Chapter 3

Algebras and Representations

In this chapter we develop the basic facts about finite-dimensional representations of associative algebras: Schur's Lemma, Burnside's Theorem, the theorem of Wedderburn characterizing simple algebras and direct sums of simple algebras, complete reducibility of representations, and the double commutant theorem. The duality between a semisimple algebra of endomorphisms and its commutant is a key aspect of representation theory, and its implications for representations of the classical groups will be worked out in later chapters. We study the representations of a finite group through its group algebra and characters, and we construct induced representations and calculate their characters.

3.1 Representations of Associative Algebras

3.1.1 Definitions and Examples

We know from the previous chapter that every regular representation (ρ, V) of a classical group G decomposes into a direct sum of irreducible representations. The next task is to determine the extent of uniqueness of such a decomposition and to find explicit projection operators onto irreducible subspaces of V. In the tradition of modern mathematics we will attack these problems by putting them in a more general (abstract) context. We first allow G to be an arbitrary group. We next introduce an algebra whose representation theory contains that of G.

Consider linear operators on V of the form

$$T = \sum_{i=1}^{N} a_i \rho(g_i)$$

where $a_i \in \mathbb{C}$, $g_i \in G$, and $N < \infty$. Suppose $W \subset V$ is a linear subspace. If W is invariant under G and $w \in W$, then $Tw \in W$, since $\rho(g_i)w \in W$ for all i. Conversely, if $TW \subset W$ for all such operators T, then $\rho(G)W \subset W$, since we can take $T = \rho(g)$ with g arbitrary in G. It is convenient to use the notation

$$T = \sum_{g \in G} c(g)\, \rho(g), \tag{3.1}$$

where the coefficients $c(g) \in \mathbb{C}$ and only a finite number of them are nonzero. The set of operators of the form (3.1) includes all the operators $\rho(g)$ for $g \in G$, of course. By its very definition it is a linear subspace of $\mathrm{End}(V)$. Furthermore, if $A = \sum_g a(g)\rho(g)$ and $B = \sum_g b(g)\rho(g)$ are two such operators, then

$$AB = \sum_g c(g)\rho(g), \qquad \text{with } c(g) = \sum_{xy=g} a(x)b(y).$$

Thus this set of operators is a *subalgebra* of the associative algebra $\mathrm{End}(V)$ with the same invariant subspaces as G. Furthermore, an operator $R \in \mathrm{End}(V)$ commutes with the action of G if and only if it commutes with all the operators of the form (3.1). The advantage of considering all these linear operators, instead of just the operators $\rho(G)$, is that we can use techniques from ring theory (ideals, images, kernels) to study group representations.

We proceed to study algebras of linear operators in more detail. We will first view them as *abstract* algebras, and then study their representations as algebras of linear transformations.

An *associative algebra* over the complex field $\mathbb{C}$ is a vector space $\mathcal{A}$ over $\mathbb{C}$ together with a bilinear multiplication map

$$\mu : \mathcal{A} \times \mathcal{A} \to \mathcal{A}, \quad x, y \mapsto xy = \mu(x, y),$$

such that $(xy)z = x(yz)$. The algebra $\mathcal{A}$ is said to have an *identity element* if there exists $e \in \mathcal{A}$ such that $ae = ea = a$ for all $a \in \mathcal{A}$. If $\mathcal{A}$ has an identity element it is unique and we will usually use the notation 1 for e.

Examples of Associative Algebras

1. Let V be a vector space over $\mathbb{C}$, and let $\mathcal{A} = \mathrm{End}(V)$ be the space of $\mathbb{C}$-linear transformations on V. Then $\mathcal{A}$ is an associative algebra, with multiplication the composition of transformations. When $\dim V = n < \infty$, then this algebra has a basis consisting of the n^2 *elementary matrices* E_{ij}, for $1 \le i, j \le n$, which multiply by

$$E_{ij}E_{km} = \delta_{jk}E_{im}.$$

This algebra will play a fundamental role in our study of associative algebras and their representations.

2. Let G be any group (not necessarily a linear algebraic group). We define an associative algebra $\mathbb{C}[G]$, called the *group algebra* of G, as follows. As a vector space, $\mathbb{C}[G]$ is the set of all functions $f : G \to \mathbb{C}$ such that the *support* of f (the set where $f(g) \neq 0$) is finite. This space has a basis consisting of the functions $\{\delta_g : g \in G\}$, where

$$\delta_g(x) = \begin{cases} 1 & \text{if } x = g \\ 0 & \text{otherwise} \end{cases}$$

Thus an element x of $\mathbb{C}[G]$ has a unique expression as a formal sum

$$\sum_{g \in G} x(g)\, \delta_g$$

(with only a finite number of coefficients $x(g) \neq 0$).

We identify G with the elements δ_g in $\mathbb{C}[G]$, and we define multiplication on $\mathbb{C}[G]$ as the bilinear extension of group multiplication. Thus, given functions $x, y \in \mathbb{C}[G]$, we define their product $x * y$ by

$$\Big(\sum x(g)\,\delta_g\Big) * \Big(\sum y(h)\,\delta_h\Big) = \sum x(g)y(h)\,\delta_{gh},$$

with the sum over $g, h \in G$. (We indicate the multiplication by $*$ so it will not be confused with the pointwise multiplication of functions on G.) This product is associative by the associativity of group multiplication. The identity element $1 \in G$ becomes the element δ_1 in $\mathbb{C}[G]$ and G is a subgroup of the group of invertible elements of $\mathbb{C}[G]$. The function $x * y$ is called the *convolution* of the functions x and y; from the definition it is clear that

$$(x * y)(g) = \sum_{hk=g} x(h)y(k) = \sum_{h \in G} x(h)y(h^{-1}g).$$

If H is a group and $\phi : G \to H$ is a group homomorphism, then ϕ extends uniquely to a linear map $\widetilde{\phi} : \mathbb{C}[G] \to \mathbb{C}[H]$ by the rule

$$\widetilde{\phi}\Big(\sum x(g)\delta_g\Big) = \sum x(g)\delta_{\phi(g)}\,.$$

From the definition of multiplication in $\mathbb{C}[G]$ we see that the extended map $\widetilde{\phi}$ is an associative algebra homomorphism. Furthermore, if K is another group and $\psi : H \to K$ is a group homomorphism, then $\widetilde{\psi \circ \phi} = \widetilde{\psi} \circ \widetilde{\phi}$.

An important special case occurs when G is a subgroup of H and ϕ is the inclusion map. Then $\widetilde{\phi}$ is injective (since the δ_g form a basis of $\mathbb{C}[G]$). Thus we can identify $\mathbb{C}[G]$ with the subalgebra of $\mathbb{C}[H]$ consisting of functions supported on G.

3. Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$. Just as in the case of group algebras, there is an associative algebra $U(\mathfrak{g})$ (the *universal enveloping algebra* of $\mathfrak{g}$) and an injective linear map $j : \mathfrak{g} \to U(\mathfrak{g})$ such that $j(\mathfrak{g})$ generates $U(\mathfrak{g})$ and

$$j([X, Y]) = j(X)j(Y) - j(Y)j(X)$$

(the multiplication on the right is in $U(\mathfrak{g})$; see Appendix C.2.1 and Theorem C.2.4). Since $U(\mathfrak{g})$ is uniquely determined by $\mathfrak{g}$, up to isomorphism, we will identify $\mathfrak{g}$ with $j(\mathfrak{g})$. If $\mathfrak{h} \subset \mathfrak{g}$ is a Lie subalgebra then the Poincaré-Birkoff-Witt Theorem C.2.4 allows us to identify $U(\mathfrak{h})$ with the associative subalgebra of $U(\mathfrak{g})$ generated by $\mathfrak{h}$, so we have the same situation as for the group algebra of a subgroup $H \subset G$.

Let $\mathcal{A}$ be an associative algebra over the complex field $\mathbb{C}$. A *representation* of $\mathcal{A}$ is a pair (ρ, V), where V is a vector space over $\mathbb{C}$ and $\rho : \mathcal{A} \to \mathrm{End}(V)$ is an associative algebra homomorphism. If $\mathcal{A}$ has an identity element 1, then we require that $\rho(1)$ act as the identity transformation I_V on V. When the map ρ is understood from the context, we shall call V an $\mathcal{A}$-*module* and write av for $\rho(a)v$.

If V, W are both $\mathcal{A}$-modules, then we make the vector space $V \oplus W$ into an $\mathcal{A}$-module by the action $a \cdot (v \oplus w) = av \oplus aw$.

If $U \subset V$ is a linear subspace such that $\rho(a)U \subset U$ for all $a \in \mathcal{A}$, then we say that U is *invariant* under the representation. In this case we can define a representation (ρ_U, U) by the restriction of $\rho(\mathcal{A})$ to U, and a representation $(\rho_{V/U}, V/U)$ by the natural quotient action of $\rho(\mathcal{A})$ on V/U. A representation (ρ, V) is *irreducible* if the only invariant subspaces are $\{0\}$ and V.

Define $\mathrm{Ker}(\rho) = \{x \in \mathcal{A} : \rho(x) = 0\}$. This is a two-sided ideal in $\mathcal{A}$, and V is a module for the quotient algebra $\mathcal{A}/\mathrm{Ker}(\rho)$ via the natural quotient map. A representation ρ is *faithful* if $\mathrm{Ker}(\rho) = 0$.

Let (ρ, V) and (τ, W) be representations of $\mathcal{A}$, and let $\mathrm{Hom}(V, W)$ be the space of $\mathbb{C}$-linear maps from V to W. We denote by $\mathrm{Hom}_{\mathcal{A}}(V, W)$ the set of all $T \in \mathrm{Hom}(V, W)$ such that $T\rho(a) = \tau(a)T$ for all $a \in \mathcal{A}$. Such a map is called an *intertwining operator* between the two representations or a *module homomorphism*. For example, if $U \subset V$ is an invariant subspace, then the inclusion map $U \to V$ and the quotient map $V \to V/U$ are intertwining operators. The representations (ρ, V) and (τ, W) are *equivalent* if there exists an invertible operator in $\mathrm{Hom}_{\mathcal{A}}(V, W)$. In this case we write $(\rho, V) \cong (\tau, W)$.

The composition of two intertwining operators, when defined, is again an intertwining operator. In particular, when $V = W$ and $\rho = \tau$, then $\mathrm{Hom}_{\mathcal{A}}(V, V)$ is an associative algebra, which we denote by $\mathrm{End}_{\mathcal{A}}(V)$.

Examples of Representations

1. Let $\mathcal{A} = \mathbb{C}[x]$. Let V be a finite-dimensional vector space, and let $T \in \mathrm{End}(V)$. Define a representation (ρ, V) of $\mathcal{A}$ by $\rho(f) = f(T)$ for $f \in \mathbb{C}[x]$. Then $\mathrm{Ker}(\rho)$ is the ideal in $\mathcal{A}$ generated by the *minimal polynomial* of T. The problem of finding a canonical form for this representation is the same as finding the Jordan canonical form for T (see Section B.1.2).

2. Let G be a group and let $\mathcal{A} = \mathbb{C}[G]$ be the group algebra of G. If (ρ, V) is a representation of $\mathcal{A}$, then the map $g \mapsto \rho(\delta_g)$ is a group homomorphism from G to $\mathrm{GL}(V)$. Conversely, every representation $\pi : G \to \mathrm{GL}(V)$ extends uniquely to a representation ρ of $\mathbb{C}[G]$ on V by

$$\rho(f) = \sum_{g \in G} f(g)\pi(g)$$

for $f \in \mathbb{C}[G]$. We shall use the same symbol to denote a representation of a group and the group algebra.

Two important new constructions are possible in the case of group representations. The first is the *contragredient* or *dual* representation $(\rho^*,\ V^*)$, where

$$(\rho^*(g)f)(v) = f(\rho(g^{-1})v)$$

for $g \in G,\ v \in V$ and $f \in V^*$. The second is the *tensor product* $(\rho \otimes \sigma,\ V \otimes W)$ of two representations by

$$(\rho \otimes \sigma)(g)(v \otimes w) = \rho(g)v \otimes \sigma(g)w.$$

For example, let (ρ, V) and (σ, W) be finite-dimensional representations of G. There is a representation π of G on $\mathrm{Hom}(V, W)$ by

$$\pi(g)T = \sigma(g)T\rho(g)^{-1}, \quad \text{for } T \in \mathrm{Hom}(V, W).$$

There is a natural linear isomorphism

$$\mathrm{Hom}(V, W) \cong W \otimes V^* \tag{3.2}$$

(see Section B.2.2). Here a tensor of the form $w \otimes v^*$ gives the linear transformation $Tv = \langle v^*, v\rangle w$ from V to W. Since the tensor $\sigma(g)w \otimes \rho^*(g)v^*$ gives the linear transformation

$$v \mapsto \langle \rho^*(g)v^*, v\rangle \sigma(g)w = \langle v^*, \rho(g)^{-1}v\rangle \sigma(g)w = \sigma(g)T\rho(g)^{-1}v,$$

we see that π is equivalent to $\sigma \otimes \rho^*$. In particular, the space $\mathrm{Hom}_G(V, W)$ of G-intertwining maps between V and W corresponds to the space $(W \otimes V^*)^G$ of G-fixed elements in $W \otimes V^*$.

We can iterate the tensor product construction to obtain G-modules $\bigotimes^k V = V^{\otimes k}$ (the k-fold tensor product of V with itself) with $g \in G$ acting by

$$\rho^{\otimes k}(g)(v_1 \otimes \cdots \otimes v_k) = \rho(g)v_1 \otimes \cdots \otimes \rho(g)v_k$$

on decomposable tensors. The subspaces $S^k(V)$ (symmetric tensors) and $\bigwedge^k V$ (skew-symmetric tensors) are G-invariant (see Sections B.2.3 and B.2.4). These modules are called the *symmetric* and *skew-symmetric* powers of ρ.

The contragredient and tensor product constructions for group representations are associated with the *inversion map* $g \mapsto g^{-1}$ and the *diagonal map* $g \mapsto (g, g)$. The properties of these maps can be described axiomatically using the notion of a *Hopf algebra* (see Exercises 3.1.5).

3. Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$, and let (ρ, V) be a representation of $\mathfrak{g}$. The universal mapping property implies that ρ extends uniquely to a representation of $U(\mathfrak{g})$ (see Section C.2.1), and that every representation of $\mathfrak{g}$ comes from a unique representation of $U(\mathfrak{g})$, just as in the case of group algebras.

In this case we define the *dual* representation (ρ^*, V^*) by

$$(\rho^*(X)f)(v) = -f(\rho(X)v)$$

for $X \in \mathfrak{g}$ and $f \in V^*$. We can also define the *tensor product* $(\rho \otimes \sigma,\ V \otimes W)$ of two representations by letting $X \in \mathfrak{g}$ act by

$$X \cdot (v \otimes w) = \rho(X)v \otimes w + v \otimes \sigma(X)w.$$

When $\mathfrak{g}$ is the Lie algebra of a linear algebraic group G and ρ, σ are the differentials of regular representations of G, then these constructions are the same as those in Section 1.2.3. These constructions are associated with the maps $X \mapsto -X$ and the map $X \mapsto X \otimes 1 + 1 \otimes X$. As in the case of group algebras, the properties of these maps can be described axiomatically using the notion of a *Hopf algebra* (see Exercises 3.1.5). The k-fold tensor powers of ρ and the symmetric and skew-symmetric powers are defined by analogy with the case of group representations. Here $X \in \mathfrak{g}$ acts by

$$\begin{aligned}\rho^{\otimes k}(X)(v_1 \otimes \cdots \otimes v_k) \quad &= \quad \rho(X)v_1 \otimes \cdots \otimes v_k + v_1 \otimes \rho(X)v_2 \otimes \cdots \otimes v_k \\ &\quad + \cdots + v_1 \otimes \cdots \otimes \rho(X)v_k\end{aligned}$$

on decomposable tensors. This action extends linearly to all tensors.

3.1.2 Schur's Lemma

The following observation of I. Schur is a fundamental tool in representation theory.

