Chapter 3

Algebras and Representations

3.2 Simple Associative Algebras

3.2.1 Wedderburn's Theorem

An associative algebra $\mathcal{A}$ is called *simple* if the only two-sided ideals in $\mathcal{A}$ are 0 and $\mathcal{A}$. We now show that a finite-dimensional simple algebra is completely determined by its dimension.

Theorem 3.2.1 (Wedderburn) *The algebra* $\mathrm{End}(V)$ *is simple for every finite dimensional complex vector space* V*. Conversely, if* $\mathcal{A}$ *is any finite dimensional simple algebra over* $\mathbb{C}$ *with unit, then there is a finite dimensional complex vector space* V *such that* $\mathcal{A} \cong \mathrm{End}(V)$*.*

Proof. If u, v are nonzero vectors in V, then there exists $T \in \mathrm{End}(V)$ so that $Tv = u$ (take $f \in V^*$ with $f(v) = 1$ and define $Tx = f(x)u$ for $x \in V$). Thus $\mathrm{End}(V)v = V$. Now suppose $0 \neq \mathcal{B} \subset \mathrm{End}(V)$ is a two-sided ideal and $0 \neq v \in V$. Then $\mathcal{B}v = \mathcal{B}\,\mathrm{End}(V)v = \mathcal{B}V$, since $\mathcal{B}$ is a right ideal. But $\mathcal{B}V \neq 0$ since $\mathcal{B} \neq 0$, and $\mathcal{B}V$ is invariant under $\mathrm{End}(V)$ since $\mathcal{B}$ is a left ideal. Hence

$$\mathcal{B}v = V \qquad \text{for all } 0 \neq v \in V.$$

This proves that V is an irreducible $\mathcal{B}$-module. Burnside's Theorem implies that $\mathcal{B} = \mathrm{End}(V)$. Hence $\mathrm{End}(V)$ is a simple algebra.

Now suppose $\mathcal{A}$ is a finite-dimensional simple algebra over $\mathbb{C}$ with unit. Define the left regular representation

$$\lambda : \mathcal{A} \to \mathrm{End}(\mathcal{A})$$

by $\lambda(x)y = xy$. Choose a left ideal $V \subset \mathcal{A}$ of minimal positive dimension, and define $\rho(x) = \lambda(x)|_V$ for $x \in \mathcal{A}$. Then (ρ, V) is an irreducible representation of $\mathcal{A}$. Hence $\rho(\mathcal{A}) = \mathrm{End}(V)$ by Burnside's theorem. Furthermore $\mathrm{Ker}(\rho)$ is zero, since it is a two-sided ideal. Thus $\mathcal{A} \cong \rho(\mathcal{A})$ as an algebra. $\Diamond$

3.2.2 Representations of End(V)

Let V be a finite-dimensional complex vector space. The representation of End(V) on V is irreducible (see the proof of Theorem 3.2.1). We shall prove that, up to equivalence, this is the *unique* irreducible representation of End(V). This will be a consequence of Wedderburn's Theorem once we prove that every automorphism of End(V) is inner.

Scholium 3.2.2 *Let $\phi \in \mathrm{Aut}(\mathrm{End}(V))$. Then there exists $g \in \mathrm{GL}(V)$ such that $\phi(x) = gxg^{-1}$ for all $x \in \mathrm{End}(V)$.*

Proof. Choose a basis $e_1, \ldots, e_n$ for V and let $E_{ij} \in \mathrm{End}(V)$ be the transformation that maps e_i to e_j and annihilates e_k for $k \neq i$. Set $P_i = \phi(E_{ii})$. Since ϕ is an automorphism of End(V), we have

$$P_i^2 = P_i \neq 0, \quad P_i P_j = \delta_{ij} P_j, \quad \sum_{i=1}^{n} P_i = I_V.$$

