New Exercises for

Representations and Invariants of the Classical Groups
by Roe Goodman and Nolan R. Wallach
(1998 edition)

Revised July 15, 1999

p.17, after exercise 10. INSERT THE FOLLOWING EXERCISES:

11. Assume that (ρ, V) is an irreducible regular representation of the linear algebraic group G. Fix $v^* \in V^*$ with $v^* \neq 0$. For $v \in V$ let $\varphi_v \in \mathrm{Aff}(G)$ be the representative function $\varphi_v(g) = \langle v^*, \rho(g)v \rangle$. Let $E = \{\varphi_v : v \in V\}$ and let $T : V \to E$ be the map $Tv = \varphi_v$. Prove that T is a bijective linear map and that $T\rho(g) = R(g)T$ for all $g \in G$, where $R(g)f(x) = f(xg)$ for $f \in \mathrm{Aff}(G)$. Thus every irreducible regular representation of G is equivalent to a subrepresentation of $(R, \mathrm{Aff}(G))$.

12. Let N be the group of matrices

$$\begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix}, \quad z \in \mathbb{C}$$

and let Γ be the subgroup of N consisting of the matrices with $z \in \mathbb{Z}$ an integer. Prove that Γ is Zariski-dense in N.

13. Define a multiplication μ on $\mathbb{C}^\times \times \mathbb{C}$ by $\mu([x_1, x_2], [y_1, y_2]) = [x_1 y_1,\, x_2 + x_1 y_2]$.

(a) Prove that μ satisfies the group axioms and that the inversion map is regular.

(b) Let $S = (\mathbb{C}^\times \times \mathbb{C}, \mu)$ be the linear algebraic group with $\mathrm{Aff}(S) = \mathbb{C}[x_1, x_1^{-1}, x_2]$ and multiplication μ. Let $R(y)f(x) = f(\mu(x, y))$ be the right translation representation of S on $\mathrm{Aff}(S)$. Let $V \subset \mathrm{Aff}(S)$ be the space spanned by the functions x_1 and x_2. Show that V is invariant under $R(y)$, for $y \in S$.

(c) Let $\rho(y) = R(y)|_V$ for $y \in S$. Calculate the matrix of $\rho(y)$ relative to the basis $\{x_1, x_2\}$ of V. Prove that $\rho : S \to \mathrm{GL}(2, \mathbb{C})$ is injective, and that $S \cong \rho(S)$ as an algebraic group.

p.34, after exercise 8. INSERT THE FOLLOWING EXERCISES:

9. Let $\mathcal{A} = \{a \in M_n(\mathbb{C}) : x_{ij}(a) = 0 \text{ for all } i > j\}$.

(a) Show that $\mathcal{A}$ is a subalgebra of $M_n(\mathbb{C})$ (relative to the usual matrix product).

(b) Let G be the group of invertible elements in $\mathcal{A}$. Use exercise 6. to find $\mathrm{Lie}(G)$.

10. Let G and H be connected linear algebraic groups. Suppose $\phi : G \to H$ is a surjective regular homomorphism such that $\mathrm{Ker}(\phi)$ is finite. Prove that $d\phi : \mathrm{Lie}(G) \to \mathrm{Lie}(H)$ is an isomorphism. (*Hint:* Prove that $\dim G = \dim H$.)

11. Let G be a linear algebraic group. Let Int be the representation of G on $\mathrm{Aff}(G)$ given by $\mathrm{Int}(g)f(x) = f(g^{-1}xg)$ for $f \in \mathrm{Aff}(G)$ (thus $\mathrm{Int}(g) = L(g)R(g)$). Assume that H is a Zariski closed normal subgroup of G.

(a) Let $f \in \mathcal{I}_H$. Prove that there is a finite-dimensional subspace $V \subset \mathcal{I}_H$ so that $f \in V$ and $\mathrm{Int}(g)V \subset V$.

(b) Set $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathfrak{h} = \mathrm{Lie}(H)$. Prove that $\mathrm{Ad}(G)\mathfrak{h} \subset \mathfrak{h}$. (*Hint:* Use (a) to show that $R(g)X_A R(g)^{-1}\mathcal{I}_H \subset \mathcal{I}_H$ for all $A \in \mathfrak{h}$ and all $g \in G$.)

(c) Prove that $[\mathfrak{g}, \mathfrak{h}] \subset \mathfrak{h}$, and hence $\mathfrak{h}$ is an ideal in $\mathfrak{g}$. (*Hint:* By (b), $\mathfrak{h}$ is an $\mathrm{Ad}(G)$-invariant subspace of $\mathfrak{g}$.)

p.49, l.4 (Exercise #1) REPLACE:

1. Check the assertion in (1.4.2) above.

BY:

1. Define a real form $\mathrm{Sp}(p, q)$ of $\mathrm{Sp}(p + q, \mathbb{C})$ analogous to the real form $\mathrm{U}(p, q)$ of $\mathrm{GL}(p + q, \mathbb{C})$.

p.84, after exercise 6. INSERT THE FOLLOWING EXERCISE:

7. Let G be a connected linear algebraic group and let $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ be the adjoint representation of G. Let $N = \mathrm{Ker}(\mathrm{Ad})$. The group G/N is called the *adjoint group* of G.