Lemma 3.1.1 (Schur) *If (ρ, V) and (τ, W) are irreducible representations of an associative algebra $\mathcal{A}$ with $\dim V$ and $\dim W$ finite, then*

$$\dim \operatorname{Hom}_{\mathcal{A}}(V, W) = \begin{cases} 1 & \text{if } (\rho, V) \cong (\tau, W) \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $T \in \operatorname{Hom}_{\mathcal{A}}(V, W)$. Then $\operatorname{Ker}(T)$ and $\operatorname{Range}(T)$ are invariant subspaces of V and W, respectively. If $T \neq 0$, then $\operatorname{Ker}(T) \neq V$ and $\operatorname{Range}(T) \neq 0$. Hence by the irreducibility of the representations $\operatorname{Ker}(T) = 0$ and $\operatorname{Range}(T) = W$, so that T is a linear isomorphism. Thus $\operatorname{Hom}_{\mathcal{A}}(V, W) \neq 0$ if and only if $(\rho, V) \cong (\tau, W)$.

Suppose the representations are equivalent. If $S, T \in \operatorname{Hom}_{\mathcal{A}}(V, W)$ are non-zero, then $T^{-1}S \in \operatorname{End}_{\mathcal{A}}(V)$. Since $\dim V < \infty$ there exists $\lambda \in \mathbb{C}$ such that

$$\operatorname{Ker}(T^{-1}S - \lambda I_V) \neq 0.$$

Hence $\operatorname{Ker}(T^{-1}S - \lambda I_V) = V$ since it is an $\mathcal{A}$-invariant subspace and V is irreducible. This shows that $S = \lambda T$ and hence $\dim \operatorname{Hom}_{\mathcal{A}}(V, W) = 1$. $\Diamond$

3.1.3 Burnside's Theorem

As a first step in understanding the structure of an associative algebra, we prove that the image of the algebra in a finite-dimensional irreducible representation (ρ, V) is completely determined by $\dim V$ (the *degree* of the representation).

Theorem 3.1.2 (Burnside) *Let (ρ, V) be an irreducible representation of an associative algebra $\mathcal{A}$. If $\dim V$ is finite and $\rho(\mathcal{A}) \neq 0$ then $\rho(\mathcal{A}) = \operatorname{End}(V)$.*

Proof. Set $\mathcal{B} = \rho(\mathcal{A})$. Then $\mathcal{B}$ is a non-zero finite-dimensional algebra, and V is an irreducible $\mathcal{B}$-module. We first observe that if $0 \neq v \in V$ then $\mathcal{B}v \neq 0$ (this is true for any representation in case $\mathcal{A}$ has an identity element). Indeed, the subspace

$$\{v \in V \,:\, \mathcal{B}v = 0\}$$

is not all of V, since $\mathcal{B} \neq 0$. Hence it must consist only of the zero vector, since it is $\mathcal{B}$-invariant. For any subset $S \subset V$, define its *annihilator*

$$\operatorname{Ann}(S) = \{b \in \mathcal{B} \,:\, bv = 0 \text{ for all } v \in S\}.$$

Then $\operatorname{Ann}(S)$ is a left ideal in $\mathcal{B}$. We observe that if $\mathcal{C}$ is any left ideal of $\mathcal{B}$ and $v \in V$, then $\mathcal{C}v$ is an invariant subspace of V. Hence either $\mathcal{C}v = 0$ or $\mathcal{C}v = V$, by the irreducibility of the representation.

Choose a left ideal $\mathcal{C}_1 \subset \mathcal{B}$ of minimal positive dimension, and choose a vector $v_1 \in V$ such that $\mathcal{C}_1 v_1 \neq 0$. Then

$$\mathcal{C}_1 v_1 = V \tag{3.3}$$

by the observation above. Define $\mathcal{B}_1 = \mathrm{Ann}\{v_1\}$. Then $\mathcal{B}_1 \cap \mathcal{C}_1 = 0$, since it is a left ideal properly contained in $\mathcal{C}_1$. In particular, the map

$$T_1 : \mathcal{C}_1 \to V, \quad T_1 c = cv_1$$

is bijective. Since $\mathcal{C}_1$ is a $\mathcal{B}$-module under left multiplication and $T_1 \in \mathrm{Hom}_{\mathcal{B}}(\mathcal{C}_1, V)$, we conclude that $\mathcal{C}_1 \cong V$ as a $\mathcal{B}$-module. Furthermore, if $x \in \mathcal{B}$, then by equation (3.3) there exists $y \in \mathcal{C}_1$ such that $xv_1 = yv_1$. Hence $x - y \in \mathcal{B}_1$. This proves that

$$\mathcal{B} = \mathcal{B}_1 \oplus \mathcal{C}_1.$$

If $\mathcal{B}_1 \neq 0$, we repeat this construction by choosing a left ideal $\mathcal{C}_2 \subset \mathcal{B}_1$ of minimal positive dimension, and a vector $v_2 \in V$ (as above), such that $\mathcal{C}_2 v_2 = V$. We set

$$\mathcal{B}_2 = \mathrm{Ann}\{v_1, v_2\} \subset \mathcal{B}_1.$$

If we argue in exactly the same way as we did for $\mathcal{C}_1$ and $\mathcal{B}_1$ we find that $\mathcal{B}_2 \cap \mathcal{C}_2 = 0$, $\mathcal{C}_2 \cong V$ as a $\mathcal{B}$-module under the map

$$T_2 : \mathcal{C}_2 \to V, \quad T_2 c = cv_2,$$

and $\mathcal{B}_1 = \mathcal{B}_2 \oplus \mathcal{C}_2$. Hence

$$\mathcal{B} = \mathcal{B}_2 \oplus \mathcal{C}_1 \oplus \mathcal{C}_2.$$

Since $\dim \mathcal{B} < \infty$ this procedure terminates after a finite number of steps, giving us a set of minimal left ideals $\mathcal{C}_1, \ldots, \mathcal{C}_m$ and $\mathcal{B}$-module isomorphisms $T_i : \mathcal{C}_i \cong V$ such that

$$\mathcal{B} = \mathcal{C}_1 \oplus \cdots \oplus \mathcal{C}_m. \tag{3.4}$$

Hence $\mathcal{B}$, viewed as a $\mathcal{B}$-module under left multiplication, is equivalent to the direct sum of m copies of V. In particular, $\dim \mathcal{B} = mn$, where $n = \dim V$. Since

$$\dim \mathcal{B} \leq \dim \mathrm{End}(V) = n^2,$$

we have $m \leq n$. Thus it suffices to prove that $m \geq n$.

Consider the action of right multiplication by $x \in \mathcal{B}$ on $\mathcal{C}_j$. By equation (3.4) there are linear maps $T_{ij}(x) : \mathcal{C}_j \to \mathcal{C}_i$ such that

$$yx = \sum_{i=1}^{m} T_{ij}(x) y, \qquad \text{for } y \in \mathcal{C}_j. \tag{3.5}$$

It is clear from equation (3.5) that $T_{ij}(x) \in \mathrm{Hom}_{\mathcal{B}}(\mathcal{C}_j, \mathcal{C}_i)$. But by Schur's lemma, this space is one-dimensional, and is spanned by $T_i^{-1} T_j$. Hence there are scalars $\mu_{ij}(x) \in \mathbb{C}$ such that

$$T_{ij}(x) = \mu_{ij}(x) T_i^{-1} T_j$$

for $i, j = 1, \ldots, m$. Clearly $\mu_{ij}(x)$ is a linear function of x. Define $\mu(x)$ to be the matrix $[\mu_{ij}(x)]$. Then

$$\mu : \mathcal{B} \to M_m(\mathbb{C})$$

is a linear map. If $\mu(x) = 0$, then by equation (3.5) we have $\mathcal{B}xv = 0$ for all $v \in V$. Hence $xv = 0$ for all $v \in V$, by the observation at the beginning of the proof, which implies that $x = 0$. Thus the map μ is injective. It follows that

$$mn = \dim \mathcal{B} \leq \dim M_m(\mathbb{C}) = m^2.$$

Hence $n \leq m$, so we conclude that $m = n$ and $\mathcal{B} \cong M_n(\mathbb{C})$. $\Diamond$

3.1.4 Complete Reducibility

Let (ρ, V) be a finite-dimensional representation of the associative algebra $\mathcal{A}$. Suppose $W \subset V$ is an $\mathcal{A}$-invariant subspace. By extending a basis for W to a basis for V, we obtain a vector-space isomorphism $V \cong W \oplus (V/W)$. However, this isomorphism is not necessarily an isomorphism of $\mathcal{A}$-modules. We say that the $\mathcal{A}$-module V is *completely reducible* if it is finite-dimensional and for every $\mathcal{A}$-invariant subspace $W \subset V$ there exists a complementary invariant subspace $U \subset V$ such that $V = W \oplus U$. In this case $U \cong V/W$ as an $\mathcal{A}$-module. To see this, let $P : V \to U$ be the projection operator such that $Pw = 0$ for all $w \in W$. Since U is invariant under $\rho(\mathcal{A})$, we have $\rho(a)P = P\rho(a)$. Let $\widetilde{P} : V/W \to U$ be defined by

$$\widetilde{P}(v + W) = Pv \quad \text{for } v \in V$$

Then $\widetilde{P}$ is an isomorphism of vector spaces and

$$\rho(a)\widetilde{P}(v + W) = \rho(a)Pv = P\rho(a)v = \widetilde{P}(\rho(a)v + W).$$

This shows that $\widetilde{P}$ defines an $\mathcal{A}$-module isomorphism between V/W and U.

We have already proved that every regular representation of a classical group is completely reducible. We now show that for representations of any associative algebra, the property of complete reducibility is inherited by subrepresentations and quotient representations.

Lemma 3.1.3 *Let (ρ, V) be completely reducible and suppose $W \subset V$ is an invariant subspace. Set $\sigma(x) = \rho(x)|_W$ and $\pi(x)(v + W) = \rho(x)v + W$ for $x \in \mathcal{A}$ and $v \in V$. Then the representations (σ, W) and $(\pi, V/W)$ are completely reducible.*

Proof. Write $V = W \oplus U$ for some invariant subspace U, and let $P \in \mathrm{End}_{\mathcal{A}}(V)$ be the projection onto W with kernel U. If $Y \subset W$ is an invariant subspace, then the subspace $U \oplus Y$ is invariant. Hence there is an invariant subspace $Z \subset V$ such that

$$V = (U \oplus Y) \oplus Z. \tag{3.6}$$

The subspace $P(Z) \subset W$ is invariant, and we claim that

$$W = Y \oplus P(Z). \tag{3.7}$$

We have $\dim W = \dim V/U = \dim Y + \dim Z$ by (3.6). Since $\mathrm{Ker}\, P = U$, the map $z \mapsto P(z)$ is a bijective intertwining operator from Z to $P(Z)$, so that $\dim Z = \dim P(Z)$. Hence $\dim W = \dim Y + \dim P(Z)$. Also

$$\begin{aligned} W &= P(V) = P(U) + P(Y) + P(Z) \\ &= Y + P(Z). \end{aligned}$$

Thus (3.7) holds, which proves the complete reducibility of (σ, W).

Let $M \subset V/W$ be an invariant subspace, and let $\widetilde{M} = p^{-1}(M) \subset V$, where $p : V \to V/W$ is the canonical quotient map. Then $\widetilde{M}$ is invariant, so there exists an invariant subspace $\widetilde{N}$ with $V = \widetilde{M} \oplus \widetilde{N}$. Set $N = p(\widetilde{N})$. This is an invariant subspace, and $V/W = M \oplus N$. Thus $(\pi, V/W)$ is completely reducible. $\Diamond$

The converse to Lemma 3.1.3 is not true. For example, let $\mathcal{A}$ be the algebra of matrices of the form

$$\begin{bmatrix} x & y \\ 0 & x \end{bmatrix} \quad x, y \in \mathbb{C}$$

acting on $V = \mathbb{C}^2$ (column vectors). The first column of the matrices in $\mathcal{A}$ defines an irreducible invariant subspace W. Since V/W is one-dimensional it is also irreducible. But the matrices in $\mathcal{A}$ have only one distinct eigenvalue and are not diagonal, so there is no invariant complement to W in V. Thus V is not completely reducible as an $\mathcal{A}$ module.

Proposition 3.1.4 *Let (ρ, V) be a finite-dimensional representation of the associative algebra $\mathcal{A}$. The following are equivalent:*
(1) *(ρ, V) is completely reducible.*
(2) *$V = V_1 \oplus \cdots \oplus V_d$ with each V_i invariant and irreducible.*

Proof. (1) $\Rightarrow$ (2): If $\dim V = 1$ then V is irreducible and (2) trivially holds. Assume that (1) $\Rightarrow$ (2) for all $\mathcal{A}$-modules V of dimension less than r. Let V be a module of dimension r. If V is irreducible then (2) trivially holds. Otherwise there are non-zero submodules W and U such that $V = U \oplus W$. These submodules are completely reducible by Lemma 3.1.3, and hence they decompose as the direct sum of irreducibles by the induction hypothesis. Thus (2) also holds for V.

(2) $\Rightarrow$ (1): We prove (1) by induction on the number d of irreducible summands in (2). Let $0 \neq W \subset V$ be a submodule. If $d = 1$, then $W = V$ by irreducibility, and we are done. If $d > 1$, let $P_1 : V \to V_1$ be the projection operator associated with the direct sum decomposition (2). If $P_1 W = 0$, then

$$W \subset V_2 \oplus \cdots \oplus V_d,$$

so by the induction hypothesis there is an $\mathcal{A}$-invariant complement to W. If $P_1 W \neq 0$ then $P_1 W = V_1$, since it is an $\mathcal{A}$-invariant subspace. Set

$$W' = \operatorname{Ker}(P_1|_W).$$

We have $W' \subset V_2 \oplus \cdots \oplus V_d$, so by the induction hypothesis there exists an $\mathcal{A}$-invariant subspace $U \subset V_2 \oplus \cdots \oplus V_d$ such that

$$V_2 \oplus \cdots \oplus V_d = W' \oplus U.$$

Since $P_1 U = 0$, we have $W \cap U \subset W' \cap U = 0$. Also $\dim W = \dim V_1 + \dim W'$, so

$$\begin{aligned} \dim W + \dim U &= \dim V_1 + \dim W' + \dim U \\ &= \dim V_1 + \sum_{i \geq 2} \dim V_i = \dim V. \end{aligned}$$

Hence $V = U \oplus W$. $\Diamond$

Corollary 3.1.5 *Suppose (ρ, V) and (σ, W) are completely reducible representations of $\mathcal{A}$. Then $(\rho \oplus \sigma, V \oplus W)$ is a completely reducible representation.*

Proof. By Proposition 3.1.4 V and W are direct sums of irreducible invariant subspaces. It follows that $V \oplus W$ is a direct sum of irreducible invariant subspaces, hence completely reducible. $\Diamond$

If U is a finite-dimensional irreducible $\mathcal{A}$-module, we denote by $[U]$ the equivalence class of all $\mathcal{A}$-modules equivalent to U. Let $\widehat{\mathcal{A}}$ be the set of all equivalence classes of finite-dimensional irreducible $\mathcal{A}$-modules. Suppose that V is a completely reducible $\mathcal{A}$-module. For each $\xi \in \widehat{\mathcal{A}}$ we define

$$V_{(\xi)} = \sum_{U \subset V,\, [U]=\xi} U,$$

where the subspaces U are invariant and irreducible under $\mathcal{A}$ and furnish representations of $\mathcal{A}$ in the equivalence class ξ. We call $V_{(\xi)}$ the ξ-*isotypic subspace* of V.