For $i = 1, \ldots n$ choose $0 \neq f_i \in P_i V$. Then the set $\{f_1, \ldots, f_n\}$ is linearly independent. To prove this, we first note that $P_i f_j = \delta_{ij} f_j$. If $\sum_i c_i f_i = 0$ then

$$0 = P_j \left(\sum_i c_i f_i \right) = c_j f_j.$$

Thus $c_j = 0$ for all j. Since $\dim V = n$, it follows that $\{f_1, \ldots, f_n\}$ is a basis for V. Hence there exists $x \in \mathrm{GL}(V)$ such that $xe_i = f_i$ for $i = 1, \ldots, n$. Define $\tilde{\phi} \in \mathrm{Aut}(\mathrm{End}(V))$ by

$$\tilde{\phi}(y) = x^{-1} \phi(y) x.$$

Then $\tilde{\phi}(E_{ii}) = E_{ii}$, so replacing ϕ by $\tilde{\phi}$ we may assume that $\phi(E_{ii}) = E_{ii}$ for $i = 1, \ldots, n$.

We now calculate the action of ϕ on the off-diagonal matrix units. With ϕ normalized as above, we have

$$\phi(E_{ij}) = \phi(E_{ii} E_{ij} E_{jj}) = E_{ii} \phi(E_{ij}) E_{jj}.$$

Hence $\phi(E_{ij}) e_k = 0$ for $k \neq j$, and $\phi(E_{ij}) e_j \in \mathbb{C} e_i$. This implies that

$$\phi(E_{ij}) = \lambda_{ij} E_{ij} \tag{3.1}$$

for some non-zero scalar λ_{ij}. Since $\phi(E_{ij} E_{jk}) = \phi(E_{ik})$, the scalars λ_{ij} satisfy the relations

$$\lambda_{ij} \lambda_{jk} = \lambda_{ik}.$$

Since we have normalized ϕ so that $\lambda_{ii} = 1$, it follows that $\lambda_{ij}^{-1} = \lambda_{ji}$. Set $\lambda_i = \lambda_{i1}$. Then

$$\lambda_{ij} = \lambda_{i1} \lambda_{1j} = \lambda_i \lambda_j^{-1}.$$

Set $h = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. Then

$$h E_{ij} h^{-1} = \lambda_i \lambda_j^{-1} E_{ij} = \lambda_{ij} E_{ij},$$

so by equation (3.1) we have $h^{-1} \phi(E_{ij}) h = E_{ij}$ for all i, j. Hence $h^{-1} \phi(x) h = x$ for all $x \in \mathrm{End}(V)$. Thus ϕ is the inner automorphism given by h. $\Diamond$

Proposition 3.2.3 *Up to equivalence, the only irreducible representation of* $\mathrm{End}(V)$ *is the representation* τ *on* V *given by* $\tau(x)v = xv$.

Proof. Let (ρ, W) be an irreducible representation of $\mathrm{End}(V)$. Wedderburn's theorem implies that $\mathrm{End}(V) \cong \mathrm{End}(W)$ as an algebra. Since $\dim \mathrm{End}(V) = \dim(V)^2$, we have $\dim(V) = \dim(W)$. Fix a linear bijection $T : V \to W$, and define

$$\phi(x) = T^{-1}\rho(x)T, \quad \text{for } x \in \mathrm{End}(V).$$

Then ϕ is an automorphism of $\mathrm{End}(V)$, so by Scholium 3.2.2 there exists $g \in \mathrm{End}(V)$ such that $\phi(x) = gxg^{-1}$. Set $S = (Tg)^{-1}$. Then $S : W \to V$ and

$$S\rho(x) = ST\phi(x)T^{-1} = STgxg^{-1}T^{-1} = xS$$

for $x \in \mathrm{End}(V)$. Since S is a linear bijection, we conclude that $(\rho, W) \cong (V, \tau)$. $\Diamond$

We now establish a canonical form for an arbitrary finite-dimensional representation of $\mathrm{End}(V)$. For this we will need the following differentiated version of Scholium 3.2.2. Recall that a *derivation* of an algebra $\mathcal{A}$ is a map $D \in \mathrm{End}(\mathcal{A})$ such that $D(xy) = (Dx)y + x(Dy)$ for all $x, y \in \mathcal{A}$.

Scholium 3.2.4 *Let* D *be a derivation of the associative algebra* $\mathrm{End}(V)$. *Then there exists* $A \in \mathrm{End}(V)$ *such that* $D(x) = Ax - xA$ *for all* $x \in \mathrm{End}(V)$.