(a) Suppose $\mathfrak{g}$ is a simple Lie algebra. Prove that N is finite.

(b) Suppose $G = \mathrm{SL}(n, \mathbb{C})$, so that $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$. Find N in this case. The group G/N is denoted by $\mathrm{PSL}(n, \mathbb{C})$ (the *projective linear group*).

p.109, after exercise 5. INSERT THE FOLLOWING EXERCISES:

6. Let $G = \mathrm{SL}(3, \mathbb{C})$, H the diagonal matrices in G, and let $V = \mathbb{C}^3 \otimes \mathbb{C}^3$.

(a) Find the weights of H on V. Express the weights in terms of $\varepsilon_1, \varepsilon_2, \varepsilon_3$ and for each weight determine its multiplicity. Verify that the weight multiplicities are invariant under the Weyl group W of G.

(b) Verify that each Weyl group orbit in the set of weights of V contains exactly one dominant weight. Find the *extreme* dominant weights β (those such that $\beta + \alpha$ is not a weight, for any positive root α).

(c) Write the weights of V in terms of the fundamental weights $\{\varpi_1, \varpi_2\}$ and plot the set of weights in the $\mathfrak{h}^*$ plane, as in Figure 2.5. Indicate multiplicities and W-orbits in the plot.

(d) V decomposes into G-invariant subspaces $V = V_+ \oplus V_-$, where V_+ consists of the symmetric 2-tensors, and V_- is the skew-symmetric 2-tensors. Determine the weights and multiplicities of $V_\pm$ and verify that the weight multiplicities are invariant under W.

7. Let $G = \mathrm{Sp}(\mathbb{C}^4, \Omega)$, where $\Omega = \begin{bmatrix} 0 & s_0 \\ -s_0 & 0 \end{bmatrix}$ and s_0 has antidiagonal 1 as usual. Let H be the diagonal matrices in G, and let $V = \bigwedge^2 \mathbb{C}^4$.

(a) Find all the weights of H on V. Express the weights in terms of $\varepsilon_1, \varepsilon_2$ and for each weight determine its multiplicity (note that $\varepsilon_3 = -\varepsilon_2$ and $\varepsilon_4 = -\varepsilon_1$ as elements of $\mathfrak{h}^*$). Verify that the weight multiplicities are invariant under the Weyl group W of G.

(b) Verify that each Weyl group orbit in the set of weights of V contains exactly one dominant weight. Find the *extreme* dominant weights β (those such that $\beta + \alpha$ is not a weight, for any positive root α).

(c) Write the weights of V in terms of the fundamental weights $\{\varpi_1, \varpi_2\}$ and plot the set of weights in the $\mathfrak{h}^*$ plane, as in Figure 2.6. Indicate multiplicities and W orbits in the plot.

p.198, Exercises 4.3.3 REPLACE EXERCISE 1 BY

1. Let $V = \mathbb{C}^n$ and $G = \mathrm{GL}(n, \mathbb{C})$. For $v \in V$ and $v^* \in V^*$, let $T(v \otimes v^*) = vv^* \in M_n$. This defines the canonical isomorphism $u \mapsto T(u)$ between $V \otimes V^*$ and M_n. Let $T_k = T^{\otimes k}$ be the canonical isomorphism $(V \otimes V^*)^{\otimes k} \to (M_n)^{\otimes k}$. Let $g \in G$ act on $x \in M_n$ by $g \cdot x = gxg^{-1}$.

(a) Show that T_k intertwines the action of G on $(V \otimes V^*)^{\otimes k}$ and $(M_n)^{\otimes k}$.

(b) Let $\sigma \in \mathfrak{S}_k$ be a cyclic permutation $m_1 \to m_2 \to \cdots \to m_k \to m_{k+1} = m_1$. Let $C_\sigma : (V \otimes V^*)^{\otimes k} \to \mathbb{C}$ be the G-invariant contraction

$$C_\sigma(v_1 \otimes v_1^* \otimes \cdots \otimes v_k \otimes v_k^*) = \prod_{j=1}^{k} \langle v_{m_j}^*, v_{m_{j+1}} \rangle$$

Set $X_j = T(v_j \otimes v_j^*)$. Prove that

$$C_\sigma(v_1 \otimes v_1^* \otimes \cdots \otimes v_k \otimes v_k^*) = \mathrm{tr}(X_{m_1} X_{m_2} \cdots X_{m_k}).$$

(*Hint:* Note that for $X \in M_n$, one has $T(v^* \otimes Xv) = XT(v^* \otimes v)$ and $\mathrm{tr}(T(v^* \otimes v)) = v^*v$.)