For each $\xi \in \widehat{\mathcal{A}}$ fix a module E_ξ in the class ξ. There is a linear map

$$S_\xi : \mathrm{Hom}_{\mathcal{A}}(E_\xi, V) \otimes E_\xi \to V, \qquad S_\xi(u \otimes w) = u(w)$$

for $u \in \mathrm{Hom}_{\mathcal{A}}(E_\xi, V)$ and $w \in E_\xi$. If we make $\mathrm{Hom}_{\mathcal{A}}(E_\xi, V) \otimes E_\xi$ into an $\mathcal{A}$-module with action $x \cdot (u \otimes w) = u \otimes (x \cdot w)$ for $x \in \mathcal{A}$, then S_ξ is an $\mathcal{A}$-intertwining map. If $0 \neq u \in \mathrm{Hom}_{\mathcal{A}}(E_\xi, V)$ then Schur's Lemma implies that $u(E_\xi)$ is an irreducible $\mathcal{A}$-submodule of V isomorphic to E_ξ. Hence

$$S_\xi(\mathrm{Hom}_{\mathcal{A}}(E_\xi, V) \otimes E_\xi) \subset V_{(\xi)}$$

for every $\xi \in \widehat{\mathcal{A}}$.

Proposition 3.1.6 *Let V be a completely reducible $\mathcal{A}$-module. Let*

$$V = V_1 \oplus \cdots \oplus V_d \tag{3.8}$$

be any decomposition with each V_i invariant and irreducible. Then

$$V_{(\xi)} = \bigoplus_{[V_j]=\xi} V_j \tag{3.9}$$

for all $\xi \in \widehat{\mathcal{A}}$, and hence

$$V = \bigoplus_{\xi \in \widehat{\mathcal{A}}} V_{(\xi)}. \tag{3.10}$$

The map S_ξ gives an $\mathcal{A}$-module isomorphism

$$\mathrm{Hom}_{\mathcal{A}}(E_\xi, V) \otimes E_\xi \cong V_{(\xi)}$$

for each $\xi \in \widehat{\mathcal{A}}$.

Proof. Suppose $\xi \in \widehat{\mathcal{A}}$ and $V_{(\xi)} \neq \{0\}$. Let $U \subset V$ be any irreducible, invariant subspace such that $[U] = \xi$. Since $U \cap V_j$ is invariant under $\mathcal{A}$, it must be either $\{0\}$ or V_j. Hence

$$U \cap V_j = \{0\} \quad \text{if} \quad [V_j] \neq \xi.$$

Hence $V_j \cap V_{(\xi)} = \{0\}$ if $[V_j] \neq \xi$, by definition of $V_{(\xi)}$. Since $V_i \subset V_{(\xi)}$ for all i such that $[V_i] = \xi$, this result implies (3.9), which implies (3.10).

To prove the last statement of the proposition, we take $\xi \in \widehat{\mathcal{A}}$ so that $V_{(\xi)} \neq 0$. Then we may assume that

$$V_{(\xi)} = \underbrace{E_\xi \oplus \cdots \oplus E_\xi}_{m \text{ copies}}.$$

Let $U = \operatorname{Hom}_{\mathcal{A}}(E_\xi, V)$ and let $\phi_j \in U$ be the map

$$\phi_j(w) = (0, \ldots, 0, \underbrace{w}_{j\text{th}}, 0, \ldots, 0).$$

If $u \in U$ then the range of u is contained in $V_{(\xi)}$, so we can write

$$u(w) = (u_1(w), \ldots, u_m(w)) \quad \text{for } w \in E_\xi,$$

where $u_j \in \operatorname{End}_{\mathcal{A}}(E_\xi)$. But $\operatorname{End}_{\mathcal{A}}(E_\xi) = \mathbb{C}I$ by Schur's Lemma, so it follows that

$$u(w) = (c_1 w, \ldots, c_m w) = \sum_{i=1}^{m} c_i \phi_i(w)$$

for some $c_i \in \mathbb{C}$. Hence $\{\phi_1, \ldots, \phi_m\}$ spans U. Since this set is clearly linearly independent, it is a basis for U. Hence

$$m = \dim \operatorname{Hom}_{\mathcal{A}}(E_\xi, V).$$

For $v_i \in E_\xi$ we have

$$S_\xi\Big(\sum_{i=1}^{m} \phi_i \otimes v_i\Big) = (v_1, \ldots, v_m)$$

Hence S_ξ is an $\mathcal{A}$-module isomorphism between $\operatorname{Hom}_{\mathcal{A}}(E_\xi, V) \otimes E_\xi$ and $V_{(\xi)}$. $\Diamond$

We call (3.10) the *primary decomposition* of V. The cardinality $m_V(\xi)$ of the set $\{j : [V_j] = \xi\}$ is called the *multiplicity* of ξ in V. We have

$$m_V(\xi) = \dim \operatorname{Hom}_{\mathcal{A}}(E_\xi, V) = \dim \operatorname{Hom}_{\mathcal{A}}(V, E_\xi).$$

Here the first equality was proved in Proposition 3.1.6. To obtain the second inequality, we may assume that

$$V = W \oplus \underbrace{E_\xi \oplus \cdots \oplus E_\xi}_{m \text{ copies}},$$

where W is the sum of the isotypic subspaces for representations not equivalent to ξ. If $T \in \operatorname{Hom}_{\mathcal{A}}(V, E_\xi)$ then by Schur's Lemma $T(W) = 0$ and T is a linear combination of the operators $\{T_1, \ldots, T_m\}$, where

$$T_i(w \oplus v_1 \oplus \cdots \oplus v_m) = v_i.$$

Since these operators are linearly independent, they furnish a basis for $\mathrm{Hom}_{\mathcal{A}}(V, E_\xi)$.

Let U and V be completely reducible $\mathcal{A}$-modules. Define

$$\langle U, V\rangle = \dim \mathrm{Hom}_{\mathcal{A}}(U, V).$$

Then from Proposition 3.1.6 we have

$$\langle U, V\rangle = \sum_{\xi \in \widehat{\mathcal{A}}} m_U(\xi) m_V(\xi). \tag{3.11}$$

It follows that

$$\langle U, V\rangle = \langle V, U\rangle, \qquad \langle U, V \oplus W\rangle = \langle U, V\rangle + \langle U, W\rangle$$

for any completely reducible $\mathcal{A}$-modules U, V, W.

The multiplicities $m_V(\xi)$ have the following monotonicity properties.

Proposition 3.1.7 *Let U and V be completely reducible $\mathcal{A}$-modules.*

(1) *If $T : U \to V$ is a surjective $\mathcal{A}$-module map, then $m_U(\xi) \geq m_V(\xi)$ for each $\xi \in \widehat{\mathcal{A}}$.*

(2) *If $T : U \to V$ is an injective $\mathcal{A}$-module map, then $m_U(\xi) \leq m_V(\xi)$ for each $\xi \in \widehat{\mathcal{A}}$.*

Proof. (1) Let $S \in \mathrm{Hom}_{\mathcal{A}}(V, E_\xi)$. Then $ST \in \mathrm{Hom}_{\mathcal{A}}(U, E_\xi)$. Since T is surjective, the map $S \mapsto ST$ is an injection from $\mathrm{Hom}_{\mathcal{A}}(V, E_\xi)$ into $\mathrm{Hom}_{\mathcal{A}}(U, E_\xi)$. Thus

$$m_U(\xi) = \dim \mathrm{Hom}_{\mathcal{A}}(U, E_\xi) \geq \dim \mathrm{Hom}_{\mathcal{A}}(V, E_\xi) = m_V(\xi).$$

(2) Let $S \in \mathrm{Hom}_{\mathcal{A}}(E_\xi, U)$. Then $TS \in \mathrm{Hom}_{\mathcal{A}}(E_\xi, V)$. Since T is injective, the map $S \mapsto TS$ is an injection from $\mathrm{Hom}_{\mathcal{A}}(E_\xi, U)$ into $\mathrm{Hom}_{\mathcal{A}}(E_\xi, V)$. Thus $m_U(\xi) \leq m_V(\xi)$ in this case. $\Diamond$

Using these general results we can determine the finite-dimensional irreducible representations of the product of two groups. Let (ρ, V) and (σ, W) be representations of groups G and H respectively. Their *outer tensor product* is the representation $(\rho \widehat{\otimes} \sigma, V \otimes W)$ of $G \times H$ given by $(\rho \widehat{\otimes} \sigma)(g, h) = \rho(g) \otimes \sigma(h)$. Notice that when $G = H$, then the restriction of the outer tensor product $\rho \widehat{\otimes} \sigma$ to the diagonal subgroup $\{(g, g) : g \in G\}$ of $G \times G$ is the tensor product $\rho \otimes \sigma$.

Proposition 3.1.8 *Suppose (ρ, V) and (σ, W) are finite-dimensional and irreducible. Then the outer tensor product $(\rho \widehat{\otimes} \sigma, V \otimes W)$ is an irreducible representation of $G \times H$, and every finite-dimensional irreducible representation of $G \times H$ is of this form.*

Proof. By Theorem 3.1.2 we have

$$\rho(\mathbb{C}[G]) = \mathrm{End}(V), \quad \sigma(\mathbb{C}[H]) = \mathrm{End}(W).$$

Hence $(\rho \widehat{\otimes} \sigma)(\mathbb{C}[G \times H])$ contains all operators $T \otimes S$, for arbitrary $T \in \mathrm{End}(V)$ and $S \in \mathrm{End}(W)$. These operators span $\mathrm{End}(V \otimes W)$, so $\rho \widehat{\otimes} \sigma$ is irreducible.

Let (τ, U) be an irreducible representation of $G \times H$. Set $\tau_1(g) = \tau(g, 1)$ and $\tau_2(h) = \tau(1, h)$ for $g \in G$ and $h \in H$. Since $\dim U < \infty$, there exists a nonzero subspace $V \subset U$

that is invariant and irreducible for τ_1. Set $\rho = \tau_1|_V$. Since $\tau_2(h)$ commutes with $\tau_1(g)$, the subspaces $\tau_2(h)V$ are also invariant and irreducible for τ_1, for every $h \in H$, and as G modules they are all equivalent to (ρ, V). Suppose $V' \subset U$ is any subspace invariant under G. Then by Schur's lemma

$$(\tau_2(h)V) \cap V' = \begin{cases} \tau_2(h)V & \text{if } (\tau_2(h)V) \cap V' \neq 0 \\ 0 & \text{otherwise.} \end{cases} \tag{3.12}$$

By the irreducibility of τ

$$U = \mathbb{C}[G \times H] \cdot V = \mathbb{C}[H] \cdot V. \tag{3.13}$$

It follows by (3.12) and (3.13) that there are $h_1, \ldots, h_d \in H$ such that

$$U = \bigoplus_{j=1}^{d} \tau_2(h_j)V.$$

This shows that (τ_1, U) is a completely reducible G module. Since it is the sum of d copies of (ρ, V), there is a space W with $\dim W = d$, and a G-module isomorphism

$$U \cong V \otimes W, \quad \tau_1 \cong \rho \otimes I_W. \tag{3.14}$$

By Burnside's Theorem, $\tau_1(\mathbb{C}[G])$ maps onto $\mathrm{End}(V) \otimes I_W$ under this isomorphism. Thus the operators $\tau_2(h)$, for $h \in H$, give operators that commute with all the operators $X \otimes I_W$. We now use the following lemma.

Lemma 3.1.9 *Let V, W be finite-dimensional vector spaces and let $T \in \mathrm{End}(V \otimes W)$. Suppose that*

$$T(X \otimes I_W) = (X \otimes I_W)T$$

for all $X \in \mathrm{End}(V)$. Then there exists $Y \in \mathrm{End}(W)$ such that $T = I_V \otimes Y$.

Proof. We determine Y as follows: Given $u^* \in U^*$, define a linear map $B_{u^*} : V \otimes U \to V$ by

$$B_{u^*}(v \otimes u) = u^*(u)v \quad \text{for } u \in U,\ v \in V.$$

Given $u \in U$, define a linear map $A_u : V \to V \otimes U$ by

$$A_u(v) = v \otimes u.$$

Set $S_{u^*,u} = B_{u^*} \circ T \circ A_u$. Then $S_{u^*,u} \in \mathrm{End}(V)$ and it satisfies

$$\begin{aligned} S_{u^*,u}Xv &= B_{u^*}T(Xv \otimes u) \\ &= B_{u^*}(X \otimes I_U)T(v \otimes u) \\ &= XB_{u^*}T(A_uv) = XS_{u^*,u}v \end{aligned}$$

for all $X \in \mathrm{End}(V)$. By Schur's lemma there is a scalar $c(u^*, u) \in \mathbb{C}$ such that $S_{u^*,u} = c(u^*, u)I_V$. Clearly $c(u^*, u)$ is a bilinear form on $U^* \times U$. Since U is finite-dimensional, there exists $Y \in \mathrm{End}(U)$ such that

$$c(u^*, u) = u^*(Yu), \qquad \text{for all } u \in U,\ u^* \in U^*.$$

Going back to the definition of $S_{u^*,u}$, we find that

$$B_{u^*}T(v \otimes u) = u^*(Yu)v = B_{u^*}(v \otimes Yu).$$

Since this holds for all $u^* \in U^*$, we have

$$T(v \otimes u) = v \otimes Yu$$

and hence $T = I_V \otimes Y$. $\Diamond$

Completion of proof of Proposition 3.1.8:

From Lemma 3.1.9 we see that there exists a representation σ of H on W so that

$$\tau_2 \cong I_V \otimes \sigma.$$

Hence the original representation τ of $G \times H$ is equivalent to $\rho \widehat{\otimes} \sigma$. The representation σ must be irreducible. Indeed, if $W_1 \subset W$ is invariant under $\sigma(H)$ then $V \otimes W_1$ is invariant under $\rho \widehat{\otimes} \sigma$ and is hence either 0 or $V \otimes W$ by the irreducibility of τ. $\Diamond$

Proposition 3.1.10 *Let $(R, \mathrm{Aff}(G))$ be the right regular representation of a linear algebraic group G. Suppose every finite-dimensional $R(G)$-invariant subspace of* $\mathrm{Aff}(G)$ *is completely reducible. Then G is reductive.*

Proof. Let (σ, V) be a regular representation. For $\lambda \in V^*$ define

$$T_\lambda : V \to \mathrm{Aff}(G), \quad T_\lambda(v)(g) = \lambda(\sigma(g)v) \text{ for } v \in V,\ g \in G.$$

Then $T_\lambda \circ \sigma(g) = R(g) \circ T_\lambda$, so $W_\lambda = T_\lambda V$ is a finite-dimensional G submodule of $\mathrm{Aff}(G)$. Also, if $\{\lambda_1, \ldots, \lambda_n\}$ is a basis for V^* and we set

$$W = W_{\lambda_1} \oplus \cdots \oplus W_{\lambda_n},$$

then the map $T : V \to W$ given by $T(v) = T_{\lambda_1}(v) \oplus \cdots \oplus T_{\lambda_n}(v)$ is injective and intertwines the G actions on V and W. By hypothesis each subspace W_λ is completely reducible under $R(G)$, so W is completely reducible. Since (σ, V) is equivalent to a subrepresentation of (R, W), it is also completely reducible. $\Diamond$

Proposition 3.1.11 *If G and H are reductive algebraic groups, then $G \times H$ is reductive.*

Proof. Let $U \subset \mathrm{Aff}(G \times H)$ be a finite-dimensional subspace invariant under $R(G \times H)$. We have $\mathrm{Aff}(G \times H) = \mathrm{Aff}(G) \otimes \mathrm{Aff}(H)$ both as an algebra and as a $G \times H$ module, by Lemma A.1.10. Hence by taking a basis for U we see that there are finite-dimensional invariant subspaces $V \subset \mathrm{Aff}(G)$ and $W \subset \mathrm{Aff}(H)$ with $U \subset V \otimes W$. Since G and H are reductive, V is the direct sum of irreducible submodules V_i and W is the direct sum of irreducible submodules W_i. Thus

$$V \otimes W \cong \bigoplus_{i,j} V_i \otimes W_j.$$

By Proposition 3.1.8 $V_i \otimes W_j$ is an irreducible $G \times H$ module. This shows that $V \otimes W$ is completely reducible, and hence so is U, by Lemma 3.1.3. From Proposition 3.1.10 we conclude that $G \times H$ is reductive. $\Diamond$

Theorem 3.1.12 *Let H be an algebraic torus. Then H is reductive. Furthermore, if (ρ, V) is a regular representation of H then there is a finite set $\mathfrak{X}(V) \subset \mathfrak{X}(H)$ such that*

$$V = \bigoplus_{\lambda \in \mathfrak{X}(V)} V(\lambda), \tag{3.15}$$

where $V(\lambda) = \{v \in V \,:\, \rho(h)v = h^\lambda v \text{ for all } h \in H\}$.

