Chapter 3

Algebras and Representations

3.2 Simple Associative Algebras

3.2.1 Wedderburn’s Theorem

An associative algebra $A$ is called simple if the only two-sided ideals in $A$ are 0 and $A$. We now show that a finite-dimensional simple algebra is completely determined by its dimension.

**Theorem 3.2.1 (Wedderburn)** The algebra $\text{End}(V)$ is simple for every finite dimensional complex vector space $V$. Conversely, if $A$ is any finite dimensional simple algebra over $\mathbb{C}$ with unit, then there is a finite dimensional complex vector space $V$ such that $A \cong \text{End}(V)$.

**Proof.** If $u, v$ are nonzero vectors in $V$, then there exists $T \in \text{End}(V)$ so that $Tv = u$ (take $f \in V^*$ with $f(v) = 1$ and define $Tx = f(x)u$ for $x \in V$). Thus $\text{End}(V)v = V$. Now suppose $0 \neq \mathcal{B} \subset \text{End}(V)$ is a two-sided ideal and $0 \neq v \in V$. Then $\mathcal{B}v = \mathcal{B}\text{End}(V)v = \mathcal{BV}$, since $\mathcal{B}$ is a right ideal. But $\mathcal{BV} \neq 0$ since $\mathcal{B} \neq 0$, and $\mathcal{BV}$ is invariant under $\text{End}(V)$ since $\mathcal{B}$ is a left ideal. Hence

$$\mathcal{B}v = V \quad \text{for all } 0 \neq v \in V.$$  

This proves that $V$ is an irreducible $\mathcal{B}$-module. Burnside’s Theorem implies that $\mathcal{B} = \text{End}(V)$. Hence $\text{End}(V)$ is a simple algebra.

Now suppose $A$ is a finite-dimensional simple algebra over $\mathbb{C}$ with unit. Define the left regular representation

$$\lambda : A \to \text{End}(A)$$

by $\lambda(x)y = xy$. Choose a left ideal $V \subset A$ of minimal positive dimension, and define $\rho(x) = \lambda(x)|_V$ for $x \in A$. Then $(\rho, V)$ is an irreducible representation of $A$. Hence $\rho(A) = \text{End}(V)$ by Burnside’s theorem. Furthermore $\text{Ker}(\rho)$ is zero, since it is a two-sided ideal. Thus $A \cong \rho(A)$ as an algebra. ♦
3.2.2 Representations of $\text{End}(V)$

Let $V$ be a finite-dimensional complex vector space. The representation of $\text{End}(V)$ on $V$ is irreducible (see the proof of Theorem 3.2.1). We shall prove that, up to equivalence, this is the unique irreducible representation of $\text{End}(V)$. This will be a consequence of Wedderburn’s Theorem once we prove that every automorphism of $\text{End}(V)$ is inner.

**Scholium 3.2.2** Let $\phi \in \text{Aut}(\text{End}(V))$. Then there exists $g \in \text{GL}(V)$ such that $\phi(x) = gxg^{-1}$ for all $x \in \text{End}(V)$.

**Proof.** Choose a basis $e_1, \ldots, e_n$ for $V$ and let $E_{ij} \in \text{End}(V)$ be the transformation that maps $e_i$ to $e_j$ and annihilates $e_k$ for $k \neq i$. Set $P_i = \phi(E_{ii})$. Since $\phi$ is an automorphism of $\text{End}(V)$, we have

$$P_i^2 = P_i \neq 0, \quad P_i P_j = \delta_{ij} P_j, \quad \sum_{i=1}^n P_i = I_V.$$  