(c) Let $\sigma \in \mathfrak{S}_k$ be a product of disjoint cyclic permutations $c_1, \ldots, c_r$, where c_i is the cycle $m_{1,i} \to m_{2,i} \to \cdots \to m_{p_i,i} \to m_{1,i}$. Let $C_\sigma : (V \otimes V^*)^{\otimes k} \to \mathbb{C}$ be the G-invariant contraction

$$C_\sigma(v_1 \otimes v_1^* \otimes \cdots \otimes v_k \otimes v_k^*) = \prod_{i=1}^{r} \prod_{j=1}^{p_i} \langle v_{m_{j,i}}^*, v_{m_{j+1,i}} \rangle$$

Set $X_j = T(v_j \otimes v_j^*)$. Prove that

$$C_\sigma(v_1 \otimes v_1^* \otimes \cdots \otimes v_k \otimes v_k^*) = \prod_{i=1}^{r} \mathrm{tr}(X_{m_{1,i}} X_{m_{2,i}} \cdots X_{m_{p_i,i}}).$$

p.198, Exercises 4.3.3 INSERT AFTER EXERCISE #3:

4. Let $G = \mathrm{GL}(n\mathbb{C})$

(a) Use Exercise #1 to find a basis for the G-invariant linear functionals on $M_n^{\otimes 2}$ (assume $n \geq 2$).

(b) Prove that there are no nonzero skew-symmetric G invariant bilinear forms on M_n. (*Hint:* Use the result in (a) and the projection from $(M_n)^{\otimes 2}$ onto $(M_n)^{\wedge 2}$.)

5. Let $G = \mathrm{GL}(n\mathbb{C})$

(a) Find a spanning set for the G-invariant linear functionals on $M_n^{\otimes 3}$.

(b) Define $\omega(X_1, X_2, X_3) = \mathrm{tr}([X_1, X_2]X_3)$ for $X_i \in M_n$. Prove that ω is skew-symmetric and G invariant.

(c) Prove that ω is the unique G invariant skew-symmetric linear functional on $M_n^{\otimes 3}$, up to a scalar multiple. (*Hint:* Use the result in (a) and the projection from $(M_n)^{\otimes 3}$ onto $(M_n)^{\wedge 3}$.)

6. Let $G = \mathrm{O}(V, B)$, where B is a symmetric bilinear form on V (assume $\dim V \geq 3$). Let $\{e_i\}$ be a basis for V such that $B(e_i, e_j) = \delta_{ij}$.

(a) Let $R \in \left(V^{\otimes 4}\right)^G$. Show that there are constants $a, b, c \in \mathbb{C}$ so that

$$R = \sum_{i,j,k,l} \{a\delta_{ij}\delta_{kl} + b\delta_{ik}\delta_{jl} + c\delta_{il}\delta_{jk}\}\, e_i \otimes e_j \otimes e_k \otimes e_l$$

(*Hint:* Determine all the two-partitions of $\{1, 2, 3, 4\}$).

(b) Use (a) to find a basis for the space $\left[S^2(V) \otimes S^2(V)\right]^G$. (*Hint:* Symmetrize relative to tensor positions 1, 2 and positions 3, 4.)

(c) Use (b) to show that $\dim \mathrm{End}_G(S^2(V)) = 2$ and that $S^2(V)$ decomposes into the sum of two inequivalent irreducible G modules. (*Hint:* $S^2(V) \cong S^2(V)^*$ as G modules.)

(d) Find the dimensions of the irreducible modules in (c). (*Hint*: There is an obvious irreducible submodule in $S^2(V)$.)

7. Let $G = \mathrm{O}(V, B)$ as in the previous exercise.

(a) Use part (a) of the previous exercise to find a basis for the space $\left[\bigwedge^2 V \otimes \bigwedge^2 V\right]^G$. (*Hint:* Skew-symmetrize relative to tensor positions 1, 2 and positions 3, 4.)

(b) Use (a) to show that $\dim \mathrm{End}_G(\bigwedge^2 V) = 1$ and hence $\bigwedge^2 V$ is irreducible under G. (*Hint:* $\bigwedge^2 V \cong \bigwedge^2 V^*$ as G modules.)

8. Let $G = \mathrm{Sp}(V, \Omega)$, where Ω is a nonsingular skew form on V (assume $\dim V \geq 4$ is even). Let $\{f_i\}$ and $\{f^j\}$ be bases for V such that $\Omega(f_i, f^j) = \delta_{ij}$.

(a) Show that $\left(V^{\otimes 4}\right)^G$ is spanned by the tensors

$$\sum_{i,j} f_i \otimes f^i \otimes f_j \otimes f^j, \quad \sum_{i,j} f_i \otimes f_j \otimes f^i \otimes f^j, \quad \sum_{i,j} f_i \otimes f_j \otimes f^j \otimes f^i.$$

(b) Use (a) to find a basis for the space $\left[\bigwedge^2 V \otimes \bigwedge^2 V\right]^G$. (*Hint:* Skew-symmetrize relative to tensor positions 1, 2 and positions 3, 4.)

(c) Use (b) to show that $\dim \operatorname{End}_G(\bigwedge^2 V) = 2$ and that $\bigwedge^2 V$ decomposes into the sum of two inequivalent irreducible G modules. (*Hint:* $\bigwedge^2 V \cong \bigwedge^2 V^*$ as a G-module.)

(d) Find the dimensions of the irreducible modules in (c). (*Hint*: There is an obvious irreducible submodule in $\bigwedge^2 V$.)

