n}_\xi = \sum_{\alpha \in \Phi^+(\xi)} \mathfrak{g}_\alpha$$

(the ξ eigenspace of $\mathfrak{a}$ in $\mathfrak{n}$). The *multiplicity* of ξ is

$$m_\xi = \dim \mathfrak{n}_\xi = \left|\Phi^+(\xi)\right|.$$

If $\alpha \in \Phi^+(\xi)$, $\beta \in \Psi$, and $\alpha + \beta \in \Phi$, then $\alpha + \beta \in \Phi^+(\xi)$. Hence the subspace $\mathfrak{n}_\xi$ is invariant under the adjoint action of $\mathfrak{m}$, and there is a decomposition

$$\mathfrak{n} = \bigoplus_{\xi \in \Sigma^+} \mathfrak{n}_\xi$$

as a module relative to the adjoint action of $\mathfrak{l} = \mathfrak{m} \oplus \mathfrak{a}$.

Let $\langle \cdot \mid \cdot \rangle$ be a positive multiple of the Killing form on $\mathfrak{t}$, which we use to identify $\mathfrak{t}$ with $\mathfrak{t}^*$ and to identify $\mathfrak{a}^*$ with a subspace of $\mathfrak{t}^*$. Then $\Psi \perp \mathfrak{a}^*$ and $\mathfrak{a}^* = \operatorname{Span} \Gamma(G/H)$. We normalize this form to make $\langle \alpha \mid \alpha \rangle = 2$ for $\alpha \in \Delta_0$ when these roots all have the same length. When $\mathfrak{m}'$ has roots of two lengths, then it follows by the classification of simple Lie algebras that these roots occur in only one simple ideal, say $\mathfrak{q}$, of $\mathfrak{m}'$. Thus we can normalize the form so that when

[1] When H is not a symmetric subgroup of G, the set Σ is usually not a root system in $\mathfrak{a}^*_{\mathbb{R}}$.

$\mathfrak{q}$ is of type B (*resp.* type C) then the long roots (*resp.* short roots) in Δ_0 have squared length two. Given $\alpha \in \Phi$, we write

$$\alpha^\vee = \frac{2}{\langle \alpha \mid \alpha \rangle}\alpha$$

for the coroot to α. Define

$$\rho_{\mathfrak{g}} = \frac{1}{2}\sum_{\alpha\in\Phi^+} \alpha\,, \qquad \rho_{\mathfrak{m}} = \frac{1}{2}\sum_{\beta\in\Psi^+} \beta\,, \qquad \delta = \frac{1}{2}\sum_{\xi\in\Sigma^+} m_\xi\,\xi\,.$$

Then $\delta \perp \rho_{\mathfrak{m}}$. Since $\rho_{\mathfrak{m}}$ is the sum of the fundamental highest weights of $\mathfrak{m}$, we have

$$\langle \rho_{\mathfrak{m}} \mid \alpha^\vee \rangle = 1 \quad \text{for all } \alpha \in \Delta_0\,. \tag{18}$$

Lemma 5.1. *There is an orthogonal decomposition*

$$\rho_{\mathfrak{g}} = \delta + \rho_{\mathfrak{m}}\,. \tag{19}$$

Proof. From the definitions it is clear that the restriction of $\rho_{\mathfrak{g}} - \rho_{\mathfrak{m}}$ to $\mathfrak{a}$ coincides with δ. Since $\delta \in \mathfrak{a}^*$ it suffices to prove that

$$\rho_{\mathfrak{g}} - \rho_{\mathfrak{m}} \in \mathfrak{a}^*\,. \tag{20}$$

To establish (20), we note from (18) that $\langle \rho_{\mathfrak{g}} - \rho_{\mathfrak{m}} \mid \alpha^\vee \rangle = 0$ for all $\alpha \in \Delta_0$, since $\langle \rho_{\mathfrak{g}} \mid \alpha^\vee \rangle = 1$ for all $\alpha \in \Delta$. Hence $\rho_{\mathfrak{g}} - \rho_{\mathfrak{m}} \in \mathfrak{c}^*$. So we must show that $\rho_{\mathfrak{g}} - \rho_{\mathfrak{m}} \perp \mathfrak{c}_0$. Since $\rho_{\mathfrak{m}} \perp \mathfrak{c}_0$, it only remains to show that $\rho_{\mathfrak{g}} \perp \mathfrak{c}_0$.

From Lemma 4.1 and classification we find that $\mathfrak{c}_0 \neq 0$ only for the non-symmetric pair $(\mathbf{SL}_{p+q}\,, \mathbf{SL}_p \times \mathbf{SL}_q)$ and for the symmetric spaces of types **A III**, **A IV**, **D I** ($\ell = r+1$), **D III** ($\ell = 2r+1$), **E II**, and **E III**. These are exactly the cases in which $|\operatorname{Supp}\mu_i| = 2$ for some i. In all these cases there is a non-trivial symmetry of the Dynkin diagram that comes from an automorphism σ of $\mathfrak{g}$ that preserves $\mathfrak{t}$, the Killing form, the set of positive roots, the set Δ_0, and the set $\{\mu_1, \ldots, \mu_r\}$. Hence σ fixes $\rho_{\mathfrak{g}}$. We calculate in each case that σ acts by -1 on $\mathfrak{c}_0$ (see Section 6, Case 1; Section 7, Case 1; Section 8, Cases 2, 5, 6, 7, 8). Hence $\langle \rho_{\mathfrak{g}} \mid X \rangle = -\langle \rho_{\mathfrak{g}} \mid X \rangle$ for $X \in \mathfrak{c}_0$. This proves that $\rho_{\mathfrak{g}} \perp \mathfrak{c}_0$. ∎

Let $\lambda \in \Gamma(G/H)$. For $\xi \in \Sigma^+$ define

$$d_\xi(\lambda) = \prod_{\alpha\in\Phi^+(\xi)} \frac{\langle \lambda + \rho_{\mathfrak{g}} \mid \alpha \rangle}{\langle \rho_{\mathfrak{g}} \mid \alpha \rangle}\,.$$

Let $d(\lambda)$ be the dimension of the irreducible G-module E_λ. Then the Weyl dimension formula gives

$$d(\lambda) = \prod_{\xi\in\Sigma^+} d_\xi(\lambda)\,. \tag{21}$$

For $\alpha \in \Phi^+(\xi)$ we have $\langle \lambda \mid \alpha \rangle = \langle \lambda \mid \xi \rangle$. Hence by (19) we can write

$$d_\xi(\lambda) = \prod_{\alpha\in\Phi^+(\xi)} \frac{\langle \lambda + \delta \mid \xi \rangle + \langle \rho_{\mathfrak{m}} \mid \alpha \rangle}{\langle \delta \mid \xi \rangle + \langle \rho_{\mathfrak{m}} \mid \alpha \rangle}\,. \tag{22}$$

We now determine the shifts $\langle \rho_{\mathfrak{m}} \mid \alpha \rangle$ in formula (22) using the representations of $\mathfrak{sl}_2$. Let $h^0_{\mathfrak{m}}$ be the element of $\mathrm{Span}\{h_\alpha : \alpha \in \Delta_0\}$ such that

$$\langle h^0_{\mathfrak{m}} \mid \alpha \rangle = 2 \quad \text{for all } \alpha \in \Delta_0 .$$

Then $h^0_{\mathfrak{m}}$ is a regular element in $\mathfrak{m}'$, and there exist elements $e^0_{\mathfrak{m}}$ and $f^0_{\mathfrak{m}}$ in $\mathfrak{m}'$ such that $\{e^0_{\mathfrak{m}}, f^0_{\mathfrak{m}}, h^0_{\mathfrak{m}}\}$ is a principal $\mathfrak{sl}_2$ triple in $\mathfrak{m}'$ (see [Ko] and [Bo2, Ch. VIII, §11]). Denote the span of these elements by $\mathfrak{s}$ and let $S \subset M$ be the connected subgroup with Lie algebra $\mathfrak{s}$.

Suppose $g \in G$ and $\mathrm{Ad}(g)\mathfrak{a} = \mathfrak{a}$. If ξ is a restricted root, then $g \cdot \xi$ is also a restricted root, where $g \cdot \xi \in \mathfrak{a}^*$ is defined by $\langle g \cdot \xi \mid x \rangle = \langle \xi \mid \mathrm{Ad}(g)^{-1} x \rangle$ for $x \in \mathfrak{a}$.

Lemma 5.2. *If $g \in G$ and $\mathrm{Ad}(g)$ preserves $\mathfrak{a}$, then the restricted root spaces $\mathfrak{n}_\xi$ and $\mathfrak{n}_{g\cdot\xi}$ are isomorphic as $\mathfrak{s}$-modules.*

Proof. Since $\mathfrak{l}$ is the zero weight space of $\mathrm{Ad}(\mathfrak{a})$ in $\mathfrak{g}$ by (16), we have $\mathrm{Ad}(g)\mathfrak{l} = \mathfrak{l}$. Hence $\mathrm{Ad}(g)\mathfrak{m}' = \mathfrak{m}'$ since $\mathfrak{m}' = \mathfrak{l}'$. Now $\mathrm{Ad}(g)\mathfrak{s}$ is a principal $\mathfrak{sl}_2$ subalgebra of $\mathfrak{m}'$, so there exists $m \in M'$ such that $\mathrm{Ad}(mg)\mathfrak{s} = \mathfrak{s}$ ([Bo2, Ch. VIII, §11, Prop. 9]). Since $m \cdot \xi = \xi$ we may replace g by mg and assume that $\mathrm{Ad}(g)\mathfrak{s} = \mathfrak{s}$. But all automorphisms of $\mathfrak{sl}_2$ come from its adjoint group, so there exists $s \in S$ such that $\mathrm{Ad}(s)|_{\mathfrak{s}} = \mathrm{Ad}(g)|_{\mathfrak{s}}$. Then $\mathrm{Ad}(s^{-1}g)|_{\mathfrak{s}}$ is the identity, $s \cdot \xi = \xi$, and $\mathrm{Ad}(s^{-1}g) : \mathfrak{n}_\xi \to \mathfrak{n}_{g\cdot\xi}$ gives an isomorphism of $\mathfrak{s}$ modules. ■

Let $\xi \in \Sigma^+$ be a restricted positive root. Define

$$k_\xi = \max\{k : \ k \text{ is an eigenvalue of } \mathrm{ad}\, h^0_{\mathfrak{m}} \text{ on } \mathfrak{n}_\xi \} .$$

By the representation theory of $\mathfrak{sl}_2$ we know that k_ξ is a non-negative integer and

$$k_\xi = -\min\{\langle h^0_{\mathfrak{m}} \mid \alpha \rangle : \alpha \in \Phi^+(\xi)\} . \tag{23}$$

Definition 5.3. A root $\alpha \in \Phi^+(\xi)$ is a *basic root* if α gives the minimum value in (23).

Let $\Phi(x, y\,;\,t)$ be the function defined in (2); recall that we write $\Phi(x, y) = \Phi(x, y\,;\,0)$.

Proposition 5.4. *The eigenvalues of $\mathrm{ad}\, h^0_{\mathfrak{m}}$ on $\mathfrak{n}_\xi$ are integers between $-k_\xi$ and k_ξ. They include $-k_\xi, -k_\xi+2, \ldots, k_\xi-2, k_\xi$. In particular, $\dim \mathfrak{n}_\xi \geq k_\xi + 1$.*

Let $b = \frac{1}{2}k_\xi$, let $\lambda \in \Gamma(G/H)$, and assume all roots in Δ_0 have the same length. Then the following hold.

1. *The shifts $\langle \rho_{\mathfrak{m}} \mid \alpha \rangle$ in (22) are the eigenvalues (with multiplicities) of $\frac{1}{2}\,\mathrm{ad}\, h^0_{\mathfrak{m}}$ on $\mathfrak{n}_\xi$. In particular, the shifts include $-b, -b+1, \ldots, b-1, b$.*

2. *Suppose $b = 0$. Then $d_\xi(\lambda) = [\Phi(\langle \lambda \mid \xi \rangle, \langle \delta \mid \xi \rangle)]^{m_\xi}$.*

3. *Suppose* $b > 0$, $m_\xi = (2b+1)p$ *for some integer* $p \geq 1$, *and* $\Phi^+(\xi)$ *has* p *basic roots. Then* $d_\xi(\lambda) = [\Phi(\langle \lambda \mid \xi \rangle, \langle \delta \mid \xi \rangle; b)]^p$.

4. *Suppose* $b > 0$ *and* $m_\xi = 2b+2$. *Then*

$$d_\xi(\lambda) = \Phi(\langle \lambda \mid \xi \rangle, \langle \delta \mid \xi \rangle)\, \Phi(\langle \lambda \mid \xi \rangle, \langle \delta \mid \xi \rangle; b).$$

Proof. The statement about eigenvalues follows from the invariance of $\mathfrak{n}_\xi$ under $\{e^0_{\mathfrak{m}}, f^0_{\mathfrak{m}}, h^0_{\mathfrak{m}}\}$ and the representation theory of $\mathfrak{sl}_2$.

Now assume that all roots in Δ_0 have the same length. Then $\rho_{\mathfrak{m}} = \frac{1}{2}h^0_{\mathfrak{m}}$, so assertion (1) in the proposition follows from the eigenvalue property just proved. If $b = 0$, then all eigenvalues of $\operatorname{ad} h^0_{\mathfrak{m}}$ on $\mathfrak{n}_\xi$ are zero, and hence the shifts are all zero, proving assertion (2). When there are p basic roots in $\Phi^+(\xi)$ and $m_\xi = p(k_\xi + 1)$, then each basic root vector is a lowest weight vector. Thus $\mathfrak{n}_\xi$ is the sum of p copies of the irreducible representation of $\{e^0_{\mathfrak{m}}, f^0_{\mathfrak{m}}, h^0_{\mathfrak{m}}\}$ of dimension $k_\xi + 1$, and hence the shifts are precisely $-b, -b+1, \ldots, b-1, b$ with multiplicity p, proving (3). Finally, when $m_\xi = k_\xi + 2$, then $\mathfrak{n}_\xi$ must be the sum of the irreducible representation of $\{e^0_{\mathfrak{m}}, f^0_{\mathfrak{m}}, h^0_{\mathfrak{m}}\}$ of dimension $k_\xi + 1$ and the trivial representation. Hence the shifts are $-b, -b+1, \ldots, b-1, b$ together with zero, proving (4). ■

Let W_G be the Weyl group of G.

Lemma 5.5. *Suppose* $\xi = \overline{\alpha}$ *and* $\eta = \overline{\beta}$ *are restricted roots and that* α *and* β *are the unique basic roots in the nests* $\Phi^+(\xi)$ *and* $\Phi^+(\eta)$. *Assume that there exists* $w \in W_G$ *such that* $w\alpha = \beta$ *and* $w\Delta_0 = \Delta_0$. *Then* $w\Phi^+(\xi) = \Phi^+(\eta)$ *and* $\langle \rho_{\mathfrak{m}} \mid w\mu \rangle = \langle \rho_{\mathfrak{m}} \mid \mu \rangle$ *for all* $\mu \in \Phi^+(\xi)$.

Proof. Since α is the unique basic root, $\Phi^+(\xi)$ consists of all elements of Φ^+ of the form $\alpha + \sum_{\gamma \in \Delta_0} m_\gamma \gamma$, where $m_\gamma \geq 0$. A similar statement holds for $\Phi^+(\eta)$. Since w permutes the elements of Δ_0 and maps roots to roots, it is clear that $w\Phi^+(\xi) = \Phi^+(\eta)$ and $w\rho_{\mathfrak{m}} = \rho_{\mathfrak{m}}$. Thus $\langle \rho_{\mathfrak{m}} \mid w\mu \rangle = \langle w\rho_{\mathfrak{m}} \mid w\mu \rangle = \langle \rho_{\mathfrak{m}} \mid \mu \rangle$ for all $\mu \in \Phi^+(\xi)$. ■

When Δ_0 has two root lengths, then the shifts $\langle \rho_{\mathfrak{m}} \mid \alpha \rangle$ in the dimension formula cannot be determined just using $h^0_{\mathfrak{m}}$. Define

$$\varpi^0_{\mathfrak{m}} = \rho_{\mathfrak{m}} - \frac{1}{2} h^0_{\mathfrak{m}}.$$

Then $\varpi^0_{\mathfrak{m}} \in \operatorname{Span} \Delta_0$ and the shifts are

$$\langle \rho_{\mathfrak{m}} \mid \alpha \rangle = \frac{1}{2}\langle h^0_{\mathfrak{m}} \mid \alpha \rangle + \langle \varpi^0_{\mathfrak{m}} \mid \alpha \rangle. \tag{24}$$

From (18) we have

$$\langle \varpi^0_{\mathfrak{m}} \mid \alpha \rangle = \frac{1}{2}\langle \alpha \mid \alpha \rangle - 1.$$

If $\alpha = \alpha^\vee$ for all $\alpha \in \Delta_0$, then $\varpi^0_{\mathfrak{m}} = 0$ and $h^0_{\mathfrak{m}} = 2\rho_{\mathfrak{m}}$.

When there are two root lengths and $\mathfrak{m}$ contains a simple ideal whose Dynkin diagram is of type C, then from (24) and our normalization of the Killing form we calculate that

$$\langle\, \varpi^0_{\mathfrak{m}} \mid \alpha \,\rangle = \begin{cases} 0 & \text{for all short roots } \alpha \in \Delta_0\,, \\ 1 & \text{for the long root } \alpha \in \Delta_0\,. \end{cases} \tag{25}$$

In this case $2\varpi^0_{\mathfrak{m}}$ is the fundamental dominant weight of $\mathfrak{m}'$ associated with the long simple root. Likewise, when $\mathfrak{m}$ has a simple ideal whose Dynkin diagram is of type B, then we calculate that

$$\langle\, \varpi^0_{\mathfrak{m}} \mid \alpha \,\rangle = \begin{cases} 0 & \text{for all long roots } \alpha \in \Delta_0\,, \\ -1/2 & \text{for the short root } \alpha \in \Delta_0\,. \end{cases} \tag{26}$$

In this case $-\varpi^0_{\mathfrak{m}}$ is the fundamental dominant weight of $\mathfrak{m}'$ associated with the short simple root (the highest weight of the spin representation of $\mathfrak{m}'$).

Remark 5.6. The two situations just described suffice, since the Dynkin diagrams of type F_4 and G_2 do not occur as subdiagrams of connected Dynkin diagrams.

Proposition 5.7. *If $\xi \in \Sigma^+$ and $\dim \mathfrak{n}_\xi = 1$, then $d_\xi(\lambda) = \Phi(\langle\, \lambda \mid \xi \,\rangle\,, \langle\, \delta \mid \xi \,\rangle)$ for all $\lambda \in \Gamma(G/H)$.*

Proof. Let $\Phi^+(\xi) = \{\alpha\}$. Since $\mathfrak{n}_\xi$ is a trivial $\mathfrak{m}'$ module, we have $\langle\, h_\beta \mid \alpha \,\rangle = 0$ for all $\beta \in \Delta_0$. Hence $\langle\, h^0_{\mathfrak{m}} \mid \alpha \,\rangle = 0$ and $\langle\, \varpi^0_{\mathfrak{m}} \mid \alpha \,\rangle = 0$. Thus by (24) the $\rho_{\mathfrak{m}}$ shift in the dimension formula is zero. ∎

We create the *marked Satake diagram* for the pair (G, H) as follows.

(i) In the Dynkin diagram for Δ indicate the vertices corresponding to elements of Δ_0 by $\bullet$ and indicate the vertices corresponding to the other simple roots by $\circ$. Join vertices corresponding to roots with the same restriction to $\mathfrak{a}$ by a double-pointed arrow.

(ii) When all roots in Δ_0 have the same length and $\alpha \in \Delta \setminus \Delta_0$ is adjacent to an element of Δ_0, put the number $\langle\, h^0_{\mathfrak{m}} \mid \alpha \,\rangle$ at the vertex for α.

(iii) When the roots in Δ_0 have two lengths and $\alpha \in \Delta \setminus \Delta_0$ is adjacent to an element of Δ_0, put the pair of numbers $(\langle\, h^0_{\mathfrak{m}} \mid \alpha \,\rangle\,, \langle\, \varpi_{\mathfrak{m}} \mid \alpha \,\rangle)$ at the vertex for α.

Here *adjacent* refers to the corresponding vertices and edges in the Dynkin diagram for the root system of $\mathfrak{g}$.

Remark 5.8. Let $\alpha \in \Delta \setminus \Delta_0$. Since $h^0_{\mathfrak{m}}$ is a linear combination with positive coefficients of the simple coroots of $\mathfrak{m}$, we have $\langle\, h^0_{\mathfrak{m}} \mid \alpha \,\rangle < 0$ if α is adjacent to Δ_0. Furthermore, $\langle\, h^0_{\mathfrak{m}} \mid \alpha \,\rangle = \langle\, \varpi^0_{\mathfrak{m}} \mid \alpha \,\rangle = 0$ if α is not adjacent to Δ_0. Thus the marked Satake diagram and formulas (25) and (26) determine the values $\langle\, h^0_{\mathfrak{m}} \mid \alpha \,\rangle$ and $\langle\, \varpi^0_{\mathfrak{m}} \mid \alpha \,\rangle$ for all $\alpha \in \Delta$.

6. Rank-One Affine Spherical Spaces

From É. Cartan's classification (see [He1, Ch. X Table VI]) the irreducible affine spherical pairs (G, H) of rank one with G semisimple and H connected and symmetric (the fixed points of an involution of G) are as follows (in types **B II** and **D II** the spherical representations are single-valued on the orthogonal groups, so the spin groups are not needed).

A IV: $G = \mathbf{SL}_{\ell+1}(\mathbb{C})$ with $\ell \geq 1$ and $H = \mathbf{GL}_\ell(\mathbb{C})$ embedded by $h \mapsto h \oplus \det h^{-1}$.

B II: $G = \mathbf{SO}_{2\ell+1}(\mathbb{C})$ with $\ell \geq 2$ and $H = \mathbf{SO}_{2\ell}(\mathbb{C})$ embedded by $h \mapsto h \oplus 1$.

D II: $G = \mathbf{SO}_{2\ell}(\mathbb{C})$ with $\ell \geq 2$ and $H = \mathbf{SO}_{2\ell-1}(\mathbb{C})$ embedded by $h \mapsto h \oplus 1$.

C II: $G = \mathbf{Sp}_{2\ell}(\mathbb{C})$ with $\ell \geq 3$ and $H = \mathbf{Sp}_2(\mathbb{C}) \times \mathbf{Sp}_{2\ell-2}(\mathbb{C})$ embedded in block-diagonal form.

F II: $G = \mathbf{F}_4(\mathbb{C})$ and $H = \mathbf{Spin}_9(\mathbb{C})$ embedded as described in [Ba, §4.2].

Type **B II** with $\ell = 1$ is isomorphic to Type **A IV** with $\ell = 1$, and Type **C II** with $\ell = 2$ is isomorphic to Type **B II** with $\ell = 2$, and so these are omitted from this list.

From Krämer's classification [Kr] there are two irreducible affine spherical pairs (G, H) of rank one with G simple and H not a symmetric subgroup of G. The groups involved form a descending chain

$$\mathbf{Spin}_7(\mathbb{C}) \supset \mathbf{G}_2(\mathbb{C}) \supset \mathbf{SL}_3(\mathbb{C})$$

and are of dimensions 21, 14, 8 and ranks 3, 2, 2 respectively. The embeddings of the groups are described in [Ad, Ch. 5] and [Wo1, §8.10]. The compact forms of the corresponding homogeneous spaces G/H are constant positive curvature spheres of dimensions 7 and 6 (cf. [Wo2, §12.7]).

Case 1. **The pair $(\mathbf{SL}_{\ell+1}, \mathbf{GL}_\ell)$.** Here G has rank ℓ. We take the diagonal matrices in $\mathfrak{g}$ as a Cartan subalgebra and use simple roots $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $i = 1, \ldots, \ell$. The fundamental H-spherical weight is $\mu_1 = \varpi_1 + \varpi_\ell$ (see [GW, Ch. 12.3.3]). Assume for the moment that $\ell \geq 2$. If $\ell = 2$ then Δ_0 is empty, while if $\ell \geq 3$ then $\Delta_0 = \{\alpha_2, \ldots, \alpha_{\ell-1}\}$; in both cases $\dim \mathfrak{c} = 2$. Since $|\operatorname{Supp} \mu_1| = 2$, we have $\dim \mathfrak{c}_0 = 1$ and $\dim \mathfrak{a} = 1$ by Lemma 4.1. Thus $\mathfrak{m} \cong \mathfrak{c}_0 \oplus \mathfrak{sl}_{\ell-1}$.

To determine $\mathfrak{c}_0$, we identify $\mathfrak{t}$ with $\mathfrak{t}^*$ using the form $\langle \cdot \mid \cdot \rangle$ and write $\mathbf{x} \in \mathfrak{t}$ as $c_1\alpha_1 + \cdots + c_\ell\alpha_\ell$. Then $\langle \mu_1 \mid \mathbf{x} \rangle = 0$ gives the relation $c_\ell = -c_1$. It is easy to check that the vector

$$\begin{aligned}\mathbf{y} =& (\ell-1)\alpha_1 + (\ell-3)\alpha_2 + (\ell-5)\alpha_4 + \cdots \\ &\qquad + (5-\ell)\alpha_{\ell-2} + (3-\ell)\alpha_{\ell-1} + (1-\ell)\alpha_\ell \\ =& (\ell-1)\varepsilon_1 - 2\big(\varepsilon_2 + \cdots + \varepsilon_\ell\big) + (\ell-1)\varepsilon_{\ell+1}\end{aligned} \tag{27}$$

is orthogonal to Δ_0, and hence gives a basis for $\mathfrak{c}_0$. Note that $\mathbf{y}$ is transformed to $-\mathbf{y}$ under the diagram automorphism interchanging α_i with $\alpha_{\ell+1-i}$ for $i = 1, \ldots, \ell$, verifying the claim in the proof of Lemma 5.1.