For $i = 1, \ldots, n$ choose $0 \neq f_i \in P_i V$. Then the set $\{f_1, \ldots, f_n\}$ is linearly independent. To prove this, we first note that $P_i f_j = \delta_{ij} f_j$. If $\sum_i c_i f_i = 0$ then

$$0 = P_j \left( \sum_i c_i f_i \right) = c_j f_j.$$  

Thus $c_j = 0$ for all $j$. Since $\dim V = n$, it follows that $\{f_1, \ldots, f_n\}$ is a basis for $V$. Hence there exists $x \in \text{GL}(V)$ such that $xe_i = f_i$ for $i = 1, \ldots, n$. Define $\tilde{\phi} \in \text{Aut}(\text{End}(V))$ by

$$\tilde{\phi}(y) = x^{-1} \phi(y) x.$$  

Then $\tilde{\phi}(E_{ii}) = E_{ii}$, so replacing $\phi$ by $\tilde{\phi}$ we may assume that $\phi(E_{ii}) = E_{ii}$ for $i = 1, \ldots, n$.

We now calculate the action of $\phi$ on the off-diagonal matrix units. With $\phi$ normalized as above, we have

$$\phi(E_{ij}) = \phi(E_{ii}E_{ij}E_{jj}) = E_{ii} \phi(E_{ij}) E_{jj}.$$  

Hence $\phi(E_{ij}) e_k = 0$ for $k \neq j$, and $\phi(E_{ij}) e_j \in \mathbb{C} e_i$. This implies that

$$\phi(E_{ij}) = \lambda_{ij} E_{ij} \quad (3.1)$$  

for some non-zero scalar $\lambda_{ij}$. Since $\phi(E_{ij} E_{jk}) = \phi(E_{ik})$, the scalars $\lambda_{ij}$ satisfy the relations

$$\lambda_{ij} \lambda_{jk} = \lambda_{ik}.$$  

Since we have normalized $\phi$ so that $\lambda_{ii} = 1$, it follows that $\lambda_{ij}^{-1} = \lambda_{ji}$. Set $\lambda_i = \lambda_{ii}$. Then

$$\lambda_{ij} = \lambda_i \lambda_{1j} = \lambda_j \lambda_{ij}^{-1}.$$  

Set $h = \text{diag}(\lambda_1, \ldots, \lambda_n)$. Then

$$h E_{ij} h^{-1} = \lambda_i \lambda_j^{-1} E_{ij} = \lambda_i E_{ij},$$  

so by equation (3.1) we have $h^{-1} \phi(E_{ij}) h = E_{ij}$ for all $i, j$. Hence $h^{-1} \phi(x) h = x$ for all $x \in \text{End}(V)$. Thus $\phi$ is the inner automorphism given by $h$. $\diamondsuit$
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**Proposition 3.2.3** Up to equivalence, the only irreducible representation of \( \text{End}(V) \) is the representation \( \tau \) on \( V \) given by \( \tau(x)v = xv \).

**Proof.** Let \( (\rho, W) \) be an irreducible representation of \( \text{End}(V) \). Wedderburn’s theorem implies that \( \text{End}(V) \cong \text{End}(W) \) as an algebra. Since \( \dim \text{End}(V) = \dim(V)^2 \), we have \( \dim(V) = \dim(W) \). Fix a linear bijection \( T : V \to W \), and define

\[
\phi(x) = T^{-1} \rho(x)T, \quad \text{for } x \in \text{End}(V).
\]

Then \( \phi \) is an automorphism of \( \text{End}(V) \), so by Scholium 3.2.2 there exists \( g \in \text{End}(V) \) such that \( \phi(x) = gxg^{-1} \). Set \( S = (Tg)^{-1} \). Then \( S : W \to V \) and

\[
S\rho(x) = ST\phi(x)T^{-1} = STgxg^{-1}T^{-1} = xS
\]

for \( x \in \text{End}(V) \). Since \( S \) is a linear bijection, we conclude that \( (\rho, W) \cong (V, \tau) \). \( \diamond \)

We now establish a canonical form for an arbitrary finite-dimensional representation of \( \text{End}(V) \). For this we will need the following differentiated version of Scholium 3.2.2. Recall that a derivation of an algebra \( A \) is a map \( D \in \text{End}(A) \) such that \( D(xy) = (Dx)y + x(Dy) \) for all \( x, y \in A \).