9. Let $G = \mathrm{Sp}(V, \Omega)$ as in the previous exercise.

(a) Use part (a) of the previous exercise to find a basis for the space $\left[S^2(V) \otimes S^2(V)\right]^G$. *(Hint:* Symmetrize relative to tensor positions 1, 2 and positions 3, 4.)

(b) Use (a) to show that $\dim \operatorname{End}_G(S^2(V)) = 1$ and hence $S^2(V)$ is irreducible under G. (*Hint:* $S^2(V) \cong S^2(V)^*$ as a G-module.)

Insert after line 5 on p. 226 (before 4.6 Notes)

4.5.8 Exercises

1. Let $G = \mathrm{GL}(k, \mathbb{C})$ and $V = M_{k,p}(\mathbb{C}) \oplus M_{k,q}(\mathbb{C})$. Let $g \in G$ act on V by $g \cdot [x \quad y] = [gx \quad (g^t)^{-1}y]$ for $x \in M_{k,p}(\mathbb{C})$ and $y \in M_{k,q}(\mathbb{C})$. Note that the columns x_i of x transform as vectors in $\mathbb{C}^n$ and the columns y_j of y transform as covectors in $(\mathbb{C}^n)^*$

(a) Let $\mathfrak{p}_-$ be the subspace of $\mathbb{D}(V)$ spanned by the operators of multiplication by $(x_i)^t \cdot y_j$ for $1 \le i \le p$, $1 \le j \le q$. Let $\mathfrak{p}_+$ be the subspace of $\mathbb{D}(V)$ spanned by the operators $\Delta_{ij} = \sum_{r=1}^k \frac{\partial}{\partial x_{ri}} \frac{\partial}{\partial y_{rj}}$ for $1 \le i \le p$, $1 \le j \le q$. Prove that $\mathfrak{p}_\pm \subset \mathbb{D}(V)^G$.

(b) Let $\mathfrak{k}$ be the subspace of $\mathbb{D}(V)$ spanned by the operators $E_{ij}^{(x)} + \frac{k}{2}\delta_{ij}$ (with $1 \le i, j \le p$) and $E_{ij}^{(y)} + \frac{k}{2}\delta_{ij}$ (with $1 \le i, j \le q$), where $E_{ij}^{(x)}$ is defined by equation (4.5.27) and $E_{ij}^{(y)}$ is similarly defined with x_{ij} replaced by y_{ij}. Prove that $\mathfrak{k} \subset \mathbb{D}(V)^G$.

(c) Prove the commutation relations $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}_\pm] = \mathfrak{p}_\pm$, $[\mathfrak{p}_-, \mathfrak{p}_+] \subset \mathfrak{k}$.

(d) Set $\mathfrak{g}' = \mathfrak{p}_- + \mathfrak{k} + \mathfrak{p}_+$. Prove that $\mathfrak{g}'$ is isomorphic to $\mathfrak{gl}(p + q, \mathbb{C})$, and that $\mathfrak{k} \cong \mathfrak{gl}(p, \mathbb{C}) \oplus \mathfrak{gl}(q, \mathbb{C})$.

(e) Prove that $\mathbb{D}(V)^G$ is generated by $\mathfrak{g}'$. (Hint: Use Theorem 4.2.1 and note that there are four possibilities for contractions to obtain G-invariant polynomials on $V \oplus V^*$: (1) vector and covector in V; (2) vector and covector in V^*; (3) vector from V and covector from V^*; (4) covector from V and vector from V^*. Show that the contractions of types (1) and (2) furnish symbols for $\mathfrak{p}_\pm$, and that contractions of type (3) and (4) furnish symbols for $\mathfrak{k}$. Now apply Theorem 4.5.16.)

p.248, after exercise 11. Insert the following exercises:

12. Let $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{C})$. Fix the positive roots $\Phi^+ = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \varepsilon_1 - \varepsilon_3\}$ as usual. Let $\pi = \mathrm{ad}$ be the adjoint representation on $\mathfrak{g}$.

(a) Express the highest weight λ of π in terms of the fundamental weights ϖ_1 and ϖ_2. What is the highest weight vector?

(b) Find all $\beta \in P_{++}(\mathfrak{g})$ of the form $\beta = \lambda - \gamma$, where $\gamma \in Q_+(\mathfrak{g})$. (Here $P_{++}(\mathfrak{g})$ are the dominant weights, and $Q_+(\mathfrak{g})$ are the sums of positive roots.) Verify that for every such β, the corresponding weight space $\mathfrak{g}_\beta \neq 0$.

(c) Find the orbit $W \cdot \beta$ of each weight β in (b), where W is the Weyl group of $\mathfrak{g}$. Verify that the union of these orbits is the set of weights of π.

(d) Plot the set of weights of π as points in the $\mathfrak{h}^*$ plane. Observe that this set is in the convex hull of the orbit $W \cdot \lambda$ of the highest weight.

13. Let $\mathfrak{g} = \mathfrak{sp}(2, \mathbb{C})$. Fix the positive roots $\Phi^+ = \{\varepsilon_1 - \varepsilon_2, \varepsilon_1 + \varepsilon_2, 2\varepsilon_1, 2\varepsilon_2\}$ as usual. Let $\pi = \mathrm{ad}$ be the adjoint representation on $\mathfrak{g}$. Carry out parts (a), (b), (c), (d) of the previous exercise in this case.