Since $\mathfrak{c} = \mathfrak{c}_0 \oplus \mathfrak{a}$ and $\dim \mathfrak{c} = 2$, we see from (27) that

$$\mathfrak{a} = \{\mathbf{x} = \mathrm{diag}[\,t\,, \underbrace{0\,, \ldots\,, 0}_{\ell-1}\,, -t\,] \,:\, t \in \mathbb{C}\}\,. \tag{28}$$

Let $\xi_1 \in \mathfrak{a}^*$ take the value t on the element $\mathbf{x}$ in (28). Then $\xi_1 = \overline{\alpha_1} = \overline{\alpha_\ell} = \frac{1}{2}(\varepsilon_1 - \varepsilon_{\ell+1})$.

The multiplicities of the restricted positive roots are as follows when $\ell \geq 2$ (details below).

restricted root	multiplicity
ξ_1	$2\ell - 2$
$2\xi_1$	1

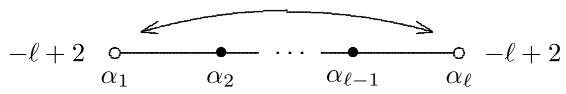


Figure 1: Marked Satake diagram for $\mathbf{SL}_{\ell+1}\,/\,\mathbf{GL}_\ell$ with $\ell \geq 2$

From the table we see that $\delta = \ell\xi_1$. Since all the roots in Δ_0 have the same length, $h^0_{\mathfrak{m}} = 2\rho_{\mathfrak{m}} = (\ell-2)\alpha_2 + \cdots + (j-1)(\ell-j)\alpha_j + \cdots + (\ell-2)\alpha_{\ell-1}$. Hence

$$\langle\, h^0_{\mathfrak{m}} \mid \alpha_i \,\rangle = \begin{cases} 2-\ell & \text{if } i = 1 \text{ or } \ell\,, \\ 2 & \text{if } i = 2, \ldots, \ell-1\,. \end{cases} \tag{29}$$

We now determine the nests of restricted roots, the basic roots, and the dimension factors $d_\xi(\lambda)$ for $\xi \in \Sigma^+$ and $\lambda \in \Gamma(G/H)$.

(i) Let $\xi = \xi_1$. Then $\Phi^+(\xi) = \{\beta_1\,, \ldots\,, \beta_{\ell-1}\} \cup \{\gamma_1\,, , \ldots\ \gamma_{\ell-1}\}$, where

$$\begin{aligned} \beta_j &= \varepsilon_1 - \varepsilon_{j+1} = \alpha_1 + \cdots + \alpha_j\,, \\ \gamma_j &= \varepsilon_{\ell-j+1} - \varepsilon_{\ell+1} = \alpha_{\ell-j+1} + \cdots + \alpha_\ell\,. \end{aligned}$$

Thus $m_\xi = 2(\ell-1)$ and the basic roots are β_1 and γ_1. From (29) we have $k_\xi = -\langle\, h^0_{\mathfrak{m}}\,, \beta_1 \,\rangle = -\langle\, h^0_{\mathfrak{m}}\,, \gamma_1 \,\rangle = \ell - 2$. Since $m_\xi = 2(k_\xi + 1)$, Proposition 5.4 (3) gives

$$d_\xi(\lambda) = \left[\Phi(\langle\, \lambda + \delta \mid \xi \,\rangle\,, \langle\, \delta \mid \xi \,\rangle\,;\, \tfrac{1}{2}(\ell-2))\right]^2\,.$$

(ii) Let $\xi = 2\xi_1$. Then $\Phi^+(\xi) = \{\alpha_1 + \cdots + \alpha_\ell\}$. Hence by Proposition 5.7 we have $d_\xi(\lambda) = \Phi(\langle\, \lambda \mid \xi \,\rangle\,, \langle\, \delta \mid \xi \,\rangle)$.

From cases (i), (ii) and (21) we obtain the dimension formula

$$\begin{aligned} d(\lambda) &= \Phi(\langle\, \lambda \mid \xi_1 \,\rangle\,, \langle\, \delta \mid \xi_1 \,\rangle)\left[\Phi\big(\langle\, \lambda \mid \xi_1 \,\rangle, \langle\, \delta \mid \xi_1 \,\rangle\,;\, \tfrac{1}{2}(\ell-2)\big)\right]^2 \\ &= W\big(\langle\, \lambda \mid \xi_1 \,\rangle\,, \langle\, \delta \mid \xi_1 \,\rangle\,;\, m_{\xi_1}\,, m_{2\xi_1}\big)\,. \end{aligned} \tag{30}$$

Here $\langle \xi_1 \mid \xi_1 \rangle = \frac{1}{2}$, so that $\langle \delta \mid \xi_1 \rangle = \frac{1}{2}\ell$ and $\langle \mu_1 \mid \xi_1 \rangle = 1$. Note that when $\ell = 2$ this formula becomes

$$d(\lambda) = \left[\Phi\big(\langle \lambda + \delta \mid \xi_1 \rangle, \langle \delta \mid \xi_1 \rangle; 0\big)\right]^3 .$$

Thus $d(\mu_1) = 2^3$ as expected, since μ_1 is the highest weight of the adjoint representation of G when $\ell = 2$.

The dimension formula when $\ell = 1$ (so $G = \mathbf{SL}_2(\mathbb{C})$ and H is a maximal torus in G) is different. In this case the fundamental H-spherical highest weight is $\mu_1 = 2\varpi_1$, the multiplicities are $m_{\xi_1} = 1$, $m_{2\xi_1} = 0$, and

$$d\big(\lambda\big) = \Phi(\langle \lambda \mid \xi_1 \rangle, \langle \delta \mid \xi_1 \rangle\big) = W\big(\langle \lambda \mid \xi_1 \rangle, \langle \delta \mid \xi_1 \rangle; m_{\xi_1}\big). \tag{31}$$

In this case $\langle \xi_1 \mid \xi_1 \rangle = 2$, so that $\langle \delta \mid \xi_1 \rangle = 1$ and $d(k\mu_1) = 2k + 1$ as expected.

***Case 2.* The pair $(\mathbf{SO}_{2\ell+1}, \mathbf{SO}_{2\ell})$.** Take G in the matrix form of [GW, §2.1.2] and the diagonal matrices in $\mathfrak{g}$ as a Cartan subalgebra.

Consider first the case $\ell \geq 3$. We choose simple roots $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $i = 1, \ldots, \ell-1$, and $\alpha_\ell = \varepsilon_\ell$. The fundamental H-spherical weight is $\mu_1 = \varpi_1 = \varepsilon_1$ (see [GW, Ch. 12.3.3]). Thus $\Delta_0 = \{\alpha_2, \ldots, \alpha_\ell\}$ and hence $\dim \mathfrak{c} = 1$. Since $|\operatorname{Supp}\mu_1| = 1$, we have $\mathfrak{a} = \mathfrak{c}$ by Lemma 4.1. Thus $\mathfrak{m} \cong \mathfrak{so}_{2\ell-1}$ and

$$\mathfrak{a} = \{\mathbf{x} = \operatorname{diag}[\,t\,, \underbrace{0\,, \ldots, 0}_{2\ell-1}, -t\,] \,:\, t \in \mathbb{C}\}. \tag{32}$$

Let $\xi_1 \in \mathfrak{a}^*$ take the value t on the element $\mathbf{x}$ in (32). Then $\xi_1 = \overline{\alpha_1} = \varepsilon_1$.

The multiplicities of the restricted positive roots for $\ell \geq 2$ are as follows (details below).

restricted root	multiplicity
ξ_1	$2\ell - 1$

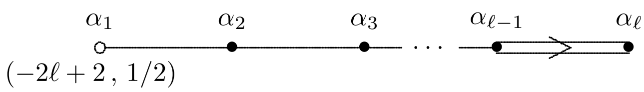


Figure 2: Marked Satake diagram for $\mathbf{SO}_{2\ell+1}\,/\,\mathbf{SO}_{2\ell}$ with $\ell \geq 3$

From the table we obtain $\delta = \big(\ell - \frac{1}{2}\big)\xi_1$. Using the basis $\{\varepsilon_1, \ldots, \varepsilon_\ell\}$ for $\mathfrak{t}$ and the identification of $\mathfrak{t}^*$ with $\mathfrak{t}$, we can write

$$\begin{aligned} 2\rho_{\mathfrak{m}} &= (2\ell-3)\varepsilon_2 + (2\ell-5)\varepsilon_3 + \cdots + 3\varepsilon_{\ell-1} + \varepsilon_\ell\,, \\ h^0_{\mathfrak{m}} &= (2\ell-2)\varepsilon_2 + (2\ell-4)\varepsilon_3 + \cdots + 4\varepsilon_{\ell-1} + 2\varepsilon_\ell\,. \end{aligned}$$

Hence $\varpi^0_{\mathfrak{m}} = \rho_{\mathfrak{m}} - \frac{1}{2}h^0_{\mathfrak{m}} = -\frac{1}{2}\big(\varepsilon_2 + \cdots + \varepsilon_\ell\big)$. From these formulas we see that

$$\langle h^0_{\mathfrak{m}} \mid \alpha_i \rangle = \begin{cases} -2\ell+2 & \text{if } i = 1\,, \\ 2 & \text{if } i = 2, \ldots, \ell\,, \end{cases} \tag{33}$$

and

$$\langle \varpi_{\mathfrak{m}}^0 \mid \alpha_i \rangle = \begin{cases} 1/2 & \text{if } i = 1\,, \\ 0 & \text{if } i = 2, \dots, \ell - 1\,, \\ -1/2 & \text{if } i = \ell\,. \end{cases} \tag{34}$$

(The last two cases in (34) were already given in (26).)

The nest of positive restricted roots is

$$\begin{aligned} \Phi^+(\xi_1) &= \{\varepsilon_1 - \varepsilon_j \,:\, 2 \le j \le \ell\} \cup \{\varepsilon_1 + \varepsilon_j \,:\, 2 \le j \le \ell\} \cup \{\varepsilon_1\} \\ &= \{\beta_j \,:\, 2 \le j \le \ell\} \cup \{\gamma_j \,:\, 2 \le j \le \ell\} \cup \{\alpha_1 + \cdots + \alpha_\ell\}\,, \end{aligned}$$

where $\beta_j = \alpha_1 + \cdots + \alpha_{j-1}$ and $\gamma_j = \beta_j + 2\alpha_j + \cdots + 2\alpha_\ell$. Thus $|\Phi^+(\xi_1)| = 2\ell - 1$ as indicated in the table.

From (33) we see that $k_{\xi_1} = 2\ell - 2$ and the only basic root in the nest is $\beta_2 = \alpha_1$. Hence the eigenvalues of $\operatorname{ad} h_{\mathfrak{m}}^0$ on $\mathfrak{n}_{\xi_1}$ are

$$-2\ell + 2,\ \dots\ , -2,\ 0,\ 2,\ \dots\ , 2\ell - 2\,,$$

each with multiplicity one, with the negative eigenvalues coming from $\{\beta_j\}$ and the positive eigenvalues from $\{\gamma_j\}$. From (34) we have $\langle \varpi_{\mathfrak{m}}^0 \mid \beta_j \rangle = 1/2$, $\langle \varpi_{\mathfrak{m}}^0 \mid \gamma_j \rangle = -1/2$, and $\langle \varpi_{\mathfrak{m}}^0 \mid \alpha_1 + \cdots + \alpha_\ell \rangle = 0$. Hence the $\rho_{\mathfrak{m}}$ shifts in the dimension formula are

$$-\ell + \tfrac{3}{2},\ \dots\ , -\tfrac{1}{2},\ 0,\ \tfrac{1}{2},\ \dots\ , \ell - \tfrac{3}{2}\,. \tag{35}$$

Note that the positive and negative shifts have unit spacing, but the space around 0 has size 1/2 because of the additional shift from $\varpi_{\mathfrak{m}}^0$, which decreases the positive eigenvalues and increases the negative eigenvalues. Let $\lambda \in \Gamma(G/H)$ be an H-spherical highest weight. It follows from (35) and (21) that

$$\begin{aligned} d(\lambda) &= \Phi(\langle \lambda \mid \xi_1 \rangle, \langle \delta \mid \xi_1 \rangle)\, \Phi\big(\langle \lambda \mid \xi_1 \rangle, \langle \delta \mid \xi_1 \rangle\,;\, \ell - \tfrac{3}{2}\big) \\ &= W\big(\langle \lambda \mid \xi_1 \rangle, \langle \delta \mid \xi_1 \rangle\,;\, m_{\xi_1}\big)\,. \end{aligned} \tag{36}$$

Here $\langle \xi_1 \mid \xi_1 \rangle = \langle \mu_1 \mid \xi_1 \rangle = 1$, so that $\langle \delta \mid \xi_1 \rangle = \ell - \frac{1}{2}$.

Remark 6.1. If $\lambda = k\mu_1$, then (36) gives $d(\lambda) = 2\ell + 1$ when $k = 1$ and

$$d(\lambda) = \frac{(k + 2\ell - 2)!}{k!(2\ell - 1)!}(2k + 2\ell - 1) = \binom{2\ell + k}{k} - \binom{2\ell + k - 2}{k - 2} \tag{37}$$

when $k \ge 2$. This is the well-known formula for the dimension of the space of spherical harmonics of degree k in $2\ell + 1$ variables.

The case $\ell = 2$ ($G = \mathbf{SO}_5(\mathbb{C})$ and $H = \mathbf{SO}_4(\mathbb{C})$) is different. The fundamental H-spherical weight is still ϖ_1, but $\Delta_0 = \{\alpha_2\}$ only has one root length, $\mathfrak{m} \cong \mathfrak{sl}_2$, and $h_{\mathfrak{m}}^0 = 2\alpha_2$. The root nest for $\xi_1 = \overline{\alpha_1}$ is

$$\Phi^+(\xi_1) = \{\, \alpha_1\,,\ \alpha_1 + \alpha_2\,,\ \alpha_1 + 2\alpha_2 \,\}$$

with basic root α_1, and $k_{\xi_1} = -\langle 2\alpha_2 \mid \alpha_1 \rangle = 2$. Hence Proposition 5.4 (3) gives

$$d(\lambda) = \Phi\big(\langle\, \lambda \mid \xi_1 \,\rangle\,,\, \langle\, \delta \mid \xi_1 \,\rangle\,;\, 1\big) = W\big(\langle\, \lambda \mid \xi_1 \,\rangle, \langle\, \delta \mid \xi_1 \,\rangle\,;\, m_{\xi_1}\big)\,. \tag{38}$$

Case 3. **The pair** $(\mathbf{SO}_{2\ell}, \mathbf{SO}_{2\ell-1})$**.** Assume that $\ell \geq 2$ and take G in the matrix form of [GW, §2.1.2], with the diagonal matrices in $\mathfrak{g}$ as a Cartan subalgebra. Use the simple roots $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $i = 1, \ldots, \ell - 1$ and $\alpha_\ell = \varepsilon_{\ell-1} + \varepsilon_\ell$.

Consider first the case $\ell \geq 3$. The fundamental H-spherical weight is $\mu_1 = \varpi_1 = \varepsilon_1$ (see [GW, Ch. 12.3.3]). Thus $\Delta_0 = \{\alpha_2, \ldots, \alpha_\ell\}$ and hence $\dim \mathfrak{c} = 1$. Since $|\operatorname{Supp} \mu_1| = 1$, we have $\mathfrak{a} = \mathfrak{c}$ by Lemma 4.1; thus $\mathfrak{m} = \mathfrak{m}' \cong \mathfrak{so}_{2\ell-2}$. For this choice of Cartan subalgebra

$$\mathfrak{a} = \{\mathbf{x} = \operatorname{diag}[\,t\,,\, \underbrace{0\,,\, \ldots\,,\, 0}_{2\ell-2},\, -t\,] \;:\; t \in \mathbb{C}\}\,. \tag{39}$$

Let $\xi_1 \in \mathfrak{a}^*$ take the value t on the element $\mathbf{x}$ in (39). Then $\xi_1 = \overline{\alpha_1} = \varepsilon_1$ and $\langle \xi_1 \mid \xi_1 \rangle = 1$.

The multiplicities of the restricted positive roots are as follows (details below).

restricted root	multiplicity
ξ_1	$2\ell - 2$

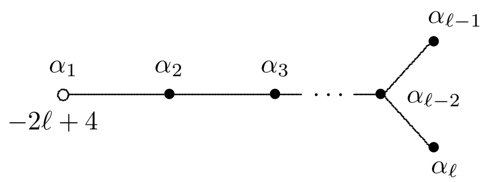


Figure 3: Marked Satake diagram for $\mathbf{SO}_{2\ell}\,/\,\mathbf{SO}_{2\ell-1}$ with $\ell \geq 3$

From the table we obtain $\delta = (\ell - 1)\xi_1$. Using the basis $\{\varepsilon_1, \ldots, \varepsilon_\ell\}$ for $\mathfrak{t}$ and the identification of $\mathfrak{t}^*$ with $\mathfrak{t}$, we can write

$$h^0_{\mathfrak{m}} = 2(\ell - 2)\varepsilon_2 + 2(\ell - 3)\varepsilon_3 + \cdots + 2\varepsilon_{\ell-1}\,.$$

Hence

$$\langle\, h^0_{\mathfrak{m}} \mid \alpha_i \,\rangle = \begin{cases} -2\ell + 4 & \text{if } i = 1\,, \\ 2 & \text{if } i = 2, \ldots, \ell\,, \end{cases} \tag{40}$$

The nest of positive restricted roots is

$$\begin{aligned} \Phi^+(\xi_1) &= \{\varepsilon_1 - \varepsilon_j \;:\; 2 \leq j \leq \ell\} \cup \{\varepsilon_1 + \varepsilon_j \;:\; 2 \leq j \leq \ell\} \\ &= \{\beta_j \;:\; 2 \leq j \leq \ell\} \cup \{\gamma_j \;:\; 2 \leq j \leq \ell - 1\} \cup \{\alpha_1 + \cdots + \alpha_{\ell-2} + \alpha_\ell\}\,, \end{aligned}$$

where $\beta_j = \alpha_1 + \cdots + \alpha_{j-1}$ and $\gamma_j = \beta_j + 2\alpha_j + \cdots + 2\alpha_{\ell-2} + \alpha_{\ell-1} + \alpha_\ell$ (the roots with coefficient 2 are omitted when $j = \ell - 1$). Thus $|\Phi^+(\xi_1)| = 2\ell - 2$ as indicated in the table. From (40) we see that the only basic root in the nest is $\beta_2 = \alpha_1$ and that $k_{\xi_1} = 2\ell - 4$.

Let $\lambda \in \Gamma(G/H)$ be an H-spherical highest weight. Since all the roots in Δ_0 have the same length and $|\Phi^+(\xi)| = k_{\xi_1} + 2$, it follows from Proposition 5.4 (4) that

$$\begin{aligned} d(\lambda) &= \Phi(\langle \lambda \mid \xi_1 \rangle, \langle \delta \mid \xi_1 \rangle)\, \Phi(\langle \lambda \mid \xi_1 \rangle, \langle \delta \mid \xi_1 \rangle\,;\, \ell - 2\big) \\ &= W\big(\langle \lambda \mid \xi_1 \rangle, \langle \delta \mid \xi_1 \rangle\,;\, m_{\xi_1}\big)\,. \end{aligned} \tag{41}$$

Here $\langle \delta \mid \xi_1 \rangle = (\ell - 1)\langle \xi_1 \mid \xi_1 \rangle = \ell - 1$.

Consider the case $\ell = 2$. Now $\mathfrak{h} \cong \mathfrak{sl}_2$ and $\mathfrak{g} \cong \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ is not simple. In this case the fundamental H-spherical highest weight is $\mu_1 = \varpi_1 + \varpi_2$ and Δ_0 is empty. Hence $\mathfrak{c} = \mathfrak{t} = \mathfrak{a} + \mathfrak{c}_0$, where $\mathfrak{c}_0 = \{\, \mathrm{diag}[\,0\,,\, t\,,\, -t\,,\, 0\,] \,:\, t \in \mathbb{C}\}$ and

$$\mathfrak{a} = \{\, \mathbf{x} = \mathrm{diag}[\,t\,,\, 0\,,\, 0\,,\, -t\,] \,:\, t \in \mathbb{C}\}\,. \tag{42}$$

Let $\xi_1 \in \mathfrak{a}^*$ take the value t on the element $\mathbf{x}$ in (42). Then $\xi_1 = \overline{\alpha_1} = \overline{\alpha_2} = \varepsilon_1$ is the positive restricted root with multiplicity two. Since $\mathfrak{m} = \mathfrak{c}_0$, we have $\rho_{\mathfrak{m}} = 0$ and $\rho_{\mathfrak{g}} = \delta = \xi_1$. Hence by the Weyl character formula

$$d(\lambda) = \big[\Phi\big(\langle \lambda \mid \xi_1 \rangle\,,\, \langle \delta \mid \xi_1 \rangle\big)\big]^2 = W\big(\langle \lambda \mid \xi_1 \rangle, \langle \delta \mid \xi_1 \rangle\,;\, m_{\xi_1}\big)\,.$$

Case 4. **The pair** $(\mathbf{Sp}_{2\ell}, \mathbf{Sp}_2 \times \mathbf{Sp}_{2\ell-2})$**.** Assume that $\ell \geq 3$ and take G in the matrix form of [GW, §2.1.2], with the Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ the matrices $\mathbf{x} = \mathrm{diag}[\mathbf{y}, -\check{\mathbf{y}}]$, where $\mathbf{y} = [\,\varepsilon_1(\mathbf{y}), \ldots, \varepsilon_\ell(\mathbf{y})\,]$ and $\check{\mathbf{y}} = [\,\varepsilon_\ell(\mathbf{y}), \ldots, \varepsilon_1(\mathbf{y})\,]$. The roots of $\mathfrak{t}$ on $\mathfrak{g}$ are $\pm\varepsilon_i \pm \varepsilon_j$ for $1 \leq i < j \leq \ell$ and $\pm 2\varepsilon_i$ for $1 \leq i \leq \ell$. Take the simple roots as $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $i = 1, \ldots, \ell - 1$ and $\alpha_\ell = 2\varepsilon_\ell$.

The semigroup $\Gamma(G/H)$ is free of rank 1 with generator

$$\mu_1 = \varpi_2 = \varepsilon_1 + \varepsilon_2 \tag{43}$$

(cf. [GW, §12.3.3]). Thus $\Delta_0 = \Delta \setminus \{\alpha_2\}$. Since $|\,\mathrm{Supp}\,\mu_1| = 1$ we know from Lemma 4.1 that $\mathfrak{a} = \mathfrak{c}$, $\mathfrak{c}_0 = 0$, and hence $\mathfrak{m} \cong \mathfrak{sp}_2(\mathbb{C}) \oplus \mathfrak{sp}_{2\ell-4}(\mathbb{C})$. Thus

$$\mathfrak{a} = \{\mathbf{x} = \mathrm{diag}[\,\mathbf{y}, -\check{\mathbf{y}}\,] \text{ with } \; \mathbf{y} = [t, t, 0, \ldots, 0] \text{ and } t \in \mathbb{C}\ \}\,. \tag{44}$$

Let $\xi_1 \in \mathfrak{a}^*$ take the value t on the element $\mathbf{x}$ in (44). Then $\xi_1 = \overline{\alpha_2} = \frac{1}{2}(\varepsilon_1 + \varepsilon_2)$. The restricted positive roots and their multiplicities are as follows (details given below).

restricted root	multiplicity
ξ_1	$4(\ell - 2)$
$2\xi_1$	3

From the table we obtain $\delta = (\ell - 1)\xi_1$. Here Δ_0 has two root lengths, with $\alpha_i^\vee = \alpha_i$ for $i < \ell$ and $\alpha_\ell^\vee = \varepsilon_\ell$. Furthermore

$$\begin{aligned} \rho_{\mathfrak{m}} &= \tfrac{1}{2}\big[\varepsilon_1 - \varepsilon_2\big] + (\ell - 2)\varepsilon_3 + \cdots + 2\varepsilon_{\ell-1} + \varepsilon_\ell\,, \\ h^0_{\mathfrak{m}} &= \big[\varepsilon_1 - \varepsilon_2\big] + \big[(2\ell - 5)\varepsilon_3 + \cdots + 3\varepsilon_{\ell-1} + \varepsilon_\ell\big] = h^0_{\mathfrak{m}_1} + h^0_{\mathfrak{m}_2}\,, \end{aligned}$$

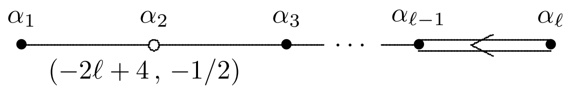


Figure 4: Marked Satake diagram for $\mathbf{Sp}_{2\ell}/(\mathbf{Sp}_2 \times \mathbf{Sp}_{2\ell-2})$ with $\ell \geq 3$

corresponding to the decomposition $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$ with $\mathfrak{m}_1 = \mathfrak{sp}_2(\mathbb{C})$ and $\mathfrak{m}_2 = \mathfrak{sp}_{2\ell-4}(\mathbb{C})$. We have

$$\langle\, h^0_{\mathfrak{m}_1} \mid \alpha_i \,\rangle = \begin{cases} 2 & \text{if } i = 1 \\ -1 & \text{if } i = 2 \\ 0 & \text{if } i > 2 \end{cases} \quad \text{and} \quad \langle\, h^0_{\mathfrak{m}_2} \mid \alpha_i \,\rangle = \begin{cases} 0 & \text{if } i = 1 \\ -2\ell+5 & \text{if } i = 2 \\ 2 & \text{if } 3 \leq i \leq \ell\,. \end{cases} \tag{45}$$

Thus $\varpi^0_{\mathfrak{m}} = \rho_{\mathfrak{m}} - \frac{1}{2}h^0_{\mathfrak{m}} = \frac{1}{2}\big[\varepsilon_3 + \cdots + \varepsilon_\ell\big]$ and hence

$$\langle\, \varpi^0_{\mathfrak{m}} \mid \alpha_i \,\rangle = \begin{cases} 0 & \text{for } i = 1 \text{ and } 3 \leq i \leq \ell - 1\,, \\ -1/2 & \text{if } i = 2\,, \\ 1 & \text{if } i = \ell\,, \end{cases} \tag{46}$$

as in (25). The nests of positive roots are as follows.