Proof. For $x, y \in \mathrm{End}(V)$,

$$\begin{aligned} D([x,y]) &= (Dx)y + x(Dy) - (Dy)x - y(Dx) \\ &= [Dx, y] + [x, Dy], \end{aligned}$$

where $[x, y] = xy - yx$ is the commutator. Thus D is also a derivation of $\mathrm{End}(V)$ as a Lie algebra. Write $I = I_V$. Then $D(x) = D(Ix) = D(I)x + D(x)$ for all $x \in \mathrm{End}(V)$, and hence $D(I) = 0$. Let $\mathfrak{g} = \mathfrak{sl}(V)$. Since $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ we also have $D\mathfrak{g} \subset \mathfrak{g}$.

Let $\mathrm{Der}(\mathfrak{g}) \subset \mathrm{End}(\mathfrak{g})$ be the vector space of all linear transformations T on $\mathfrak{g}$ such that

$$T([X, Y]) = [TX, Y] + [X, TY] \quad \text{for all } X, Y \in \mathfrak{g}.$$

If $Z \in \mathfrak{g}$ then $\mathrm{ad}\, Z \in \mathrm{Der}(\mathfrak{g})$ by the Jacobi identity. Furthermore, if $T \in \mathrm{Der}(\mathfrak{g})$ then

$$[T, \mathrm{ad}(Z)]X = T([Z, X]) - [Z, T(X)] = [T(Z), X] = \mathrm{ad}(T(Z))X \quad \text{for all } X, Z \in \mathfrak{g}.$$

Hence $[T, \mathrm{ad}(Z)] = \mathrm{ad}(T(Z))$. This shows that

$$[\mathrm{ad}(\mathfrak{g}), \mathrm{Der}(\mathfrak{g})] \subset \mathrm{ad}(\mathfrak{g}). \tag{3.2}$$

Thus we can obtain a representation ρ of $\mathfrak{g}$ on $\mathrm{Der}(\mathfrak{g})$ by

$$\rho(Z)T = [\mathrm{ad}(Z), T] \quad \text{for } T \in \mathrm{Der}(\mathfrak{g}).$$

Since the subspace $\mathrm{ad}(\mathfrak{g})$ of $\mathrm{Der}(\mathfrak{g})$ is invariant under $\rho(\mathfrak{g})$ and every representation of $\mathfrak{g}$ is completely reducible (Theorem 2.4.6), there is a subspace $U \subset \mathrm{Der}(\mathfrak{g})$ so that

$$\mathrm{Der}(\mathfrak{g}) = \mathrm{ad}(\mathfrak{g}) \oplus U, \quad [\mathrm{ad}(\mathfrak{g}), U] \subset U.$$

On the other hand, $[\mathrm{ad}(\mathfrak{g}), U] \subset \mathrm{ad}(\mathfrak{g})$ by (3.2). Hence $U = 0$. This proves that

$$\mathrm{Der}(\mathfrak{g}) = \mathrm{ad}(\mathfrak{g}). \tag{3.3}$$

Returning to the derivation D of $\mathrm{End}(V)$, we conclude that there exists $Z \in \mathfrak{g}$ so that $D(X) = [X, Z]$ for all $X \in \mathfrak{g}$. Since $D(I) = 0$, this equation holds for all $X \in \mathrm{End}(V)$. Thus we may take $A = -Z$. ◊

We now obtain a canonical form for the representations of $\mathrm{End}(V)$. We use the notation

$$V^m = \underbrace{V \oplus \cdots \oplus V}_{m \text{ copies}}$$

to denote the direct sum of m copies of the representation of $\mathrm{End}(V)$ on V.

Theorem 3.2.5 *Let $\mathcal{A} = \mathrm{End}(V)$ and suppose (ρ, W) is a finite-dimensional representation of $\mathcal{A}$. Then $\dim W = m \dim V$, where $m = \dim \mathrm{Hom}_{\mathcal{A}}(V, W)$, and there exists a linear bijection*

$$T : W \to V^m, \qquad \text{with } Tw = (v_1, \ldots, v_m),$$

such that $T\rho(x)w = (xv_1, \ldots, xv_m)$ for $x \in \mathcal{A}$ and $w \in W$. Hence W is equivalent to the $\mathcal{A}$-module $\mathrm{Hom}_{\mathcal{A}}(V, W) \otimes V$, where $x \in \mathcal{A}$ acts by $x \cdot (u \otimes v) = u \otimes (xv)$ for $u \in \mathrm{Hom}_{\mathcal{A}}(V, W)$ and $v \in V$.