Proof. The group $\mathbb{C}^\times$ is reductive by Lemma 1.3.3. Thus $H \cong (\mathbb{C}^\times)^n$ is reductive by Proposition 3.1.11. Also by Lemma 1.3.3 and Proposition 3.1.8 the irreducible representations of H are one-dimensional. Hence the primary decomposition of (ρ, V) is of the form (3.15). ♢

The characters occurring in Theorem 3.1.12 are called the *weights* of the representation ρ. We define

$$m_\rho(\lambda) = \dim V(\lambda)$$

(the *multiplicity* of λ in ρ). If all the multiplicities are one, then the representation is *multiplicity free*. For example, the defining representation of a classical group G is multiplicity free for the diagonal subgroup $H \subset G$ (cf. Section 2.1.1).

3.1.5 Exercises

1. Let $\mathcal{A}$ be an associative algebra over $\mathbb{C}$ with unit element 1. Then $\mathcal{A} \otimes \mathcal{A}$ is an associative algebra with unit element $1 \otimes 1$, where the multiplication is defined by $(a \otimes b)(c \otimes d) = (ac) \otimes (bc)$ on decomposable tensors, and extended to be bilinear. A *bialgebra structure* on $\mathcal{A}$ consists of an algebra homomorphism $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ (called the *comultiplication)* and an algebra homomorphism $\epsilon : \mathcal{A} \to \mathbb{C}$ (called the *counit*) which satisfy the following:

   (coassociativity) The maps $\Delta \otimes I_{\mathcal{A}}$ and $I_{\mathcal{A}} \otimes \Delta$ from $\mathcal{A}$ to $\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$ coincide:

   $$(\Delta \otimes I_{\mathcal{A}})(\Delta(a)) = (I_{\mathcal{A}} \otimes \Delta)(\Delta(a)) \quad \text{for all } a \in \mathcal{A},$$

   where $(\mathcal{A} \otimes \mathcal{A}) \otimes \mathcal{A}$ is identified with $\mathcal{A} \otimes (\mathcal{A} \otimes \mathcal{A})$ as usual and $I_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$ is the identity map.

   (counit) The maps $(I_{\mathcal{A}} \otimes \epsilon) \circ \Delta$ and $(\epsilon \otimes I_{\mathcal{A}}) \circ \Delta$ from $\mathcal{A}$ to $\mathcal{A}$ coincide:

   $$(I_{\mathcal{A}} \otimes \epsilon)(\Delta(a)) = (\epsilon \otimes I_{\mathcal{A}})(\Delta(a)) \quad \text{for all } a \in \mathcal{A},$$

   where we identify $\mathbb{C} \otimes \mathcal{A}$ with $\mathcal{A}$ as usual.

   (a) Let G be a group and let $\mathcal{A} = \mathbb{C}[G]$ with convolution product. Define Δ and ϵ on the basis elements δ_x for $x \in G$ by

   $$\Delta(\delta_x) = \delta_x \otimes \delta_x, \quad \epsilon(\delta_x) = 1$$

   and extend these maps by linearity. Show that Δ and ϵ satisfy the conditions for a bialgebra structure on $\mathcal{A}$ and that

   $$\langle \Delta(f), g \otimes h \rangle = \langle f, gh \rangle \quad \text{for } f, g, h \in \mathbb{C}[G].$$

Here we write $\langle \phi, \psi \rangle = \sum_{x \in X} \phi(x)\psi(x)$ for complex-valued functions ϕ, ψ on a set X, and gh denotes the pointwise product of the functions g and h.

(b) Let G be a group and consider $\mathbb{C}[G]$ as the commutative algebra of $\mathbb{C}$-valued function on G with pointwise multiplication of functions and the constant function 1 as identity element. Identify $\mathbb{C}[G] \otimes \mathbb{C}[G]$ with $\mathbb{C}[G \times G]$ by $\delta_x \otimes \delta_y \leftrightarrow \delta_{(x,y)}$ for $x, y \in G$. Define Δ by $\Delta(f)(x, y) = f(xy)$ and define $\epsilon(f) = f(1)$, where $1 \in G$ is the identity element. Show that this defines a bialgebra structure on $\mathbb{C}[G]$ and that

$$\langle \Delta(f), g \otimes h \rangle = \langle f, g * h \rangle \quad \text{for } f, g, h \in \mathbb{C}[G],$$

where $\langle \phi, \psi \rangle$ is defined as in (a), and $g * h$ denotes the convolution product of the functions g and h.

(c) Let G be a linear algebraic group consider $\mathrm{Aff}(G)$ as a (commutative) algebra with pointwise multiplication of functions and the constant function 1 as the identity element. Identify $\mathcal{A} \otimes \mathcal{A}$ with $\mathrm{Aff}(G \times G)$ as in Lemma A.1.10 and define Δ and ϵ by the same formulas as in (b). Show that this defines a bialgebra structure on $\mathrm{Aff}[G]$.

(d) Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$ and let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. Define

$$\Delta(X) = X \otimes 1 + 1 \otimes X \quad \text{for } X \in \mathfrak{g}.$$

Show that $\Delta([X, Y]) = \Delta(X)\Delta(Y) - \Delta(Y)\Delta(X)$, and conclude that Δ extends uniquely to an algebra homomorphism $\Delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$. Let $\epsilon : U(\mathfrak{g}) \to \mathbb{C}$ be the unique homomorphism such that $\epsilon(X) = 0$ for all $X \in \mathfrak{g}$ (note that by the Poincaré-Birkhoff-Witt theorem $U(\mathfrak{g}) = \mathbb{C}1 \oplus \mathfrak{g}U(\mathfrak{g})$, so $\epsilon(u) = \lambda$ if $u - \lambda 1 \in \mathfrak{g}U(\mathfrak{g})$). Show that Δ and ϵ define a bialgebra structure on $U(\mathfrak{g})$.

(e) Suppose G is a linear algebraic group. Let $\mathfrak{g} = \mathrm{Lie}(G)$. Define a bilinear form on $U(\mathfrak{g}) \times \mathrm{Aff}(G)$ by $\langle T, f \rangle = Tf(1)$ for $T \in U(\mathfrak{g})$ and $f \in \mathrm{Aff}(G)$, where the action of $U(\mathfrak{g})$ on $\mathrm{Aff}(G)$ comes from the action of $\mathfrak{g}$ as left-invariant vector fields. Show that

$$\langle \Delta(T), f \otimes g \rangle = \langle T, fg \rangle \quad \text{for all } T \in U(\mathfrak{g}) \text{ and } f, g \in \mathrm{Aff}(G),$$

where Δ is defined as in (d). (This shows that the comultiplication on $U(\mathfrak{g})$ is dual to the pointwise multiplication on $\mathrm{Aff}(G)$).

2. Let $\mathcal{A}$ be an associative algebra over $\mathbb{C}$, and suppose Δ and ϵ give $\mathcal{A}$ the structure of a bialgebra, in the sense of the previous exercise. Let (V, ρ) and (W, σ) be representations of $\mathcal{A}$.

(a) Show that the map $(a, b) \mapsto \rho(a) \otimes \sigma(b)$ extends to a representation of $\mathcal{A} \otimes \mathcal{A}$ on $V \otimes W$, denoted by $\rho \widehat{\otimes} \sigma$

(b) Define $(\rho \otimes \sigma)(a) = (\rho \widehat{\otimes} \sigma)(\Delta(a))$ for $a \in \mathcal{A}$. Show that $\rho \otimes \sigma$ is a representation of $\mathcal{A}$, called the *tensor product* $\rho \otimes \sigma$ of the representations ρ and σ.

(c) When $\mathcal{A}$ and Δ are given as in (a) or (d) of the previous exercise, verify that the tensor product defined via the map Δ is the same as the tensor product defined in Section 3.1.1.

3. Let $\mathcal{A}$ be a bialgebra, in the sense of the previous exercises with comultiplication map Δ and counit ϵ. Let $S : \mathcal{A} \to \mathcal{A}$ be an *antiautomorphsim* ($S(xy) = S(y)S(x)$ for all $x, y \in \mathcal{A}$). Then S is called an *antipode* if

$$\mu((S \otimes I_{\mathcal{A}})(\Delta(a))) = \epsilon(a)1, \qquad \mu((I_{\mathcal{A}} \otimes S)(\Delta(a))) = \epsilon(a)1 \quad \text{for all } a \in \mathcal{A},$$

where $\mu : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is the multiplication map. A bialgebra with an antipode is called a *Hopf algebra*.

(a) Let G be a group, and let $\mathcal{A} = \mathbb{C}[G]$ with convolution multiplication. Let Δ and ϵ be defined as in the exercise above, and let $Sf(x) = f(x^{-1})$ for $f \in \mathbb{C}[G]$ and $x \in G$. Show that S is an antipode.

(b) Let G be a group, and let $\mathcal{A} = \mathbb{C}[G]$ with pointwise multiplication. Let Δ and ϵ be defined as in the exercise above, and let $Sf(x) = f(x^{-1})$ for $f \in \mathbb{C}[G]$ and $x \in G$. Show that S is an antipode (the same holds when G is a linear algebraic group and $\mathcal{A} = \mathrm{Aff}(G)$).

(c) Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$. Define the maps Δ and ϵ on $U(\mathfrak{g})$ as in the exercise above. Let $S(X) = -X$ for X in $\mathfrak{g}$. Show that S extends to an antiautomorphism of $U(\mathfrak{g})$ and satisfies the conditions for an antipode.

4. Let $\mathcal{A}$ be a Hopf algebra over $\mathbb{C}$ with antipode S.

(a) Given a representation (ρ, V) of $\mathcal{A}$, define $\rho^S(x) = \rho(Sx)^*$ for $x \in \mathcal{A}$. Show that (ρ^S, V^*) is a representation of $\mathcal{A}$.

(b) Show that the representation (ρ^S, V^*) is the *dual* representation to (ρ, V) when $\mathcal{A}$ is either $\mathbb{C}[G]$ with convolution multiplication or $U(\mathfrak{g})$ (where $\mathfrak{g}$ is a Lie algebra) and the antipode is defined as in the exercise above.

5. Let $\mathcal{A} = \mathbb{C}[x]$ and let $T \in M_n[\mathbb{C}]$. Define a representation ρ of $\mathcal{A}$ on $\mathbb{C}^n$ by $\rho(x) = T$.

(a) Suppose T has n distinct eigenvalues. Prove that ρ is completely reducible.

(b) Is the representation ρ always completely reducible? (*Hint:* Put T into Jordan canonical form.)

6. Let $\mathcal{A}$ be an associative algebra and let V be a completely reducible finite-dimensional $\mathcal{A}$-module.

(a) Show that V is irreducible if and only if $\dim \mathrm{Hom}_{\mathcal{A}}(V, V) = 1$.

(b) Does (a) hold if V is not completely reducible? (*Hint:* Consider the algebra of all upper-triangular 2×2 matrices.)

7. Let G be a linear algebraic group and (ρ, V) a regular representation of G. Define a representation π of $G \times G$ on $\mathrm{End}(V)$ by

$$\pi(x, y)T = \rho(x)T\rho(y^{-1}), \qquad \text{for } T \in \mathrm{End}(V),\ x, y \in G.$$

(a) Show that the space E^ρ of representative functions (see Section 1.1.3) is invariant under $G \times G$ (acting by left and right translations), and the map $B \mapsto f_B$ from $\mathrm{End}(V)$ to E^ρ intertwines the actions π and $L \widehat{\otimes} R$ of $G \times G$.

(b) Suppose ρ is irreducible. Prove that the map $B \mapsto f_B$ from $\mathrm{End}(V)$ to $\mathrm{Aff}(G)$ is injective (*Hint*: Use Burnside's theorem).

3.2 Simple Associative Algebras

3.2.1 Wedderburn's Theorem

An associative algebra $\mathcal{A}$ is called *simple* if the only two-sided ideals in $\mathcal{A}$ are 0 and $\mathcal{A}$. We now show that a finite-dimensional simple algebra is completely determined by its dimension.

Theorem 3.2.1 (Wedderburn) *The algebra* $\mathrm{End}(V)$ *is simple for every finite dimensional complex vector space* V*. Conversely, if* $\mathcal{A}$ *is any finite dimensional simple algebra over* $\mathbb{C}$ *with unit, then there is a finite dimensional complex vector space* V *such that* $\mathcal{A} \cong \mathrm{End}(V)$*.*

Proof. If u, v are nonzero vectors in V, then there exists $T \in \mathrm{End}(V)$ so that $Tv = u$ (take $f \in V^*$ with $f(v) = 1$ and define $Tx = f(x)u$ for $x \in V$). Thus $\mathrm{End}(V)v = V$. Now suppose $0 \neq \mathcal{B} \subset \mathrm{End}(V)$ is a two-sided ideal and $0 \neq v \in V$. Then $\mathcal{B}v = \mathcal{B}\,\mathrm{End}(V)v = \mathcal{B}V$, since $\mathcal{B}$ is a right ideal. But $\mathcal{B}V \neq 0$ since $\mathcal{B} \neq 0$, and $\mathcal{B}V$ is invariant under $\mathrm{End}(V)$ since $\mathcal{B}$ is a left ideal. Hence

$$\mathcal{B}v = V \qquad \text{for all } 0 \neq v \in V.$$

This proves that V is an irreducible $\mathcal{B}$-module. Burnside's Theorem (Theorem 3.1.2) implies that $\mathcal{B} = \mathrm{End}(V)$. Hence $\mathrm{End}(V)$ is a simple algebra.

Now suppose $\mathcal{A}$ is a finite-dimensional simple algebra over $\mathbb{C}$ with unit. Define the left regular representation

$$\lambda : \mathcal{A} \to \mathrm{End}(\mathcal{A})$$

by $\lambda(x)y = xy$. Choose a left ideal $V \subset \mathcal{A}$ of minimal positive dimension, and define $\rho(x) = \lambda(x)|_V$ for $x \in \mathcal{A}$. Then (ρ, V) is an irreducible representation of $\mathcal{A}$. Hence $\rho(\mathcal{A}) = \mathrm{End}(V)$ by Burnside's theorem. Furthermore $\mathrm{Ker}(\rho)$ is zero, since it is a two-sided ideal. Thus $\mathcal{A} \cong \rho(\mathcal{A})$ as an algebra. $\Diamond$

3.2.2 Representations of $\mathrm{End}(V)$

Let V be a finite-dimensional complex vector space. The representation of $\mathrm{End}(V)$ on V is irreducible (see the proof of Theorem 3.2.1). We shall prove that, up to equivalence, this is the *unique* irreducible representation of $\mathrm{End}(V)$. This will be a consequence of Wedderburn's Theorem once we prove that every automorphism of $\mathrm{End}(V)$ is inner.

Scholium 3.2.2 *Let* $\phi \in \mathrm{Aut}(\mathrm{End}(V))$*. Then there exists* $g \in \mathrm{GL}(V)$ *such that* $\phi(x) = gxg^{-1}$ *for all* $x \in \mathrm{End}(V)$*.*

Proof. Choose a basis $e_1, \ldots, e_n$ for V and let $E_{ij} \in \mathrm{End}(V)$ be the transformation that maps e_i to e_j and annihilates e_k for $k \neq i$. Set $P_i = \phi(E_{ii})$. Since ϕ is an automorphism of $\mathrm{End}(V)$, we have

$$P_i^2 = P_i \neq 0, \quad P_i P_j = \delta_{ij} P_j, \quad \sum_{i=1}^{n} P_i = I_V.$$

For $i = 1, \dots n$ choose $0 \neq f_i \in P_i V$. Then the set $\{f_1, \dots, f_n\}$ is linearly independent. To prove this, we first note that $P_i f_j = \delta_{ij} f_j$. If $\sum_i c_i f_i = 0$ then

$$0 = P_j \left(\sum_i c_i f_i \right) = c_j f_j.$$

Thus $c_j = 0$ for all j. Since $\dim V = n$, it follows that $\{f_1, \dots, f_n\}$ is a basis for V. Hence there exists $x \in \mathrm{GL}(V)$ such that $xe_i = f_i$ for $i = 1, \dots, n$. Define $\tilde{\phi} \in \mathrm{Aut}(\mathrm{End}(V))$ by

$$\tilde{\phi}(y) = x^{-1} \phi(y) x.$$

Then $\tilde{\phi}(E_{ii}) = E_{ii}$, so replacing ϕ by $\tilde{\phi}$ we may assume that $\phi(E_{ii}) = E_{ii}$ for $i = 1, \dots, n$.