(i) Let $\xi = \xi_1$. Then

$$\begin{aligned} \Phi^+(\xi) &= \{\varepsilon_2 \pm \varepsilon_j \,:\, 3 \leq j \leq \ell\} \cup \{\varepsilon_1 \pm \varepsilon_j \,:\, 3 \leq j \leq \ell\} \\ &= \{\beta_j \,:\, 3 \leq j \leq \ell\} \cup \{\alpha_1 + \beta_j \,:\, 3 \leq j \leq \ell\} \\ &\quad \cup \{\gamma_j \,:\, 3 \leq j \leq \ell\} \cup \{\alpha_1 + \gamma_j \,:\, 3 \leq j \leq \ell\}\,, \end{aligned}$$

where $\beta_j = \alpha_2 + \cdots + \alpha_{j-1}$ and $\gamma_j = \beta_j + 2\alpha_j + \cdots + 2\alpha_{\ell-1} + \alpha_\ell$ (here we take $\gamma_\ell = \beta_\ell + \alpha_\ell$). The basic root is $\beta = \alpha_2$ and $\dim \mathfrak{n}_\xi = 4(\ell - 2)$.

From (45) we see that $\langle\, h^0_{\mathfrak{m}_1} \mid \beta \,\rangle = -1$ and $\langle\, h^0_{\mathfrak{m}_2} \mid \beta \,\rangle = -2\ell + 5$. Hence $\mathfrak{n}_{\xi_1} \cong \mathbb{C}^2 \otimes \mathbb{C}^{2(\ell-2)}$ as a representation of $\mathfrak{m}_1 \oplus \mathfrak{m}_2$ (the tensor product of the defining representations). Thus the eigenvalues of $\operatorname{ad}\mathfrak{h}^0_{\mathfrak{m}_1}$ on $\mathfrak{n}_{\xi_1}$ are ± 1 with multiplicity $2(\ell - 2)$, whereas the eigenvalues of $\operatorname{ad}\mathfrak{h}^0_{\mathfrak{m}_2}$ on $\mathfrak{n}_{\xi_1}$ are $-2\ell + 5\,, \ldots, 2\ell - 5$ with multiplicity 2. It follows that $\frac{1}{2}\operatorname{ad}\mathfrak{h}^0_{\mathfrak{m}} = \frac{1}{2}\big(\operatorname{ad} h^0_{\mathfrak{m}_1} + \operatorname{ad} h^0_{\mathfrak{m}_2}\big)$ has eigenvalues

$$-\ell + 2\,,\, \underbrace{-\ell + 3\,,\, \ldots\,,\, -1\,,\, 0\,,\, 1\,,\, \ldots\,,\, \ell - 3}_{\text{multiplicity } 2}\,,\, \ell - 2$$

on $\mathfrak{n}_{\xi_1}$, with the highest and lowest eigenvalues of multiplicity one. The eigenvalues ≤ 0 come from the roots β_j and $\alpha_1 + \beta_j$, while the eigenvalues ≥ 0 come from the roots γ_j and $\alpha_1 + \gamma_j$. By (46) $\varpi^0_{\mathfrak{m}}$ takes the value $-1/2$ on the first set of roots and the value $1/2$ on the second set. Since $\rho_{\mathfrak{m}} = \frac{1}{2}h^0_{\mathfrak{m}} + \varpi^0_{\mathfrak{m}}$, it follows that the values of $\langle\, \rho_{\mathfrak{m}} \mid \alpha \,\rangle$ for $\alpha \in \Phi^+(\xi_1)$ are

$$-\ell + \tfrac{3}{2}\,,\, \underbrace{-\ell + \tfrac{5}{2}\,,\, \ldots\,,\, -\tfrac{3}{2}}_{\text{multiplicity } 2}\,,\, -\tfrac{1}{2}\,,\, \tfrac{1}{2}\,,\, \underbrace{\tfrac{3}{2}\,,\, \ldots\,,\, \ell - \tfrac{5}{2}}_{\text{multiplicity } 2}\,,\, \ell - \tfrac{3}{2}\,.$$

Notice that the double eigenvalue 0 for $\frac{1}{2}\operatorname{ad} h^0_{\mathfrak{m}}$ is shifted to $\pm\frac{1}{2}$.

(ii) Let $\xi = 2\xi_1$. Then

$$\Phi^+(\xi) = \{2\varepsilon_2\,,\ \varepsilon_1+\varepsilon_2\,,\ 2\varepsilon_1\} = \{\beta\,,\ \beta+\alpha_1\,,\ \beta+2\alpha_1\}\,,$$

where $\beta = 2\varepsilon_2 = 2\alpha_2 + \cdots + 2\alpha_{\ell-1} + \alpha_\ell$ is the basic root. By (45) we have $\langle\, h^0_{\mathfrak{m}_1} \mid \beta\,\rangle = -2$ and $\langle\, h^0_{\mathfrak{m}_2} \mid \beta\,\rangle = 0$. Thus $\mathfrak{n}_{2\xi_1}$ is the 3-dimensional irreducible representation of $\mathfrak{m}_1$, with $\mathfrak{m}_2$ acting by zero. From (46) we have $\langle\, \varpi^0_{\mathfrak{m}} \mid \alpha\,\rangle = 0$ for all $\alpha \in \Phi^+(2\xi_1)$. Hence the values of $\langle\, \rho_{\mathfrak{m}} \mid \alpha\,\rangle$ are $-1\,,0\,,1$.

From the calculation of the shifts $\langle\, \rho_{\mathfrak{m}} \mid \alpha\,\rangle$ in cases (i) and (ii) and using (21), we obtain the dimension formula

$$\begin{aligned} d(\lambda) &= \frac{\Phi\big(\langle\, \lambda \mid \xi_1\,\rangle, \langle\, \delta \mid \xi_1\,\rangle\,;\, \ell-\frac{5}{2}\big)\,\Phi\big(\langle\, \lambda \mid \xi_1\,\rangle, \langle\, \delta \mid \xi_1\,\rangle\,;\, \ell-\frac{3}{2}\big)}{\Phi\big(\langle\, \lambda \mid \xi_1\,\rangle, \langle\, \delta \mid \xi_1\,\rangle\,;\, \frac{1}{2}\big)} \\ &\qquad \times \Phi(\langle\, \lambda \mid 2\xi_1\,\rangle, \langle\, \delta \mid 2\xi_1\,\rangle\,;\, 1) \\ &= W\big(\langle\, \lambda \mid \xi_1\,\rangle, \langle\, \delta \mid \xi_1\,\rangle\,;\, m_{\xi_1}, m_{2\xi_1}\big)\,. \end{aligned} \tag{47}$$

Here $\langle\, \delta \mid \xi_1\,\rangle = (2\ell-1)\langle\, \xi_1 \mid \xi_1\,\rangle = \ell - \frac{1}{2}$.

Remark 6.2. Let $\lambda = k\mu_1 = 2k\xi_1$. Then $\langle \lambda+\delta \mid \xi_1\rangle = k+\ell-\frac{1}{2}$. Taking $k=1$ in (47) gives $d(\mu_1) = (2\ell+1)(\ell-1) = \binom{2\ell}{2} - \binom{2\ell}{0}$. In this case μ_1 is the highest weight of the traceless (*harmonic*) subspace in $\bigwedge^2 \mathbb{C}^{2\ell}$ (see [GW, Cor. 5.5.17]).

Case 5. **The pair ($\mathbf{F}_4$, $\mathbf{Spin}_9$).** For the Cartan subalgebra $\mathfrak{t} \cong \mathbb{C}^4$ in $\mathfrak{g}$ we follow [Bo1, Planche VIII] and use simple roots $\alpha_1 = \varepsilon_2 - \varepsilon_3$, $\alpha_2 = \varepsilon_3 - \varepsilon_4$, $\alpha_3 = \varepsilon_4$, and $\alpha_4 = (1/2)(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4)$. The fundamental H-spherical weight is $\mu_1 = \varpi_4 = \varepsilon_1$ (see [Kr]). Thus $\Delta_0 = \{\alpha_1, \alpha_2, \alpha_3\}$ and hence $\dim \mathfrak{c} = 1$. Since $|\operatorname{Supp}\mu_1| = 1$, we have $\mathfrak{a} = \mathfrak{c}$ by Lemma 4.1. Thus $\mathfrak{m} \cong \mathfrak{so}_7(\mathbb{C})$. When $\mathfrak{t}$ is identified with $\mathbb{C}^4$ using the basis dual to $\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$ then $\mathfrak{a}$ consists of all elements

$$\{\mathbf{x} = [\,t\,,\,0\,,\,0\,,\,0\,]\ :\ t \in \mathbb{C}\}\,. \tag{48}$$

Let $\xi_1 \in \mathfrak{a}^*$ take the value t on the element $\mathbf{x}$ in (48). Then $\xi_1 = \overline{\alpha_4} = \frac{1}{2}\varepsilon_1$.

The multiplicities of the restricted positive roots are as follows (details given below).

restricted root	multiplicity
ξ_1	8
$2\xi_1$	7

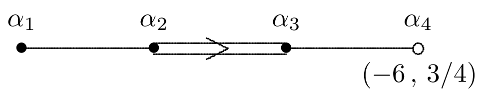


Figure 5: Marked Satake diagram for $\mathbf{F}_4/\,\mathbf{Spin}_9$

From the table we calculate that $\delta = 11\xi_1$. Using the basis $\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$ for $\mathfrak{t}$ and the identification of $\mathfrak{t}^*$ with $\mathfrak{t}$, we can write

$$2\rho_{\mathfrak{m}} = 5\varepsilon_2 + 3\varepsilon_3 + \varepsilon_4\,, \qquad h^0_{\mathfrak{m}} = 6\varepsilon_2 + 4\varepsilon_3 + 2\varepsilon_4\,.$$

Hence $\varpi^0_{\mathfrak{m}} = \rho_{\mathfrak{m}} - \frac{1}{2}h^0_{\mathfrak{m}} = -\frac{1}{2}\big(\varepsilon_2 + \varepsilon_3 + \varepsilon_4\big)$. From these formulas we see that

$$\langle\, h^0_{\mathfrak{m}} \mid \alpha_i \,\rangle = \begin{cases} 2 & \text{if } i = 1,2,3\,, \\ -6 & \text{if } i = 4\,, \end{cases} \tag{49}$$

and

$$\langle\, \varpi^0_{\mathfrak{m}} \mid \alpha_i \,\rangle = \begin{cases} 0 & \text{if } i = 1,2\,, \\ -1/2 & \text{if } i = 3\,, \\ 3/4 & \text{if } i = 4\,. \end{cases} \tag{50}$$

Note that the first two cases in (50) also follow from (26).

We now find the nests of positive restricted roots, the basic roots, and the dimension factors $d_\xi(\lambda)$ for $\xi \in \Sigma^+$ and $\lambda \in \Gamma(G/H)$.

(i) Let $\xi = \xi_1 = \overline{\alpha_4}$. Then from [Bo1, Planche VIII]

$$\Phi^+(\xi) = \{\,\beta_4\,, \beta_3\,, \beta_2\,, \beta_1\,\} \cup \{\,\gamma_1\,, \gamma_2\,, \gamma_3\,, \gamma_4\,\}\,,$$

where $\beta_j = \alpha_j + \cdots + \alpha_4$, $\gamma_1 = \beta_2 + \alpha_3$, $\gamma_2 = \beta_1 + \alpha_3$, $\gamma_3 = \beta_1 + \alpha_3 + \alpha_2$, and $\gamma_4 = \beta_1 + 2\alpha_3 + \alpha_2$. The basic root in the nest is β_4 and $\dim \mathfrak{n}_\xi = 8$. From (49) the eigenvalues of $\frac{1}{2}\,\mathrm{ad}\, h^0_{\mathfrak{m}}$ on $\mathfrak{n}_\xi$ are

$$-3\,, -2\,, -1\,, 0\,, 0\,, 1\,, 2\,, 3$$

corresponding to the roots $\beta_4\,, \ldots, \beta_1\,, \gamma_1\,, \ldots, \gamma_4$ enumerated in increasing length (relative to the simple roots). Thus $\mathfrak{n}_\xi$ decomposes as the one-dimensional plus the seven-dimensional representation of $\{e^0_{\mathfrak{m}}\,, f^0_{\mathfrak{m}}\,, h^0_{\mathfrak{m}}\}$. From (50) we have

$$\langle\, \varpi^0_{\mathfrak{m}} \mid \beta_j \,\rangle = \begin{cases} 1/4 & \text{if } i = 1,2,3\,, \\ 3/4 & \text{if } i = 4\,, \end{cases}$$

$$\langle\, \varpi^0_{\mathfrak{m}} \mid \gamma_j \,\rangle = \begin{cases} -1/4 & \text{if } i = 1,2,3\,, \\ -3/4 & \text{if } i = 4\,. \end{cases}$$

Since $\rho_{\mathfrak{m}} = \frac{1}{2}h^0_{\mathfrak{m}} + \varpi^0_{\mathfrak{m}}$, it follows that the shifts in the formula for $d_\xi(\lambda)$ are

$$-\tfrac{9}{4}\,, -\tfrac{7}{4}\,, -\tfrac{3}{4}\,, -\tfrac{1}{4}\,, \tfrac{1}{4}\,, \tfrac{3}{4}\,, \tfrac{7}{4}\,, \tfrac{9}{4}\,. \tag{51}$$

Thus the shifts are symmetric about 0 but are not in arithmetic progression.

(ii) Let $\xi = 2\xi_1 = \overline{\beta}$, where $\beta = \alpha_2 + 2\alpha_3 + 2\alpha_4$. Then [Bo1, Planche VIII] gives

$$\begin{aligned}\Phi^+(\xi) = \{\, \beta\,,\ \beta + \alpha_1\,,\ \beta + \alpha_1 + \alpha_2\,,\ \beta + \alpha_1 + \alpha_2 + \alpha_3 \,\}\\ \cup \{\, \beta + \alpha_1 + \alpha_2 + 2\alpha_3\,,\ \beta + \alpha_1 + 2\alpha_2 + 2\alpha_3\,,\ \beta + 2\alpha_1 + 2\alpha_2 + 2\alpha_3 \,\}\,.\end{aligned}$$

Thus β is the basic root in the nest and $\dim \mathfrak{n}_\xi = 7$. From (49) the eigenvalues of $\frac{1}{2}\operatorname{ad} h^0_{\mathfrak{m}}$ on $\mathfrak{n}_\xi$ are

$$-3\,,\ -2\,,\ -1\,,\ 0\,,\ 1\,,\ 2\,,\ 3$$

corresponding to the roots $\alpha \in \Phi^+(\xi)$ enumerated by increasing length (relative to the simple roots). From (50) we calculate that the corresponding sequence of values of $\langle\, \varpi^0_{\mathfrak{m}} \mid \alpha \,\rangle$ is

$$\tfrac{1}{2}\,,\ \tfrac{1}{2}\,,\ \tfrac{1}{2}\,,\ 0\,,\ -\tfrac{1}{2}\,,\ -\tfrac{1}{2}\,,\ -\tfrac{1}{2}\,.$$

Since $\rho_{\mathfrak{m}} = \frac{1}{2}h^0_{\mathfrak{m}} + \varpi^0_{\mathfrak{m}}$, it follows that the shifts in the formula for $d_\xi(\lambda)$ are

$$-\tfrac{5}{2}\,,\ -\tfrac{3}{2}\,,\ -\tfrac{1}{2}\,,\ 0\,,\ \tfrac{1}{2}\,,\ \tfrac{3}{2}\,,\ \tfrac{5}{2}\,. \tag{52}$$

Thus the shifts are symmetric about 0 but are not in arithmetic progression, and are the same as for $\mathbf{SO}_9\,/\,\mathbf{SO}_8$ in Case 2.

From (51), (52), and (21) we obtain the complete dimension formula

$$\begin{aligned} d(\lambda) = c\,\langle \lambda + \delta \mid 2\xi_1\rangle \prod_{j=1,3,7,9} \Big\{ \langle \lambda + \delta \mid \xi_1\rangle^2 - \Big(\frac{j}{4}\Big)^2 \Big\} \\ \times \prod_{j=1,3,5} \Big\{ \langle \lambda + \delta \mid 2\xi_1\rangle^2 - \Big(\frac{j}{2}\Big)^2 \Big\}\,, \end{aligned} \tag{53}$$

where c is the normalizing constant to make $d(0) = 1$. Regrouping the terms, we can write this formula in terms of the normalized Weyl dimension function as

$$\begin{aligned} d(\lambda) &= \Phi(\langle \lambda \mid \xi_1\rangle, \langle \delta \mid \xi_1\rangle)\, \Phi\big(\langle \lambda \mid 2\xi_1\rangle, \langle \delta \mid 2\xi_1\rangle\,;\, \tfrac{3}{2}\big)\, \Phi\big(\langle \lambda \mid 2\xi_1\rangle, \langle \delta \mid 2\xi_1\rangle\,;\, \tfrac{9}{2}\big) \\ &= W\big(\langle\, \lambda \mid \xi_1 \,\rangle, \langle\, \delta \mid \xi_1 \,\rangle\,;\, m_{\xi_1}, m_{2\xi_1}\big)\,. \end{aligned} \tag{54}$$

Remark 6.3. If $\lambda = k\mu_1$ with k a nonnegative integer, then $\langle \lambda + \delta \mid 2\xi_1\rangle = k + \frac{11}{2}$ since $\lambda + \delta = (k + \frac{11}{2})2\xi_1$ and $\langle\, 2\xi_1 \mid 2\xi_1 \,\rangle = 1$. Taking $k = 1$ in (54) we obtain $d(\mu_1) = 26$ (the representation of the compact form of F_4 on the traceless 3×3 hermitian matrices over the octonians; see [Ba, §4.2]).

Case 6. **The pair** $(\mathbf{Spin}_7, \mathbf{G}_2)$**.** We take the matrix realization of $\mathfrak{g}$ as in [GW, §2.1.2], with Cartan subalgebra $\mathfrak{t}$ consisting of diagonal matrices $\mathbf{x} = \operatorname{diag}[\,\mathbf{y}, 0, -\check{\mathbf{y}}\,]$ with $\mathbf{y} \in \mathbb{C}^3$. The simple roots are $\alpha_1 = \varepsilon_1 - \varepsilon_2$, $\alpha_2 = \varepsilon_2 - \varepsilon_3$, and $\alpha_3 = \varepsilon_3$.

From [Kr] we know that (G, H) is a spherical pair and $\Gamma(G/H)$ has generator $\mu_1 = \varpi_3 = \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)$ (the spin representation on $\mathbb{C}^8$). Hence

$\Delta_0 = \{\alpha_1, \alpha_2\}$. If $\mathbf{x} = \mathrm{diag}[\,\mathbf{y}, 0, -\check{\mathbf{y}}\,]$ and $\langle \alpha_1, \mathbf{x}\rangle = \langle \alpha_2, \mathbf{x}\rangle = 0$, then $\mathbf{y} = [a, a, a]$. Thus $\dim \mathfrak{c} = 1$. By Lemma 4.1 we have $\mathfrak{a} = \mathfrak{c}$ and $\mathfrak{m} \cong \mathfrak{sl}_3(\mathbb{C})$. We take the restricted root $\xi_1 = \overline{\alpha}_3 = \frac{1}{3}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)$ as a basis for $\mathfrak{a}^*$.

The nests of restricted roots are

$$\Phi^+(\xi_1) = \{\alpha_3\,, \alpha_2 + \alpha_3\,, \alpha_1 + \alpha_2 + \alpha_3\}$$

with basic root $\beta_1 = \alpha_3$, and

$$\Phi^+(2\xi_1) = \{\alpha_2 + 2\alpha_3\,, \alpha_1 + \alpha_2 + 2\alpha_3\,, \alpha_1 + 2\alpha_2 + 2\alpha_3\}$$

with basic root $\beta_2 = \alpha_2 + 2\alpha_3$. Hence the multiplicities of the restricted positive roots are as follows.

restricted root	multiplicity
ξ_1	3
$2\xi_1$	3

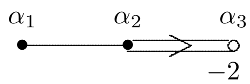


Figure 6: Marked Satake diagram for $\mathbf{Spin}_7 / \mathbf{G}_2$

From the table we obtain $\delta = \frac{9}{2}\xi_1$. Since $\{\alpha_1\,, \alpha_2\}$ are the simple roots for $\mathfrak{m}$, we have $h^0_{\mathfrak{m}} = 2\rho_{\mathfrak{m}} = 2\alpha_1 + 2\alpha_2$. Hence

$$\langle\, h^0_{\mathfrak{m}} \mid \alpha_i \,\rangle = \begin{cases} -2 & \text{if } i = 3\,, \\ 2 & \text{if } i = 1, 2\,. \end{cases} \tag{55}$$

We now obtain the dimension formula. From (55) we have $\langle\, h^0_{\mathfrak{m}} \mid \beta_1 \,\rangle = \langle\, h^0_{\mathfrak{m}} \mid \beta_2 \,\rangle = -2$. Hence $k_\xi = 2$ and $m_\xi = 3$ for $\xi = \xi_1$ and $\xi = 2\xi_1$. Thus Proposition 5.4 (3) and formula (21) give

$$\begin{aligned} d(\lambda) &= \Phi(\langle\, \lambda \mid \xi_1 \,\rangle\,, \langle\, \delta \mid \xi_1 \,\rangle\,;\, 1)\, \Phi(\langle\, \lambda \mid 2\xi_1 \,\rangle\,, \langle\, \delta \mid 2\xi_1 \,\rangle\,;\, 1) \\ &= W\big(\langle\, \lambda \mid \xi_1 \,\rangle\,, \langle\, \delta \mid \xi_1 \,\rangle\,;\, m_{\xi_1}, m_{2\xi_1}\big)\,. \end{aligned} \tag{56}$$

Remark 6.4. Let $\lambda = k\mu_1$, where k is a nonnegative integer. Then $\langle\, \lambda + \delta \mid \xi_1 \,\rangle = \frac{1}{2}(k+1)$. Using this in (56), we obtain

$$d(\lambda) = \frac{k+3}{3} \prod_{j=1}^{5} \frac{k+j}{j}\,.$$

For $k = 1$ the formula gives $d(\mu_1) = 8$ (the spin representation). For $k = 2$ the formula gives $d(2\mu_1) = 35$.

Remark 6.5. The representation of $\mathbf{Spin}_7(\mathbb{C})$ on $\bigwedge^3 \mathbb{C}^7$ has highest weight $2\mu_1$. A fundamental property of $\mathbf{G}_2(\mathbb{C})$ is that it has a unique one-dimensional subspace of fixed vectors in $\bigwedge^3 \mathbb{C}^7$ (see [Ba, §4.1] and [Ag]).

Case 7. **The pair** $(\mathbf{G}_2, \mathbf{SL}_3)$**.** We take $\mathbf{G}_2$ root system as a subset of the integer vectors in $\mathbb{R}^3$ with coordinates summing to zero, with simple roots $\alpha_1 = \varepsilon_1 - \varepsilon_2$ and $\alpha_2 = -2\varepsilon_1 + \varepsilon_2 + \varepsilon_3$ (see [Bo1, Planche IX]). The remaining positive roots are $\alpha_1 + \alpha_2$, $2\alpha_1 + \alpha_2$, $3\alpha_1 + \alpha_2$, $3\alpha_1 + 2\alpha_2$.

From [Kr] we know that (G, H) is a spherical pair and $\Gamma(G/H)$ has generator $\mu_1 = \varpi_1 = 2\alpha_1 + \alpha_2$. Hence $\Delta_0 = \{\alpha_2\}$ (see [Bo1, Planche X]). It follows from Lemma 4.1 that $\mathfrak{a} = \mathfrak{c} = \{\alpha_2\}^\perp$ and $\mathfrak{m} \cong \mathfrak{sl}_2(\mathbb{C})$. For the normalized inner product we define

$$\langle\, \varepsilon_i \mid \varepsilon_j \,\rangle = \begin{cases} 1/3 & \text{if } i = j\,, \\ 0 & \text{if } i \neq j\,. \end{cases}$$

Then $\langle\, \alpha_2 \mid \alpha_2 \,\rangle = 2$ as required in Section 5, and the root $2\alpha_1 + \alpha_2 = -\varepsilon_2 + \varepsilon_3$ is a basis for $\mathfrak{a}$.