Proof. Since $\dim W$ is finite, W contains an irreducible submodule W_1. If $W_1 \neq W$ then there is a submodule $W_2 \supset W_1$ such that the representation of $\mathrm{End}(V)$ on W_2/W_1 is irreducible. Continuing in this way, we obtain a *Jordan-Hölder series*

$$W_1 \subset W_2 \subset \cdots \subset W_m = W$$

of submodules with each quotient W_{i+1}/W_i irreducible and hence isomorphic to V by Proposition 3.2.3. In particular,

$$\dim W = m \dim V.$$

We prove the existence of the map T by induction on m. When $W = W_1$ we may take $T = I$. Thus we may assume inductively that there are intertwining maps

$$T_1 : W_1 \cong V, \quad T_2 : W/W_1 \cong V^{\otimes(m-1)}.$$

Let $\pi : W \to W/W_1$ be the canonical projection. Choose a subspace $Z \subset W$ so that $W = W_1 \oplus Z$, and let

$$P : W \to Z, \quad Q : W \to W_1$$

be the corresponding projections. (Since Z is not necessarily a ρ-invariant subspace, these projections are generally not intertwining operators.) Define a linear bijection

$$T : W \to V^m, \quad T(w_1 + z) = (T_1 w_1, T_2 \pi(z))$$

for $w_1 \in W_1$ and $z \in Z$. Since T_1, T_2 and π are intertwining maps and $\pi P = \pi$, we have

$$\begin{aligned} T\rho(x)(w_1 + z) &= T(\rho(x)w_1 + Q\rho(x)z + P\rho(x)z) \\ &= (xT_1 w_1 + T_1 Q\rho(x)z, xT_2 \pi z) \end{aligned}$$

for $x \in \mathrm{End}(V)$. Thus if $w \in W$ and we write $T(w) = (v_1, \ldots, v_m)$ with $v_i \in V$, then

$$T\rho(x)w = (xv_1 + \sum_{i=2}^{m} \mu_i(x)v_i,\ xv_2, \ldots, xv_m), \tag{3.4}$$

where $\mu_i(x) \in \mathrm{End}(V)$.

Obviously the maps $\mu_i(x)$ depend linearly on x. From the equation $\rho(xy) = \rho(x)\rho(y)$ and equation (3.4) we find that

$$\sum_{i=2}^{m} \mu_i(xy)v_i = \sum_{i=2}^{m} x\mu_i(y)v_i + \sum_{i=2}^{m} \mu_i(x)yv_i$$

for all $v_i \in V$ and $x, y \in \mathrm{End}(V)$. Hence for $i = 1, \ldots, m$ we have

$$\mu_i(xy) = x\mu_i(y) + \mu_i(x)y.$$

Thus μ_i is a derivation of $\mathrm{End}(V)$. By Scholium 3.2.4 there exists $A_i \in \mathrm{End}(V)$ so that $\mu_i(x) = [A_i, x]$.

We have now shown that ρ is equivalent to the representation $\tilde{\rho}$ on V^m given by

$$\tilde{\rho}(x)(v_1, \ldots, v_m) = (xv_1 + \sum_{i=2}^{m} [A_i, x]v_i,\ xv_2, \ldots, xv_m).$$

Define a linear transformation g on V^m by

$$g \cdot (v_1, \ldots, v_m) = (v_1 + \sum_{i=2}^{m} A_i v_i,\ v_2, \ldots, v_m).$$

Then g is a linear bijection, with inverse

$$g^{-1} \cdot (v_1, \ldots, v_m) = (v_1 - \sum_{i=2}^{m} A_i v_i,\ v_2, \ldots, v_m).$$

It follows that

$$g^{-1}\tilde{\rho}(x)g(v_1, \ldots, v_m) = (xv_1, \ldots, xv_m).$$

Thus ρ is equivalent to the direct sum of m copies of the representation of $\mathrm{End}(V)$ on V. $\Diamond$