We now calculate the action of ϕ on the off-diagonal matrix units. With ϕ normalized as above, we have

$$\phi(E_{ij}) = \phi(E_{ii} E_{ij} E_{jj}) = E_{ii} \phi(E_{ij}) E_{jj}.$$

Hence $\phi(E_{ij}) e_k = 0$ for $k \neq j$, and $\phi(E_{ij}) e_j \in \mathbb{C} e_i$. This implies that

$$\phi(E_{ij}) = \lambda_{ij} E_{ij} \tag{3.16}$$

for some non-zero scalar λ_{ij}. Since $\phi(E_{ij} E_{jk}) = \phi(E_{ik})$, the scalars λ_{ij} satisfy the relations

$$\lambda_{ij} \lambda_{jk} = \lambda_{ik}.$$

Since we have normalized ϕ so that $\lambda_{ii} = 1$, it follows that $\lambda_{ij}^{-1} = \lambda_{ji}$. Set $\lambda_i = \lambda_{i1}$. Then

$$\lambda_{ij} = \lambda_{i1} \lambda_{1j} = \lambda_i \lambda_j^{-1}.$$

Set $h = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$. Then

$$h E_{ij} h^{-1} = \lambda_i \lambda_j^{-1} E_{ij} = \lambda_{ij} E_{ij},$$

so by equation (3.16) we have $h^{-1} \phi(E_{ij}) h = E_{ij}$ for all i, j. Hence $h^{-1} \phi(x) h = x$ for all $x \in \mathrm{End}(V)$. Thus ϕ is the inner automorphism given by h. $\Diamond$

Proposition 3.2.3 *Up to equivalence, the only irreducible representation of* $\mathrm{End}(V)$ *is the representation* τ *on* V *given by* $\tau(x)v = xv$.

Proof. Let (ρ, W) be an irreducible representation of $\mathrm{End}(V)$. Wedderburn's theorem implies that $\mathrm{End}(V) \cong \mathrm{End}(W)$ as an algebra. Since $\dim \mathrm{End}(V) = \dim(V)^2$, we have $\dim(V) = \dim(W)$. Fix a linear bijection $T : V \to W$, and define

$$\phi(x) = T^{-1} \rho(x) T, \qquad \text{for } x \in \mathrm{End}(V).$$

Then ϕ is an automorphism of $\mathrm{End}(V)$, so by Scholium 3.2.2 there exists $g \in \mathrm{End}(V)$ such that $\phi(x) = gxg^{-1}$. Set $S = (Tg)^{-1}$. Then $S : W \to V$ and

$$S\rho(x) = ST\phi(x)T^{-1} = STgxg^{-1}T^{-1} = xS$$

for $x \in \mathrm{End}(V)$. Since S is a linear bijection, we conclude that $(\rho, W) \cong (V, \tau)$. $\Diamond$

We now establish a canonical form for an arbitrary finite-dimensional representation of $\mathrm{End}(V)$. For this we will need the following differentiated version of Scholium 3.2.2. Recall that a *derivation* of an algebra $\mathcal{A}$ is a map $D \in \mathrm{End}(\mathcal{A})$ such that $D(xy) = (Dx)y + x(Dy)$ for all $x, y \in \mathcal{A}$.

Scholium 3.2.4 *Let D be a derivation of the associative algebra* $\mathrm{End}(V)$. *Then there exists* $A \in \mathrm{End}(V)$ *such that* $D(x) = Ax - xA$ *for all* $x \in \mathrm{End}(V)$.

Proof. For $x, y \in \mathrm{End}(V)$,

$$\begin{aligned} D([x,y]) &= (Dx)y + x(Dy) - (Dy)x - y(Dx) \\ &= [Dx, y] + [x, Dy], \end{aligned}$$

where $[x,y] = xy - yx$ is the commutator. Thus D is also a derivation of $\mathrm{End}(V)$ as a Lie algebra. Write $I = I_V$. Then $D(x) = D(Ix) = D(I)x + D(x)$ for all $x \in \mathrm{End}(V)$, and hence $D(I) = 0$. Let $\mathfrak{g} = \mathfrak{sl}(V)$. Since $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ we also have $D\mathfrak{g} \subset \mathfrak{g}$.

Let $\mathrm{Der}(\mathfrak{g}) \subset \mathrm{End}(\mathfrak{g})$ be the vector space of all linear transformations T on $\mathfrak{g}$ such that

$$T([X,Y]) = [TX, Y] + [X, TY] \quad \text{for all } X, Y \in \mathfrak{g}.$$

If $Z \in \mathfrak{g}$ then $\mathrm{ad}\, Z \in \mathrm{Der}(\mathfrak{g})$ by the Jacobi identity. Furthermore, if $T \in \mathrm{Der}(\mathfrak{g})$ then

$$[T, \mathrm{ad}(Z)]X = T([Z,X]) - [Z, T(X)] = [T(Z), X] = \mathrm{ad}(T(Z))X \quad \text{for all } X, Z \in \mathfrak{g}.$$

Hence $[T, \mathrm{ad}(Z)] = \mathrm{ad}(T(Z))$. This shows that

$$[\mathrm{ad}(\mathfrak{g}), \mathrm{Der}(\mathfrak{g})] \subset \mathrm{ad}(\mathfrak{g}). \tag{3.17}$$

Thus we can obtain a representation ρ of $\mathfrak{g}$ on $\mathrm{Der}(\mathfrak{g})$ by

$$\rho(Z)T = [\mathrm{ad}(Z), T] \quad \text{for } T \in \mathrm{Der}(\mathfrak{g}).$$

Since the subspace $\mathrm{ad}(\mathfrak{g})$ of $\mathrm{Der}(\mathfrak{g})$ is invariant under $\rho(\mathfrak{g})$ and every representation of $\mathfrak{g}$ is completely reducible (Theorem 2.4.6), there is a subspace $U \subset \mathrm{Der}(\mathfrak{g})$ so that

$$\mathrm{Der}(\mathfrak{g}) = \mathrm{ad}(\mathfrak{g}) \oplus U, \quad [\mathrm{ad}(\mathfrak{g}), U] \subset U.$$

On the other hand, $[\mathrm{ad}(\mathfrak{g}), U] \subset \mathrm{ad}(\mathfrak{g})$ by (3.17). Hence $U = 0$. This proves that

$$\mathrm{Der}(\mathfrak{g}) = \mathrm{ad}(\mathfrak{g}). \tag{3.18}$$

Returning to the derivation D of $\mathrm{End}(V)$, we conclude that there exists $Z \in \mathfrak{g}$ so that $D(X) = [X, Z]$ for all $X \in \mathfrak{g}$. Since $D(I) = 0$, this equation holds for all $X \in \mathrm{End}(V)$. Thus we may take $A = -Z$. $\Diamond$

We now obtain a canonical form for the representations of $\mathrm{End}(V)$. We use the notation

$$V^m = \underbrace{V \oplus \cdots \oplus V}_{m \text{ copies}}$$

to denote the direct sum of m copies of the representation of $\mathrm{End}(V)$ on V.

Theorem 3.2.5 *Let* $\mathcal{A} = \mathrm{End}(V)$ *and suppose* (ρ, W) *is a finite-dimensional representation of* $\mathcal{A}$. *Then* $\dim W = m \dim V$, *where* $m = \dim \mathrm{Hom}_{\mathcal{A}}(V, W)$, *and there exists a linear bijection*

$$T : W \to V^m, \qquad \textit{with } Tw = (v_1, \ldots, v_m),$$

such that $T\rho(x)w = (xv_1, \ldots, xv_m)$ *for* $x \in \mathcal{A}$ *and* $w \in W$. *Hence* W *is equivalent to the* $\mathcal{A}$*-module* $\mathrm{Hom}_{\mathcal{A}}(V, W) \otimes V$, *where* $x \in \mathcal{A}$ *acts by* $x \cdot (u \otimes v) = u \otimes (xv)$ *for* $u \in \mathrm{Hom}_{\mathcal{A}}(V, W)$ *and* $v \in V$.

Proof. Since $\dim W$ is finite, W contains an irreducible submodule W_1. If $W_1 \neq W$ then there is a submodule $W_2 \supset W_1$ such that the representation of $\mathrm{End}(V)$ on W_2/W_1 is irreducible. Continuing in this way, we obtain a *Jordan-Hölder series*

$$W_1 \subset W_2 \subset \cdots \subset W_m = W$$

of submodules with each quotient W_{i+1}/W_i irreducible and hence isomorphic to V by Proposition 3.2.3. In particular,

$$\dim W = m \dim V.$$

We prove the existence of the map T by induction on m. When $W = W_1$ we may take $T = I$. Thus we may assume inductively that there are intertwining maps

$$T_1 : W_1 \cong V, \quad T_2 : W/W_1 \cong V^{\otimes(m-1)}.$$

Let $\pi : W \to W/W_1$ be the canonical projection. Choose a subspace $Z \subset W$ so that $W = W_1 \oplus Z$, and let

$$P : W \to Z, \quad Q : W \to W_1$$

be the corresponding projections. (Since Z is not necessarily a ρ-invariant subspace, these projections are generally not intertwining operators.) Define a linear bijection

$$T : W \to V^m, \quad T(w_1 + z) = (T_1 w_1, T_2\pi(z))$$

for $w_1 \in W_1$ and $z \in Z$. Since T_1, T_2 and π are intertwining maps and $\pi P = \pi$, we have

$$\begin{aligned} T\rho(x)(w_1 + z) &= T(\rho(x)w_1 + Q\rho(x)z + P\rho(x)z) \\ &= (xT_1w_1 + T_1Q\rho(x)z, xT_2\pi z) \end{aligned}$$

for $x \in \mathrm{End}(V)$. Thus if $w \in W$ and we write $T(w) = (v_1, \ldots, v_m)$ with $v_i \in V$, then

$$T\rho(x)w = (xv_1 + \sum_{i=2}^{m} \mu_i(x)v_i,\ xv_2, \ldots, xv_m), \tag{3.19}$$

where $\mu_i(x) \in \mathrm{End}(V)$.

Obviously the maps $\mu_i(x)$ depend linearly on x. From the equation $\rho(xy) = \rho(x)\rho(y)$ and equation (3.19) we find that

$$\sum_{i=2}^{m} \mu_i(xy)v_i = \sum_{i=2}^{m} x\mu_i(y)v_i + \sum_{i=2}^{m} \mu_i(x)yv_i$$

for all $v_i \in V$ and $x, y \in \mathrm{End}(V)$. Hence for $i = 1, \ldots, m$ we have

$$\mu_i(xy) = x\mu_i(y) + \mu_i(x)y.$$

Thus μ_i is a derivation of $\mathrm{End}(V)$. By Scholium 3.2.4 there exists $A_i \in \mathrm{End}(V)$ so that $\mu_i(x) = [A_i, x]$.

We have now shown that ρ is equivalent to the representation $\tilde{\rho}$ on V^m given by

$$\tilde{\rho}(x)(v_1, \ldots, v_m) = (xv_1 + \sum_{i=2}^{m} [A_i, x]v_i,\ xv_2, \ldots, xv_m).$$

Define a linear transformation g on V^m by

$$g \cdot (v_1, \ldots, v_m) = (v_1 + \sum_{i=2}^{m} A_i v_i,\ v_2, \ldots, v_m).$$

Then g is a linear bijection, with inverse

$$g^{-1} \cdot (v_1, \ldots, v_m) = (v_1 - \sum_{i=2}^{m} A_i v_i,\ v_2, \ldots, v_m).$$

It follows that

$$g^{-1}\tilde{\rho}(x)g(v_1, \ldots, v_m) = (xv_1, \ldots, xv_m).$$

Thus ρ is equivalent to the direct sum of m copies of the representation of $\mathrm{End}(V)$ on V.

We have now proved, in particular, that V is the only irreducible $\mathcal{A}$-module, up to equivalence. Hence the last statement of the theorem follows by Proposition 3.1.6. $\Diamond$

3.2.3 Exercises

1. Let $\mathbb{H}$ be the algebra (over $\mathbb{R}$) of quaternions (see Section 1.4.4).

   (a) Let $\mathcal{A} = \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification of $\mathbb{H}$. Show that $\mathcal{A}$ is a simple algebra and is isomorphic to $M_2(\mathbb{C})$.

   (b) Let $M_n(\mathbb{H})$ be the algebra (over $\mathbb{R}$) of $n \times n$ matrices with coefficients in $\mathbb{H}$. Let $\mathcal{A} = M_n(\mathbb{H}) \otimes_{\mathbb{R}} \mathbb{C}$ be complexification of $M_n(\mathbb{H})$. Show that $\mathcal{A}$ is a simple algebra and is isomorphic to $M_{2n}(\mathbb{C})$.

2. Let $\mathcal{A}$ be an algebra over $\mathbb{C}$ (not assumed associative) and let D_1, D_2 be derivations of $\mathcal{A}$. Prove that the operator $D_1D_2 - D_2D_1$ is a derivation of $\mathcal{A}$.

3. Let $\mathcal{A}$ be an associative algebra over $\mathbb{C}$ with unit 1. Let (ρ, U) and (σ, V) be representations of $\mathcal{A}$.

   (a) Let $S \in \mathrm{Hom}(V, U)$ be any linear map. Define $\tau(a) = \rho(a)S - S\sigma(a)$ for $a \in \mathcal{A}$. Define $\pi(a) \in \mathrm{End}(U \oplus V)$ by

   $$\pi(a)(u \oplus v) = (\rho(a)u + \tau(a)v) \oplus \sigma(a)v$$

   for $u \in U$ and $v \in V$. Show that τ satisfies the identity

   $(*)$ $\quad \tau(ab) = \tau(a)\sigma(b) + \rho(a)\tau(b)$ for all $a, b \in \mathcal{A}$, and that π is a representation of $\mathcal{A}$ equivalent to the representation $\rho \oplus \sigma$. (*Hint:* Show that the operator $T(u \oplus v) = (u + Sv) \oplus v$ gives an equivalence between π and $\rho \oplus \sigma$.)

(b) Suppose $\tau : \mathcal{A} \to \mathrm{Hom}(V, U)$ is a linear map. Define a linear map $\pi : \mathcal{A} \to \mathrm{End}(U \oplus V)$ by

$$\pi(a)(u \oplus v) = (\rho(a)u + \tau(a)v) \oplus \sigma(a)v \quad \text{for } a \in \mathcal{A},\ u \in U,\ v \in V.$$

Show that π is a representation of $\mathcal{A}$ if and only if $\tau(1) = 0$ and τ satisfies $(*)$.

(c) Suppose every linear map $\tau : \mathcal{A} \to \mathrm{Hom}(V, U)$ which satisfies $\tau(1) = 0$ and $(*)$ is of the form $\tau(a) = \rho(a)S - S\sigma(a)$ for some $S \in \mathrm{Hom}(V, U)$. Show that the representation π in (b) is equivalent to the representation $\rho \oplus \sigma$.

(d) Rephrase the proof of Theorem 3.2.5 in this framework, when $\mathcal{A} = \mathrm{End}(V)$.