**Scholium 3.2.4** Let \( D \) be a derivation of the associative algebra \( \text{End}(V) \). Then there exists \( A \in \text{End}(V) \) such that \( D(x) = Ax - xA \) for all \( x \in \text{End}(V) \).

**Proof.** For \( x, y \in \text{End}(V) \),

\[
D([x, y]) = (Dx)y + x(Dy) - (Dy)x - y(Dx)
= [Dx, y] + [x, Dy],
\]

where \([x, y] = xy - yx\) is the commutator. Thus \( D \) is also a derivation of \( \text{End}(V) \) as a Lie algebra. Write \( I = I_Y \). Then \( D(x) = D(Ix) = D(I)x + D(x) \) for all \( x \in \text{End}(V) \), and hence \( D(I) = 0 \). Let \( g = sl(V) \). Since \( g = [g, g] \) we also have \( Dg \subset g \).

Let \( \text{Der}(g) \subset \text{End}(g) \) be the vector space of all linear transformations \( T \) on \( g \) such that

\[
T([X, Y]) = [TX, Y] + [X, TY] \quad \text{for all } X, Y \in g.
\]

If \( Z \in g \) then \( \text{ad} Z \in \text{Der}(g) \) by the Jacobi identity. Furthermore, if \( T \in \text{Der}(g) \) then

\[
[T, \text{ad}(Z)]X = T([Z, X]) - [Z, T(X)] = [T(Z), X] = \text{ad}(T(Z))X \quad \text{for all } X, Z \in g.
\]

Hence \( [T, \text{ad}(Z)] = \text{ad}(T(Z)) \). This shows that

\[
[\text{ad}(g), \text{Der}(g)] \subset \text{ad}(g). \tag{3.2}
\]

Thus we can obtain a representation \( \rho \) of \( g \) on \( \text{Der}(g) \) by

\[
\rho(Z)T = [\text{ad}(Z), T] \quad \text{for } T \in \text{Der}(g).
\]

Since the subspace \( \text{ad}(g) \) of \( \text{Der}(g) \) is invariant under \( \rho(g) \) and every representation of \( g \) is completely reducible (Theorem 2.4.6), there is a subspace \( U \subset \text{Der}(g) \) so that

\[
\text{Der}(g) = \text{ad}(g) \oplus U, \quad [\text{ad}(g), U] \subset U.
\]
On the other hand, $[\text{ad}(\mathfrak{g}), U] \subset \text{ad}(\mathfrak{g})$ by (3.2). Hence $U = 0$. This proves that

$$\text{Der}(\mathfrak{g}) = \text{ad}(\mathfrak{g}).$$

(3.3)

Returning to the derivation $D$ of $\text{End}(V)$, we conclude that there exists $Z \in \mathfrak{g}$ so that $D(X) = [X, Z]$ for all $X \in \mathfrak{g}$. Since $D(I) = 0$, this equation holds for all $X \in \text{End}(V)$. Thus we may take $A = -Z$. $\blacklozenge$

We now obtain a canonical form for the representations of $\text{End}(V)$. We use the notation $V^m = V \oplus \cdots \oplus V$ to denote the direct sum of $m$ copies of the representation of $\text{End}(V)$ on $V$.

**Theorem 3.2.5** Let $A = \text{End}(V)$ and suppose $(\rho, W)$ is a finite-dimensional representation of $A$. Then $\dim W = m \dim V$, where $m = \dim \text{Hom}_A(V, W)$, and there exists a linear bijection

$$T : W \to V^m, \quad \text{with } Tw = (v_1, \ldots, v_m),$$

such that $T \rho(x)w = (xv_1, \ldots, xv_m)$ for $x \in A$ and $w \in W$. Hence $W$ is equivalent to the $A$-module $\text{Hom}_A(V, W) \otimes V$, where $x \in A$ acts by $x \cdot (u \otimes v) = u \otimes (xv)$ for $u \in \text{Hom}_A(V, W)$ and $v \in V$.

