Preface

H. Weyl, in his landmark book *The Classical Groups, Their Invariants and Representations* (Weyl [1946]), coined the phrases "classical groups" and "classical invariant theory". Weyl's book and C. Chevalley's *The Theory of Lie Groups I* (Chevalley [1946]) were major influences in establishing Lie group theory and (later) algebraic group theory as basic fields of research in mathematics, building on the fundamental work of S. Lie, W. Killing, É. Cartan and the invariant theorists of the nineteenth century. In both books the leading players are the classical matrix groups (the group of all invertible linear transformations in n-dimensions and the subgroups that leave invariant a non-degenerate symmetric or skew-symmetric bilinear form). Lie theory has progressed at an astonishing rate since the publication of these remarkable books. In our book we present a more modern introduction to the field that these classics opened to the mathematical community.

We look upon classical groups as both a fundamental class of objects for study and important examples for the general theory. The text can be read at several levels. The basic level gives a modern treatment of the material in Weyl's book and an introductory course in the representation theory of semi-simple groups and Lie algebras. The next level is an introduction to algebraic group theory with emphasis on characteristic zero and the classical groups. In many cases general theorems are proved in full detail only for the classical groups. However, the notation is such that the more sophisticated reader can read the book as a study of reductive algebraic groups over algebraically closed fields. Even experts in the field should find new perspectives and some unfamiliar results. Finally, the book can be used as a reference for the main results in classical invariant theory (vector, tensor and polynomial invariants) and the finite-dimensional representation theory of the classical groups.

In the theory, there are two basic threads. The first is to study the classical groups (and more generally reductive groups) over $\mathbb{C}$ as complexifications of compact Lie groups. One can thus apply methods of analysis (e.g. invariant measures) and special properties of real and complex vector spaces (e.g. positive-definite bilinear or Hermitian forms) to derive the main theorems. The second is to study the classical groups as special cases of reductive algebraic groups over $\mathbb{C}$ using techniques of algebraic geometry. In this book most of the fundamental results are studied from both perspectives. Whenever possible we state the main results early in each chapter in order to allow the reader to choose the style of proof (if any) desired. We feel that the care that we show in maintaining the algebraic purity is rewarded by the power of the results in the later chapters.

The fundamental results of classical invariant theory involve the duality of the general linear group and its subgroups with the group algebra of the symmetric group and its extensions (e.g. the Brauer algebras). Most expositions in other texts follow the lead of Weyl deriving the basic results from combinatorial theorems for the symmetric group. Our perspective reverses the roles of the finite groups and the continuous groups. Thus the Frobenius character formula for the irreducible representations of the symmetric group becomes a special case of a general theorem (first suggested by Verma in a lecture at Rutgers in 1987) about reductive algebraic groups which is derived from the Weyl character formula. The delicate combinatorial results then become corollaries of the representation theory of reductive algebraic groups (or compact Lie groups). The duality between an algebra of linear transformations and its commuting algebra is one of the main unifying themes of the book.

In Lie theory the examples are, in many cases, more difficult than the general theorems. In this book every new concept is detailed with its meaning for each of the classical groups. For example, in Chapter 11 every classical symmetric pair is described and a model is given for the corresponding affine variety, and in Chapter 12 the (complexified) Iwasawa decomposition is worked out explicitly. Also in Chapter 12 a new proof of the Kostant-Rallis theorem is given and every implication for the invariant theory of classical groups is explained.

This book developed out of graduate courses taught by the authors over a period of many years at Rutgers University (New Brunswick), University of California (San Diego), Université de Metz (France), and European summer schools in Poland and Italy. It has been extensively revised after this classroom exposure, and every effort has been made to make the exposition complete and accessible to students. Each chapter has an introduction that describes the principal results and techniques used. The "Dependency Chart among Chapters" indicates the internal structure of the book. A chapter (or section) in the chart depends on the chapters to which it is connected by a rising line. This chart has a central spine; to the right are the more geometric aspects of the subject and on the left the more algebraic aspects. Notice that there are several terminal nodes in this chart (such as §5.2, Ch. 6 or Ch. 8) which can serve as goals for courses or self study.

This book can serve for several different courses. An introductory one-term course in Lie algebras and algebraic groups with emphasis on the classical groups and their representations can be based on Chapter 1 (through §1.3), Chapter 2 (omitting §2.4.4), Chapter 3 (through §3.3.2), Chapter 4 (emphasizing §4.2.1 and §4.5.1 and omitting the proof of the FFT), Chapter 5 (through §5.2.2) and Chapter 6 (through §6.3), together with the necessary material from Appendices A, B, and C. An alternate path through this material emphasizing Lie groups and the compact forms of the complex classical groups (instead of algebraic groups) can also be followed, using Appendix D. Chapters 1, 2, and 11 (with Appendix A covered thoroughly) can be the core of a one-term introductory course on algebraic groups in characteristic zero. For students who have already had a one-term course in Lie algebras and Lie groups, chapters 7-10 would make a sequel emphasizing representations, character formulas, and their applications. For the main result in chapter 7 (the Weyl character formula) we give an algebraic proof (using Lie algebra cohomology) and an analytic proof (using compact forms and Weyl integral formula). An alternate (more advanced) second-term course emphasizing the geometric side of the subject would include Chapters 11 and 12. A year-long course on representations and invariant theory along the lines of Weyl's book would follow Chapters 1, 2, 3, 4, 5, Sections 7.1 and 7.5, Chapter 9 and Chapter 10.

In the end-of-chapter notes we have attempted to give credits for the results in the book and some idea of the historical development of the subject. We apologize to those whose works we have neglected to cite and for incorrect attributions. We are indebted to many people for finding errors and misprints in the preliminary versions of this book and for suggesting ways to improve the exposition. In particular we would like to thank Laura Barberis, Bachir Bekka, Friedrich Knop, Hanspeter Kraft, Tomasz Przebinda and Enriqueta

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