3.3 Commutants and Characters

3.3.1 Representations of Semisimple Algebras

A finite-dimensional associative algebra $\mathcal{A}$ with unit is said to be *semisimple* if it is the direct sum of simple algebras. Throughout this section we assume that $\mathcal{A}$ is semisimple with unit $1_{\mathcal{A}}$. By Wedderburn's theorem, there exist finite-dimensional vector spaces V^λ, with λ running over some finite set L, and an algebra isomorphism

$$\Phi : \mathcal{A} \xrightarrow{\cong} \bigoplus_{\lambda \in L} \mathrm{End}(V^\lambda). \tag{3.20}$$

Conversely, every direct sum of matrix algebras is semisimple. Let $E_\lambda \in \bigoplus_{\lambda \in L} \mathrm{End}(V^\lambda)$ denote the element

$$0 \oplus \cdots \oplus I_{V_\lambda} \oplus \cdots \oplus 0.$$

Set $e_\lambda = \Phi^{-1}(E_\lambda)$. Then $e_\lambda x = x e_\lambda$ for all $x \in \mathcal{A}$, so e_λ is in the center of $\mathcal{A}$. Clearly

$$\sum_{\lambda \in L} e_\lambda = 1_{\mathcal{A}}, \qquad e_\lambda e_\mu = \begin{cases} e_\lambda, & \text{if } \lambda = \mu \\ 0, & \text{otherwise.} \end{cases}$$

Since the center of $\mathrm{End}(V^\lambda)$ is $\mathbb{C} I_{V^\lambda}$, the set $\{e_\lambda\}_{\lambda \in L}$ is a basis for the center of $\mathcal{A}$. The property $e_\lambda^2 = e_\lambda$ is described by saying that e_λ is an *idempotent*. These central idempotents are *minimal*: if u is any idempotent element in the center of $\mathcal{A}$, then

$$u = \sum_{\lambda \in M} \pm e_\lambda \tag{3.21}$$

for some subset $M \subset L$. Thus $u = \pm e_\lambda$ if and only if M reduces to a single element λ (the proof is left as an exercise).

Given the isomorphism Φ in (3.20), we can describe all the representations of $\mathcal{A}$, as follows. We identify

$$\Phi(\mathcal{A}) E_\lambda = \mathrm{End}(V^\lambda).$$

This gives a representation (π^λ, V^λ) of $\mathcal{A}$, where

$$\pi^\lambda(x) = \Phi(x)E_\lambda \quad \text{for } x \in \mathcal{A}.$$

Proposition 3.3.1 *The representations (π^λ, V^λ) are irreducible and mutually inequivalent. Every irreducible representation of $\mathcal{A}$ is equivalent to some π^λ.*

Proof. Since $\pi^\lambda(\mathcal{A}) = \mathrm{End}(V^\lambda)$, the representation π^λ is irreducible. Suppose $T : V^\lambda \to V^\mu$ intertwines π^λ and π^μ for some $\mu \neq \lambda$. Since $\pi^\lambda(x) = \pi^\lambda(e_\lambda x)$ for all $x \in \mathcal{A}$, we have

$$T\pi^\lambda(x) = T\pi^\lambda(e_\lambda x) = \pi^\mu(e_\lambda x)T = \pi^\mu(e_\mu e_\lambda x)T = 0.$$

Hence $T = 0$, so $\pi^\lambda \not\cong \pi^\mu$.

Let (π, V) be an irreducible representation of $\mathcal{A}$. Since $\sum_\lambda e_\lambda = 1$, there is some $\lambda \in L$ so that $\pi(e_\lambda)V \neq 0$. Since

$$\pi(x)\pi(e_\lambda)V = \pi(xe_\lambda)V = \pi(e_\lambda)\pi(x)V \subset \pi(e_\lambda)V,$$

the subspace $\pi(e_\lambda)V$ is invariant under $\mathcal{A}$. The irreducibility of V implies that $\pi(e_\lambda)V = V$. If $\mu \neq \lambda$, then

$$\pi(e_\mu)V = \pi(e_\mu)\pi(e_\lambda)V = \pi(e_\mu e_\lambda)V = 0.$$

Thus $\pi(x) = \pi(e_\lambda x)$ for all $x \in \mathcal{A}$. But the map $e_\lambda x \mapsto \pi^\lambda(x)$ gives an algebra isomorphism

$$e_\lambda\mathcal{A} \cong \mathrm{End}(V^\lambda),$$

so we may view (π, V) as an irreducible representation of $\mathrm{End}(V^\lambda)$. By Proposition 3.2.3 there is a linear isomorphism $T : V \to V^\lambda$ such that

$$T\pi(e_\lambda x) = \pi^\lambda(e_\lambda x)T \quad \text{for } x \in \mathcal{A}.$$

Since $\pi(e_\lambda x) = \pi(x)$ and $\pi^\lambda(e_\lambda x) = \pi^\lambda(x)$ for $x \in \mathcal{A}$, it follows that the representations (π, V) and (π^λ, V^λ) of $\mathcal{A}$ are equivalent. $\Diamond$

From this proposition we may identify the index set L with $\widehat{\mathcal{A}}$ (the set of equivalence classes of irreducible representations of $\mathcal{A}$). In particular, we see that $\widehat{\mathcal{A}}$ is finite, and we may write (3.20) as

$$\Phi : \mathcal{A} \xrightarrow{\cong} \bigoplus_{\lambda\in\widehat{\mathcal{A}}} \mathrm{End}(V^\lambda). \tag{3.22}$$

An arbitrary representation of $\mathcal{A}$ can be described as follows.

Proposition 3.3.2 *Let $\mathcal{A}$ be given by (3.22) and suppose (ρ, W) is a finite-dimensional representation of $\mathcal{A}$. Set $U^\lambda = \mathrm{Hom}_\mathcal{A}(V^\lambda, W)$ for $\lambda \in \widehat{\mathcal{A}}$ and define a linear map*

$$S : \bigoplus_{\lambda\in\widehat{\mathcal{A}}} U^\lambda \otimes V^\lambda \to W, \qquad S\big(\sum_{\lambda\in\widehat{\mathcal{A}}} u_\lambda \otimes v_\lambda\big) = \sum_{\lambda\in\widehat{\mathcal{A}}} u_\lambda(v_\lambda).$$

Then S is an $\mathcal{A}$-module isomorphism and

$$S^{-1}\rho(x)S = \bigoplus_{\lambda\in\widehat{\mathcal{A}}} I_{U^\lambda} \otimes \pi^\lambda(x). \tag{3.23}$$

Proof. We use an argument similar to that of Proposition 3.3.1. Set $P_\lambda = \rho(e_\lambda)$. Then since ρ is a representation, we have

$$P_\lambda P_\mu = \delta_{\lambda\mu} P_\lambda, \quad \sum_{\lambda \in \widehat{\mathcal{A}}} P_\lambda = I_W.$$

Thus $W = \bigoplus_\lambda W^\lambda$, where $W^\lambda = P_\lambda W$, and

$$\rho(e_\lambda \mathcal{A}) W^\mu = \delta_{\lambda\mu} W^\lambda,$$

so the subspace W^λ is a module for $e_\lambda \mathcal{A} \cong \mathrm{End}(V^\lambda)$. Let $S_\lambda : U^\lambda \otimes V^\lambda \to W^\lambda$ be the $\mathrm{End}(V^\lambda)$-module isomorphism from Theorem 3.2.5. Since $S = \bigoplus_\lambda S_\lambda$, it follows that S is an $\mathcal{A}$-module isomorphism. $\Diamond$

Recall from Section 3.1.4 that a completely reducible representation of an algebra $\mathcal{A}$ decomposes uniquely into the direct sum of its isotypic components. For representations of semisimple algebras we can obtain the projections onto the isotypic components from the minimal central idempotents.

Corollary 3.3.3 *Suppose $\mathcal{A}$ is a semisimple algebra. For each $\lambda \in \widehat{\mathcal{A}}$ let $e_\lambda \in \mathcal{A}$ be the associated central idempotent. For any finite-dimensional representation (ρ, V) of $\mathcal{A}$ the λ-isotypic subspace is $\rho(e_\lambda)V$ and the primary decomposition is $V = \bigoplus_{\lambda \in \widehat{\mathcal{A}}} \rho(e_\lambda)V$.*

Proof. This follows from the definition of the primary decomposition and Proposition 3.3.2 $\Diamond$

Every finite-dimensional representation ρ of a semisimple algebra $\mathcal{A}$ is completely reducible (this follows from Proposition 3.3.2). We now show that this property characterizes semisimple algebras.

Proposition 3.3.4 *Let $\mathcal{A}$ be an associative algebra with unit. Suppose (ρ, V) is a completely reducible representation of $\mathcal{A}$. Then the algebra $\mathcal{B} = \rho(\mathcal{A})$ is semisimple.*

Proof. We shall prove by induction on $\dim V$ that the algebra $\mathcal{B}$ is isomorphic to a direct sum of matrix algebras. This is trivial when $\dim V = 1$, since $\mathcal{B} = M_1(\mathbb{C})$ in that case. Consider the general case. If $\mathcal{B}$ acts irreducibly on V then $\mathcal{B} = \mathrm{End}(V)$ by Burnside's Theorem (Theorem 3.1.2) and the result is true. Otherwise, by the hypothesis of complete reducibility there is a decomposition $V = U \oplus W$ into non-zero $\mathcal{B}$-invariant subspaces, with U irreducible under $\mathcal{B}$. Set

$$\mathcal{B}_0 = \{T \in \mathcal{B} \,:\, T|_W = 0\}.$$

Case 1: If $\mathcal{B}_0 = 0$, then as an algebra $\mathcal{B}$ is isomorphic to $\mathcal{B}|_W$. Since $\dim W < \dim V$ and W is completely reducible as a $\mathcal{B}$-module by Lemma 3.1.3, the induction hypothesis implies that $\mathcal{B}$ is isomorphic to a direct sum of matrix algebras.

Case 2: $\mathcal{B}_0 \neq 0$. Since $B_0 W = 0$, we must have $\mathcal{B}_0 U \neq 0$. Let $0 \neq u \in U$. Since U is irreducible, we have $\mathcal{B}u = U$. But $\mathcal{B}_0$ is a two-sided ideal in $\mathcal{B}$, so $\mathcal{B}_0 u = \mathcal{B}_0 \mathcal{B} u = \mathcal{B}_0 U$ is a non-zero $\mathcal{B}$-submodule. Hence $\mathcal{B}_0 u = U$, which shows that U is irreducible as a $\mathcal{B}_0$ module also. By Burnside's Theorem, $\mathcal{B}_0|_U = \mathrm{End}(U)$. Set

$$\mathcal{B}_1 = \{T \in \mathcal{B} \,:\, T|_U = 0\}.$$

Then $\mathcal{B}_1$ is a two-sided ideal in $\mathcal{B}$. Clearly $\mathcal{B}_0 \cap \mathcal{B}_1 = 0$ and hence $\mathcal{B}_1\mathcal{B}_0 = \mathcal{B}_0\mathcal{B}_1 = 0$. Furthermore, given $X \in \mathcal{B}$, we can find $Y \in \mathcal{B}_0$ such that $Y|_U = X|_U$ by the result just proved. Hence $X - Y \in \mathcal{B}_1$. This shows that $\mathcal{B} = \mathcal{B}_0 \oplus \mathcal{B}_1$ as an algebra. In particular, $\mathcal{B}|_W = \mathcal{B}_1|_W$, so the action of $\mathcal{B}_1$ on W is completely reducible. Since the map $T \mapsto T|_W$ is injective on $\mathcal{B}_1$, the induction hypothesis implies that $\mathcal{B}_1$ is isomorphic to a direct sum of matrix algebras. Thus $\mathcal{B}$ is semisimple. $\Diamond$

Corollary 3.3.5 (1) *Let G be a reductive linear algebraic group and let (ρ, V) be a regular representation of G. Then $\rho(\mathbb{C}[G])$ is a semisimple algebra.*

(2) *Let $\mathfrak{g}$ be the Lie algebra of a classical group, and let $\mathfrak{z}(\mathfrak{g})$ be the center of $\mathfrak{g}$. Let (π, V) be a finite-dimensional representation of $\mathfrak{g}$ and assume that $\pi(Z)$ is diagonalizable for all $Z \in \mathfrak{z}(\mathfrak{g})$. Then $\pi(U(\mathfrak{g}))$ is a semisimple algebra.*

Proof. (1) This is an immediate consequence of the definition of reductive group and Proposition 3.3.4.

(2) As in the proof of Theorem 2.5.7, we can write $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{g}'$, where $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ is a semisimple Lie algebra (in fact, $\mathfrak{g}'$ is simple except for $G = \mathrm{SO}(4, \mathbb{C})$). Hence the representation π is completely reducible by Theorem 2.4.6 and the assumption on $\mathfrak{z}(\mathfrak{g})$. Now apply Proposition 3.3.4. $\Diamond$

We now assemble these results to obtain the promised representation-theoretic characterization of semisimple algebras.

Theorem 3.3.6 *Suppose $\mathcal{A}$ is a finite-dimensional associative algebra with unit. The following are equivalent:*

(1) *The left regular representation $(L, \mathcal{A})$ of $\mathcal{A}$ is completely reducible (where $L(x)y = xy$ for $x, y \in \mathcal{A}$).*

(2) *Every finite-dimensional representation of $\mathcal{A}$ is completely reducible.*

(3) *$\mathcal{A}$ is a semisimple algebra.*

Proof. (1) $\Rightarrow$ (3): The algebra $\mathcal{B} = \lambda(\mathcal{A})$ is semisimple, by Proposition 3.3.4. But $\mathcal{A} \cong \lambda(\mathcal{A})$ since L is a faithful representation ($1 \in \mathcal{A}$).

(3) $\Rightarrow$ (2): This follows from Propositions 3.3.2 and 3.1.4.

(2) $\Rightarrow$ (1): Note that (1) is a special case of (2). $\Diamond$

3.3.2 Double Commutant Theorem

Let V be a finite dimensional vector space. For any subset $\mathcal{S} \subset \mathrm{End}(V)$ we define

$$\mathrm{Comm}(\mathcal{S}) = \{x \in \mathrm{End}(V) \,:\, xs = sx \quad \text{for all } s \in \mathcal{S}\}$$

and call it the *commutant* of $\mathcal{S}$. We observe that $\mathrm{Comm}(\mathcal{S})$ is an associative algebra with unit I_V.

Suppose now that $\mathcal{A} \subset \mathrm{End}(V)$ is a semisimple algebra with $I_V \in \mathcal{A}$. Set $\mathcal{B} = \mathrm{Comm}(\mathcal{A})$. The vector space $\mathcal{A} \otimes \mathcal{B}$ is an associative algebra under the multiplication

$$(a \otimes b)(a' \otimes b') = aa' \otimes bb',$$

and $\mathcal{A}$ (*resp.* $\mathcal{B}$) is isomorphic to the subalgebra $\mathcal{A} \otimes 1$ (*resp.* $1 \otimes \mathcal{B}$) of $\mathcal{A} \otimes \mathcal{B}$.

By Proposition 3.3.2 there is an $\mathcal{A}$-module isomorphism

$$V \cong \bigoplus_{i=1}^{r} V_i \otimes U_i \tag{3.24}$$

where V_i is an irreducible $\mathcal{A}$-module, $V_i \ncong V_j$ for $i \neq j$ and $U_i = \mathrm{Hom}_{\mathcal{A}}(V_i, V)$. Under this isomorphism

$$\mathcal{A} \cong \bigoplus_{i=1}^{r} \mathrm{End}(V_i) \otimes I_{U_i}. \tag{3.25}$$

We now use this isomorphism to obtain the basic dual relationship between the algebras $\mathcal{A}$ and $\mathrm{Comm}(\mathcal{A})$. This duality will play a fundamental role in the invariant theory of the classical groups.

Theorem 3.3.7 (Double Commutant) *Let V be a finite-dimensional vector space and $\mathcal{A} \subset \mathrm{End}(V)$ a semisimple algebra. Then the algebra $\mathcal{B} = \mathrm{Comm}(\mathcal{A})$ is semisimple and $\mathrm{Comm}(\mathcal{B}) = \mathcal{A}$. Furthermore, relative to the isomorphisms* (3.24), (3.25), *one has*

$$\mathcal{B} \cong \bigoplus_{i=1}^{r} I_{V_i} \otimes \mathrm{End}(U_i). \tag{3.26}$$

Hence the subspaces $V_i \otimes U_i$ are irreducible and mutually inequivalent representations of the algebra $\mathcal{A} \otimes \mathcal{B}$.