Let $\xi_1 = \overline{\alpha_1}$. The nests of restricted roots are

$$\begin{aligned} \Phi^+(\xi_1) &= \{\alpha_1\,,\ \alpha_1 + \alpha_2\}\,,\ \text{basic root } \beta_1 = \alpha_1\,, \\ \Phi^+(2\xi_1) &= \{2\alpha_1 + \alpha_2\}\,, \\ \Phi^+(3\xi_1) &= \{3\alpha_1 + \alpha_2\,,\ 3\alpha_1 + 2\alpha_2\}\,,\ \text{basic root } \beta_3 = 3\alpha_1 + 2\alpha_2\,. \end{aligned}$$

Hence the multiplicities of the restricted positive roots are as follows.

restricted root	multiplicity
ξ_1	2
$2\xi_1$	1
$3\xi_1$	2

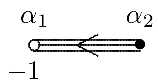


Figure 7: Marked Satake diagram for $\mathbf{G}_2/\,\mathbf{SL}_3$

From the table we obtain $\delta = 5\xi_1$. We have $h^0_{\mathfrak{m}} = \alpha_2$, so that

$$\langle\, h^0_{\mathfrak{m}} \mid \alpha_j \,\rangle = \begin{cases} -1 & \text{if } j = 1\,, \\ 2 & \text{if } j = 2\,. \end{cases} \tag{57}$$

We now obtain the dimension formula. Let $\lambda \in \Gamma(G/H)$. For the basic roots β_1 and β_3 we calculate from (57) that $\langle\, h^0_{\mathfrak{m}} \mid \beta_j \,\rangle = -1$. Let ξ be ξ_1 or $3\xi_1$. Since $\dim \mathfrak{n}_\xi = 2$, Proposition 5.4 (3) gives

$$d_\xi = \Phi\big(\langle\, \lambda \mid \xi \,\rangle\,,\, \langle\, \delta \mid \xi \,\rangle\,;\, \tfrac{1}{2}\big)\,.$$

When $\xi = 2\xi_1$, then $\dim \mathfrak{n}_\xi = 1$. Hence by Proposition 5.4 (2) and (21) it follows that

$$\begin{aligned} d(\lambda) &= \Phi\big(\langle\, \lambda \mid \xi_1 \,\rangle\,,\, \langle\, \delta \mid \xi_1 \,\rangle\,;\, \tfrac{1}{2}\big)\, \Phi(\langle\, \lambda \mid 2\xi_1 \,\rangle\,,\, \langle\, \delta \mid 2\xi_1 \,\rangle) \\ &\quad \times \Phi\big(\langle\, \lambda \mid 3\xi_1 \,\rangle\,,\, \langle\, \delta \mid 3\xi_1 \,\rangle\,;\, \tfrac{1}{2}\big) \\ &= W\big(\langle\, \lambda \mid \xi_1 \,\rangle\,, \langle\, \delta \mid \xi_1 \,\rangle\,;\, m_{\xi_1}\,,\, m_{2\xi_1}\,,\, m_{3\xi_1}\,\big)\,. \end{aligned} \tag{58}$$

Remark 6.6. Let $\lambda = k\mu_1 \in \Gamma(G/H)$, where k is a nonnegative integer. The normalized inner products are $\langle\, \xi_1 \mid \xi_1 \,\rangle = 1/6$, $\langle\, \delta \mid \xi_1 \,\rangle = 5/6$, and $\langle\, \mu_1 \mid \xi_1 \,\rangle = 1/3$ (since $\mu_1 = 2\xi_1$). Using these values, we can write (58) in terms of k as

$$d(\lambda) = \frac{2k+5}{5} \prod_{j=1}^{4} \frac{k+j}{j}\,.$$

In particular, when $k = 1$ we get $d(\mu_1) = 7$ as expected.

7. Higher Rank Non-Symmetric Excellent Affine Spherical Spaces

Here is the list due to Krämer [Kr] of excellent irreducible spherical pairs (G, H) of rank greater than one with G simple and simply-connected, H reductive, connected, and not a symmetric subgroup of G (cf. [Wo2, §12.7] for geometric descriptions). These pairs are determined by their Lie algebras $(\mathfrak{g}, \mathfrak{h})$. The enumeration below follows [Av2, Table 1], which also includes all symmetric subgroups and also includes the pairs that are not excellent.

4: $G = \mathbf{SL}_{p+q}(\mathbb{C})$ and $H = \mathbf{SL}_p(\mathbb{C}) \times \mathbf{SL}_q(\mathbb{C})$ with $1 \le p < q$. Here H is embedded in G by $(x, y) \mapsto (x^{-1})^t \oplus y$ for $x \in \mathbf{SL}_p(\mathbb{C})$ and $y \in \mathbf{SL}_q(\mathbb{C})$.

6: $G = \mathbf{SL}_{2n+1}(\mathbb{C})$ and $H \cong \mathbf{Sp}_{2n}(\mathbb{C})$ with $n \ge 1$. Here H is embedded in G by $x \mapsto x \oplus 1$ for $x \in \mathbf{Sp}_{2n}(\mathbb{C})$.

9: $G = \mathbf{Spin}_{4p+2}(\mathbb{C})$ and $H \cong \mathbf{SL}_{2p+1}(\mathbb{C})$ with $p \ge 1$. Here H is embedded in G by lifting the embedding $x \mapsto x \oplus (x^{-1})^t$ of $\mathbf{SL}_{2p+1}(\mathbb{C})$ into $\mathbf{SO}_{4p+2}(\mathbb{C}, \omega)$, where ω the symmetric bilinear form on $\mathbb{C}^{4p+2}$ with matrix 1 on the antidiagonal and zero elsewhere.

10: $G = \mathbf{Spin}_{2n+1}(\mathbb{C}, \omega)$ and H a covering of $\mathbf{GL}_n(\mathbb{C})$, with ω the symmetric bilinear form on $\mathbb{C}^{2n+1}$ with matrix 1 on the antidiagonal and zero elsewhere. Here H is the connected inverse image under the spin covering of the embedding $x \mapsto x \oplus 1 \oplus (x^{-1})^t$ of $\mathbf{GL}_n(\mathbb{C})$ into $\mathbf{SO}_{2n+1}(\mathbb{C})$.

13: $G = \mathbf{Spin}_9(\mathbb{C})$ and $H = \mathbf{Spin}_7(\mathbb{C})$. Here H embedded in G by lifting to G the homomorphism $x \mapsto \pi_3(x) \oplus 1$ of H into $\mathbf{SO}_9(\mathbb{C})$, where π_3 is the spin representation of H of degree 8.

15: $G = \mathbf{Spin}_8(\mathbb{C})$ and $H = \mathbf{G}_2(\mathbb{C})$. Here H embedded in G by $x \mapsto \pi_1(x) \oplus 1$, where π_1 is the representation of H of degree 7.

19: $G = \mathbf{Sp}_{2\ell}(\mathbb{C})$ and $H = \mathbb{C}^\times \times \mathbf{Sp}_{2\ell-2}(\mathbb{C})$ with $\ell \ge 2$. Here H is embedded by $(z, h) \mapsto \mathrm{diag}[\, z\,, h\,, z^{-1}\,]$ for $z \in \mathbb{C}^\times$ and $h \in \mathbf{Sp}_{2\ell-2}(\mathbb{C})$.

26: $G = \mathbf{E}_6(\mathbb{C})$ and $H = \mathbf{Spin}_{10}(\mathbb{C})$. Here H is embedded into the degree 27 irreducible representation of G by the map $x \mapsto 1 \oplus \pi_1(x) \oplus \pi_5(x)$, where π_1 is the vector representation of degree 10 and π_5 is a half-spin representation of degree 16.

We now proceed to calculate the restricted root systems and the restricted Weyl dimension functions for these pairs. When G is a classical group we take its matrix form such that the diagonal matrices in G give a maximal torus T. The usual inner product $\langle \cdot \mid \cdot \rangle$ making the coordinate functions $\{\varepsilon_i\}$ an orthonormal set satisfies the normalization conditions of Section 5 for these root systems. The choice of positive roots is indicated in each case.

***Case 1.* The pair $(\mathbf{SL}_{p+q}, \mathbf{SL}_p \times \mathbf{SL}_q)$.** Assume $1 \le p < q$. Let T be diagonal matrices in G and U the upper-triangular unipotent matrices. Then $B = TU$ is a Borel subgroup. The simple roots are $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ and the fundamental weights are

$$\varpi_i = \varepsilon_1 + \cdots + \varepsilon_i - \frac{i}{n}\sum_{j=1}^{n} \varepsilon_j$$

with $n = p + q$ and $i = 1, \dots, \ell$, where $\ell = n - 1$.

From [Kr] we know that (G, H) is a spherical pair and $\Gamma(G/H)$ has $r = p+1$ generators

$$\mu_1 = \varpi_1 + \varpi_\ell\,,\ \mu_2 = \varpi_2 + \varpi_{\ell-1}\,,\ \dots\,, \mu_{p-1} = \varpi_{p-1} + \varpi_{q+1}\,,$$
$$\mu_p = \varpi_p\,,\ \mu_{p+1} = \varpi_q\ .$$

Hence the support condition in Definition 3.1 is satisfied and

$$\Delta_0 = \{\alpha_{p+1}\,,\ \alpha_{p+2}\,, \dots\,, \alpha_{q-1}\}\,.$$

Since $|\operatorname{Supp}\mu_i| = 2$ for $i = 1, \dots, p-1$, we know by Lemma 4.1 that $\dim \mathfrak{c} = 2(p-1) + 2 = 2p$ and $\dim \mathfrak{c}_0 = 2p - (p+1) = p - 1$. Thus if $p = 1$ then $\mathfrak{c}_0 = 0$. Assume now $p \ge 2$ and identify $\mathfrak{t}$ with $\mathfrak{t}^*$ using the form $\langle \cdot \mid \cdot \rangle$. If $\mathbf{x} = c_1\alpha_1 + \cdots + c_\ell\alpha_\ell$, then the equations $\langle \mu_i \mid \mathbf{x} \rangle = 0$ for $i = 1, \dots, r$ give the relations

$$c_{\ell+1-i} = -c_i \quad \text{for } i = 1, \dots p-1 \text{ and } \ c_p = c_1 = 0\,.$$

Hence the linearly independent set

$$\{\,\alpha_1 - \alpha_\ell\,,\, \alpha_2 - \alpha_{\ell-1}\,,\, \cdots\,,\, \alpha_{p-1} - \alpha_{q+1}\,\}\,, \tag{59}$$

which is orthogonal to Δ_0, is a basis for $\mathfrak{c}_0$. In particular, we see from (59) that $\rho_{\mathfrak{g}} \perp \mathfrak{c}_0$ since each basis vector goes to its negative under the Dynkin diagram automorphism sending α_i to $\alpha_{\ell+1-i}$, verifying the claim in the proof of Lemma 5.1.

Consider the orthogonal set of vectors in $\mathfrak{t}$

$$\begin{cases} \mathbf{x}_1 = \operatorname{diag}[\,1\,,\,0\,,\,\dots\,,\,0\,,\,-1\,]\,, \\ \qquad\vdots \\ \mathbf{x}_p = \operatorname{diag}[\underbrace{0\,,\,\dots\,,\,0\,,\,1}_{p}\,,\,\underbrace{0\,,\,\dots\,,\,0}_{q-p}\,,\,\underbrace{-1\,,\,0\,,\,\dots\,,\,0}_{p}]\,, \\ \mathbf{x}_{p+1} = \operatorname{diag}[\underbrace{s\,,\,\dots\,,\,s}_{p}\,,\,\underbrace{-t\,,\,\dots\,,\,-t}_{q-p}\,,\,\underbrace{s\,,\,\dots\,,\,s}_{p}]\,, \end{cases}$$

where $s = (q-p)/(q+p)$ and $t = 1-s$. Here s is chosen to make $\langle\, \alpha_p \mid \mathbf{x}_{p+1} \,\rangle = 1$. These vectors are orthogonal to Δ_0 and $\mathfrak{c}_0$. Since $\dim \mathfrak{a} = \dim \mathfrak{c} - \dim \mathfrak{c}_0 = p+1$, it follows that $\{\mathbf{x}_1, \ldots, \mathbf{x}_{p+1}\}$ is a basis for $\mathfrak{a}$. Let $\{\xi_1, \ldots, \xi_{p+1}\}$ be the dual basis for $\mathfrak{a}^*$:

$$\begin{aligned} \xi_i &= \frac{1}{2}(\varepsilon_i - \varepsilon_{n+1-i}) \quad \text{for } 1 \le i \le p\,, \\ \xi_{p+1} &= \frac{1}{2p}(\varepsilon_1 + \cdots + \varepsilon_p) - \frac{1}{q-p}(\varepsilon_{p+1} + \cdots + \varepsilon_q) + \frac{1}{2p}(\varepsilon_{q+1} + \cdots + \varepsilon_n)\,. \end{aligned} \tag{60}$$

The restricted root data are as follows (details given below–the entries in the fourth and sixth columns are calculated using (61) and (63)). In the left column (r) means *regular* and (s) *singular*.

r/s	restricted root ξ	mult.	$\langle \delta \mid \xi \rangle$	# basic roots β	$\langle h^0_{\mathfrak{m}} \mid \beta \rangle$
(r)	$\xi_i - \xi_j \quad (1 \le i < j \le p)$	2	$j - i$	2	0
(r)	$\xi_i + \xi_j \quad (1 \le i < j \le p)$	2	$p + q + 1 - i - j$	2	0
(s)	$\xi_i \pm \xi_{p+1} \quad (1 \le i \le p)$	$q - p$	$\frac{1}{2}(p + q + 1 - 2i)$	1	$-(q - p - 1)$
(r)	$2\xi_i \quad (1 \le i \le p)$	1	$p + q + 1 - 2i$	1	0

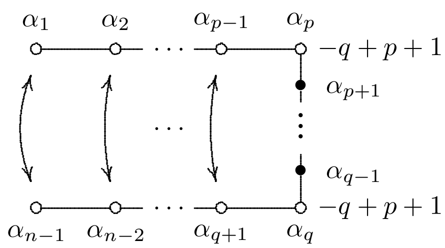


Figure 8: Marked Satake diagram for $\mathbf{SL}_{p+q} \,/\, \mathbf{SL}_p \times \mathbf{SL}_q$

From the table we calculate that

$$\delta = \sum_{i=1}^{p} \big(n - 2i + 1\big)\xi_i\,. \tag{61}$$

Note that $\langle\, \delta \mid \xi_{p+1} \,\rangle = 0$.

From the Satake diagram we see that $\mathfrak{m}' \cong \mathfrak{sl}_{q-p}(\mathbb{C})$ with simple roots Δ_0. Hence

$$h^0_{\mathfrak{m}} = \sum_{i=1}^{q-p} (q - p - 2i + 1)\varepsilon_{p+i}\,. \tag{62}$$

Thus we have

$$\langle h^0_{\mathfrak{m}}\,,\, \alpha_i \rangle = \begin{cases} -q + p + 1 & \text{when } i = p \text{ or } i = q\,, \\ 0 & \text{when } i < p \text{ or } i > q\,, \\ 2 & \text{when } p + 1 \le i \le q - 1\,, \end{cases} \tag{63}$$

which gives the markings on the Satake diagram.

The nests of positive roots, basic roots, and the dimension factors $d_\xi(\lambda)$ for $\xi \in \Sigma^+$ and $\lambda \in \Gamma(G/H)$ are as follows (cases **(i)** and **(ii)** are empty when $p = 1$).

(i) For $1 \le i < j \le p$ let $\xi = \xi_i - \xi_j$. From (60) we see that $\overline{\alpha_i} = \overline{\alpha_{n-i+1}} = \xi_i - \xi_{i+1}$ for $i = 1, \ldots, p-1$. Hence

$$\begin{aligned}\Phi^+(\xi) &= \{\, \varepsilon_i - \varepsilon_j \,,\, \varepsilon_{n+1-j} - \varepsilon_{n+1-i} \,\} \\ &= \{\, \alpha_i + \cdots + \alpha_{j-1} \,,\, \alpha_{n+1-j} + \cdots + \alpha_{n-i} \,\}\end{aligned}$$

with both roots basic.

(ii) For $1 \le i < j \le p$ let $\xi = \xi_i + \xi_j$. From (60) we have

$$\begin{aligned}\Phi^+(\xi) &= \{\, \varepsilon_i - \varepsilon_{n+1-j} \,,\, \varepsilon_j - \varepsilon_{n+1-i} \,\} \\ &= \{\, \alpha_i + \cdots + \alpha_{n-j} \,,\, \alpha_j + \cdots + \alpha_{n-i} \,\}\end{aligned}$$

with both roots basic.

From (63) the eigenvalues of $h^0_{\mathfrak{m}}$ on $\mathfrak{n}_\xi$ are

$$\begin{aligned}\langle h^0_{\mathfrak{m}} \mid \alpha_i + \cdots + \alpha_{n-j}\rangle &= \langle h^0_{\mathfrak{m}} \mid \alpha_p\rangle + \langle h^0_{\mathfrak{m}} \mid \alpha_{p+1} + \cdots + \alpha_{q-1}\rangle + \langle h^0_{\mathfrak{m}} \mid \alpha_q\rangle \\ &= -(q-p-1) + 2(q-p-1) - (q-p-1) = 0\,.\end{aligned}$$

Likewise $\langle h^0_{\mathfrak{m}} \mid \alpha_j + \cdots + \alpha_{n-i}\rangle = 0$.

(iii) For $1 \le i \le p$ let $\xi = \xi_i - \xi_{p+1}$. From (60) we see that $\overline{\alpha_q} = \xi_p - \xi_{p+1}$. Hence

$$\begin{aligned}\Phi^+(\xi) &= \{\, \varepsilon_{p+j} - \varepsilon_{n+1-i} \,:\, 1 \le j \le q-p \,\} \\ &= \{\alpha_{p+j} + \cdots + \alpha_{n-i} \,:\; 1 \le j \le q-p\}\,.\end{aligned}$$

From (63) the lowest eigenvalue of $h^0_{\mathfrak{m}}$ on $\mathfrak{n}_\xi$ is $-q+p+1$, and the basic root is $\beta = \alpha_q + \alpha_{q+1} + \cdots + \alpha_{n-i}$.

(iv) For $1 \le i \le p$ let $\xi = \xi_i + \xi_{p+1}$. From (60) we see that $\overline{\alpha_p} = \xi_p + \xi_{p+1}$. Hence

$$\begin{aligned}\Phi^+(\xi) &= \{\, \varepsilon_i - \varepsilon_{p+j} \,:\, 1 \le j \le q-p \,\} \\ &= \{\alpha_i + \cdots + \alpha_{p+j-1} \,:\; 1 \le j \le q-p\}\,.\end{aligned}$$

From (63) we see that the lowest eigenvalue of $h^0_{\mathfrak{m}}$ on $\mathfrak{n}_\xi$ is $-q+p+1$, and the basic root is $\beta = \alpha_i + \alpha_{i+1} + \cdots + \alpha_p$.

(v) For $1 \le i \le p$ let $\xi = 2\xi_i$. Then by (60) we have

$$\Phi^+(\xi) = \{\varepsilon_i - \varepsilon_{n+1-i}\} = \{\alpha_i + \cdots + \alpha_{n-i}\}\,.$$

In cases (i), (ii) we have $d_\xi(\lambda) = [\Phi(\langle\, \lambda \mid \xi\,\rangle \,,\, \langle\, \delta \mid \xi\,\rangle)]^2$ by Proposition 5.4 (2). In cases (iii) and (iv), since $m_\xi = q-p$, we conclude by Proposition 5.4 (3) that

$$d_\xi(\lambda) = \Phi\big(\langle\, \lambda \mid \xi\,\rangle \,,\, \langle\, \delta \mid \xi\,\rangle \,;\, \tfrac{1}{2}(q-p-1)\big)\,.$$

In case (v) we have $d_\xi(\lambda) = \Phi(\langle\lambda \mid \xi\rangle \,,\, \langle\delta \mid \xi\rangle)$ by Proposition 5.4 (2).

From cases (i)–(v) and (21) we obtain the full dimension formula. Let

$$\Xi_0 = \{\xi_i \pm \xi_j \,:\, 1 \le i < j \le p\}\,, \quad \Xi_1 = \{\xi_i \pm \xi_{p+1} \,:\, 1 \le i \le p\}\,,$$
$$\Xi_2 = \{2\xi_i \,:\, 1 \le i \le p\}\,.$$

Then $\Sigma^+_{\mathrm{reg}} = \Xi_0 \cup \Xi_2$ and $\Sigma^+_{\mathrm{sing}} = \Xi_1$. The dimension formula is

$$\begin{aligned} d(\lambda) &= \prod_{\xi \in \Xi_0^+} \{\Phi(\langle \lambda \mid \xi \rangle\,, \langle \delta \mid \xi \rangle)\}^2 \prod_{\xi \in \Xi_1^+} \Phi\big(\langle\, \lambda \mid \xi \,\rangle\,, \langle\, \delta \mid \xi \,\rangle\,;\, \tfrac{1}{2}(q-p-1)\big) \\ &\quad \times \prod_{\xi \in \Xi_2^+} \Phi(\langle \lambda \mid \xi \rangle\,, \langle \delta \mid \xi \rangle) \\ &= \prod_{\xi \in \Sigma^+_{\mathrm{reg}}} W\big(\langle \lambda \mid \xi \rangle\,, \langle \delta \mid \xi \rangle\,;\, m_\xi\big) \prod_{\xi \in \Sigma^+_{\mathrm{sing}}} W_{\mathrm{sing}}\big(\langle \lambda \mid \xi \rangle\,, \langle \delta \mid \xi \rangle\,;\, m_\xi\big)\,. \end{aligned} \tag{64}$$

Case 2. **The pair** $(\mathbf{SL}_{2n+1}, \mathbf{Sp}_{2n})$**.** From [Kr] we know that (G, H) is a spherical pair and $\Gamma(G/H) = \mathfrak{X}_+(B)$. Thus every irreducible representation of G has a one-dimensional space of H-fixed vectors, $\Delta_0 = \emptyset$, $\mathfrak{a} = \mathfrak{c} = \mathfrak{t}$, and the restricted roots are the same as the roots. Thus every restricted root is regular and has multiplicity one. The dimension function is given by Weyl's formula (21).

Case 3. **The pair** $(\mathbf{Spin}_{4p+2}, \mathbf{SL}_{2p+1})$**.** Let $\mathfrak{g} = \mathfrak{so}(\mathbb{C}^{2\ell}, \omega)$, where $\ell = 2p+1$ is odd and ω is the symmetric bilinear form on $\mathbb{C}^{2\ell}$ with matrix 1 on the antidiagonal and 0 elsewhere. Define the involution θ as in [GW, §12.3, Type DIII]. Then $\mathfrak{g}^\theta \cong \mathfrak{gl}_\ell(\mathbb{C})$; we take $\mathfrak{h}$ to be the subalgebra corresponding to $\mathfrak{sl}_\ell(\mathbb{C})$ under this isomorphism. Take $\mathfrak{t}$ the diagonal matrices in $\mathfrak{g}$. Then $\mathbf{x} \in \mathfrak{t}$ is given by $\mathbf{x} = \mathrm{diag}[\,\mathbf{y},\, -\check{\mathbf{y}}\,]$, where $\mathbf{y} = [\,\varepsilon_1(\mathbf{y})\,, \ldots, \varepsilon_\ell(\mathbf{y})\,]$. The roots of $\mathfrak{t}$ on $\mathfrak{g}$ are $\pm\varepsilon_i \pm \varepsilon_j$ for $1 \le i < j \le \ell$. For making calculations in this case it is convenient to use the ordered basis for $\mathfrak{t}^*$

$$\varepsilon_1 > -\varepsilon_2 > \varepsilon_3 > \cdots > -\varepsilon_{2p} > \varepsilon_{2p+1}\,,$$

as in [GW, §12.3.2, Type DIII]. Let Φ^+ be the positive roots relative to this order (obtained from the usual system of positive roots by $\varepsilon_i \mapsto -(-1)^i \varepsilon_i$). The simple roots in Φ^+ are then

$$\begin{aligned} &\alpha_1 = \varepsilon_1 + \varepsilon_2\,,\ \alpha_2 = -\varepsilon_2 - \varepsilon_3\,,\ \ldots\,,\ \alpha_{2p-1} = \varepsilon_{2p-1} + \varepsilon_{2p}\,, \\ &\alpha_{2p} = -\varepsilon_{2p} - \varepsilon_{2p+1}\,,\ \alpha_{2p+1} = -\varepsilon_{2p} + \varepsilon_{2p+1}\,. \end{aligned} \tag{65}$$

Let $G = \mathbf{Spin}_{2\ell}(\mathbb{C})$ and H the connected subgroup of G with Lie algebra $\mathfrak{h}$. From [Kr] we know that (G, H) is a spherical pair and $\Gamma(G/H)$ is free of rank $p+1$ with generators

$$\mu_1 = \varpi_2\,,\ \mu_2 = \varpi_4\,,\ \ldots\,,\mu_p = \varpi_{2p}\,,\ \mu_{p+1} = \varpi_{2p+1}$$

(μ_p and μ_{p+1} are the highest weights for the half-spin representations of G). Hence the support condition in Definition 3.1 is satisfied and $\Delta_0 = \{\alpha_1\,, \alpha_3\,, \ldots\,, \alpha_{2p-1}\}$.

Since $|\operatorname{Supp}\mu_i| = 1$ for $i = 1, \ldots, p+1$, Lemma 4.1 gives $\mathfrak{a} = \mathfrak{c}$. For the choice (65) of simple roots we see that $\mathfrak{a}$ consists of all $\mathbf{x} = \operatorname{diag}[\mathbf{y}, -\check{\mathbf{y}}] \in \mathfrak{t}$ with

$$\mathbf{y} = [\, a_1\,, -a_1\,, \ldots, a_p\,, -a_p\,, a_{p+1}\,]\,. \tag{66}$$

Thus an orthogonal basis for $\mathfrak{a}^*$ is given by

$$\xi_i = \tfrac{1}{2}(\varepsilon_{2i-1} - \varepsilon_{2i}) \quad \text{for } 1 \le i \le p\,, \qquad \xi_{p+1} = \varepsilon_{2p+1}\,. \tag{67}$$

Note that $\varepsilon_i = \overline{\alpha_{2i}}$ for $1 \le i \le p$ and $\varepsilon_{p+1} = -\overline{\alpha_{2p}} = \overline{\alpha_{2p+1}}$.

The restricted root data are as follows (details given below–the entries in the fourth and sixth columns are calculated using (68) and (69)).

r/s	restricted root ξ	mult.	$\langle \delta \mid \xi \rangle$	# basic roots β	$\langle h^0_{\mathfrak{m}} \mid \beta \rangle$
(r)	$\xi_i - \xi_j \quad (1 \le i < j \le p)$	4	$\frac{1}{2}(j-i)$	1	-2
(r)	$\xi_i + \xi_j \quad (1 \le i < j \le p)$	4	$4p+3-2(i+j)$	1	-2
(s)	$\xi_i \pm \xi_{p+1} \quad (1 \le i \le p)$	2	$\frac{1}{2}(4p+3-4i)$	1	-1
(r)	$2\xi_i \quad (1 \le i \le p)$	1	$4p+3-4i$	1	0

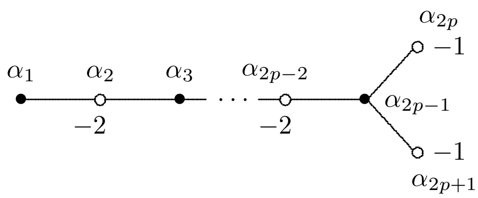


Figure 9: Marked Satake diagram for $\mathbf{Spin}_{4p+2} \,/\, \mathbf{SL}_{2p+1}$ with $p \ge 1$

From the table we calculate that the element δ in this case is

$$\delta = \sum_{i=1}^{p} \big(4(p-i) + 3\big)\xi_i\,. \tag{68}$$

Note that $\langle\, \delta \mid \xi_{p+1} \,\rangle = 0$.