**Proof.** Since $\dim W$ is finite, $W$ contains an irreducible submodule $W_1$. If $W_1 \neq W$ then there is a submodule $W_2 \supset W_1$ such that the representation of $\text{End}(V)$ on $W_2/W_1$ is irreducible. Continuing in this way, we obtain a Jordan-Hölder series

$$W_1 \subset W_2 \subset \cdots \subset W_m = W$$

of submodules with each quotient $W_{i+1}/W_i$ irreducible and hence isomorphic to $V$ by Proposition 3.2.3. In particular,

$$\dim W = m \dim V.$$

We prove the existence of the map $T$ by induction on $m$. When $W = W_1$ we may take $T = I$. Thus we may assume inductively that there are intertwining maps

$$T_1 : W_1 \cong V, \quad T_2 : W/W_1 \cong V^\otimes(m-1).$$

Let $\pi : W \to W/W_1$ be the canonical projection. Choose a subspace $Z \subset W$ so that $W = W_1 \oplus Z$, and let

$$P : W \to Z, \quad Q : W \to W_1$$

be the corresponding projections. (Since $Z$ is not necessarily a $\rho$-invariant subspace, these projections are generally not intertwining operators.) Define a linear bijection

$$T : W \to V^m, \quad T(w_1 + z) = (T_1 w_1, T_2 \pi(z))$$

for $w_1 \in W_1$ and $z \in Z$. Since $T_1, T_2$ and $\pi$ are intertwining maps and $\pi P = \pi$, we have

$$T \rho(x)(w_1 + z) = T(\rho(x)w_1 + Q \rho(x)z + P \rho(x)z) = (xT_1 w_1 + T_1 Q \rho(x)z, xT_2 \pi z)$$
for \( x \in \text{End}(V) \). Thus if \( w \in W \) and we write \( T(w) = (v_1, \ldots, v_m) \) with \( v_i \in V \), then

\[
T \rho(x)w = (xv_1 + \sum_{i=2}^{m} \mu_i(x)v_i, xv_2, \ldots, xv_m), \tag{3.4}
\]

where \( \mu_i(x) \in \text{End}(V) \).

Obviously the maps \( \mu_i(x) \) depend linearly on \( x \). From the equation \( \rho(xy) = \rho(x)\rho(y) \) and equation (3.4) we find that

\[
\sum_{i=2}^{m} \mu_i(xy)v_i = \sum_{i=2}^{m} x\mu_i(y)v_i + \sum_{i=2}^{m} \mu_i(x)yv_i
\]

for all \( v_i \in V \) and \( x, y \in \text{End}(V) \). Hence for \( i = 1, \ldots, m \) we have

\[
\mu_i(xy) = x\mu_i(y) + \mu_i(x)y.
\]

Thus \( \mu_i \) is a derivation of \( \text{End}(V) \). By Scholium 3.2.4 there exists \( A_i \in \text{End}(V) \) so that \( \mu_i(x) = [A_i, x] \).

We have now shown that \( \rho \) is equivalent to the representation \( \tilde{\rho} \) on \( V^m \) given by

\[
\tilde{\rho}(x)(v_1, \ldots, v_m) = (xv_1 + \sum_{i=2}^{m} [A_i, x]v_i, xv_2, \ldots, xv_m).
\]

Define a linear transformation \( g \) on \( V^m \) by

\[
g \cdot (v_1, \ldots, v_m) = (v_1 + \sum_{i=2}^{m} A_i v_i, v_2, \ldots, v_m).
\]

Then \( g \) is a linear bijection, with inverse

\[
g^{-1} \cdot (v_1, \ldots, v_m) = (v_1 - \sum_{i=2}^{m} A_i v_i, v_2, \ldots, v_m).
\]

It follows that

\[
g^{-1} \tilde{\rho}(x)g(v_1, \ldots, v_m) = (xv_1, \ldots, xv_m).
\]

Thus \( \rho \) is equivalent to the direct sum of \( m \) copies of the representation of \( \text{End}(V) \) on \( V \). 
\( \diamond \)