14. Let $\mathfrak{g} = \mathfrak{sp}(2, \mathbb{C})$. Suppose (π, V) is the irreducible representation of $\mathfrak{g}$ with highest weight $\rho = \varpi_1 + \varpi_2$ (the smallest regular dominant weight).

(a) Show that there is exactly one $\beta \in P_{++}(\mathfrak{g})$ of the form $\beta = \rho - \gamma$, where $0 \neq \gamma \in Q_+(\mathfrak{g})$. Show that $V_\beta \neq 0$ and find a *spanning set* for it. (*Hint*: Use the representation theory of $\mathfrak{sl}(2, \mathbb{C})$ and the action of $U(\mathfrak{g})$ on the highest weight vector.)

(b) Find the orbits $W \cdot \rho$ and $W \cdot \beta$, where W is the Weyl group of $\mathfrak{g}$.

(c) Plot the weights of π in the $\mathfrak{h}^*$ plane. Observe that all the weights are contained in the convex hull of the orbit $W \cdot \rho$ of the highest weight.

(d) The Weyl dimension formula implies that $\dim V = 2^{|\Phi^+|} = 16$. Use this result to determine the dimension of the weight space V_β in (a).

15. Let $G = \mathrm{Sp}(\mathbb{C}^4, \Omega)$. Let $\{e_{\pm 1}, e_{\pm 2}\}$ be a basis for $\mathbb{C}^4$ so that $\Omega(e_1, e_{-1}) = \Omega(e_2, e_{-2}) = 1$ and $\Omega(e_i, e_j) = 0$ otherwise. Here $e_{\pm i}$ has weight $\varepsilon_{\pm i}$. Consider the representation ρ of G on $\bigwedge^2 \mathbb{C}^4$.

(a) Find the weights and a basis of weight vectors for ρ. Express the weights in terms of the basis ε_1, ε_2 and verify that the set of weights is invariant under the Weyl group of G.

(b) Set $X = \iota(e_{-1})\iota(e_1) + \iota(e_{-2})\iota(e_2)$, where $\iota(x)$ is the graded derivation of $\bigwedge \mathbb{C}^4$ such that $\iota(x)y = \Omega(x, y)$ for $x, y \in \mathbb{C}^4$. Show that

$$X(u \wedge v) = \sum_{p=1}^{2} \begin{vmatrix} \Omega(e_p, u) & \Omega(e_{-p}, u) \\ \Omega(e_p, v) & \Omega(e_{-p}, v) \end{vmatrix}$$

for $u, v \in \mathbb{C}^4$.

(c) Let $\mathcal{H}^2 = \mathrm{Ker}(X) \subset \bigwedge^2 \mathbb{C}^4$ (this is an irreducible G module with highest weight ϖ_2). Use the formula in (b) to find a basis for $\mathcal{H}^2$. (*Hint:* $\mathcal{H}^2$ is the sum of weight spaces.)

p.279, Exercises 6.1.4 REPLACE EXERCISE #4 BY:

4. Let V be a complex vector space with a symmetric bilinear form β. Let $\{e_1, \ldots, e_n\}$ be a basis for V such that $\beta(e_i, e_j) = \delta_{ij}$.

(a) Show that if i, j, k are distinct, then

$$e_i e_j e_k = e_j e_k e_i = e_k e_i e_j,$$

where the product is in the Clifford algebra for (V, β).

(b) Show that if $A = [a_{ij}]$ is a symmetric $n \times n$ matrix, then

$$\sum_{i,j=1}^{n} a_{ij}\, e_i\, e_j = \frac{1}{2}\mathrm{tr}(A)$$

(product in the Clifford algebra for (V, β)).

(c) Show that if $A = [a_{ij}]$ is a skew-symmetric $n \times n$ matrix, then

$$\sum_{i,j=1}^{n} a_{ij}\, e_i\, e_j = 2 \sum_{1 \le i < j \le n} a_{ij}\, e_i\, e_j$$

(product in the Clifford algebra for (V, β)).

(d) Let $R_{ijkl} \in \mathbb{C}$ for $1 \le i, j, k, l \le n$ be such that

(i) $R_{ijkl} = R_{klij}$,

(ii) $R_{jikl} = -R_{ijkl}$,

(iii) $R_{ijkl} + R_{kijl} + R_{jkil} = 0$.

Show that $\sum R_{ijkl} e_i e_j e_k e_l = (1/2) \sum R_{ijji}$, where the multiplication of the e_i is in the Clifford algebra for (V, β). (*Hint*: Use part (a) to show that for each l, the sum over distinct triples i, j, k is zero. Then use the anticommutation relations to show that the sum with $i = j$ is also zero. Finally, use part (b) to simplify the remaining sum.)