Since $\mathfrak{c}_0 = 0$, we see from the Satake diagram that $\mathfrak{m} \cong \mathfrak{sl}_2(\mathbb{C}) \times \cdots \times \mathfrak{sl}_2(\mathbb{C})$ (p factors) and $h^0_{\mathfrak{m}} = \alpha_1 + \alpha_3 + \cdots + \alpha_{2p-1}$ when we identify $\mathfrak{t}$ with $\mathfrak{t}^*$ using the form $\langle \cdot \mid \cdot \rangle$. Thus

$$\langle\, h^0_{\mathfrak{m}}\,,\, \alpha_i \,\rangle = \begin{cases} 2 & \text{if } i = 2k-1 \text{ for } 1 \le k \le p\,, \\ -2 & \text{if } i = 2k \text{ for } 1 \le k \le p\,, \\ -1 & \text{if } i = 2p \text{ or } i = 2p+1\,, \end{cases} \tag{69}$$

which gives the markings in the Satake diagram.

The nests of positive roots, the basic roots, and the dimension factors $d_\xi(\lambda)$ for $\xi \in \Sigma^+$ and $\lambda \in \Gamma(G/H)$ are as follows.

(i) For $1 \le i < j \le p$ let $\xi = \xi_i - \xi_j$. Then

$$\begin{aligned}\Phi^+(\xi) &= \{\varepsilon_{2i-1} - \varepsilon_{2j-1}\,, \varepsilon_{2i-1} + \varepsilon_{2j}\,, -\varepsilon_{2i} - \varepsilon_{2j-1}\,, -\varepsilon_{2i} + \varepsilon_{2j}\} \\ &= \{\beta\,,\, \alpha_{2i-1} + \beta\,,\, \beta + \alpha_{2j-1}\,,\, \alpha_{2i-1} + \beta + \alpha_{2j-1}\}\,,\end{aligned}$$

where $\beta = \alpha_{2i} + \alpha_{2i+1} + \cdots + \alpha_{2j-2}$. From (69) we see that β is the basic root in $\Phi^+(\xi)$.

(ii) For $1 \le i < j \le p$ let $\xi = \xi_i + \xi_j$. Then

$$\begin{aligned}\Phi^+(\xi) &= \{\varepsilon_{2i-1} + \varepsilon_{2j-1}\,, \varepsilon_{2i-1} - \varepsilon_{2j}\,, -\varepsilon_{2i} + \varepsilon_{2j-1}\,, -\varepsilon_{2i} - \varepsilon_{2j}\} \\ &= \{\beta\,,\, \alpha_{2i-1} + \beta\,,\, \beta + \alpha_{2j-1}\,,\, \alpha_{2i-1} + \beta + \alpha_{2j-1}\}\,,\end{aligned}$$

where $\beta = \alpha_{2i} + \cdots + \alpha_{2j-1} + 2\alpha_{2j} + \cdots + 2\alpha_{2p-1} + \alpha_{2p} + \alpha_{2p+1}$ is the basic root.

(iii) For $1 \le i \le p$ let $\xi = \xi_i - \xi_{p+1}$. Then

$$\Phi^+(\xi) = \{\varepsilon_{2i-1} - \varepsilon_{2p+1}\,, -\varepsilon_{2i} - \varepsilon_{2p+1}\} = \{\beta\,,\, \alpha_{2i-1} + \beta\}\,,$$

where $\beta = \alpha_{2i} + \cdots + \alpha_{2p}$ is the basic root.

(iv) For $1 \le i \le p$ let $\xi = \xi_i + \xi_{p+1}$. Then

$$\Phi^+(\xi) = \{\varepsilon_{2i-1} + \varepsilon_{2p+1}\,,\, -\varepsilon_{2i} + \varepsilon_{2p+1}\} = \{\beta\,,\, \beta + \alpha_{2p-1}\}\,,$$

where $\beta = \alpha_{2i} + \cdots + \alpha_{2p-1} + \alpha_{2p+1}$ is the basic root.

(v) For $1 \le i \le p$ let $\xi = 2\xi_i$. Then

$$\Phi^+(\xi) = \{\varepsilon_{2i-1} - \varepsilon_{2i}\} = \{\beta\}\,,$$

where $\beta = \alpha_{2i-1} + 2\alpha_{2i} + \cdots + 2\alpha_{2p-1} + \alpha_{2p} + \alpha_{2p+1}$.

In cases (i) and (ii) since $k_\xi = 2$ and $m_\xi = k_\xi + 2$, Proposition 5.4 (4) gives

$$d_\xi(\lambda) = \Phi(\langle\, \lambda \mid \xi\,\rangle\,,\, \langle\, \delta \mid \xi\,\rangle)\, \Phi(\langle\, \lambda \mid \xi\,\rangle\,,\, \langle\, \delta \mid \xi\,\rangle\,;\, 1)\,.$$

In cases (iii) and (iv) since $k_\xi = 1$ and $m_\xi = k_\xi + 1$, Proposition 5.4 (3) gives

$$d_\xi(\lambda) = \Phi\big(\langle\, \lambda \mid \xi\,\rangle\,,\, \varphi(\langle\, \delta \mid \xi\,\rangle\,;\, \tfrac{1}{2})\,.$$

In case (v) since $k_\xi = 0$, Proposition 5.4 (2) gives

$$d_\xi(\lambda) = \Phi(\langle\, \lambda \mid \xi\,\rangle\,,\, \langle\, \delta \mid \xi\,\rangle)\,.$$

From cases (i)–(v) and (21) we obtain the full dimension formula. Let

$$\Xi_0^+ = \{\xi_i \pm \xi_j \,:\, 1 \le i < j \le p\}\,, \quad \Xi_1^+ = \{\xi_i \pm \xi_{p+1} \,:\, 1 \le i \le p\}\,,$$
$$\Xi_2^+ = \{2\xi_i \,:\, 1 \le i \le p\}\,.$$

Then $\Sigma^+_{\mathrm{reg}} = \Xi^+_0 \cup \Xi^+_2$ and $\Sigma^+_{\mathrm{sing}} = \Xi^+_1$. The dimension formula is

$$\begin{aligned} d(\lambda) &= \prod_{\xi\in\Xi^+_0} \Phi(\langle \lambda \mid \xi \rangle, \langle \delta \mid \xi \rangle)\, \Phi(\langle \lambda \mid \xi \rangle, \langle \delta \mid \xi \rangle\,;\, 1) \\ &\qquad \times \prod_{\xi\in\Xi^+_1} \Phi\big(\langle \lambda \mid \xi \rangle, \langle \delta \mid \xi \rangle\,;\, \tfrac{1}{2}\big) \prod_{\xi\in\Xi^+_2} \Phi(\langle \lambda \mid \xi \rangle, \langle \delta \mid \xi \rangle) \\ &= \prod_{\xi\in\Sigma^+_{\mathrm{reg}}} W\big(\langle\lambda \mid \xi\rangle, \langle\delta \mid \xi\rangle, \,;\, m_\xi\big) \prod_{\xi\in\Sigma^+_{\mathrm{sing}}} W_{\mathrm{sing}}\big(\langle\lambda \mid \xi\rangle, \langle\delta \mid \xi\rangle, \,;\, m_\xi\big)\,. \end{aligned} \tag{70}$$

Case 4. **The pair** $(\mathbf{Spin}_{2\ell+1}, \mathbf{GL}_\ell)$**.** By [Kr] the fundamental H-spherical highest weights are ϖ_1, ϖ_2, $\ldots$, ϖ_{n-1}, $2\varpi_n$. Thus $\Gamma(G/H)$ consists of all dominant weights with the coefficient of ϖ_n even. Hence $\Delta_0 = \emptyset$, $\mathfrak{a} = \mathfrak{t}$, $\mathfrak{m} = 0$, and $N = U$. The restricted roots coincide with the roots, and all root multiplicities are 1. Thus every restricted root is regular and has multiplicity one. The dimension function is given by Weyl's product formula (21).

Case 5. **The pair** $(\mathbf{Spin}_9, \mathbf{Spin}_7)$**.** With the matrix form of $\mathfrak{g}$ and $\mathfrak{t}$ chosen as in [GW, §3.1], the Cartan subalgebra consists of diagonal matrices $\mathbf{x} = \mathrm{diag}[\,\mathbf{y}\,, 0\,, -\check{\mathbf{y}}\,]$ with $\mathbf{y} \in \mathbb{C}^4$. The simple roots are $\alpha_1 = \varepsilon_1 - \varepsilon_2$, $\alpha_2 = \varepsilon_2 - \varepsilon_3$, $\alpha_3 = \varepsilon_3 - \varepsilon_4$, and $\alpha_4 = \varepsilon_4$. From [Kr] one knows that (G, H) is a spherical pair with fundamental H-spherical highest weights $\mu_1 = \varpi_1$ and $\mu_2 = \varpi_4$. Hence the support condition in Definition 3.1 is satisfied and $\Delta_0 = \{\alpha_2, \alpha_3\}$. We can write

$$\mu_1 = \varepsilon_1\,, \quad \mu_2 = \tfrac{1}{2}\big(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4\big)\,. \tag{71}$$

Since $|\,\mathrm{Supp}\,\mu_i| = 1$ for $i = 1, 2$, Lemma 4.1 gives $\mathfrak{a} = \mathfrak{c}$ and $\mathfrak{m} \cong \mathfrak{sl}_3(\mathbb{C})$. Using $\alpha_2 = \varepsilon_2 - \varepsilon_3$ and $\alpha_3 = \varepsilon_3 - \varepsilon_4$ we thus obtain

$$\mathfrak{a} = \mathrm{Ker}\,\alpha_1 \cap \mathrm{Ker}\,\alpha_2 = \{\mathrm{diag}[\,\mathbf{y}, 0, -\check{\mathbf{y}}\,] \,:\, \mathbf{y} = [\,a, b, b, b\,]\}\,. \tag{72}$$

We take

$$\xi_1 = \varepsilon_1 \quad \text{and} \quad \xi_2 = \tfrac{1}{3}(\varepsilon_2 + \varepsilon_3 + \varepsilon_4) \tag{73}$$

as an orthogonal basis for $\mathfrak{a}^*$.

The restricted root data are as follows (details given below; the fourth and sixth columns are calculated from (74) and (75)).

r/s	restricted root ξ	mult.	$\langle \delta \mid \xi \rangle$	# basic roots β	$\langle h^0_{\mathfrak{m}} \mid \beta \rangle$
(r)	ξ_1	1	7/2	1	0
(r)	ξ_2	3	3/2	1	−2
(r)	$\xi_1 - \xi_2$	3	2	1	−2
(r)	$\xi_1 + \xi_2$	3	5	1	−2
(r)	$2\xi_2$	3	3	1	−2

From the table we calculate that

$$\delta = \frac{1}{2}\big(7\xi_1 + 9\xi_2\big)\,. \tag{74}$$

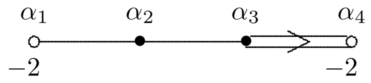


Figure 10: Marked Satake diagram for $\mathbf{Spin}_9 \,/\, \mathbf{Spin}_7$

Since $\mathfrak{m} \cong \mathfrak{sl}_3$ and the positive roots of $\mathfrak{m}$ are α_2, α_3, $\alpha_2 + \alpha_3$, we have $h^0_{\mathfrak{m}} = 2\rho_{\mathfrak{m}} = 2\alpha_2 + 2\alpha_3$. Hence

$$\langle\, h^0_{\mathfrak{m}} \mid \alpha_i \,\rangle = \begin{cases} -2 & \text{if } i = 1, 4\,, \\ 2 & \text{if } i = 2, 3\,, \end{cases} \tag{75}$$

which gives the markings in the Satake diagram.

The nests of positive roots, the basic roots, and the dimension factors $d_\xi(\lambda)$ for $\xi \in \Sigma^+$ and $\lambda \in \Gamma(G/H)$ are as follows.

(i) Let $\xi = \xi_1$. Then $\Phi^+(\xi) = \{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4\}$.

(ii) Let $\xi = \xi_1 - \xi_2$. Then

$$\Phi^+(\xi) = \{\varepsilon_1 - \varepsilon_i \,:\, 2 \le i \le 4\} = \{\beta,\, \beta + \alpha_3,\, \beta + \alpha_3 + \alpha_4\}\,,$$

where $\beta = \alpha_1 + \alpha_2$ is the basic root.

(iii) Let $\xi = \xi_2$. Then

$$\Phi^+(\xi) = \{\varepsilon_i \,:\, 2 \le i \le 4\} = \{\beta,\, \beta + \alpha_3,\, \beta + \alpha_3 + \alpha_2\}\,,$$

where $\beta = \alpha_4$ is the basic root.

(iv) Let $\xi = 2\xi_2$. Then

$$\Phi^+(\xi) = \{\varepsilon_i + \varepsilon_j \,:\, 2 \le i < j \le 4\} = \{\beta,\, \beta + \alpha_2,\, \beta + \alpha_2 + \alpha_3\}\,,$$

where $\beta = \alpha_3 + 2\alpha_4$ is the basic root.

(v) Let $\xi = \xi_1 + \xi_2$. Then

$$\Phi^+(\xi) = \{\varepsilon_1 + \varepsilon_j \,:\, 2 \le j \le 4\} = \{\beta,\, \beta + \alpha_3,\, \beta + \alpha_2 + \alpha_3\}\,,$$

where $\beta = \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4$ is the basic root.

In case (i) Proposition 5.4 (2) gives $d_\xi(\lambda) = \Phi(\langle\, \lambda \mid \xi \,\rangle\,,\, \langle\, \delta \mid \xi \,\rangle)$. In cases (ii)-(iv) we have $k_\xi = 2$ and $\dim \mathfrak{n}_\xi = k_\xi + 1$. Hence Proposition 5.4 (3) gives

$$d_\xi(\lambda) = \Phi(\langle\, \lambda \mid \xi \,\rangle\,,\, \langle\, \delta \mid \xi \,\rangle\,;\, 1)$$

in all these cases. Thus from (21) the dimension formula is

$$\begin{aligned} d(\lambda) &= \Phi(\langle \lambda \mid \xi_1 \rangle \langle \delta \mid \xi_1 \rangle)\, \Phi(\langle \lambda \mid \xi_2 \rangle, \langle \delta \mid \xi_2 \rangle\,;\,1) \\ &\qquad \times \Phi(\langle \lambda \mid 2\xi_2 \rangle, \langle \delta \mid 2\xi_2 \rangle\,;\,1) \prod_{\xi=\xi_1\pm\xi_2} \Phi(\langle \lambda \mid \xi \rangle, \langle \delta \mid \xi \rangle\,;\,1) \\ &= \prod_{\xi\in\Sigma_0^+} W\big(\langle \lambda \mid \xi \rangle, \langle \delta \mid \xi \rangle\,;\, m_\xi\,, m_{2\xi}\big)\,. \end{aligned} \tag{76}$$

Here $\Sigma_0^+ = \Sigma_{\mathrm{reg}}^+ = \{\, \xi_1\,,\, \xi_2\,,\, \xi_1 \pm \xi_2 \,\}$ and there are no singular roots. However some of the factors in the dimension formula occur for rank-one symmetric spaces, while some of the factors occur for the rank-one non-symmetric space $\mathbf{Spin}_7\,/\mathbf{G}_2$.

Remark 7.1. If $\lambda = k_1\mu_1 + k_2\mu_2$, then formula (76) can be written as

$$d(\lambda) = c\,(2k_1 + k_2 + 7)(k_2 + 3)\binom{k_2 + 5}{5} \prod_{j=1}^{3}(k_1 + k_2 + j + 3)\,, \tag{77}$$

where the normalizing constant $c = 2/7!$. For example, when $\lambda = \mu_1$ formula (77) gives $d(\varpi_1) = 9$ (the vector representation on $\mathbb{C}^9$). When $\lambda = \mu_2$, the formula gives $d(\varpi_4) = 16$ (the spin representation).

Case 6. **The pair** $(\mathbf{Spin}_8,\, \mathbf{G}_2)$**.** We take the matrix realization of $\mathfrak{g}$ with Cartan subalgebra $\mathfrak{t}$ consisting of diagonal matrices $\mathbf{x} = \mathrm{diag}[\,\mathbf{y}, -\check{\mathbf{y}}\,]$ with $\mathbf{y} \in \mathbb{C}^4$. The simple roots are $\alpha_1 = \varepsilon_1 - \varepsilon_2\,,\ \ \alpha_2 = \varepsilon_2 - \varepsilon_3\,,\ \ \alpha_3 = \varepsilon_3 - \varepsilon_4\,,$ and $\alpha_4 = \varepsilon_3 + \varepsilon_4\,$.

From [Kr] we know that (G, H) is a spherical pair and $\Gamma(G/H)$ has generators

$$\begin{aligned} \mu_1 &= \varpi_1 = \varepsilon_1\,, \\ \mu_2 &= \varpi_3 = \tfrac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_4)\,, \\ \mu_3 &= \varpi_4 = \tfrac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)\,. \end{aligned} \tag{78}$$

Hence the support condition in Definition 3.1 is satisfied and $\Delta_0 = \{\alpha_2\}$. Since $|\,\mathrm{Supp}\,\mu_i| = 1$ for $i = 1, 2, 3$, Lemma 4.1 gives $\dim \mathfrak{c} = 3$ and $\mathfrak{a} = \mathfrak{c}$.

If $\mathbf{x} = \mathrm{diag}[\,\mathbf{y}, -\check{\mathbf{y}}\,] \in \mathfrak{t}$ and $\langle \alpha_2, \mathbf{x} \rangle = 0$, then $\mathbf{y} = [\,a, b, b, c\,]$. We take

$$\xi_1 = \varepsilon_1\,, \quad \xi_2 = \tfrac{1}{2}(\varepsilon_2 + \varepsilon_3)\,, \quad \text{and} \quad \xi_3 = \varepsilon_4 \tag{79}$$

as an orthogonal basis for $\mathfrak{a}^*$. The restricted root data are as follows (details given below–the entries in the fourth and sixth columns are calculated using (80) and (81)).

r/s	restricted root ξ	mult.	$\langle \delta \mid \xi \rangle$	# basic roots β	$\langle h^0_{\mathfrak{m}} \mid \beta \rangle$
(s)	$\xi_1 - \xi_2\,,\ \xi_2 - \xi_3\,,\ \xi_2 + \xi_3$	2	3/2	1	-1
(r)	$\xi_1 - \xi_3\,,\ \xi_1 + \xi_3\,,\ 2\xi_2$	1	3	1	0
(s)	$\xi_1 + \xi_2$	2	9/2	2	-1

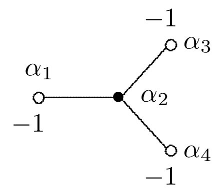


Figure 11: Marked Satake diagram for $\mathbf{Spin}_8/\mathbf{G}_2$

From the table we obtain

$$\delta = 3\xi_1 + 3\xi_2 \,. \tag{80}$$

Note that $\langle\, \delta \mid \xi_3 \,\rangle = 0$.

From the Satake diagram we see that $\mathfrak{m} \cong \mathfrak{sl}(2, \mathbb{C})$ and $h^0_{\mathfrak{m}} = \alpha_2$. Thus

$$\langle\, h^0_{\mathfrak{m}} \mid \alpha_i \,\rangle = \begin{cases} -1 & \text{if } i = 1, 3, 4 \ , \\ 2 & \text{if } i = 2 \ , \end{cases} \tag{81}$$

which gives the markings in the diagram. The nests of positive roots, basic roots, and the dimension factors $d_\xi(\lambda)$ for $\xi \in \Sigma^+$ and $\lambda \in \Gamma(G/H)$ are as follows.

(i) Let ξ be $\xi_1 - \xi_2$, $\xi_2 - \xi_3$, or $\xi_2 + \xi_3$. Then $\Phi^+(\xi) = \{\beta\, , \, \beta + \alpha_2\}$, where the basic root β is α_1, α_3, or α_4, respectively.

(ii) Let $\xi = \xi_1 + \xi_2$. Then $\Phi^+(\xi) = \{\beta\, , \, \beta + \alpha_2\}$, where the basic root $\beta = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$.

(iii) Let $\xi = \xi_1 - \xi_3$, $\xi_1 + \xi_3$, or $2\xi_2$. Then $\Phi^+(\xi) = \{\alpha_i + \alpha_j + \alpha_2\}$ with (i, j) equal to $(1, 3)$, $(2, 4)$, or $(1, 4)$, respectively.

In cases (i) and (ii) we have $k_\xi = 1$ and $m_\xi = k_\xi + 1$, so Proposition 5.4 (3) gives

$$d_\xi(\lambda) = \Phi\big(\langle\, \lambda \mid \xi \,\rangle\, , \langle\, \delta \mid \xi \,\rangle\, ; \, \tfrac{1}{2}\big)$$

in these cases. In case (iii) Proposition 5.4 (2) gives $d_\xi(\lambda) = \Phi(\langle\, \lambda \mid \xi \,\rangle\, , \langle\, \delta \mid \xi \,\rangle)$.

Let $\Sigma^+_{\mathrm{reg}} = \{\xi_1 \pm \xi_3\, , \, 2\xi_2\,\}$ and $\Sigma^+_{\mathrm{sing}} = \{\xi_1 \pm \xi_2\, , \, \xi_2 \pm \xi_3\}$. From (21) we obtain

$$\begin{aligned} d(\lambda) &= \prod_{\xi \in \Sigma^+_{\mathrm{reg}}} \Phi(\langle\, \lambda \mid \xi \,\rangle, \langle\, \delta \mid \xi \,\rangle) \prod_{\xi \in \Sigma^+_{\mathrm{sing}}} \Phi\big(\langle\, \lambda \mid \xi \,\rangle, \langle\, \delta \mid \xi \,\rangle\, ; \, \tfrac{1}{2}\big) \\ &= \prod_{\xi \in \Sigma^+_{\mathrm{reg}}} W(\langle\, \lambda \mid \xi \,\rangle\, , \langle\, \delta \mid \xi \,\rangle\, ; \, m_\xi\,) \prod_{\xi \in \Sigma^+_{\mathrm{sing}}} W_{\mathrm{sing}}(\langle\, \lambda \mid \xi \,\rangle\, , \langle \delta \mid \xi \,\rangle\, ; \, m_\xi\,)\, . \end{aligned} \tag{82}$$

In this formula the regular factors occur for rank-one symmetric spaces.

Remark 7.2. If $\lambda = k_1\mu_1 + k_2\mu_2 + k_3\mu_3$, then formula (82) can be written as

$$d(\lambda) = c_1 \prod_{i=1}^{3} \binom{k_i + 2}{2} \prod_{1 \le i < j \le 3} (k_i + k_j + 3) \prod_{j=1}^{2} (k_1 + k_2 + k_3 + j + 3)\, , \tag{83}$$

where $c_1 = 1/(3^3 \cdot 4 \cdot 5)$. This version clearly exhibits the symmetry in k_1, k_2, k_3 that comes from the triality outer automorphisms of G associated with the symmetries of the Dynkin diagram. Taking $\lambda = \mu_i$ $(i = 1, 2, 3)$ in (83) gives $d(\lambda) = 8$ (the vector and the two half-spin representations), whereas taking $\lambda = \mu_2 + \mu_3$ gives $d(\lambda) = 56 = \binom{8}{3}$ (the representation of $\mathbf{SO}_8(\mathbb{C})$ on $\bigwedge^3 \mathbb{C}^8$).

Case 7. **The pair** $(\mathbf{Sp}_{2\ell}, \mathbb{C}^\times \times \mathbf{Sp}_{2\ell-2})$**.** We take G in the matrix form of [GW, §2.1.2], with Cartan subalgebra $\mathfrak{t}$ the diagonal matrices $\mathbf{x} = \mathrm{diag}[\,\mathbf{y}, -\check{\mathbf{y}}\,]$, where $\mathbf{y} = [\,\varepsilon_1(\mathbf{y}), \ldots, \varepsilon_\ell(\mathbf{y})\,]$ and $\check{\mathbf{y}} = [\,\varepsilon_\ell(\mathbf{y}), \ldots, \varepsilon_1(\mathbf{y})\,]$. The roots of $\mathfrak{t}$ on $\mathfrak{g}$ are $\pm\varepsilon_i \pm \varepsilon_j$ for $1 \le i < j \le \ell$ and $\pm 2\varepsilon_i$ for $1 \le i \le \ell$. Take the simple roots as $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $i = 1, \ldots, \ell - 1$ and $\alpha_\ell = 2\varepsilon_\ell$.