Proof. We first prove (3.26). We may assume that $V = \sum_i V_i \otimes U_i$ as in (3.24). Clearly the right side of (3.26) is contained in $\mathcal{B}$. For the opposite inclusion, let $P_i : V \to V_i \otimes U_i$, for $i = 1, \ldots, r$ be the projections associated with this decomposition. Then $P_i \in \mathcal{A}$ by (3.25). Hence if $T \in \mathcal{B}$ then

$$P_i T = T P_i, \quad T = \sum_{i=1}^{r} T|_{V_i \otimes U_i}.$$

Thus it suffices to prove (3.26) when $r = 1$, where it follows by Lemma 3.1.9.

From (3.26) we see that

$$\mathcal{B} \cong \bigoplus_{i=1}^{r} \mathrm{End}(U_i),$$

as an associative algebra. This implies that $\mathcal{B}$ is semisimple and $U_i \ncong U_j$ as a $\mathcal{B}$-module if $i \neq j$. Repeating the argument just given, with the roles of $\mathcal{A}$ and $\mathcal{B}$ interchanged, we conclude that $\mathcal{A} = \mathrm{Comm}(\mathcal{B})$. Finally, since

$$\mathrm{End}(V_i \otimes U_i) = \mathrm{End}(V_i) \otimes \mathrm{End}(U_i),$$

we see that $V_i \otimes U_i$ is an irreducible $\mathcal{A} \otimes \mathcal{B}$-module and these modules are mutually inequivalent. $\Diamond$

We can view (3.24) in two ways: as a decomposition of V into isotypic subspaces for $\mathcal{A}$ (where the representation V_i occurs with multiplicity $\dim U_i$), or as a decomposition of V

into isotypic subspaces for $\mathcal{B}$ (where the representation U_i occurs with multiplicity $\dim V_i$). This *dual* point of view sets up a correspondence between irreducible representations of $\mathcal{A}$ and irreducible representations of $\mathcal{B}$, where V_i is paired with U_i.

We now apply the Double Commutant Theorem to obtain a result that will play a central role in our study of tensor and polynomial invariants for the classical groups. Let V be a finite-dimensional vector space and ρ the defining representation of $\mathrm{GL}(V)$. For all integers $k \geq 0$ we have the representations $\rho_k = \rho^{\otimes k}$ on $\bigotimes^k V$. Since

$$\rho_k(g)(v_1 \otimes \cdots \otimes v_k) = gv_1 \otimes \cdots \otimes gv_k$$

for $g \in \mathrm{GL}(V)$, we can permute the positions of the vectors in the tensor product without changing the G-action. Thus there is the following algebra of operators that commute with $\rho_k(\mathrm{GL}(V))$. Let $\mathfrak{S}_k$ be the group of permutations of $\{1, 2, \ldots, k\}$. We define a representation σ_k of $\mathfrak{S}_k$ on $\bigotimes^k V$ by

$$\sigma_k(s)(v_1 \otimes \cdots \otimes v_k) = v_{s^{-1}(1)} \otimes \cdots \otimes v_{s^{-1}(k)}.$$

for $s \in \mathfrak{S}_k$. Hence $\sigma_k(s)$ moves the vector in the ith position in the tensor product to the position $s(i)$. It is clear from this description that $\sigma_k(s)\sigma_k(t) = \sigma_k(st)$ for $s, t \in \mathfrak{S}_k$, so σ_k is a representation of $\mathfrak{S}_k$.

Theorem 3.3.8 *Set* $\mathcal{A} = \rho_k(\mathbb{C}[\mathrm{GL}(V)])$ *and* $\mathcal{B} = \sigma_k(\mathbb{C}[\mathfrak{S}_k])$*. Then* $\mathrm{Comm}(\mathcal{B}) = \mathcal{A}$ *and* $\mathrm{Comm}(\mathcal{A}) = \mathcal{B}$*.*

Proof. Since the algebra $\mathcal{B}$ is semisimple, as the group algebra of a finite group, it suffices by Theorem 3.3.7 to prove that $\mathrm{Comm}(\mathcal{B}) = \mathcal{A}$.

It is clear that $\sigma_k(s)$ commutes with $\rho_k(g)$ for all $s \in \mathfrak{S}_k$ and $g \in \mathrm{GL}(V)$. Thus $\mathcal{A} \subset \mathrm{Comm}(\mathcal{B})$. To prove the opposite inclusion, we fix a basis $\{e_1, \ldots, e_n\}$ for V. For an ordered k-tuple

$$I = (i_1, \ldots, i_k) \quad \text{with } 1 \leq i_j \leq n,$$

set $\#I = k$ and

$$e_I = e_{i_1} \otimes \cdots \otimes e_{i_k}.$$

The elements e_I form a basis for $\bigotimes^k V$ as I ranges over the finite set of all such k-tuples. The group $\mathfrak{S}_k$ permutes this basis by the action $\sigma_k(s)e_I = e_{s\cdot I}$, where for $I = (i_1, \ldots, i_k)$ we set

$$s \cdot (i_1, \ldots, i_k) = (i_{s^{-1}(1)}, \ldots, i_{s^{-1}(k)})$$

for $s \in \mathfrak{S}_k$. Note that s changes the *positions* (1 to k) of the indices, not their values (1 to n). We have $(st) \cdot I = s \cdot (t \cdot I)$ for $s, t \in \mathfrak{S}_k$.

Suppose $T \in \mathrm{End}(\bigotimes^k V)$ has matrix $[a_{I,J}]$ relative to the basis $\{e_I\}$:

$$Te_J = \sum_I a_{I,J}\, e_I.$$

We have

$$T(\sigma_k(s)e_J) = T(e_{s\cdot J}) = \sum_I a_{I,s\cdot J}\, e_I$$

for $s \in \mathfrak{S}_k$, while

$$\sigma_k(s)(Te_J) = \sum_I a_{I,J}\, e_{s\cdot I} = \sum_I a_{s^{-1}\cdot I,J}\, e_I\,.$$

Thus $T \in \mathrm{Comm}(\mathcal{B})$ if and only if $a_{I,s\cdot J} = a_{s^{-1}\cdot I,J}$ for all multi-indices I, J and all $s \in \mathfrak{S}_k$. Replacing I by $s \cdot I$, we can write this condition as

$$a_{s\cdot I,s\cdot J} = a_{I,J} \quad \text{for all } I, J \text{ and all } s \in \mathfrak{S}_k. \tag{3.27}$$

Consider the non-degenerate bilinear form

$$(X, Y) = \mathrm{tr}(XY)$$

on $\mathrm{End}(\bigotimes^k V)$. We claim that the restriction of this form to $\mathrm{Comm}(\mathcal{B})$ is non-degenerate. Indeed, we have a projection $X \mapsto X^\natural$ of $\mathrm{End}(\bigotimes^k V)$ onto $\mathrm{Comm}(\mathcal{B})$ given by averaging over $\mathfrak{S}_k$:

$$X^\natural = \frac{1}{k!} \sum_{s\in\mathfrak{S}_k} \sigma_k(s) X \sigma_k(s)^{-1}.$$

If $T \in \mathrm{Comm}(\mathcal{B})$ then

$$(X^\natural, T) = \frac{1}{k!} \sum_{s\in\mathfrak{S}_k} \mathrm{tr}(\sigma_k(s) X \sigma_k(s)^{-1} T) = (X, T),$$

since $\sigma_k(s)T = T\sigma_k(s)$. Thus $(\mathrm{Comm}(\mathcal{B}), T) = 0$ implies that $(X, T) = 0$ for all $X \in \mathrm{End}(\bigotimes^k V)$, and so $T = 0$. This proves the non-degeneracy of the trace form on $\mathrm{Comm}(\mathcal{B})$.

Thus to prove that $\mathcal{A} = \mathrm{Comm}(\mathcal{B})$, it suffices to show that if $T \in \mathrm{Comm}(\mathcal{B})$ is orthogonal to $\mathcal{A}$ then $T = 0$. Now if $g \in \mathrm{GL}(V)$ and $ge_j = \sum_i g_{ij}\, e_i$, then $\rho_k(g)$ has matrix

$$g_{I,J} = g_{i_1 j_1} \cdots g_{i_k j_k}\,.$$

Thus we assume that

$$(T, \rho_k(g)) = \sum_{I,J} a_{I,J}\, g_{j_1 i_1} \cdots g_{j_k i_k} = 0$$

for all $g \in \mathrm{GL}(n, \mathbb{C})$, where $[a_{I,J}]$ is the matrix of T. But the function $g \mapsto (T, \rho_k(g))$ on $\mathrm{GL}(n, \mathbb{C})$ extends to a polynomial function on $M_n(\mathbb{C})$. Since this function vanishes on $\mathrm{GL}(n, \mathbb{C})$, it must be identically zero. Hence for all $X = [x_{ij}] \in M_n(\mathbb{C})$ we have

$$\sum_{I,J} a_{I,J}\, x_{j_1 i_1} \cdots x_{j_k i_k} = 0. \tag{3.28}$$

We now show that (3.27) and (3.28) imply that $a_{I,J} = 0$ for all I, J. We begin by grouping the terms in (3.28) according to distinct monomials in the matrix entries $\{x_{ij}\}$. Introduce the notation

$$x_{I,J} = x_{i_1 j_1} \cdots x_{i_k j_k}\,,$$

and view these monomials as polynomial functions on $M_n(\mathbb{C})$. Let Ξ be the set of all ordered pairs (I, J) of multi-indices with $\#I = \#J = k$. The group $\mathfrak{S}_k$ acts on Ξ by $s \cdot (I, J) = (s \cdot I, s \cdot J)$, and from (3.27) we see that T commutes with $\mathfrak{S}_k$ if and only if the function $(I, J) \mapsto a_{I,J}$ is constant on the orbits of $\mathfrak{S}_k$ in Ξ. The action of $\mathfrak{S}_k$ on Ξ defines an equivalence relation on Ξ, where $(I, J) \equiv (I', J')$ if $(I', J') = (s \cdot I, s \cdot J)$ for some $s \in \mathfrak{S}_k$. This gives a decomposition of Ξ into disjoint equivalence classes. Choose a set Γ of representatives for the equivalence classes. Then every monomial $x_{I,J}$ with $\#I = \#J = k$ can be written as x_γ for some $\gamma \in \Gamma$. Indeed, since the variables x_{ij} mutually commute, we have

$$x_\gamma = x_{s\cdot\gamma}$$

for all $s \in \mathfrak{S}_k$ and $\gamma \in \Gamma$. Suppose $x_{I,J} = x_{I',J'}$. Then there must be an integer p such that

$$x_{i'_1 j'_1} = x_{i_p j_p}.$$

Call $p = 1'$. Similarly, there must be an integer $q \neq p$ such that

$$x_{i'_2 j'_2} = x_{i_q j_q}.$$

Call $q = 2'$. Continuing this way, we obtain a permutation $s : (1, 2, \ldots, k) \to (1', 2', \ldots, k')$ such that $I = s \cdot I'$ and $J = s \cdot J'$. This proves that γ is uniquely determined by x_γ. For $\gamma \in \Gamma$ let $n_\gamma = |\mathfrak{S}_k \cdot \gamma|$ be the cardinality of the corresponding orbit.

Assume that the coefficients $a_{I,J}$ satisfy (3.27) and (3.28). Since $a_{I,J} = a_\gamma$ for all $(I, J) \in \mathfrak{S}_k \cdot \gamma$, equation (3.28) implies that

$$\sum_{\gamma\in\Gamma} n_\gamma\, a_\gamma\, x_\gamma = 0.$$

But the set of functions $\{x_\gamma : \gamma \in \Gamma\}$ is linearly independent, so this implies that $a_{I,J} = 0$ for all $(I, J) \in \Xi$. $\Diamond$

3.3.3 Characters

Let $\mathcal{A}$ be an associative algebra with 1. If (ρ, V) is a finite-dimensional representation of $\mathcal{A}$, then the *character* of the representation is the linear functional $\mathrm{ch}\, V$ on $\mathcal{A}$ given by

$$\mathrm{ch}V(a) = \mathrm{tr}_V(\rho(a)) \qquad \text{for } a \in \mathcal{A}.$$

Proposition 3.3.9 *The following properties hold for characters:*

(1) $\mathrm{ch}V(ab) = \mathrm{ch}V(ba)$ *for all* $a, b \in \mathcal{A}$.

(2) $\mathrm{ch}V(1) = \dim V$

(3) *If* $U \subset V$ *is a submodule, and* $W = V/U$ *is the quotient module, then* $\mathrm{ch}V = \mathrm{ch}U + \mathrm{ch}W$.

Remark. In (3) we do not assume that U has a complementary submodule in V.

Proof. Properties (1) and (2) are obvious from the definition. As for (3), we pick a subspace $Z \subset V$ complementary to U. Then the matrix of $\rho(a), a \in \mathcal{A}$ relative to the decomposition $V = U \oplus Z$ is in block triangular form. Its trace is the sum of the trace on U and on $Z \mod U$. But the action of $\rho(a)$ on $Z \mod U$ is the same as the action on W, so the traces coincide. $\Diamond$

The use of characters in representation theory is a powerful tool, as will become apparent in Chapters 7, 8 and 9. Let us find the extent to which a representation is determined by its character.

Lemma 3.3.10 *Suppose $(\rho_1, V_1), \ldots, (\rho_r, V_r)$ are finite-dimensional irreducible representations of $\mathcal{A}$ such that ρ_i is not equivalent to ρ_j when $i \neq j$. Then the set $\{\mathrm{ch}V_1, \ldots, \mathrm{ch}V_r\}$ of linear functionals on $\mathcal{A}$ is linearly independent.*

Proof. Set $V = V_1 \oplus \cdots \oplus V_r$ and $\rho = \rho_1 \oplus \cdots \oplus \rho_r$. Since the representations V_i are mutually inequivalent, Theorem 3.3.7 implies that

$$\rho(\mathcal{A}) = \bigoplus_{i=1}^{r} \mathrm{End}(V_i)$$

Let $I_i \in \mathrm{End}(V_i)$ be the identity operator on V_i, and pick $Q_i \in \mathcal{A}$ with $\rho(Q_i) = I_i$. Then

$$\mathrm{ch}\, V_i(Q_j) = \mathrm{tr}(I_j|_{V_i}) = \delta_{ij} \dim V_i\,.$$

Thus given a linear relation $\sum a_i \,\mathrm{ch}V_i = 0$, we may evaluate on Q_j to conclude that $a_j \dim V_j = 0$. Hence $a_j = 0$ for all j. $\Diamond$

Let (ρ, V) be a finite-dimensional $\mathcal{A}$-module. A *composition series* (or *Jordan-Hölder series*) for V is a sequence of submodules

$$(0) = V_0 \subset V_1 \subset \cdots \subset V_r = V$$

such that $0 \neq W_i = V_i/V_{i-1}$ is irreducible for $i = 1, \ldots, r$. It is clear by induction on $\dim V$ that a composition series always exists. We define the *semi-simplification* of V to be the module

$$V_{ss} = \bigoplus W_i\,.$$

By (3) of Proposition 3.3.9 and the obvious induction, we see that

$$\mathrm{ch}\, V = \sum_{i=1}^{r} \mathrm{ch}(V_i/V_{i-1}) = \mathrm{ch}\, V_{ss}. \tag{3.29}$$

Theorem 3.3.11 *Let (ρ, V) be a finite-dimensional $\mathcal{A}$-module.*

(1) *The irreducible factors in a composition series for V are unique up to isomorphism and order of appearance.*

(2) *The module V_{ss} is uniquely determined by* $\mathrm{ch}V$ *up to isomorphism. In particular, if V is completely reducible, then V is uniquely determined up to isomorphism by* $\mathrm{ch}V$.