(e) Let $\mathfrak{g}$ be a Lie algebra and B a symmetric non-degenerate bilinear form on $\mathfrak{g}$ such that $B([x, y], z) = -B(y, [x, z])$. Let $e_1, ..., e_n$ be an orthonormal basis of $\mathfrak{g}$ relative to B. Show that $R_{ijkl} = B([e_i, e_j], [e_k, e_l])$ satsifies (i), (ii), and (iii) of part (d).

p.290, after exercise 4. INSERT THE FOLLOWING EXERCISE:

5. Let $V = \mathbb{C}^n$ with nondegenerate bilinear form β. Let $\mathcal{C} = \mathrm{Cliff}(V, \beta)$ and identify V with $\gamma(V) \subset \mathcal{C}$ by the canonical map γ. Let α be the automorphism of $\mathcal{C}$ such that $\alpha(v) = -v$ for $v \in V$, let τ be the antiautomorphism of $\mathcal{C}$ such that $\tau(v) = v$ for $v \in V$, and let $x \mapsto x^*$ be the antiautomorphism $\alpha \circ \tau$ of $\mathcal{C}$. Define the *norm function* $\Delta : \mathcal{C} \to \mathcal{C}$ by $\Delta(x) = x^* x$. Let $\mathcal{L} = \{x \in \mathcal{C} \,:\, \Delta(x) \in \mathbb{C}\}$.

(a) Show that $\lambda + v \in \mathcal{L}$ for all $\lambda \in \mathbb{C}$ and $v \in V$.

(b) Show that if $x, y \in \mathcal{L}$ and $\lambda \in \mathbb{C}$ then $\lambda x \in \mathcal{L}$ and

$$\Delta(xy) = \Delta(x)\Delta(y), \quad \Delta(\tau(x)) = \Delta(\alpha(x)) = \Delta(x^*) = \Delta(x).$$

Hence $xy \in \mathcal{L}$ and $\mathcal{L}$ is invariant under τ and α. Prove that $x \in \mathcal{L}$ is invertible if and only if $\Delta(x) \neq 0$. In this case $x^{-1} = \Delta(x)^{-1} x^*$ and $\Delta(x^{-1}) = 1/\Delta(x)$.

(c) Let $\Gamma(V, \beta) \subset \mathcal{L}$ be the set of all products $w_1 \cdots w_k$, where $w_j \in \mathbb{C} + V$ and $\Delta(w_j) \neq 0$ for all $1 \le j \le k$ (k arbitrary). Prove that $\Gamma(V, \beta)$ is a group (under multiplication) that is stable under α and τ.

(d) Prove that if $g \in \Gamma(V, \beta)$ then $\alpha(g)(\mathbb{C} + V)g^* = \mathbb{C} + V$. ($\Gamma(V, \beta)$ is called the *Clifford group*; note that it contains $\mathrm{Pin}(V, \beta)$.)

p.499, after exercise 4. INSERT THE FOLLOWING EXERCISES:

5. Let $G = \mathrm{SL}(2, \mathbb{C})$ act on $\mathbb{C}^2$ by left multiplication as usual. This gives an action on $\mathbb{P}^1(\mathbb{C})$. Let $H = \{\mathrm{diag}[z, z^{-1}] \, : \, z \in \mathbb{C}^\times\}$ be the diagonal subgroup, let N be the subgroup of upper-triangular unipotent matrices $\begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix}$, $z \in \mathbb{C}$, and let $B = HN$ be the upper triangular subgroup.

(a) Show that G acts transitively on $\mathbb{P}(\mathbb{C})$. Find a point whose stabilizer is B.

(b) Show that H has one open dense orbit and two closed orbits on $\mathbb{P}(\mathbb{C})$. Show that N has one open dense orbit and one closed orbit on $\mathbb{P}(\mathbb{C})$.

(c) Identify $\mathbb{P}(\mathbb{C})$ with the two-sphere $\mathbf{S}^2$ by stereographic projection and give geometric descriptions of the orbits in (b).

6. (*Same notation as previous exercise)* Let G act on $\mathfrak{g} = \{x \in M_2(\mathbb{C}) \, : \, \mathrm{tr}(x) = 0\}$ by the adjoint representation $\mathrm{Ad}(g)x = gxg^{-1}$. For $\mu \in \mathbb{C}$ define $X_\mu = \{x \in \mathfrak{g} \, : \, \mathrm{tr}(x^2) = 2\mu\}$. Use the Jordan canonical form to prove the following.

(a) If $\mu \neq 0$ then X_μ is a G orbit and $X_\mu \cong G/H$ as a G-space.

(b) If $\mu = 0$ then $X_0 = \{0\} \cup Y$ is the union of two G orbits, where Y is the orbit of $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Show that $Y \cong G/\{\pm 1\}N$ and that Y is not closed in $\mathfrak{g}$.

7. *(Same notation as previous exercise)* Let $Z = \mathbb{P}(\mathfrak{g}) \cong \mathbb{P}^2(\mathbb{C})$ be the projective space of $\mathfrak{g}$, and let $\pi : \mathfrak{g} \to Z$ be the canonical mapping.

(a) Show that G has two orbits on Z, namely $Z_1 = \pi(X_1)$ and $Z_0 = \pi(Y)$.

