ALICE THROUGH LOOKING GLASS AFTER LOOKING GLASS: THE MATHEMATICS OF MIRRORS AND KALEIDOSCOPES

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1. Alice and the Mirrors

Let us imagine that Lewis Carroll stopped condensing determinants long enough to write a third Alice book called *Alice Through Looking Glass After Looking Glass*. The book opens with Alice in her chamber in front of several looking glasses. She enters one of them and discovers that she is in a new *virtual chamber* that looks almost like her own. On closer examination she discovers that she is now left-handed and her books are all written backwards. There are also *virtual mirrors* in this chamber. Stepping through one of them, she continues her trip through many virtual chambers until, to her great relief, she suddenly finds herself back in her own real chamber. Eager to have new adventures, Alice wonders how many different ways the mirrors could be arranged so that she could have other trips through the looking glasses.

Alice's problem was solved (for all dimensions) by H.S.M. Coxeter [4], who classified all possible systems of n mirrors in n-dimensional Euclidean space whose reflections generate a finite group of orthogonal matrices. In this paper we describe Coxeter's results, emphasizing the connection with kaleidoscopes. The mathematical tools involved are some linear algebra (including determinants), basic group theory, and a bit of graph theory. We also give plans for three-dimensional kaleidoscopes that exhibit the symmetries of the three types of Platonic solids.

The mathematics of mirrors in n dimensions is the study of those finite groups of orthogonal $n \times n$ real matrices that are generated by reflection matrices. These groups appeared in many parts of mathematics in the late nineteenth and early twentieth century, in connection with geometry, invariant theory, and Lie groups, especially in the work of W. Killing, E. Cartan and H. Weyl [8]. As abstract groups, almost all of them turn out to be very familiar: dihedral groups, the symmetric group of all permutations, the group of all signed permutations, and the group of all evenly-signed permutations. There are also six *exceptional* groups that occur in dimensions three to eight.

2. Mirrors and Kaleidoscopes

Mirrors and mirror symmetries have been part of human culture for millenia (see [13]), but using several mirrors to obtain attractive multiple reflection patterns seems to be a more recent development. The familiar cylindrical kaleidoscope, usually attributed to Sir David Brewster around 1819 [1], uses two mirrors joined along an edge at an angle $180°/m$ (for some small integer $m \geq 3$) to produce images of remarkable beauty when the mirrors are

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[1]Coxeter [6, Remarks 1.9] also mentions A. Kircher in 1646

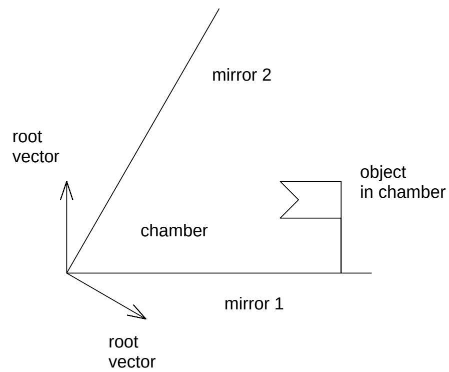

FIGURE 1. Type A_2 Kaleidoscope (60° dihedral angle)

properly aligned.[2] Some real object (typically pieces of colored crystals) is placed in the V-shaped chamber between the two mirrors, as shown in Figure 1 (each mirror is shown with a *root vector* perpendicular to the mirror). The image seen in the kaleidoscope then consists of the real object together with the *virtual objects* in the *virtual chambers* that are the multiple reflections of the real chamber and object, as shown in Figure 2 (the reflections of the root vectors are also shown). The real chamber together with all the virtual chambers fill the plane evenly, and the possible cylindrical kaleidoscopes correspond to the infinitely many regular polygons (equilateral triangle, square, pentagon, hexagon, . . .).