Proof. Let (ρ_i, U_i), for $i = 1, \ldots, n$, be the pairwise inequivalent irreducible representations that occur in the composition series for V, with corresponding multiplicities m_i. Then

$$\mathrm{ch}\, V = \sum_{i=1}^{n} m_i \,\mathrm{ch}(U_i)$$

by (3.29). Lemma 3.3.10 implies that the multiplicities m_i are uniquely determined by $\mathrm{ch}\, V$. This implies (1) and (2). $\Diamond$

Example

Let $G = \mathrm{SL}(2, \mathbb{C})$ and let

$$d(q) = \begin{bmatrix} q & 0 \\ 0 & q^{-1} \end{bmatrix} \qquad \text{for } q \in \mathbb{C}^\times.$$

If (ρ, V) is a regular representation of G, then $\mathrm{ch}(V)$ (as a character of the group algebra $\mathbb{C}[G]$) is completely determined by the rational function $q \mapsto \mathrm{ch}(V)(d(q))$ for $q \in \mathbb{C}^\times$, since the set

$$\{g d(q) g^{-1} \,:\, g \in \mathrm{SL}(2,\mathbb{C}),\, q \in \mathbb{C}^\times\}$$

is Zariski-dense in $\mathrm{SL}(2, \mathbb{C})$. For example, let (ρ_k, V_k) be the $(k+1)$-dimensional irreducible regular representation of $\mathrm{SL}(2, \mathbb{C})$ (see Proposition 2.2.3). Then

$$\mathrm{ch}(V_k)(d(q)) = q^k + q^{k-2} + \cdots + q^{-k+2} + q^{-k}.$$

For n a positive integer we define

$$[n]_q = q^{n-1} + q^{n-3} + \cdots + q^{-n+3} + q^{-n+1} = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

as a rational function of q. Thus we can write

$$\mathrm{ch}(V_k)(d(q)) = [k+1]_q$$

Define $[0]_q = 1$ and $[n]_q! = \prod_{j=0}^{n} [n-j]_q$ for n a positive integer and set

$$\begin{bmatrix} m+n \\ n \end{bmatrix}_q = \frac{[m+n]_q!}{[m]_q! [n]_q!}$$

(the *q-binomial coefficients*). Note the symmetry $\begin{bmatrix} m+n \\ n \end{bmatrix}_q = \begin{bmatrix} m+n \\ m \end{bmatrix}_q$.

Theorem 3.3.12 (Hermite Reciprocity) *Let V_k be the $(k+1)$-dimensional irreducible representation of $\mathrm{SL}(2, \mathbb{C})$. Let $S^j(V_k)$ be the jth symmetric power of V_k. Then for $q \in \mathbb{C}^\times$,*

$$\mathrm{ch}(S^j(V_k))(d(q)) = \begin{bmatrix} k+j \\ k \end{bmatrix}_q. \tag{3.30}$$

In particular, $S^j(V_k) \cong S^k(V_j)$ as representations of $\mathrm{SL}(2, \mathbb{C})$.

Proof. Since the representation $S^j(V_k)$ is completely reducible, it is determined up to equivalence by its character, by Theorem 3.3.11. Hence by the remarks above, it suffices to prove (3.30).

We fix k and write

$$F_j(q) = \mathrm{ch}(S^j(V_k))(d(q))$$

for $q \in \mathbb{C}^\times$. Let $\{x_0, \ldots, x_k\}$ be a basis for V_k such that

$$\rho_k(d(q)) x_j = q^{k-2j} x_j.$$

Then the monomials

$$x_0^{m_0} x_1^{m_1} \cdots x_k^{m_k}, \qquad m_0 + \cdots + m_k = j$$

give a basis for $S^j(V_k)$, and $d(q)$ acts on such a monomial by the scalar q^r with

$$r = km_0 + (k-2)m_1 + \cdots + (2-k)m_{k-1} - km_k.$$

Hence

$$F_j(q) = \sum_{m_0,\ldots,m_k} q^{km_0+(k-2)m_1+\cdots+(2-k)m_{k-1}-km_k}$$

with the sum over all nonnegative integers $m_0, \ldots, m_k$ such that $m_0 + \cdots + m_k = j$.

We form the *generating function*

$$F(t,q) = \sum_{j=0}^{\infty} t^j F_j(q),$$

which we view as a formal power series in the indeterminate t with coefficients in the ring $\mathbb{C}[q, q^{-1}]$ of rational functions of q.

Lemma 3.3.13

$$F(t,q) = \prod_{j=0}^{k} (1 - tq^{k-2j})^{-1} \tag{3.31}$$

Proof. By definition $(1 - tq^{k-2j})^{-1}$ is the formal power series

$$\sum_{m=0}^{\infty} t^m q^{m(k-2j)}.$$

Hence the right side of (3.31) is

$$\sum_{m_0,\ldots,m_k} t^{m_0+\cdots+m_k}\, q^{km_0+(k-2)m_1+\cdots+(2-k)m_{k-1}-km_k}$$

with the sum over all nonnegative integers $m_0, \ldots, m_k$. Thus the coefficient of t^j is $F_j(q)$. $\Diamond$

To complete the proof of Theorem 3.3.12, it now suffices to prove the following result.

Lemma 3.3.14

$$\prod_{j=0}^{k} (1 - tq^{k-2j})^{-1} = \sum_{j=0}^{\infty} t^j \begin{bmatrix} k+j \\ k \end{bmatrix}_q.$$

Proof. By induction on k. If $k = 0$ then the equation says that $(1-t)^{-1} = \sum_{j=0}^{\infty} t^j$. We set

$$H_k(t,q) = \sum_{j=0}^{\infty} t^j \begin{bmatrix} k+j \\ k \end{bmatrix}_q$$

and we assume that

$$H_k(t,q) = \prod_{j=0}^{k}(1 - tq^{k-2j})^{-1}$$

Now

$$\left[\begin{array}{c} k+1+j \\ k+1 \end{array}\right]_q = \frac{q^{k+1+j} - q^{-k-1-j}}{q^{k+1} - q^{-k-1}} \left[\begin{array}{c} k+j \\ k \end{array}\right]_q .$$

Thus

$$\begin{aligned} H_{k+1}(t,q) &= \frac{q^{k+1}}{q^{k+1} - q^{-k-1}} H_k(tq,\, q) - \frac{q^{-k-1}}{q^{k+1} - q^{-k-1}} H_k(t/q, q) \\ &= \frac{q^{k+1}}{(q^{k+1} - q^{-k-1}) \prod_{j=0}^{k}(1 - tq^{k+1-2j})} - \frac{q^{-k-1}}{(q^{k+1} - q^{-k-1}) \prod_{j=0}^{k}(1 - tq^{k-1-2j})} \\ &= \frac{1}{(q^{k+1} - q^{-k-1}) \prod_{j=1}^{k}(1 - tq^{k+1-2j})} \left(\frac{q^{k+1}}{1 - tq^{k+1}} - \frac{q^{-k-1}}{1 - tq^{-k-1}} \right) \\ &= \prod_{j=0}^{k+1}(1 - tq^{k+1-2j})^{-1} \end{aligned}$$

$\Diamond$

3.3.4 Exercises

1. Let $\mathcal{A}$ be an associative algebra with 1 and let $L : \mathcal{A} \to \mathrm{End}(\mathcal{A})$ be the *left regular representation* $L(a)x = ax$. Suppose $T \in \mathrm{End}(\mathcal{A})$ commutes with $L(\mathcal{A})$. Prove that there is an element $b \in \mathcal{A}$ so that $T(a) = ab$ for all $a \in \mathcal{A}$. (*Hint:* Consider the action of T on 1.)

2. Let G be a group. Suppose $T \in \mathrm{End}(\mathbb{C}[G])$ commutes with left translations by G. Show that there is a function $\phi \in \mathbb{C}[G]$ so that $Tf = f * \phi$ (convolution product) for all $f \in \mathbb{C}[G]$. (*Hint*: Use the previous exercise.)

3. Prove (3.21).

4. Let $\mathcal{A} = \mathrm{End}(V)$ with V a finite-dimensional complex vector space. Suppose $S \in \mathcal{A}^*$ and $S(xy) = S(yx)$ for all $x, y \in \mathcal{A}$. Prove that $S(x) = c\ \mathrm{tr}(x)$ for some scalar c. (*Hint:* Write $S(x) = \mathrm{tr}(zx)$ where $z \in \mathrm{End}(V)$. Then $\mathrm{tr}(z[x,y]) = 0$ for all $x, y \in \mathrm{End}(V)$. Now write $z = cI_V + w$ with $\mathrm{tr}(w) = 0$ and use Theorem 2.3.1, (5).)

5. (*Notation as in Section 3.3.1*) Let $\mathcal{A}$ be a finite-dimensional semisimple associative algebra over $\mathbb{C}$ with unit.

   (a) Let $S \in \mathcal{A}^*$. Show that there exist operators $z_\lambda \in \mathrm{End}(V^\lambda)$ so that

$$S(x) = \sum_{\lambda \in \widehat{\mathcal{A}}} \mathrm{tr}(z_\lambda \pi^\lambda(x)) \qquad \text{for all } x \in \mathcal{A}.$$

(*Hint:* Use (3.22) and the nondegeneracy of the bilinear form $\mathrm{tr}_{V^\lambda}(uv)$ on $\mathrm{End}(V^\lambda)$.)

(b) Define a linear functional T on $\mathcal{A}$ by

$$T(x) = \sum_{\lambda \in \widehat{\mathcal{A}}} \mathrm{tr}(\pi^\lambda(x)).$$

Show that $T(xy) = T(yx)$ for all $x, y \in \mathcal{A}$.

(c) Suppose $S \in \mathcal{A}^*$ and $S(xy) = S(yx)$ for all $x, y \in \mathcal{A}$. Show that there exists an element z in the center of $\mathcal{A}$ so that $S(x) = T(zx)$ for all $x \in \mathcal{A}$, where T is the linear functional in (b). (*Hint:* Consider first the case that $\mathcal{A}$ is simple, and use the previous exercise. Then apply (a).)

(d) Let S and z be as in (c). Show that the bilinear form $B(x, y) = S(xy)$ on $\mathcal{A}$ is nondegenerate if and only if z is invertible in $\mathcal{A}$, and that this condition is the same as $\pi^\lambda(z) \neq 0$ for all $\lambda \in \widehat{\mathcal{A}}$.

6. Let (ρ, V) and (σ, W) be finite-dimensional representations of a group G, and let $g \in G$.

(a) Show that $\mathrm{ch}(V \otimes W)(g) = \mathrm{ch}(V)(g) \cdot \mathrm{ch}(W)(g)$.

(b) Show that $\mathrm{ch}(\bigwedge^2 V)(g) = \frac{1}{2}\left\{\mathrm{ch}(V)(g)^2 - \mathrm{ch}(V)(g^2)\right\}$.

(c) Show that $\mathrm{ch}(S^2(V))(g) = \frac{1}{2}\left\{\mathrm{ch}(V)(g)^2 + \mathrm{ch}(V)(g^2)\right\}$.

(*Hint:* Let $\{\lambda_i\}$ be the eigenvalues of $\rho(g)$ on V. Then $\{\lambda_i\lambda_j\}_{i<j}$ are the eigenvalues of g on $\bigwedge^2 V$ and $\{\lambda_i\lambda_j\}_{i\leq j}$ are the eigenvalues of g on $S^2(V)$.)

The following exercises use the notation in Section 3.3.3.

7. Let (σ, W) be a regular representation of $\mathrm{SL}(2, \mathbb{C})$. For $q \in \mathbb{C}^\times$ let $f(q) = \mathrm{ch}(W)(d(q))$. Write $f(q) = f_{\mathrm{even}}(q) + f_{\mathrm{odd}}(q)$, where $f_{\mathrm{even}}(-q) = f_{\mathrm{even}}(q)$ and $f_{\mathrm{odd}}(-q) = -f_{\mathrm{odd}}(q)$.

(a) Show that $f_{\mathrm{even}}(q) = f_{\mathrm{even}}(q^{-1})$ and $f_{\mathrm{odd}}(q) = f_{\mathrm{odd}}(q^{-1})$.

(b) Let $f_{\mathrm{even}}(q) = \sum_{k\in\mathbb{Z}} a_k\, q^{2k}$ and $f_{\mathrm{odd}}(q) = \sum_{k\in\mathbb{Z}} b_k\, q^{2k+1}$. Show that the sequences $\{a_k\}$ and $\{b_k\}$ are unimodal.

(*Hint:* See Exercises 2.4.5.)

8. Let (σ, W) be a regular representation of $\mathrm{SL}(2, \mathbb{C})$ and let $W \cong \bigoplus_{k\geq 0} m_k V_k$ be the decomposition of W into isotypic components. Say that W is *even* if $m_k = 0$ for all odd integers k, and say that W is *odd* if $m_k = 0$ for all even integers.

(a) Show W is even (*resp.* odd) if and only if $\mathrm{ch}(W)(d(-q)) = \mathrm{ch}(W)(d(q))$ (*resp.* $\mathrm{ch}(W)(d(-q)) = -\,\mathrm{ch}(W)(d(q))$). (*Hint:* Use Proposition 2.2.3.)

(b) Show that $S^j(V_k)$ is even if jk is even, and is odd if jk is odd. (*Hint:* Use the model for V_k from Section 2.2.2 to show that $-I \in \mathrm{SL}(2, \mathbb{C})$ acts on V_k by $(-1)^k$ and hence acts by $(-1)^{jk}$ on $S^j(V_k)$.)

9. Set $f(q) = \begin{bmatrix} m+n \\ m \end{bmatrix}_q$ for $q \in \mathbb{C}^\times$ and positive integers m, n.

(a) Show that $f(q) = f(q^{-1})$.

(b) Show that $f(q) = \sum_{k\in\mathbb{Z}} a_k\, q^{2k+\epsilon}$, where $\epsilon = 0$ when mn is even and $\epsilon = 1$ when mn is odd.

(c) Show that the sequence $\{a_k\}$ in (b) is unimodal.

(*Hint:* Use the previous exercises and Theorem 3.3.12.)

10. (a) Show (by a computer algebra system or otherwise) that

$$\begin{bmatrix} 4+3 \\ 3 \end{bmatrix}_q = q^{12} + q^{10} + 2q^8 + 3q^6 + 4q^4 + 4q^2 + 5 + \cdots$$

(where $\cdots$ indicates terms in negative powers of q).

(b) Use (a) to prove that

$$S^3(V_4) \cong S^4(V_3) \cong V_{12} \oplus V_8 \oplus V_6 \oplus V_4 \oplus V_0.$$

(*Hint:* Use Proposition 2.2.3 and Theorem 3.3.12.)

11. (a) Show (by a computer algebra system or otherwise) that

$$\begin{bmatrix} 5+3 \\ 3 \end{bmatrix}_q = q^{15} + q^{13} + 2q^{11} + 3q^9 + 4q^7 + 5q^5 + 6q^3 + 6q + \cdots$$

(where $\cdots$ indicates terms in negative powers of q).

(b) Use (a) to prove that

$$S^3(V_5) \cong S^5(V_3) \cong V_{15} \oplus V_{11} \oplus V_9 \oplus V_7 \oplus V_5 \oplus V_3.$$

(*Hint:* Use Proposition 2.2.3 and Theorem 3.3.12.)

12. For $n \in \mathbb{N}$ and $q \in \mathbb{C}$ define

$$\{n\}_q = q^{n-1} + q^{n-2} + \cdots + q + 1 = \frac{q^n - 1}{q - 1}$$

(some authors write $[n]_q$ for $\{n\}_q$).

(a) Show that $\{n\}_{q^2} = q^{n-1}[n]_q$.

(b) Define

$$C_{n+m,m}(q) = \frac{\{m+n\}_q!}{\{m\}_q!\{n\}_q!}$$

(this is an alternate version of the q-binomial coefficient which also gives the ordinary binomial coefficient when $q = 1$). Let F be the field with q elements ($q = p^n$ with p a prime). Prove that $C_{m+n,m}(q)$ is the number of m-dimensional subspaces in the vector space F^{m+n}. (*Hint:* The number of nonzero elements of F^{m+n} is $q^{n+m} - 1$. If $v \in F^{m+n} - \{0\}$ then the number of elements that are not multiples of v is $q^{n+m} - q$. Continuing in this way we find that the cardinality of the set of all linearly independent m-tuples $\{v_1, ..., v_m\}$ is $(q^{n+m} - 1)(q^{n+m-1} - 1) \cdots (q^{n+1} - 1) = a_{n,m}$. The desired cardinality is thus $a_{n,m}/a_{0,m} = C_{n+m,m}(q)$.)