(b) Find subgroups L_1 and L_0 of G so that $Z_i \cong G/L_i$ as a G space. (*Hint:* Be careful; from the previous problem you know that $H \subset L_1$ and $N \subset L_0$, but these inclusions are proper.)

(c) Prove (without calculation) that one orbit must be closed in Z and one orbit must be dense in Z. Then calculate $\dim Z_i$ and identify the closed orbit. Find equations defining the closed orbit.

8. Let $X = \mathbb{C}^2 \setminus \{0\}$ with its structure as a quasiprojective algebraic set. Then $X = X_1 \cup X_2$, where $X_1 = \mathbb{C}^\times \times \mathbb{C}$ and $X_2 = \mathbb{C} \times \mathbb{C}^\times$ are affine open subsets. Also $f \in \mathcal{O}(X)$ (the ring of regular functions on X) if and only if $f|_{X_i} \in \mathrm{Aff}(X_i)$ for $i = 1, 2$.

(a) Prove that $\mathcal{O}(X) = \mathbb{C}[x_1, x_2]$, where x_i are the coordinate functions on $\mathbb{C}^2$. (*Hint:* Let $f \in \mathcal{O}(X)$. Write $f|_{X_1}$ as a polynomial in x_1, x_1^{-1}, x_2 and write $f|_{X_2}$ as a polynomial in x_1, x_2, x_2^{-1}. Then compare these expressions on $X_1 \cap X_2$.)

(b) Prove that X is not a projective algebraic set. (*Hint:* Consider $\mathcal{O}(X)$.)

(c) Prove that X is not an affine algebraic set. (*Hint:* By (a) there is a homomorphism $f \mapsto f(0)$ of $\mathcal{O}(X)$.)

(d) Let $G = \mathrm{SL}(2, \mathbb{C})$ and N the upper-triangular unipotent matrices in G. Prove that $G/N \cong \mathbb{C}^2 \setminus \{0\}$, with G acting as usual on $\mathbb{C}^2$. (*Hint:* Find a vector in $\mathbb{C}^2$ whose stabilizer is N.)

9. Let $G = \mathrm{SL}(2, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C})$. Let ρ be the representation of G on $\mathbb{M}_2 = M_2(\mathbb{C})$ given by $\rho(g, h)z = gzh^t$. Let $\pi : \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{M}_2$ by $\pi(x, y) = xy^t$. Identify $\mathbb{P}^3$ with $\mathbb{P}(\mathbb{M}_2)$ and let $\tilde{\pi} : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ be the map induced by π (the standard imbedding of $\mathbb{P}^m \times \mathbb{P}^n$ in $\mathbb{P}^{mn+m+n}$).

(a) Show that the image of $\tilde{\pi}$ is $\{[z] : z \in \mathbb{M}_2 \setminus \{0\} \text{ and } \det(z) = 0\}$.

(b) Let G act on $\mathbb{P}^1 \times \mathbb{P}^1$ by the natural action on $\mathbb{C}^2 \times \mathbb{C}^2$ and let G act on $\mathbb{P}^3$ by the representation ρ on $\mathbb{M}_2$. Show that $\tilde{\pi}$ intertwines the G actions.

(c) Show that G has two orbits on $\mathbb{P}^3$ and describe the closed orbit.

10. (*Notation as in previous exercise*) Consider the subspaces $V_1 = \mathbb{C}E_{11} + \mathbb{C}E_{12}$ and $V_2 = \mathbb{C}E_{11} + \mathbb{C}E_{21}$ of $\mathbb{M}_2$, where E_{ij} are the usual elementary matrices.

(a) Show that V_i are totally isotropic for the bilinear form B.

(b) Let $B_i = \{g \in G : \rho(g)V_i = V_i\}$ for $i = 1, 2$. Describe B_1, B_2 and $B = B_1 \cap B_2$ in matrix form.

(c) Show that $B = H \cdot N$ where H is a maximal torus in G and N is a connected unipotent normal subgroup of B.

11. Let $X = \{x \in M_{4\times 2}(\mathbb{C}) : \mathrm{rank}(x) = 2\}$. For $J = (i_1, i_2)$ with $1 \le i_1 < i_2 \le 4$ let $X_J = \{x \in X : \xi_J(x) \neq 0\}$, where

$$\xi_J(x) = \det \begin{bmatrix} x_{i_1 1} & x_{i_1 2} \\ x_{i_2 1} & x_{i_2 2} \end{bmatrix}$$

is the Plücker coordinate corresponding to J.

(a) Let $A_{\{1,2\}} = \{x \in X : x_{ij} = \delta_{ij} \text{ for } 1 \le i, j \le 2\}$. Calculate the restrictions of the Plücker coordinates to $A_{\{1,2\}}$.

(b) Let $\mathrm{GL}(2, \mathbb{C})$ act by right multiplication on X. Show that $X_{\{1,2\}}$ is invariant under $\mathrm{GL}(2, \mathbb{C})$ and $A_{\{1,2\}}$ is a cross-section for the $\mathrm{GL}(2, \mathbb{C})$ orbits.