From [Kr] we know that (G, H) is a spherical pair and $\Gamma(G/H)$ is free of rank 2 with generators

$$\mu_1 = 2\varpi_1 = 2\varepsilon_1\,, \quad \mu_2 = \varpi_2 = \varepsilon_1 + \varepsilon_2\,. \tag{84}$$

Hence the support condition in Definition 3.1 is satisfied. If $\ell = 2$, then Δ_0 is empty, $\mathfrak{c} = \mathfrak{t}$, and $\lambda = k_1\varpi_1 + k_2\varpi_2$ is a spherical highest weight if and only k_1 is even.[2] Thus we may assume that $\ell \ge 3$ in the following. Then $\Delta_0 = \{\alpha_3, \ldots, \alpha_\ell\}$ and

$$\mathfrak{c} = \{\mathbf{x} = \mathrm{diag}[\,\mathbf{y}, -\check{\mathbf{y}}\,] \text{ with } \ \mathbf{y} = [\xi_1, \xi_2, 0, \ldots, 0]\ \}\,. \tag{85}$$

Since $|\,\mathrm{Supp}\,\mu_i| = 1$ for $i = 1, 2$, we know from Lemma 4.1 that $\mathfrak{a} = \mathfrak{c}$ and $\mathfrak{m} = \mathfrak{sp}_{2\ell-4}(\mathbb{C})$. The restricted root data are as follows (details given below–the entries in the fourth and sixth columns are calculated using (86) and (87)).

r/s	restricted root ξ	mult.	$\langle\delta \mid \xi\rangle$	# basic roots β	$\langle h^0_{\mathfrak{m}} \mid \beta\rangle$
(r)	$\xi_1 - \xi_2$	1	1	1	0
(r)	$\xi_1 + \xi_2$	1	$2\ell - 1$	1	0
(s)	$\xi_i \quad (i = 1, 2)$	$2\ell - 4$	$\ell + 1 - i$	1	$-2\ell + 5$
(s)	$2\xi_i \quad (i = 1, 2)$	1	$2(\ell + 1 - i)$	1	0

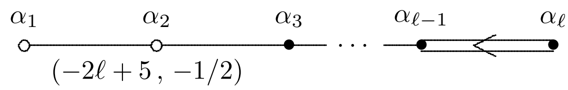


Figure 12: Marked Satake diagram for $\mathbf{Sp}_{2\ell}/(\mathbb{C}^\times \times \mathbf{Sp}_{2\ell-2})$

From the table we obtain

$$\delta = \ell\xi_1 + (\ell - 1)\xi_2\,. \tag{86}$$

In this case Δ_0 is of type $C_{\ell-2}$ and has two root lengths when $\ell \ge 4$. We have $\alpha_\ell^\vee = \varepsilon_\ell$. With the choice of positive roots for $\mathfrak{g}$ above, we have $\rho_{\mathfrak{m}} = (\ell - 2)\varepsilon_3 + (\ell - 3)\varepsilon_4 + \cdots + \varepsilon_\ell$ and $h^0_{\mathfrak{m}} = (2\ell - 5)\varepsilon_3 + (2\ell - 7)\varepsilon_4 + \cdots + \varepsilon_\ell$. Thus

$$\varpi^0_{\mathfrak{m}} = \rho_{\mathfrak{m}} - \frac{1}{2}h^0_{\mathfrak{m}} = \frac{1}{2}\big(\varepsilon_3 + \ldots + \varepsilon_\ell\big)$$

[2]This is the same as Case 5 since $\mathbf{Spin}_5 \cong \mathbf{Sp}_4$ and $\mathbb{C}^\times \times \mathbf{Sp}_2 \cong \mathbf{GL}_2$.

From these formulas we see that

$$\langle\, h^0_{\mathfrak{m}} \mid \alpha_i \,\rangle = \begin{cases} 0 & \text{if } i = 1\,, \\ -2\ell + 5 & \text{if } i = 2\,, \end{cases} \tag{87}$$

and

$$\langle\, \varpi^0_{\mathfrak{m}} \mid \alpha_i \,\rangle = \begin{cases} 0 & \text{for } i = 1 \text{ and } 3 \le i \le \ell - 1\,, \\ -1/2 & \text{if } i = 2\,, \\ 1 & \text{if } i = \ell\,, \end{cases} \tag{88}$$

as in (25). This furnishes the markings in the Satake diagram.

The nests of positive roots and the dimension factors $d_\xi(\lambda)$ for $\xi \in \Sigma^+$ and $\lambda \in \Gamma(G/H)$ are as follows.

(i) Let $\xi = \xi_1 - \xi_2$. Then $\Phi^+(\xi) = \{\alpha_1\}$.

(ii) Let $\xi = \xi_1 + \xi_2$. Then $\Phi^+(\xi) = \{\beta\}$, where $\beta = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-1} + \alpha_\ell$.

(iii) Let $\xi = \xi_i$ with $i = 1, 2$. Then

$$\Phi^+(\xi) = \{\varepsilon_i \pm \varepsilon_j \,:\, 3 \le j \le \ell\} = \{\beta_j\,,\, \gamma_j \,:\, 3 \le j \le \ell\}\,.$$

Here $\beta_j = \beta + \alpha_3 + \cdots + \alpha_{j-1}$ and $\gamma_j = \beta_j + 2\alpha_j + \cdots + 2\alpha_{n-1} + \alpha_\ell$, with the basic root $\beta = \alpha_1 + \alpha_2$ when $i = 1$ and $\beta = \alpha_2$ when $i = 2$. (In these formulas $\beta_3 = \beta$ and $\gamma_\ell = \beta_\ell + \alpha_\ell$.) From (87) we see that $\langle\, h^0_{\mathfrak{m}}\,,\, \beta\,\rangle = -2\ell + 5$. Since $\dim \mathfrak{n}_\xi = 2\ell - 4$, it follows from Proposition 5.7 that the eigenvalues of $\operatorname{ad} h^0_{\mathfrak{m}}$ on $\mathfrak{n}_\xi$ are precisely

$$-2\ell + 5\,,\ \ldots\,,\ -1\,,\ 1\,,\ \ldots\,,\ 2\ell - 5$$

with multiplicity one. The negative eigenvalues come from the roots $\beta_j = \varepsilon_i - \varepsilon_j$, while the positive eigenvalues come from the roots $\gamma_j = \varepsilon_i + \varepsilon_j$. Since $\rho_{\mathfrak{m}} = (1/2)h^0_{\mathfrak{m}} + \varpi^0_{\mathfrak{m}}$ and $\langle\, \varpi^0_{\mathfrak{m}}\,,\, \beta_j \,\rangle = -1/2$, $\langle\, \varpi^0_{\mathfrak{m}}\,,\, \gamma_j \,\rangle = 1/2$ for $3 \le j \le \ell$, it follows that the shifts $\langle\, \rho_{\mathfrak{m}} \mid \alpha \,\rangle$ in $d_\xi(\lambda)$ are

$$-\ell + 2\,,\ \ldots\,,\ -2\,,\ -1\,,\ 1\,,\ 2\ \ldots\,,\ \ell - 2$$

(an arithmetic progression of step one with a gap at zero[3]). Hence

$$d_\xi(\lambda) = \frac{\Phi(\langle\, \lambda \mid \xi \,\rangle\,,\, \langle\, \delta \mid \xi \,\rangle\,;\, \ell - 2)}{\Phi(\langle\, \lambda \mid \xi \,\rangle\,,\, \langle\, \delta \mid \xi \,\rangle)}\,,$$

where the factor in the denominator creates the gap at zero.

(iv) Let $\xi = 2\xi_i$ with i=1, 2. Then $\Phi^+(\xi)$=$\{\beta\}$, where $\beta = 2\alpha_i + \cdots + 2\alpha_{\ell-1} + \alpha_\ell$.

[3] A direct calculation of these shifts from the formula for $\rho_{\mathfrak{m}}$ does not explain the gap at zero.

In cases (i), (ii), and (iv) Proposition 5.7 gives $d_\xi(\lambda) = \Phi(\langle\lambda \mid \xi\rangle, \langle\delta \mid \xi\rangle)$. Combining this with the formula from case (iii) and using (21), we can obtain the full dimension formula. Let $\Sigma^+_{\mathrm{reg}} = \{\xi_1 + \xi_2, \xi_1 - \xi_2\}$ and $\Sigma^+_{\mathrm{sing}} = \{\xi_1, \xi_2\}$. Then

$$\begin{aligned} d(\lambda) &= \prod_{\xi\in\Sigma^+_{\mathrm{reg}}} \Phi(\langle\lambda \mid \xi\rangle, \langle\delta \mid \xi\rangle) \prod_{\xi\in\Sigma^+_{\mathrm{sing}}} \Phi(\langle\lambda \mid \xi\rangle, \langle\delta \mid \xi\rangle; \ell - 2) \\ &= \prod_{\xi\in\Sigma^+_{\mathrm{reg}}} W\big(\langle\lambda \mid \xi\rangle, \langle\delta \mid \xi\rangle; m_\xi\big) \\ &\qquad \times \prod_{\xi\in\Sigma^+_{\mathrm{sing}}} W_{\mathrm{sing}}\big(\langle\lambda \mid \xi\rangle, \langle\delta \mid \xi\rangle; m_\xi, m_{2\xi}\big). \end{aligned} \tag{89}$$

In this formula the regular dimension factors occur for rank-one symmetric spaces. Note that the denominator for ξ_i in case (iii) is cancelled by the dimension factor for $2\xi_i$ from case (iv).

Remark 7.3. If $\lambda = k_1\mu_1 + k_2\mu_2$, then formula (89) can be written as

$$d(\lambda) = c\,(2k_1+1)(2k_1+2k_2+2\ell-1)\binom{k_2+2\ell-3}{2\ell-3}\binom{2k_1+k_2+2\ell-2}{2\ell-3}, \tag{90}$$

where $c = 1/[(2\ell-1)(2\ell-2)]$ is the normalizing constant to make $d(0) = 1$. Taking $\lambda = \mu_1$ gives $d(\mu_1) = 2\ell(2\ell+1)/2$. Here μ_1 is the highest weight of the irreducible G-module $S^2(\mathbb{C}^{2\ell})$ (see [GW, §10.2.3, Example 2]). Taking $\lambda = \mu_2$ gives $d(\mu_2) = (2\ell+1)(\ell-1) = \binom{2\ell}{2} - \binom{2\ell}{0}$. In this case μ_2 is the highest weight of the traceless (*harmonic*) subspace in $\bigwedge^2 \mathbb{C}^{2\ell}$ (see [GW, Cor. 5.5.17]).

Case 8. **The pair** $(\mathbf{E}_6, \mathbf{Spin}_{10})$**.** We take the root system and the simple roots $\alpha_1, \ldots, \alpha_6$ for $\mathfrak{g}$ as in [Bo1, Planche V]. We identify the Cartan subalgebra $\mathfrak{t}$ with the vectors $\mathbf{x} \in \mathbb{C}^8$ such that $\langle\varepsilon_6 \mid \mathbf{x}\rangle = \langle\varepsilon_7 \mid \mathbf{x}\rangle = -\langle\varepsilon_8 \mid \mathbf{x}\rangle$.

From [Kr] we know that (G, H) is a spherical pair and $\Gamma(G/H)$ has generators

$$\begin{aligned} \mu_1 &= \varpi_1 = \frac{2}{3}\big(\varepsilon_8 - \varepsilon_7 - \varepsilon_6\big), \\ \mu_2 &= \varpi_2 = \frac{1}{2}\big(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 - \varepsilon_6 - \varepsilon_7 + \varepsilon_8\big), \\ \mu_3 &= \varpi_6 = \frac{1}{3}\big(\varepsilon_8 - \varepsilon_7 - \varepsilon_6\big) + \varepsilon_5 \end{aligned} \tag{91}$$

corresponding to the endpoints of the Dynkin diagram. Hence the support condition in Definition 3.1 is satisfied and $\Delta_0 = \{\alpha_3, \alpha_4, \alpha_5\}$, where $\alpha_3 = \varepsilon_2 - \varepsilon_1$, $\alpha_4 = \varepsilon_3 - \varepsilon_2$, $\alpha_5 = \varepsilon_4 - \varepsilon_3$. Since $|\operatorname{Supp}\mu_i| = 1$ for $i = 1, 2$, Lemma 4.1 gives $\mathfrak{a} = \mathfrak{c}$.

Let $\mathbf{x} \in \mathfrak{t}$. Then $\langle\alpha_i \mid \mathbf{x}\rangle = 0$ for $i = 3, 4, 5$ if and only if

$$\mathbf{x} = \gamma_1(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4) + \gamma_2\varepsilon_5 + \gamma_3(-\varepsilon_6 - \varepsilon_7 + \varepsilon_8), \tag{92}$$

with $\gamma_i \in \mathbb{C}$. Thus $\mathfrak{a}$ consists of all $\mathbf{x}$ in (92).

Let $\xi_1 = \alpha_1|_{\mathfrak{a}}$, $\xi_2 = \alpha_2|_{\mathfrak{a}}$, $\xi_3 = \alpha_6|_{\mathfrak{a}}$. By [Bo1, Planche V] these roots are given by $\alpha_1 = \frac{1}{2}\big(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4 - \varepsilon_5 - \varepsilon_6 - \varepsilon_7 + \varepsilon_8\big)$, $\alpha_2 = \varepsilon_1 + \varepsilon_2$, and $\alpha_6 = \varepsilon_5 - \varepsilon_4$. Hence from (92) we calculate that

$$\begin{aligned} \xi_1 &= -\frac{1}{4}\big(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4\big) - \frac{1}{2}\big(\varepsilon_5 + \varepsilon_6 + \varepsilon_7 - \varepsilon_8\big)\,, \\ \xi_2 &= \varepsilon_5\,, \qquad \xi_3 = \frac{1}{3}\big(\varepsilon_6 + \varepsilon_7 - \varepsilon_8\big)\,. \end{aligned} \tag{93}$$

The restricted root data are as follows (details given below–the entries in the fourth and sixth columns are calculated using (94) and (95)).

r/s	restricted root ξ	mult.	$\langle \delta \mid \xi \rangle$	# basic roots β	$\langle h^0_{\mathfrak{m}} \mid \beta \rangle$
(s)	$\xi_1\,,\ \xi_3$	4	5/2	1	-3
(s)	$\xi_1 + \xi_2\,,\ \xi_2 + \xi_3$	4	11/2	1	-3
(r)	ξ_2	6	3	1	-4
(r)	$\xi_1 + \xi_2 + \xi_3$	6	8	1	-4
(r)	$\xi_1 + \xi_3$	1	5	1	0
(r)	$\xi_1 + 2\xi_2 + \xi_3$	1	11	1	0

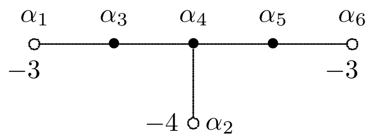


Figure 13: Marked Satake diagram for $\mathbf{E}_6/\,\mathbf{Spin}_{10}$

From the table we obtain

$$\delta = 8\xi_1 + 11\xi_2 + 8\xi_3\,. \tag{94}$$

Since $\mathfrak{m} \cong \mathfrak{sl}(4, \mathbb{C})$ with simple roots $\{\alpha_3\,,\, \alpha_4\,,\, \alpha_5\}$, we have $h^0_{\mathfrak{m}} = 3\alpha_3 + 4\alpha_4 + 3\alpha_5$. Hence

$$\langle\, h^0_{\mathfrak{m}} \mid \alpha_i \,\rangle = \begin{cases} -3 & \text{if } i = 1,\, 6\,, \\ -4 & \text{if } i = 2\,, \\ 2 & \text{if } i = 3,\, 4,\, 5\,, \end{cases} \tag{95}$$

which gives the markings in the Satake diagram.

The nests of positive roots and the dimension factors $d_\xi(\lambda)$ for $\xi \in \Sigma^+$ and $\lambda \in \Gamma(G/H)$ are as follows. We describe the positive roots in terms of their coefficients relative to the simple roots, as in [Bo1, Planche V].

(i) For the restricted root ξ_1 the nest is

$$\Phi^+(\xi_1) = \left\{ \begin{matrix} 10000 & 11000 & 11100 & 11110 \\ 0 & 0 & 0 & 0 \end{matrix} \right\}$$

and the basic root is $\beta = \alpha_1$. For the restricted root ξ_3 the nest is

$$\Phi^+(\xi_3) = \left\{ \begin{matrix} 00001 & 00011 & 00111 & 01111 \\ 0 & 0 & 0 & 0 \end{matrix} \right\}$$

and the basic root is $\beta = \alpha_6$. For the restricted root $\xi_1 + \xi_2$ the nest is

$$\Phi^+(\xi_1+\xi_2) = \left\{ \begin{matrix} 11100 & 11110 & 11210 & 12210 \\ 1 & 1 & 1 & 1 \end{matrix} \right\}$$

and the basic root is $\beta = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$. For the restricted root $\xi_2 + \xi_3$ the nest is

$$\Phi^+(\xi_2+\xi_3) = \left\{ \begin{matrix} 00111 & 01111 & 01211 & 01221 \\ 1 & 1 & 1 & 1 \end{matrix} \right\}$$

and the basic root is $\beta = \alpha_2+\alpha_4+\alpha_5+\alpha_6$. When ξ is any of these restricted roots and β is the corresponding basic root, then from (95) we calculate that $\langle\, h^0_{\mathfrak{m}} \mid \beta \,\rangle = -3$. Since $m_\xi = k_\xi + 1$ in all these cases, Proposition 5.4 (3) gives

$$d_\xi(\lambda) = \Phi\big(\langle\, \lambda \mid \xi \,\rangle\,,\, \langle\, \delta \mid \xi \,\rangle\,;\, \tfrac{3}{2}\big)\,.$$

(ii) For the restricted root ξ_2 the nest is

$$\Phi^+(\xi_2) = \left\{ \begin{matrix} 00000 & 00100 & 01100 & 00110 & 01110 & 01210 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{matrix} \right\}$$

and the basic root is $\beta = \alpha_2$. For the restricted root $\xi_1 + \xi_2 + \xi_3$ the nest is

$$\Phi^+(\xi_1+\xi_2+\xi_3) = \left\{ \begin{matrix} 11111 & 11211 & 12211 & 11221 & 12221 & 12321 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{matrix} \right\}$$

and the basic root is $\beta = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$. When ξ is either of these restricted roots and β is the corresponding basic root, then from (95) we calculate that $\langle\, h^0_{\mathfrak{m}} \mid \beta \,\rangle = -4$. Since $m_\xi = k_\xi + 2$ in both cases, Proposition 5.4 (4) gives

$$d_\xi(\lambda) = \Phi(\langle\, \lambda \mid \xi \,\rangle\,,\, \langle\, \delta \mid \xi \,\rangle)\, \Phi(\langle\, \lambda \mid \xi \,\rangle\,,\, \langle\, \delta \mid \xi \,\rangle\,;\, 2)\,.$$

(iii) For $\xi = \xi_1 + \xi_3$ the nest is $\Phi^+(\xi) = \left\{ \begin{matrix} 11111 \\ 0 \end{matrix} \right\}$, while for $\xi = \xi_1 + 2\xi_2 + \xi_3$ the nest is $\Phi^+(\xi) = \left\{ \begin{matrix} 12321 \\ 2 \end{matrix} \right\}$. Hence $d_\xi(\lambda) = \Phi(\langle\, \lambda \mid \xi \,\rangle\,,\, \langle\, \delta \mid \xi \,\rangle)$ by Proposition 5.4 (2).

Let

$\Xi_0^+ = \{\xi_2\,,\, \xi_1{+}\xi_2{+}\xi_3\}$, $\Xi_1^+ = \{\xi_1\,,\, \xi_3\,,\, \xi_1{+}\xi_2\,,\, \xi_2{+}\xi_3\}$, $\Xi_2^+ = \{\xi_1{+}\xi_3\,,\, \xi_1{+}2\xi_2{+}\xi_3\}$.

Then from cases (i)–(iii) we see that $\Sigma^+_{\mathrm{reg}} = \Xi_0^+ \cup \Xi_2^+$ and $\Sigma^+_{\mathrm{sing}} = \Xi_1^+$. Furthermore, the dimension formula is

$$\begin{aligned} d(\lambda) &= \prod_{\xi \in \Xi_0^+} \Phi(\langle\, \lambda \mid \xi \,\rangle\,,\, \langle\, \delta \mid \xi \,\rangle)\, \Phi(\langle\, \lambda \mid \xi \,\rangle\,,\, \langle\, \delta \mid \xi \,\rangle\,;\, 2) \\ &\qquad \times \prod_{\xi \in \Xi_1^+} \Phi\big(\langle\, \lambda \mid \xi \,\rangle\,,\, \langle\, \delta \mid \xi \,\rangle\,;\, \tfrac{3}{2}\big) \prod_{\xi \in \Xi_2^+} \Phi\big(\langle\, \lambda \mid \xi \,\rangle\,,\, \langle\, \delta \mid \xi \,\rangle\big) \\ &= \prod_{\xi \in \Sigma^+_{\mathrm{reg}}} W\big(\langle \lambda \mid \xi \rangle\,,\, \langle \delta \mid \xi \rangle\,;\, m_\xi\big) \prod_{\xi \in \Sigma^+_{\mathrm{sing}}} W_{\mathrm{sing}}\big(\langle \lambda \mid \xi \rangle\,,\, \langle \delta \mid \xi \rangle\,;\, m_\xi\big)\,. \end{aligned} \tag{96}$$

Remark 7.4. If $\lambda = k_1\mu_1 + k_2\mu_2 + k_3\mu_3$, then using (91), (93), and the values of $\langle\, \delta \mid \xi\,\rangle$ given in the table, we can write formula (96) as

$$\begin{aligned} d(\lambda) = c\,&(k_2+3)(k_1+k_3+5)(k_1+k_2+k_3+8)(k_1+2k_2+k_3+11) \\ &\times \prod_{j=1}^{4}(k_1+j)(k_3+j)(k_1+k_2+j+3)(k_2+k_3+j+3) \\ &\times \prod_{j=1}^{5}(k_2+j)(k_1+k_2+k_3+j+5)\,, \end{aligned} \tag{97}$$

where c is the normalizing constant to make $d(0) = 1$. Thus the dimension formula is symmetric in k_1 and k_3 (this is evident *a priori* from the outer automorphism of G associated with the Dynkin diagram symmetry). Taking $\lambda = \mu_1$ or μ_3, we calculate that $d(\lambda) = 27$ (the two mutually contragredient representations of G on the exceptional simple Jordan algebra). Taking $\lambda = \mu_2$ (the highest root of $\mathfrak{g}$), we calculate that $d(\lambda) = 78$ (the adjoint representation of G).

8. Higher Rank Symmetric Spaces

We now turn to the proof of Theorem 2.1 for an irreducible symmetric pair (G, K) of rank $r \geq 2$ that is the complexification of a compact symmetric space of type **I** (in the terminology of [He1, Ch. VIII, §5]). We use the same case-by-case method as for the higher rank nonsymmetric spherical pairs. However, there is a significant simplification: due to the Weyl group symmetry of the restricted root system it suffices to consider the root nests and dimension factors for a set of simple roots.

Lemma 8.1. *Assume that the Dynkin diagram of $\mathfrak{m}$ is simply laced. If the dimension factors satisfy*

$$d_\xi(\lambda)d_{2\xi}(\lambda) = W\big(\langle\lambda \mid \xi\rangle, \langle\delta \mid \xi\rangle\ ;\, m_\xi, m_{2\xi}\big) \quad \text{when } \lambda \in \Gamma(G/K) \tag{98}$$

for each simple indivisible restricted root ξ, then (98) holds for all $\xi \in \Sigma_0^+$. Here $d_{2\xi}(\lambda) = 1$ if 2ξ is not a restricted root.

Proof. Let $\xi \in \Sigma_0^+$ and let $\mathfrak{s}$ be the principal three-dimensional simple algebra in $\mathfrak{m}$ from Section 5. Let $\mathfrak{a}_\mathbb{R}$ be the real span of the restricted roots. Since (G, K) is a symmetric pair and $\mathrm{Ker}(\xi) \cap \mathfrak{a}_\mathbb{R}$ is a wall of some Weyl chamber in $\mathfrak{a}_\mathbb{R}$, there exists $g \in K$ such that $\mathrm{Ad}(g)$ leaves $\mathfrak{a}_\mathbb{R}$ invariant, $g \cdot \Sigma = \Sigma$, and $\mathrm{Ker}(g \cdot \xi) \cap \mathfrak{a}_\mathbb{R}$ is a wall of the Weyl chamber in $\mathfrak{a}_\mathbb{R}$ defined by the simple roots in Σ^+ (cf. [He1, Ch. VII, Theorem 2.12]). Hence $g \cdot \xi$ is a simple restricted root. Thus by Lemma 5.2 we conclude that $\mathfrak{n}_\xi$ is isomorphic to $\mathfrak{n}_{g\cdot\xi}$ as an $\mathfrak{s}$ module. Since the diagram of $\mathfrak{m}$ is simply laced, we have $h^0_\mathfrak{m} = 2\rho_\mathfrak{m}$, and hence the shifts $\langle\rho_\mathfrak{m} \mid \alpha\rangle$ in the dimension factors $d_\xi(\lambda)$ (respectively $d_{2\xi}(\lambda)$) are determined by the eigenvalues of $\mathrm{ad}(h^0_\mathfrak{m})$ on $\mathfrak{n}_\xi$ (respectively $\mathfrak{n}_{2\xi}$). ∎

Recall that for a restricted root ξ we let k_ξ be the largest eigenvalue of $\mathrm{ad}(h^0_\mathfrak{m})$ on $\mathfrak{n}_\xi$.

Proposition 8.2. *Assume that the Dynkin diagram of $\mathfrak{m}$ is simply laced and that every simple indivisible restricted root ξ satisfies one of the following.*

1. $m_\xi = 1$.
2. $m_\xi = 2$, $m_{2\xi} = 0$, $k_\xi = 0$, *and there are two basic roots in* $\Phi^+(\xi)$.
3. $m_\xi = 3$, $m_{2\xi} = 0$, *and* $k_\xi = 2$.
4. $m_\xi = k_\xi + 2 \geq 4$ *and* $m_{2\xi} = 0$.
5. $m_\xi = 2(k_\xi + 1)$, $m_{2\xi} = 1$, *and there are two basic roots in* $\Phi^+(\xi)$.

Then all restricted root nests are regular and the dimension formula (7) *is valid for all* $\lambda \in \Gamma(G/K)$.

Proof. This follows from Proposition 5.4, formulas (3) and (4),and Lemma 8.1. ∎

The symmetric spaces of rank $r \geq 2$ that satisfy condition (1) of Proposition 8.2 for all simple restricted roots are those of types **A I**, **D I** ($r = \ell$), **E I**, **E V**, **E VIII**, **F I**, and **G** (to check this it suffices to look at the Satake diagrams).

We proceed to carry out a case-by-case analysis of the remaining irreducible symmetric spaces, obtain their marked Satake diagrams and root nest data, and verify that formula (98) holds for all simple restricted roots with multiplicity greater than one (for multiplicity one, this is automatic). It turns out that all but two of the cases with rank $r \geq 2$ are covered by Proposition 8.2. The remaining two cases (with $\mathfrak{m}$ not simply laced) are type **B I** and type **C II** ($\ell \geq 2r+1$). For the simple restricted roots in these cases we use the method of Section 6, Cases 3 and 4, to prove (98), followed by a Weyl group argument using Lemma 5.5 to extend this result to all the positive restricted roots (details given in Cases 3 and 4 below).

Remark 8.3. Following [He1, Ch. X, Table VI] the simple restricted roots are labeled using the enumeration of the simple roots. Thus $\lambda_i = \overline{\alpha_i}$ when the restriction of α_i to $\mathfrak{a}$ is nonzero.

Case 1. **Type A II.** Let $G = \mathbf{SL}_{2r+2}(\mathbb{C})$ and $K = \mathbf{Sp}_{2r+2}(\mathbb{C})$ with $r \geq 2$. Then G has rank $\ell = 2r+1$ and the fundamental K-spherical highest weights are $\mu_i = \varpi_{2i}$ for $i = 1, \ldots, r$. Hence $\mathfrak{m} \cong \mathfrak{sl}_2 \oplus \cdots \oplus \mathfrak{sl}_2$ ($r+1$ copies) and $h^0_{\mathfrak{m}} = \alpha_1 + \alpha_3 + \cdots + \alpha_{2r-1} + \alpha_{2r+1}$. Thus $\langle h^0_{\mathfrak{m}} \mid \alpha_{2i} \rangle = -2$ for $1 \leq i \leq r$. The simple restricted root data are as follows.

restricted root	multiplicity	# basic roots
λ_{2i} $(1 \leq i \leq r)$	4	1

The root nests are

$$\Phi^+\big(\lambda_{2i}\big) = \{\, \alpha_{2i} \,,\ \alpha_{2i-1} + \alpha_{2i} \,,\ \alpha_{2i} + \alpha_{2i+1} \,,\ \alpha_{2i-1} + \alpha_{2i} + \alpha_{2i+1} \,\}$$

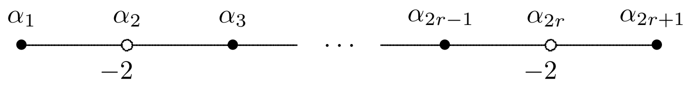


Figure 14: Marked Satake diagram for $\mathbf{SL}_{2r+2} \,/\, \mathbf{Sp}_{2r+2}$

for $i = 1, \ldots, r$, and α_{2i} is the only basic root in the nest. Since $k_\lambda + 2 = m_\lambda$ for all restricted simple positive roots, condition (4) in Proposition 8.2 is satisfied.

Case 2. **Type A III.** Let $G = \mathbf{SL}_{\ell+1}(\mathbb{C})$ and $K = \mathbf{S}(\mathbf{GL}_r(\mathbb{C}) \times \mathbf{GL}_{\ell+1-r}(\mathbb{C}))$ with $\ell \geq 2r - 1 \geq 3$. The fundamental K-spherical highest weights are $\mu_i = \varpi_i + \varpi_{\ell-i+1}$ for $i = 1, \ldots, r$. The simple restricted root data are as follows.

rest. root	mult.	# basic roots
λ_i $(1 \le i \le r-1)$	2	2
λ_r	$2(\ell - 2r + 1)$	2
$2\lambda_r$	1	1

rest. root	mult.	# basic roots
λ_i $(1 \le i \le r-1)$	2	2
λ_r	1	1

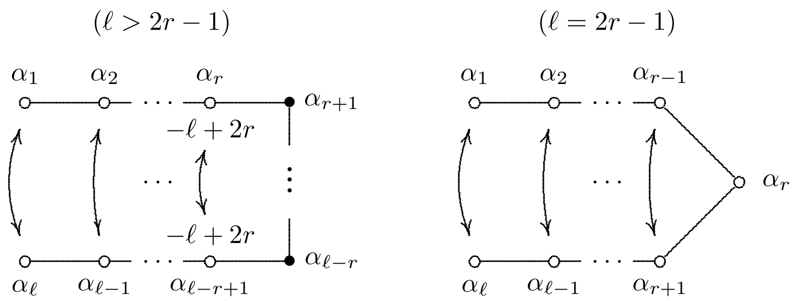


Figure 15: Marked Satake diagrams for $\mathbf{SL}_{\ell+1} / \mathbf{S}(\mathbf{GL}_r \times \mathbf{GL}_{\ell+1-r})$

When $\ell = 2r - 1$ then $\Delta_0 = \emptyset$ and $\mathfrak{m}' = 0$. Since $|\operatorname{Supp} \mu_i| = 2$ for $i = 1, \ldots, r-1$ and $|\operatorname{Supp} \mu_r| = 1$, we know by Lemma 4.1 that $\dim \mathfrak{c} = 2(r-1) + 1 = 2r - 1$ and $\dim \mathfrak{c}_0 = (2r-1) - r = r - 1$. Identify $\mathfrak{t}$ with $\mathfrak{t}^*$ using the form $\langle \cdot \mid \cdot \rangle$. If $\mathbf{x} = c_1\alpha_1 + \cdots + c_\ell\alpha_\ell$ is in $\mathfrak{t}$, then the equations $\langle \mu_i \mid \mathbf{x} \rangle = 0$ for $i = 1, \ldots, r$ become

$$c_{\ell+1-i} = -c_i \quad \text{for } i = 1, \ldots, r-1 \text{ and } \; c_r = 0\,.$$

Hence the linearly independent set

$$\{\, \alpha_1 - \alpha_\ell\,,\, \alpha_2 - \alpha_{\ell-1}\,,\, \cdots\,,\, \alpha_{r-1} - \alpha_{r+1} \,\} \tag{99}$$

is a basis for $\mathfrak{c}_0$. In particular, we see from (99) that each basis vector goes to its negative under the Dynkin diagram automorphism sending α_i to $\alpha_{\ell+1-i}$ for $i = 1, \ldots, \ell$, verifying the claim in the proof of Lemma 5.1. The root nests are $\Phi^+\big(\lambda_i\big) = \{\, \alpha_i\,,\, \alpha_{\ell+1-i} \,\}$ for $1 \le i \le r-1$ and $\Phi^+\big(\lambda_r\big) = \{\, \alpha_r \,\}$. Thus condition (2) of Proposition 8.2 is satisfied by λ_i for $1 \le i \le r-1$ and condition (1) is satisfied by λ_r.

Now assume $\ell > 2r-1$. Then $\Delta_0 = \{\, \alpha_{r+1}\,, \,\ldots\,, \,\alpha_{\ell-r} \,\}$ and $|\,\mathrm{Supp}\,\mu_i| = 2$ for $i = 1, \ldots, r$. Hence by Lemma 4.1 $\dim \mathfrak{c} = 2r$ and $\dim \mathfrak{c}_0 = 2r - r = r$. The set (99) is in $\mathfrak{c}_0$, as is the vector

$$\begin{aligned}\mathbf{y} =& (\ell - r + 1)\alpha_r + (\ell - r - 1)\alpha_{r+1} + (\ell - r - 3)\alpha_{r+2} + \cdots \\ &+ (r + 3 - \ell)\alpha_{\ell-r-1} + (r + 1 - \ell)\alpha_{\ell-r} + (r - 1 - \ell)\alpha_{\ell-r+1}\,.\end{aligned}$$

Hence (99) together with $\mathbf{y}$ give a basis for $\mathfrak{c}_0$. In particular, we see that each basis vector for $\mathfrak{c}_0$ goes to its negative under the Dynkin diagram automorphism, verifying the claim in the proof of Lemma 5.1. We have $\mathfrak{m}' \cong \mathfrak{sl}_{\ell-2r+1}$ and $h^0_{\mathfrak{m}} = (\ell - 2r)\alpha_{r+1} + \cdots + (\ell - 2r)\alpha_{\ell-r}$. Thus $\langle h^0_{\mathfrak{m}} \mid \alpha_i \rangle = -\ell + 2r$ for $i = r$ and $i = \ell - r + 1$. Furthermore, the root nests are $\Phi^+\big(\lambda_i\big) = \{\, \alpha_i\,, \,\alpha_{\ell+1-i} \,\}$ for $1 \le i \le r-1$, and

$$\begin{aligned}\Phi^+\big(\lambda_r\big) &= \{\, \alpha_r + \cdots + \alpha_{r+i} \,:\, 0 \le i \le \ell - 2r \,\} \\ &\quad \cup \{\, \alpha_{\ell-r+1} + \cdots + \alpha_{\ell-r+1-i} \,:\, 0 \le i \le \ell - 2r \,\}\,, \\ \Phi^+\big(2\lambda_r\big) &= \{\, \alpha_r + \cdots + \alpha_{\ell-r+1} \,\}\,.\end{aligned}$$

The nest $\Phi^+\big(\lambda_i\big)$ has two basic roots α_i and $\alpha_{\ell+1-i}$ for $1 \le i \le r$. Thus condition (2) of Proposition 8.2 is satisfied by λ_i for $1 \le i \le r-1$ and condition (5) is satisfied by λ_r.

Case 3. **Type B I.** Let $G = \mathbf{SO}_{2\ell+1}(\mathbb{C})$ and $K = \mathbf{SO}_r(\mathbb{C}) \times \mathbf{SO}_{2\ell+1-r}(\mathbb{C})$ with $2 \le r < \ell$. The fundamental K-spherical highest weights are $\mu_i = 2\varpi_i$ for $i = 1, \ldots, r-1$ and $\mu_r = \varpi_r$. The simple restricted root data are as follows.

restricted root	multiplicity	# basic roots
$\lambda_i \quad (1 \le i \le r-1)$	1	1
λ_r	$2(\ell - r) + 1$	1

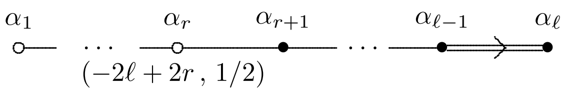


Figure 16: Marked Satake diagram for $\mathbf{SO}_{2\ell+1} \,/\, \mathbf{SO}_r \times \mathbf{SO}_{2\ell+1-r}$

From the Satake diagram and the fact that $|\,\mathrm{Supp}\,\mu_i| = 1$ for $i = 1, \ldots, r$, we know that $\mathfrak{c}_0 = 0$ and $\mathfrak{m} \cong \mathfrak{so}_{2(\ell-r)+1}$, so we have $\mathfrak{h}^0_{\mathfrak{m}} = (2\ell - 2r)\alpha_{2r+1} + \cdots + 2\alpha_\ell$ and

$$\langle\, \varpi^0_{\mathfrak{m}} \mid \alpha_i \,\rangle = \begin{cases} 1/2 & \text{if } i = r\,, \\ 0 & \text{if } i = r+1, \ldots, \ell - 1\,, \\ -1/2 & \text{if } i = \ell\,, \end{cases} \tag{100}$$

as in Section 6, Case 2. The markings on the diagram follow from this. The only nest with more than one root is

$$\Phi^+(\lambda_r) = \{\beta_j \,:\, r+1 \le j \le \ell\} \cup \{\gamma_j \,:\, r+1 \le j \le \ell\} \cup \{\alpha_r + \cdots + \alpha_\ell\}\,,$$

where $\beta_j = \alpha_r + \cdots + \alpha_{j-1}$ and $\gamma_j = \beta_j + 2\alpha_j + \cdots + 2\alpha_\ell$. Thus $|\Phi^+(\lambda_r)| = 2(\ell - r) + 1$ and there is one basic root α_r as indicated in the table.

The eigenvalues of $\operatorname{ad} h^0_{\mathfrak{m}}$ on $\mathfrak{n}_{\lambda_r}$ are

$$-2\ell + 2r, \, \dots \, , -2, \, 0, \, 2, \, \dots \, , 2\ell - 2r \, ,$$

each with multiplicity one, with the negative eigenvalues coming from $\{\beta_j\}$ and the positive eigenvalues from $\{\gamma_j\}$, as in Section 6, Case 2. From (100) we have

$$\langle\, \varpi^0_{\mathfrak{m}} \mid \beta_j \,\rangle = 1/2, \quad \langle\, \varpi^0_{\mathfrak{m}} \mid \gamma_j \,\rangle = -1/2, \quad \langle\, \varpi^0_{\mathfrak{m}} \mid \alpha_r + \cdots + \alpha_\ell \,\rangle = 0.$$

Hence the $\rho_{\mathfrak{m}}$ shifts in the dimension factor for λ_r are

$$-\ell + r + \tfrac{1}{2}, \, \dots \, , -\tfrac{1}{2}, \, 0, \, \tfrac{1}{2}, \, \dots \, , \ell - r - \tfrac{1}{2} \, . \tag{101}$$

Since $\ell - r - \frac{1}{2} = \frac{1}{2} m_{\lambda_r} - 1$, we conclude that $d_{\lambda_r}(\lambda) = W\big(\langle\, \lambda \mid \lambda_r \,\rangle, \langle\, \delta \mid \lambda_r \,\rangle \,;\, m_{\lambda_r}\big)$.

The restricted root system is of type B_r with λ_r the short simple root. Each long restricted root ξ is conjugate under the action of the Weyl group of G/K to α_1. Thus ξ has multiplicity one so by Proposition 5.7 we conclude that (98) holds for ξ.

The positive short restricted roots are $\xi_i = \lambda_i + \cdots + \lambda_r$ for $1 \le i \le r$ with basic roots $\alpha_i + \cdots + \alpha_r$. If the positive roots of G are defined relative to the standard ordered basis $\varepsilon_1 < \varepsilon_2 < \cdots < \varepsilon_\ell$ for $\mathfrak{t}^*$, then $\alpha_i + \cdots + \alpha_r = \varepsilon_i - \varepsilon_{r+1}$. The Weyl group of G consists of all signed permutations of $\{1, \dots, \ell\}$. Let w be the permutation $i \leftrightarrow r$. Then w sends $\alpha_i + \cdots + \alpha_r \to \alpha_r$ and fixes the roots in Δ_0, so we can apply Lemma 5.5 to conclude that (98) holds for ξ_i. Thus all restricted root nests are regular and the dimension formula (7) is valid for all $\lambda \in \Gamma(G/K)$.

Case 4. **Type C II.** Let $G = \mathbf{Sp}_{2\ell}(\mathbb{C})$ and $K = \mathbf{Sp}_{2r}(\mathbb{C}) \times \mathbf{Sp}_{2\ell-2r}(\mathbb{C})$ with $\ell \ge 2r + 1$ and $r \ge 2$. The K-spherical highest weights are $\mu_i = \varpi_{2i}$ for $i = 1, \dots, r$. The simple restricted root data are as follows.

restricted root	multiplicity	# basic roots
$\lambda_{2i} \quad (1 \le i \le r-1)$	4	1
λ_{2r}	$4(\ell - 2r)$	1
$2\lambda_{2r}$	3	1

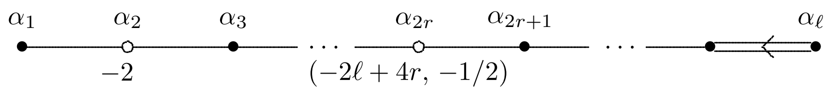


Figure 17: Marked Satake diagram for $\mathbf{Sp}_{2\ell} \,/\, \mathbf{Sp}_{2r} \times \mathbf{Sp}_{2(\ell-r)}$ $\;(\ell \ge 2r + 1)$

From the Satake diagram and the fact that $|\operatorname{Supp} \mu_i| = 1$ for $i = 1, \dots, r$, we know that $\mathfrak{c}_0 = 0$ and $\mathfrak{m} \cong \underbrace{\mathfrak{sl}_2 \oplus \cdots \oplus \mathfrak{sl}_2}_{r \text{ copies}} \oplus \mathfrak{sp}_{2(\ell-2r)}$ and

$$\mathfrak{h}^0_{\mathfrak{m}} = \alpha_1 + \alpha_3 + \cdots + \alpha_{2r-1} + (2\ell - 4r - 1)\alpha_{2r+1} + \cdots \, . \tag{102}$$

As in Section 6, Case 4, we calculate that

$$\langle\, \varpi^0_{\mathfrak{m}} \mid \alpha_i \,\rangle = \begin{cases} 0 & \text{if } 1 \le i \le 2r - 1 \text{ or } 2r + 1 \le i \le \ell - 1 \, , \\ -1/2 & \text{if } i = 2r \, , \\ 1 & \text{if } i = \ell \, . \end{cases} \tag{103}$$

The markings on the diagram follow from (102) and (103).

For the restricted root λ_{2i} with $1 \le i \le r-1$ the root nest is

$$\Phi^+\big(\lambda_{2i}\big) = \{\, \alpha_{2i}\,,\, \alpha_{2i-1} + \alpha_{2i}\,,\, \alpha_{2i} + \alpha_{2i+1}\,,\, \alpha_{2i-1} + \alpha_{2i} + \alpha_{2i+1} \,\} \tag{104}$$

with basic root α_{2i}. Thus condition (4) of Proposition 8.2 is satisfied.

For the indivisible restricted root $\xi = \lambda_{2r}$ we have

$$\begin{aligned}\Phi^+(\xi) = \{\beta_j \,:\, 2r+1 \le j \le \ell\} &\cup \{\alpha_{2r-1} + \beta_j \,:\, 2r+1 \le j \le \ell\} \\ \cup \{\gamma_j \,:\, 2r+1 \le j \le \ell\} &\cup \{\alpha_{2r-1} + \gamma_j \,:\, 2r+1 \le j \le \ell\}\,,\end{aligned}$$

where $\beta_j = \alpha_{2r} + \cdots + \alpha_{j-1}$ and $\gamma_j = \beta_j + 2\alpha_j + \cdots + 2\alpha_{\ell-1} + \alpha_\ell$ (here we take $\gamma_\ell = \beta_\ell + \alpha_\ell$). The basic root is $\beta = \alpha_{2r}$ and there are $4(\ell - 2r)$ roots in the nest. Furthermore,

$$\Phi^+(2\xi) = \{\beta'\,,\, \beta' + \alpha_{2r-1}\,,\, \beta' + 2\alpha_{2r-1}\}\,,$$

where $\beta' = 2\alpha_{2r} + \cdots + 2\alpha_{\ell-1} + \alpha_\ell$ is the basic root. Using these root nests, formula (102), and the same argument as in Section 6, Case 4, we find that the values of $\langle\, \rho_{\mathfrak{m}} \mid \alpha \,\rangle$ for $\alpha \in \Phi^+(\xi)$ are

$$-\ell + 2r - \tfrac{1}{2}\,,\, \underbrace{-\ell + 2r + \tfrac{1}{2}\,,\, \ldots\,,\, -\tfrac{3}{2}}_{\text{multiplicity } 2}\,,\, -\tfrac{1}{2}\,,\, \tfrac{1}{2}\,,\, \underbrace{\tfrac{3}{2}\,,\, \ldots\,,\, \ell - 2r - \tfrac{1}{2}}_{\text{multiplicity } 2}\,,\, \ell - 2r + \tfrac{1}{2}\,.$$

Likewise, the values of $\langle\, \rho_{\mathfrak{m}} \mid \alpha \,\rangle$ for $\alpha \in \Phi^+(2\xi)$ are $-1,\ 0,\ 1$. Hence for $\lambda \in \Gamma(G/K)$ we have

$$d_\xi(\lambda) d_{2\xi}(\lambda) = W\big(\langle\, \lambda \mid \xi \,\rangle, \langle\, \delta \mid \xi \,\rangle\,;\, m_\xi, m_{2\xi}\big)\,. \tag{105}$$

Thus all the simple indivisible restricted roots for G/K satisfy (98).

Since each root nest for the simple restricted roots has only one basic root, we can use the same strategy as for type **B I** to prove that (98) holds for all positive indivisible restricted roots. Thus it suffices to find elements of the Weyl group of G that satisfy the conditions of Lemma 5.5. It will then follow that all restricted root nests are regular and the dimension formula (7) is valid for all $\lambda \in \Gamma(G/K)$. We carry out this argument case-by-case.

Let the positive roots of G be defined relative to an ordered basis $\varepsilon_1 < \varepsilon_2 < \cdots < \varepsilon_\ell$ for $\mathfrak{t}^*$. The Weyl group of G consists of all signed permutations of $\{1, \ldots, \ell\}$. The restricted root system is of type BC_r, with λ_{2r} the short indivisible simple root.

1. Let $\xi_{ij} = \lambda_{2i} + \cdots + \lambda_{2j}$ for $1 \le i < j \le r-1$ (long positive indivisible root). The basic root for ξ_{ij} is

$$\alpha_{2i} + \alpha_{2i+1} + \cdots + \alpha_{2j} = \varepsilon_{2i} - \varepsilon_{2j+1}\,.$$

Let w be the signed permutation

$$2i+1 \to 2j+1, \quad 2i+2 \to 2j+2, \quad 2j+1 \to \overline{2i+2}, \quad 2j+2 \to \overline{2i+1}$$

that fixes all other indices (where $p \to \overline{q}$ means $\varepsilon_p \to -\varepsilon_q$). Then

$$w : \alpha_{2i} \to \varepsilon_{2i} - \varepsilon_{2j+1}\,, \quad w : \alpha_{2i+1} \leftrightarrow \alpha_{2j+1}\,,$$

and w fixes the other roots in Δ_0.

2. Let $\eta_{ij} = \lambda_{2i} + \cdots + \lambda_{2j-2} + 2\lambda_{2j} + \cdots + 2\lambda_{2r}$ for $1 \le i < j \le r$ (long positive indivisible root). The basic root for η_{ij} is
$$\alpha_{2i} + \cdots + \alpha_{2j-1} + 2\alpha_{2j} + \cdots + 2\alpha_{2\ell-1} + \alpha_\ell = \varepsilon_{2i} + \varepsilon_{2j}\,.$$
Let w be the signed permutation
$$1 \leftrightarrow 2i-1, \quad 2 \leftrightarrow 2i, \quad 3 \to \overline{2j}, \quad 4 \to \overline{2j-1}, \quad 2j-1 \to 3, \quad 2j \to 4$$
that fixes all other indices. Then
$$w : \alpha_{2i} \to \varepsilon_{2i} + \varepsilon_{2j}\,, \quad w : \alpha_1 \leftrightarrow \alpha_{2i-1}\,, \quad w : \alpha_3 \leftrightarrow \alpha_{2j-1}\,,$$
and w fixes the other roots in Δ_0.

3. Let $\xi_i = \lambda_{2i} + \cdots + \lambda_{2r}$ for $1 \le i \le r$ (short positive indivisible root). The basic root for ξ_i is
$$\alpha_{2i} + \alpha_{2i+1} + \cdots + \alpha_{2r} = \varepsilon_{2i} - \varepsilon_{2r+1}\,.$$
Let w be the permutation $2i-1 \leftrightarrow 2r-1$, $2i \leftrightarrow 2r$ that fixes all other indices. Then
$$w : \alpha_{2r} \to \varepsilon_{2i} - \varepsilon_{2r+1}\,, \quad w : \alpha_{2i-1} \leftrightarrow \alpha_{2r-1}\,,$$
and w fixes the other roots in Δ_0.

4. Let $2\xi_i = 2\lambda_{2i} + \cdots + 2\lambda_{2r}$ for $1 \le i \le r$. The basic root for $2\xi_i$ is
$$2\alpha_{2i} + 2\alpha_{2i+1} + \cdots + 2\alpha_{\ell-1} + \alpha_\ell = 2\varepsilon_{2i}\,.$$
Let w be the permutation $2i-1 \leftrightarrow 2r-1$, $2i \leftrightarrow 2r$ that fixes all other indices. Then
$$w : 2\varepsilon_{2i} \leftrightarrow 2\varepsilon_{2r}\,, \quad w : \alpha_{2i-1} \leftrightarrow \alpha_{2r-1}\,,$$
and w fixes the other roots in Δ_0.

Now let $G = \mathbf{Sp}_{4r}(\mathbb{C})$ and $K = \mathbf{Sp}_{2r}(\mathbb{C}) \times \mathbf{Sp}_{2r}(\mathbb{C})$. The K-spherical highest weights are $\mu_i = \varpi_{2i}$ for $i = 1, \ldots, r$. The simple restricted root data are as follows.

restricted root	multiplicity	# basic roots
$\lambda_{2i}\quad (1 \le i \le r-1)$	4	1
λ_{2r}	3	1

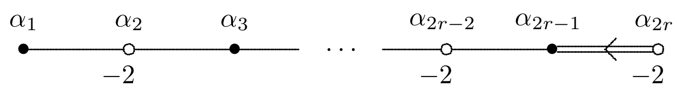


Figure 18: Marked Satake diagram for $\mathbf{Sp}_{4r} / \mathbf{Sp}_{2r} \times \mathbf{Sp}_{2r}$

From the Satake diagram and the fact that $|\operatorname{Supp}\mu_i| = 1$ for $i = 1, \dots, r$, we know that $\mathfrak{c}_0 = 0$ and $\mathfrak{m} \cong \underbrace{\mathfrak{sl}_2 \oplus \cdots \oplus \mathfrak{sl}_2}_{r \text{ copies}}$ and $\mathfrak{h}^0_{\mathfrak{m}} = \alpha_1 + \alpha_3 + \cdots + \alpha_{2r-1}$. This gives the markings on the diagram. The root nests are given by (104) with basic root α_{2i} for $1 \le i \le r-1$, and $\Phi^+\big(\lambda_{2r}\big) = \{\,\alpha_{2r}\,,\ \alpha_{2r-1} + \alpha_{2r}\,,\ 2\alpha_{2r-1} + \alpha_{2r}\,\}$ with basic root α_{2r}. Thus condition (4) of Proposition 8.2 is satisfied by λ_{2i} for $1 \le i \le r-1$ and condition (3) is satisfied by λ_{2r}.