(c) Let $\pi : X \to \mathrm{Grass}_2(\mathbb{C}^4)$ map x to its orbit under $\mathrm{GL}(2, \mathbb{C})$. Let $\mathrm{GL}(4, \mathbb{C})$ act by left multiplication on X and hence also on $\mathrm{Grass}_2(\mathbb{C}^4)$. Show that this action is transitive and calculate the stabilizer of $\pi([\ e_1 \quad e_2\ ])$, where e_i are the standard basis vectors for $\mathbb{C}^4$.

p.506, l.-14 and -13 Replace exercise 6 by the following:

6. Let $G = \mathrm{GL}(n, \mathbb{C})$, B the upper-triangular subgroup of G, and H the diagonal subgroup of G. Suppose $P \subset G$ is a closed subgroup such that $B \subset P$.

(a) Prove that $\mathrm{Lie}(P)$ is of the form

$$\mathfrak{b} + \sum_{\alpha \in S} \mathfrak{g}_{-\alpha} \qquad (*)$$

where $\mathfrak{b} = \mathrm{Lie}(B)$ and $S \subset \Phi^+$ (the positive roots of $\mathfrak{g}$). (*Hint*: $\mathrm{Lie}(P)$ is invariant under $\mathrm{Ad}(H)$.)

(b) Let $S \subset \Phi^+$ be any subset and let $\{\alpha_1, \ldots, \alpha_l\}$ be the simple roots in Φ^+. Prove that the subspace defined by $(*)$ is a Lie algebra if and only if S satisfies the properties

(P1) If $\alpha, \beta \in S$ and $\alpha + \beta \in \Phi^+$, then $\alpha + \beta \in S$.

(P2) If $\beta \in S$ and $\beta - \alpha_i \in \Phi^+$ then $\beta - \alpha_i \in S$.

(*Hint*: $\mathfrak{b}$ is generated by $\mathfrak{h}$ and $\{\mathfrak{g}_{\alpha_i} : i = 1, \ldots, l\}$.)

(c) Determine all subsets S of Φ^+ that satisfy **(P1)** and **(P2)**. (*Hint:* Write the roots in terms of simple roots.)

(c) Let R be any subset of the simple roots, and define S_R to be all the positive roots β so that no elements of R occur in β. Show that S_R satisfies **(P1)** and **(P2)**. Conversely, if S satisfies **(P1)** and **(P2)**, let R be the set of simple roots that do not occur in any $\beta \in S$. Prove that $S = S_R$.

(d) Let $G = \mathrm{GL}(n, \mathbb{C})$. Use (c) to determine all subsets S of Φ^+ that satisfy **(P1)** and **(P2)**. (*Hint:* Use Exercise 2.3.5 #2 (a).)

(e) For each subset S found in (d), show that there is a closed subgroup $P \supset B$ with $\mathrm{Lie}(P)$ given by $(*)$. (*Hint:* Show that S corresponds to a partition of n and consider the corresponding block decomposition of G.)

p.506, after exercise 6. INSERT THE FOLLOWING EXERCISE:

7. Let $G = \mathrm{GL}(n, \mathbb{C})$, $H = D_n$ the diagonal matrices in G, N the upper-triangular unipotent matrices, and $B = HN$. Let X be the space of all flags in $\mathbb{C}^n$.

(a) Suppose $x = \{V_1 \subset V_2 \subset \cdots \subset V_n\}$ is a flag that is invariant under H. Prove that there is a permutation $\sigma \in \mathfrak{S}_n$ so that

$$V_i = \mathrm{Span}\{e_{\sigma(1)}, \ldots, e_{\sigma(i)}\} \quad \text{for } i = 1, \ldots, n.$$

(*Hint:* H is reductive and its action on $\mathbb{C}^n$ is multiplicity-free.)

(b) Suppose the flag x in (a) is also invariant under N. Prove that $\sigma(i) = i$ for all i. (*Hint:* Use induction on i.)

(c) Prove that if $g \in G$ and $gBg^{-1} = B$, then $g \in B$. (*Hint:* By (a) and (b), B has exactly one fixed point on $X = G/B$.)

p.532, l.15 to l. 19 REPLACE:

1. Let L be a reductive group, and set $G = L \times L$. Let $K = \{(g, g) : g \in L\}$ be the diagonal embedding of L in G. Show that (G, K) is a spherical pair. (HINT: The irreducible representations of G are of the form $\pi = \sigma \widehat{\otimes} \mu$, where σ and μ are irreducible representations of L. Use Schur's Lemma to show that the K-spherical representations of G are the representations $\pi = \sigma \widehat{\otimes} \sigma^*$.)

BY:

1. Use the criterion of Theorem 12.2.1 to show that the following spaces are multiplicity free:

(a) $G = \mathrm{GL}(n) \times \mathrm{GL}(k)$, $X = M_{n,k}(\mathbb{C})$, $(g, h) \cdot x = gxh^{-1}$. (HINT: Lemma B.2.8.)

(b) $G = \mathrm{GL}(n)$, $X = SM_n(\mathbb{C})$, $g \cdot x = gxg^t$. (HINT: Lemma B.2.9.)

(c) $G = \mathrm{GL}(n)$, $X = AM_n(\mathbb{C})$, $g \cdot x = gxg^t$. (HINT: Lemma B.2.10.)