Case 5. **Type DI.** Let $G = \mathbf{SO}_{2\ell}(\mathbb{C})$ and $K = \mathbf{SO}_r(\mathbb{C}) \times \mathbf{SO}_{2\ell-r}(\mathbb{C})$ with $2 \le r < \ell$. The fundamental K-spherical highest weights are $\mu_i = 2\varpi_i$ for $i = 1, \dots, r-1$, together with

$$\mu_r = \begin{cases} \varpi_r & \text{if } r \le \ell - 2\,, \\ \varpi_{\ell-1} + \varpi_\ell & \text{if } r = \ell - 1\,. \end{cases}$$

The simple restricted root data are as follows.

restricted root	mult.	# basic roots
$\lambda_i \ \ (1 \le i \le r-1)$	1	1
λ_r	2	2

restricted root	mult.	# basic roots
$\lambda_i \ \ (1 \le i \le r-1)$	1	1
λ_r	$2(\ell - r)$	1

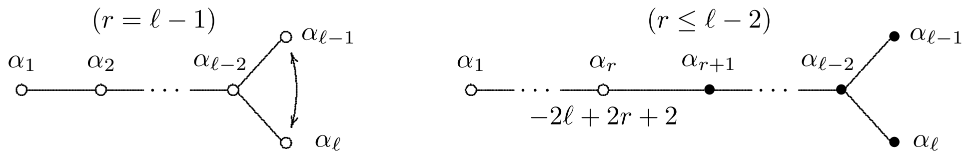


Figure 19: Marked Satake diagrams for $\mathbf{SO}_{2\ell} \,/\, \mathbf{SO}_r \times \mathbf{SO}_{2\ell-r}$

Let $r = \ell - 1$. Then $\Delta_0 = \emptyset$ and $\dim \mathfrak{c}_0 = (r+1) - r = 1$ by Lemma 4.1 since $|\operatorname{Supp}\mu_r| = 2$. Identify $\mathfrak{t}$ with $\mathfrak{t}^*$ using the form $\langle \cdot \mid \cdot \rangle$. If $\mathbf{x} = c_1\alpha_1 + \cdots + c_\ell\alpha_\ell$ is in $\mathfrak{t}$, then the equations $\langle \mu_i \mid \mathbf{x} \rangle = 0$ for $i = 1, \dots, r$ become

$$c_i = 0 \quad \text{for } i = 1, \dots, r-2 \text{ and } \ c_{\ell-1} = -c_\ell\,.$$

Hence $\mathbf{y} = \alpha_{\ell-1} - \alpha_\ell$ is a basis for $\mathfrak{c}_0$. In particular, $\mathbf{y} \mapsto -\mathbf{y}$ under the Dynkin diagram automorphism that fixes α_i for $i = 1, \dots, \ell-2$ and interchanges $\alpha_{\ell-1}$ with α_ℓ, verifying the claim in the proof of Lemma 5.1. Condition (1) of Proposition 8.2 is satisfied by λ_i for $1 \le i \le r-1$ and condition (2) is satisfied by λ_r.

Now assume $r \le \ell - 2$. Then $\dim \mathfrak{c}_0 = 0$ by Lemma 4.1 since $|\operatorname{Supp}\mu_i| = 1$ for $i = 1, \dots, r$. When $r < \ell - 2$, then from the Satake diagram we conclude that $\mathfrak{m} \cong \mathfrak{so}_{2(\ell-r)}$ and hence $\mathfrak{h}^0_{\mathfrak{m}} = 2(\ell - r - 1)\alpha_{r+1} + \cdots$. Thus $\langle \mathfrak{h}^0_{\mathfrak{m}} \mid \alpha_r \rangle = -2\ell + 2r + 1$ as indicated in Figure 19. As in Section 6, Case 3, we have

$$\Phi^+(\lambda_r) = \{\beta_j \,:\, r+1 \le j \le \ell\} \cup \{\gamma_j \,:\, r+1 \le j \le \ell - 1\} \cup \{\alpha_r + \cdots + \alpha_{\ell-2} + \alpha_\ell\}\,,$$

where $\beta_j = \alpha_r + \cdots + \alpha_{j-1}$ and $\gamma_j = \beta_j + 2\alpha_j + \cdots + 2\alpha_{\ell-2} + \alpha_{\ell-1} + \alpha_\ell$ (the roots with coefficient 2 are omitted when $j = \ell - 1$). Thus $|\Phi^+(\xi_1)| = 2\ell - 2r$ and the only basic root in the nest is α_r. Thus condition (1) of Proposition 8.2 is satisfied by λ_i for $1 \le i \le r-1$ and condition (4) is satisfied by λ_r.

Finally, let $r = \ell - 2$. Then $\mathfrak{m} \cong \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ and $\mathfrak{h}^0_{\mathfrak{m}} = \alpha_{\ell-1} + \alpha_\ell$. Hence $\langle \mathfrak{h}^0_{\mathfrak{m}} \mid \alpha_r \rangle = -2$ as indicated in Figure 19 and

$$\Phi^+(\lambda_r) = \{\beta\,,\, \beta + \alpha_{\ell-1}\,,\, \beta + \alpha_\ell\,,\, \beta + \alpha_{\ell-1} + \alpha_\ell\}$$

where $\beta = \alpha_r$ is the basic root. Thus condition (1) of Proposition 8.2 is satisfied by λ_i for $1 \le i \le r-1$ and condition (4) is satisfied by λ_r.

Case 6. **Type D III.** Let $G = \mathbf{SO}_{2\ell}(\mathbb{C})$ and $K = \mathbf{GL}_\ell(\mathbb{C})$ with $\ell \ge 4$. The fundamental K-spherical highest weights are $\mu_i = \varpi_{2i}$ for $i = 1, \ldots, r-1$ together with

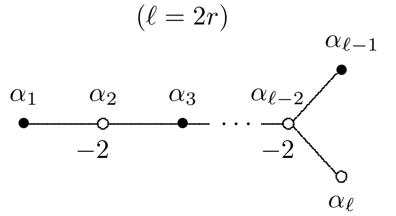
$$\mu_r = \begin{cases} 2\varpi_\ell & \text{when } \ell = 2r\,, \\ \varpi_{\ell-1} + \varpi_\ell & \text{when } \ell = 2r+1\,. \end{cases}$$

The simple restricted root data are as follows.

restricted root	mult.	# basic roots
λ_{2i} $(1 \le i \le r-1)$	4	1
λ_{2r}	1	1

restricted root	mult.	# basic roots
λ_{2i} $(1 \le i \le r-1)$	4	1
λ_{2r}	4	2
$2\lambda_{2r}$	1	1

$(\ell = 2r)$ $(\ell = 2r+1)$



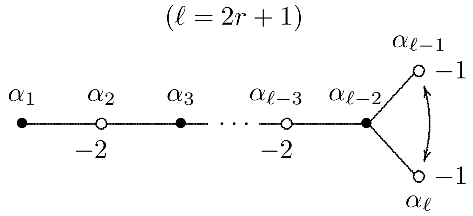


Figure 20: Marked Satake diagrams for $\mathbf{SO}_{2\ell} \,/\, \mathbf{GL}_\ell$

Assume $\ell = 2r$. Then $\dim \mathfrak{c}_0 = 0$ by Lemma 4.1 since $|\operatorname{Supp} \mu_i| = 1$ for $i = 1, \ldots, r$. Hence from the Satake diagram $\mathfrak{m} \cong \mathfrak{sl}_2 \oplus \cdots \oplus \mathfrak{sl}_2$ (r copies) and $\mathfrak{h}^0_{\mathfrak{m}} = \alpha_1 + \alpha_3 + \cdots + \alpha_{\ell-1}$. Thus $\langle \mathfrak{h}^0_{\mathfrak{m}} \mid \alpha_{2i} \rangle = -2$ for $i = 1, \ldots, \ell-1$ as indicated in Figure 20. The root nests are $\Phi^+\big(\lambda_\ell\big) = \{\alpha_\ell\}$ together with

$$\Phi^+\big(\lambda_{2i}\big) = \{\, \alpha_{2i}\,,\, \alpha_{2i-1} + \alpha_{2i}\,,\, \alpha_{2i} + \alpha_{2i+1}\,,\, \alpha_{2i-1} + \alpha_{2i} + \alpha_{2i+1} \,\} \tag{106}$$

for $i = 1, \ldots, r-1$ with basic root α_{2i}. Thus condition (4) of Proposition 8.2 is satisfied by λ_{2i} for $1 \le i \le r-1$ and condition (1) is satisfied by λ_{2r}.

Now assume $\ell = 2r+1$. Then $\dim \mathfrak{c}_0 = (r+1) - r = 1$ by Lemma 4.1 since $|\operatorname{Supp} \mu_r| = 2$. Identify $\mathfrak{t}$ with $\mathfrak{t}^*$ using the form $\langle \cdot \mid \cdot \rangle$. If $\mathbf{x} = c_1\alpha_1 + \cdots + c_\ell\alpha_\ell$ is in $\mathfrak{t}$, then the equations $\langle \mu_i \mid \mathbf{x} \rangle = 0$ for $i = 1, \ldots, r$ become

$$c_{2i} = 0 \quad \text{for } i = 1, \ldots, r-1 \text{ and } c_{\ell-1} = -c_\ell\,.$$

If $\mathbf{x}$ satisfies these equations and $\mathbf{x} \perp \Delta_0$, then $c_{2i-1} = 0$ for $i = 1, \ldots, r-1$. Hence $\mathbf{y} = \alpha_{\ell-1} - \alpha_\ell$ is a basis for $\mathfrak{c}_0$. In particular, $\mathbf{y} \mapsto -\mathbf{y}$ under the Dynkin diagram automorphism that fixes α_i for $i = 1, \dots, \ell - 2$ and interchanges $\alpha_{\ell-1}$ with α_ℓ, verifying the claim in the proof of Lemma 5.1. From the Satake diagram we obtain $\mathfrak{m}' \cong \mathfrak{sl}_2 \oplus \cdots \oplus \mathfrak{sl}_2$ $(r-1$ copies) and $\mathfrak{h}^0_{\mathfrak{m}} = \alpha_1 + \alpha_3 + \cdots \alpha_{\ell-2}$. Hence $\langle \mathfrak{h}^0_{\mathfrak{m}} \mid \alpha_{2i} \rangle = -2$ for $i = 1, \dots, \ell - 3$, while $\langle \mathfrak{h}^0_{\mathfrak{m}} \mid \alpha_i \rangle = -1$ for $i = \ell - 1$ and $i = \ell$, as indicated in Figure 20. The root nest $\Phi^+\big(\lambda_{2i}\big)$ is given by (106) for $i = 1, \dots, r-1$, while

$$\begin{cases} \Phi^+\big(\lambda_{2r}\big) = \{\, \alpha_{\ell-1}\,,\, \alpha_\ell\,,\, \alpha_\ell + \alpha_{\ell-2}\,,\, \alpha_{\ell-1} + \alpha_{\ell-2} \,\} & \text{(basic roots } \alpha_{\ell-1}\,,\, \alpha_\ell)\,, \\ \Phi^+\big(2\lambda_{2r}\big) = \{\, \alpha_{\ell-2} + \alpha_{\ell-1} + \alpha_\ell \,\}\,. \end{cases}$$

Thus condition (4) of Proposition 8.2 is satisfied by λ_{2i} for $1 \le i \le r-1$ and condition (5) is satisfied by λ_ℓ.

Case 7. **Type E II.** Let G be the complex exceptional group of type $\mathbf{E}_6$ and $K = \mathbf{SL}_6 \times \mathbf{SL}_2$. The fundamental K-spherical highest weights are $\mu_1 = \varpi_1 + \varpi_6$, $\mu_2 = \varpi_3 + \varpi_5$, $\mu_3 = 2\varpi_4$, and $\mu_4 = 2\varpi_2$. The simple restricted root data are as follows.

restricted root	multiplicity	# basic roots
$\lambda_2\,,\ \lambda_4$	1	1
$\lambda_1\,,\, \lambda_3$	2	2

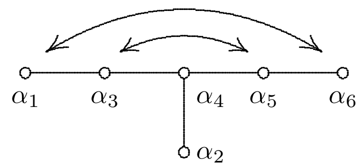


Figure 21: Satake diagram for $\mathbf{E}_6/\,\mathbf{SL}_6 \times \mathbf{SL}_2$

The root data follows from the Satake diagram. Thus condition (1) of Proposition 8.2 is satisfied by λ_2 and λ_4, while condition (2) is satisfied by λ_1 and λ_3.

We have $\Delta_0 = \emptyset$ and $\dim \mathfrak{c}_0 = 6 - 4 = 2$ by Lemma 4.1 since $|\operatorname{Supp} \mu_i| = 2$ for $i = 1, 2$. Identify $\mathfrak{t}$ with $\mathfrak{t}^*$ using the form $\langle \cdot \mid \cdot \rangle$. If $\mathbf{x} = c_1\alpha_1 + \cdots + c_6\alpha_6$ is in $\mathfrak{t}$, then the equations $\langle \mu_i \mid \mathbf{x} \rangle = 0$ for $i = 1, \dots, 4$ become

$$c_i = 0 \quad \text{for } i = 2, 4 \text{ and } \ c_1 = -c_6,\ c_3 = -c_5\,.$$

If $\mathbf{x}$ satisfies these equations then $\mathbf{x} \mapsto -\mathbf{x}$ under the Dynkin diagram automorphism that fixes α_i for $i = 2, 4$ and interchanges α_1 with α_6 and α_3 with α_5, verifying the claim in the proof of Lemma 5.1.

Case 8. **Type E III.** Let G be the complex exceptional group of type $\mathbf{E}_6$ and K the connected subgroup with Lie algebra $\mathfrak{so}_{10}(\mathbb{C}) \oplus \mathfrak{so}_2(\mathbb{C})$. The fundamental K-spherical highest weights are $\varpi_1 + \varpi_6$ and ϖ_2. The simple restricted root data are as follows.

restricted root	multiplicity	# basic roots
λ_2	6	1
λ_1	8	2
$2\lambda_1$	1	1

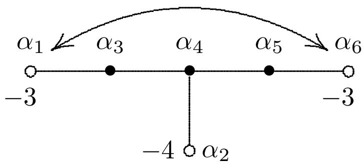


Figure 22: Marked Satake diagram for $\mathbf{E}_6/\,\mathbf{SO}_{10} \times \mathbf{SO}_2$

From the Satake diagram we see that $\mathfrak{m}' \cong \mathfrak{sl}_4$ and thus $h^0_{\mathfrak{m}} = 3\alpha_2 + 4\alpha_4 + 3\alpha_5$. This gives the markings in the Satake diagram. Using the notation of Section 7, Case 8, we obtain a Cartan subspace $\mathfrak{a}$ for G/K by the equation $\xi_1 = \xi_3$; the root nests for the simple restricted roots are

$$\Phi^+(\lambda_1) = \Phi^+(\xi_1) \cup \Phi^+(\xi_3)\,, \quad \Phi^+(\lambda_2) = \Phi^+(\xi_2)\,, \quad \Phi^+(2\lambda_1) = \Phi^+(\xi_1 + \xi_3)\,.$$

Since $h^0_{\mathfrak{m}}$ is the same for G/K and G/H (where $H = \mathbf{SO}_{10}(\mathbb{C})$), the determination of the number of basic roots follows from the calculations in Section 7, Case 8. Thus condition (5) of Proposition 8.2 is satisfied by λ_1 and condition (4) is satisfied by λ_2.

We have $\Delta_0 = \{\alpha_3, \alpha_4, \alpha_5\}$ and $\dim \mathfrak{c}_0 = 3 - 2 = 1$ by Lemma 4.1 since $|\,\mathrm{Supp}\,\mu_1| = 2$. Identify $\mathfrak{t}$ with $\mathfrak{t}^*$ using the form $\langle \cdot \mid \cdot \rangle$. If $\mathbf{x} = c_1\alpha_1 + \cdots + c_6\alpha_6$ is in $\mathfrak{t}$, then the equations $\langle \mu_i \mid \mathbf{x} \rangle = 0$ for $i = 1, 2$ become

$$c_2 = 0 \quad \text{and} \quad c_1 = -c_6\,.$$

The element $\mathbf{x} = 4\alpha_1 + 2\alpha_3 - 2\alpha_5 - 4\alpha_6$ satisfies these equations and $\mathbf{x} \perp \Delta_0$. Hence $\mathbf{x}$ gives a basis for $\mathfrak{c}_0$. We have $\mathbf{x} \mapsto -\mathbf{x}$ under the Dynkin diagram automorphism that fixes α_i for $i = 2, 4$ and interchanges α_1 with α_6 and α_3 with α_5, verifying the claim in the proof of Lemma 5.1.

Case 9. **Type E IV.** Let G be the complex exceptional group of type $\mathbf{E}_6$ and K the complex exceptional group of type $\mathbf{F}_4$. The fundamental K-spherical highest weights are $\mu_1 = \varpi_1$ and $\mu_2 = \varpi_6$. The simple restricted root data are as follows.

restricted root	multiplicity	# basic roots
$\lambda_1\,,\ \lambda_6$	8	1

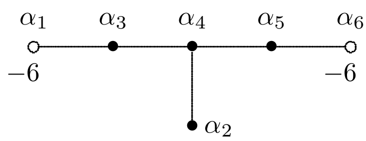


Figure 23: Marked Satake diagram for $\mathbf{E}_6/\mathbf{F}_4$

We have $\mathfrak{c}_0 = 0$ by Lemma 4.1 since $|\,\mathrm{Supp}\,\mu_i| = 1$ for $i = 1, 2$. Hence from the Satake diagram we obtain $\mathfrak{m} = \mathfrak{so}_8$ and $h^0_{\mathfrak{m}} = 6\alpha_3 + 10\alpha_4 + 6\alpha_2 + 6\alpha_5$, which gives the indicated markings. Since $m_\xi = k_\xi + 2$ for both simple restricted roots ξ, condition (4) of Proposition 8.2 is satisfied.

Case 10. **Type E VI.** Let G be the complex exceptional group of type $\mathbf{E}_7$ and let K be the connected subgroup with Lie algebra $\mathfrak{so}_{12}(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$. The fundamental K-spherical highest weights are $\mu_1 = 2\varpi_1$, $\mu_2 = 2\varpi_3$, $\mu_3 = \varpi_4$, and $\mu_4 = \varpi_6$. The simple restricted root data are as follows.

restricted root	multiplicity	# basic roots
λ_1, λ_3	1	1
λ_4, λ_6	4	1

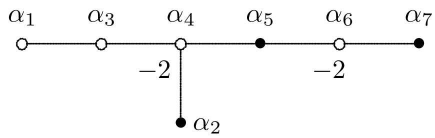


Figure 24: Marked Satake diagram for $\mathbf{E}_7/\,\mathbf{SO}_{12}^{'} \times \mathbf{SL}_2$

We have $\mathfrak{c}_0 = 0$ by Lemma 4.1 since $|\operatorname{Supp}\mu_i| = 1$ for $i = 1, \ldots, 4$. Hence from the Satake diagram we obtain $\mathfrak{m} \cong \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ and $\mathfrak{h}_{\mathfrak{m}}^0 = \alpha_2 + \alpha_5 + \alpha_7$, which gives the indicated markings. Condition (4) of Proposition 8.2 is satisfied by λ_4 and λ_6, while condition (1) is satisfied by λ_1 and λ_3.

Case 11. **Type E VII.** Let G be the complex exceptional group of type $\mathbf{E}_7$ and $K = \mathbf{E}_6 \times \mathbf{SO}_2$. The fundamental K-spherical highest weights are $\mu_1 = \varpi_1$, $\mu_2 = \varpi_6$, and $\mu_3 = 2\varpi_7$. The simple restricted root data are as follows.

restricted root	multiplicity	# basic roots
λ_1, λ_6	8	1
λ_7	1	1

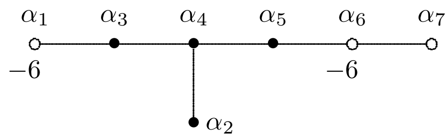


Figure 25: Marked Satake diagram for $\mathbf{E}_7/\mathbf{E}_6 \times \mathbf{SO}_2$

We have $\mathfrak{c}_0 = 0$ by Lemma 4.1 since $|\operatorname{Supp}\mu_i| = 1$ for $i = 1, 2, 3$. Hence from the Satake diagram we obtain $\mathfrak{m} = \mathfrak{so}_8$ and $h_{\mathfrak{m}}^0 = 6\alpha_3 + 10\alpha_4 + 6\alpha_2 + 6\alpha_5$, which gives the indicated markings. Condition (4) of Proposition 8.2 is satisfied by λ_1 and λ_6, while condition (1) is satisfied by λ_7.

Case 12. **Type E IX.** Let G be the complex exceptional group of type $\mathbf{E}_8$ and $K = \mathbf{E}_7 \times \mathbf{SL}_2$. The fundamental K-spherical highest weights are $\mu_1 = \varpi_1$, $\mu_2 = \varpi_6$, $\mu_3 = 2\varpi_7$, and $\mu_4 = 2\varpi_8$. The simple restricted root data are as follows.

restricted root	multiplicity	# basic roots
λ_1, λ_6	8	1
λ_7, λ_8	1	1

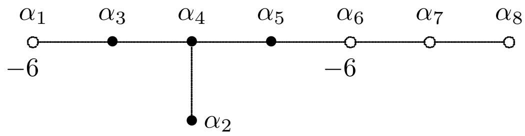


Figure 26: Marked Satake diagram for $\mathbf{E}_8/\mathbf{E}_7 \times \mathbf{SL}_2$

We have $\mathfrak{c}_0 = 0$ by Lemma 4.1 since $|\operatorname{Supp} \mu_i| = 1$ for $i = 1, \ldots, 4$. Hence from the Satake diagram $\mathfrak{m} = \mathfrak{so}_8$ and $h^0_{\mathfrak{m}} = 6\alpha_3 + 10\alpha_4 + 6\alpha_2 + 6\alpha_5$, which gives the indicated markings. Condition (4) of Proposition 8.2 is satisfied by λ_1 and λ_6, while condition (1) is satisfied by λ_7 and λ_8.

Acknowledgment. We would like to thank the referee for an extensive list of corrections and many suggestions for clarifying the exposition and the proofs.

References

[Ad] Adams, J. F. "Lectures on Exceptional Lie Groups", Z. Mahmud and M. Mimura, Eds., University of Chicago Press, 1996.

[Ag] Agricola, I., *Old and New on the Exceptional Group G_2*, Notices Amer. Math. Soc. **55** (2008), 922–929.

[Av1] Avdeev, R. S., *Extended weight semigroups of affine spherical homogeneous spaces of non-simple semisimple algebraic groups*, Izv. Ross. Akad. Nauk Ser. Mat., **71** (2010), 3–26; English transl.: Izv. Math. **74** (2010), 1103–1126.

[Av2] —, *Excellent affine spherical homogeneous spaces of semisimple algebraic groups*, Tr. Mosk. Math. Obs. **74** (2010), 235–269; English transl.: Trans. Moscow Math. Soc. **2010**, 209–240.

[Ba] Baez, J. C., *The octonions*, Bull. Amer. Math. Soc. **39** (2002), 145–205.

[Bo1] Bourbaki, N., ≪Éléments de mathématique, Fascicule XXXIV, Groupes et algèbres de Lie≫, Chapitres IV, V & VI, Hermann, Paris, 1968.

[Bo2] —, ≪Éléments de mathématique, Fascicule XXXVIII, Groupes et algèbres de Lie≫, Chapitres VII & VIII, Hermann, Paris, 1975.

[Br] Brion, M., *Vers une généralisation des espaces symétriques*, J. Algebra **134** (1990), 115–143.

[Gi1] Gindikin, S., *Horospherical Cauchy-Radon transform on compact symmetric spaces*, Mosc. Math. J. **6** (2006), 299–305.

[Gi2] —, *Harmonic analysis on symmetric Stein manifolds from the point of view of complex analysis*, Jap. J. Math. **1** (2006), 87–105.

[Gi3] —, *Horospherical transform on Riemannian symmetric manifolds of non-compact type*, Funkts. Anal. Prilozhen. **42** (2008), 50–59; English transl.: Funct. Analy. Appl. **42** (2008), 290–297.

[Go] Goodman, R., *Harmonic Analysis on Compact Symmetric Spaces: the Legacy of Élie Cartan and Hermann Weyl*, in: "Groups and Analysis: The Legacy of Hermann Weyl", K. Tent (editor), London Mathematical Society Lecture Note Series **354**, Cambridge University Press, 2008, 1–23.

[GW] Goodman, R., and N. R. Wallach, "Symmetry, Representations, and Invariants" (GTM 255), Springer, 2009.

[He1] Helgason, S., "Differential Geometry, Lie Groups, and Symmetric Spaces," Pure and Applied Mathematics **80**, Academic Press, 1978.

[He2] —, "Geometric Analysis on Symmetric Spaces," Mathematical Surveys and Monographs **39**), American Mathematical Society, 1994.

[Hu] Humphreys, J. E., "Linear Algebraic Groups," Graduate Texts in Mathematics **21**, Springer-Verlag, 1975.

[Ko] Kostant, B., *The principal three-dimensional subgroup and the Betti numbers of a complex simple Lie group*, Amer. J. of Math. **81** (1959), 973–1032.

[Kr] Krämer, M., *Sphärische Untergruppen in kompakten zusammenhängenden Liegruppen*, Compositio Math. **38** (1979), 129–153.

[VG] Vinberg, E. B., and S. G. Gindikin, *Degeneration of horospheres in spherical homogeneous spaces* (Russian), unpublished manuscript (2007), 1–15.

[VK] Vinberg, E. B., and B. N. Kimelfeld, *Homogeneous domains on flag manifolds and spherical subgroups of semisimple Lie groups*, Funkts. Anal. Prilozhen. **12** (1978), 12–19. English transl.: Func. Anal. Appl. **12** (1978), 168–174.

[VP] Vinberg, E. B., and V. L. Popov, *On a class of quasihomogeneous affine varieties*, Izv. Akad. Nauk SSR **36** (1972), 749–763. English transl.: Math. USSR Izvestija **6** (1972), 743–758.

[Vr] Vretare, L., *Elementary spherical functions on symmetric spaces*, Math. Scand. **39** (1976), 343–358.

[Wo1] Wolf, J. A., "Spaces of Constant Curvature," 5th Ed., Publish or Perish, Inc., 1984.

[Wo2] —, "Harmonic Analysis on Commutative Spaces," MSM **142**, Amer. Math. Soc., Providence, R. I., 2007.

Simon Gindikin
Department of Mathematics
Rutgers University
110 Frelinghuysen Rd
Piscataway NJ 08854-8019, USA
gindikin@math.rutgers.edu

Roe Goodman
Department of Mathematics
Rutgers University
110 Frelinghuysen Rd
Piscataway NJ 08854-8019, USA
goodman@math.rutgers.edu

Received May 9, 2012
and in final form August 9, 2012