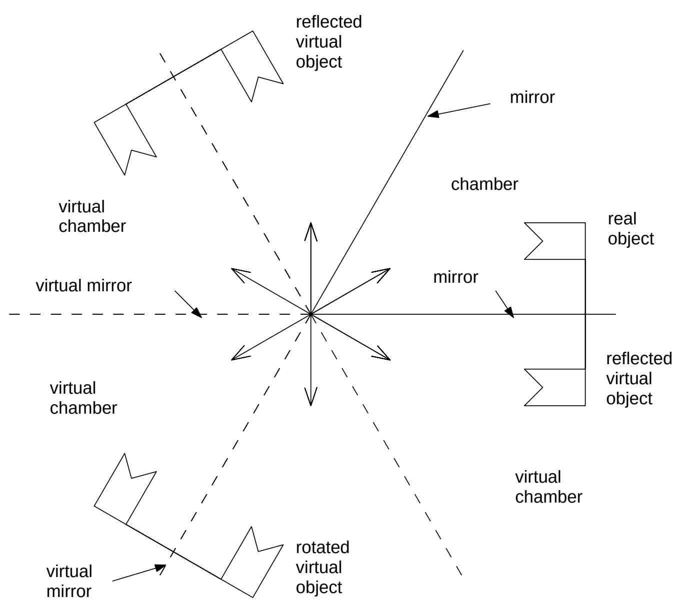

FIGURE 2. Virtual Mirrors and Virtual Objects (60° dihedral angle)

In three dimensions there are only three types of regular solids: tetrahedron (self-dual), cube (with dual octahedron), and icosahedron (with dual dodecahedron). For each type

[2]See the web page of Kaleidoscopes of America (`http://www.kaleido.com`) for examples and links.

there is an associated three-mirror conical kaleidoscope, which also produces images of remarkable beauty (when the mirrors are properly aligned) that we will describe at the end of the paper. Such mirror arrangements were apparently first studied by A. Möbius in 1852 [12]. [3]

In n-dimensional Euclidean space $\mathbb{R}^n$ for $n > 3$ similar constructions are also possible, although not easy to visualize.[4] We find a basic difference between two dimensions and higher dimensions: when $n \geq 3$ there are only a finite number of *genuine* n-dimensional kaleidoscopes. In fact, there are only three types, except when $n = 4$ (5 types) or $n = 6, 7, 8$ (4 types in each dimension). Here the term 'genuine' means that the kaleidoscope has no mirror that is perpendicular to all its other mirrors.

3. Mirrors, Reflections and Reflection Groups

3.1. **Reflection in one mirror.** A mathematical model for a *mirror* through 0 in n-dimensional Euclidean space $\mathbb{R}^n$ is an $(n-1)$ dimensional subspace M, which we can describe by a single linear equation

$$\boldsymbol{\alpha} \cdot \mathbf{v} = 0.$$

Here we take $\mathbb{R}^n$ as $n \times 1$ column vectors, and $\mathbf{u} \cdot \mathbf{v} = \mathbf{u}'\mathbf{v}$ is the usual inner product with $\mathbf{u}'$ the transposed row vector. The vector $\boldsymbol{\alpha} \in \mathbb{R}^n$ is perpendicular to M; we call it a *root vector* for the mirror. We can assume $\boldsymbol{\alpha}$ has length one, since any non-zero multiple of $\boldsymbol{\alpha}$ is also a root vector for the same mirror.

The *reflection* in the mirror is the linear transformation R of $\mathbb{R}^n$ defined by

$$R\mathbf{v} = \begin{cases} \mathbf{v} & \text{if } \mathbf{v} \in M, \\ -\mathbf{v} & \text{if } \mathbf{v} \text{ is a multiple of } \boldsymbol{\alpha}. \end{cases}$$

We write $M = M_{\boldsymbol{\alpha}}$ and $R = R_{\boldsymbol{\alpha}}$ if we want to show how M and R depend on the direction $\boldsymbol{\alpha}$.

To find a formula for R as a matrix, recall that the projection of a vector $\mathbf{v}$ onto the line $\mathbb{R}\boldsymbol{\alpha}$ is

$$\mathbf{v}_1 = (\mathbf{v} \cdot \boldsymbol{\alpha})\boldsymbol{\alpha}.$$

Thus we have an orthogonal decomposition $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_0$, where $\mathbf{v}_0 = \mathbf{v} - \mathbf{v}_1$ lies in the mirror. Hence

$$R\mathbf{v} = -\mathbf{v}_1 + \mathbf{v}_0 = \mathbf{v} - 2(\boldsymbol{\alpha} \cdot \mathbf{v})\boldsymbol{\alpha}.$$

Thus R acts on column vectors by the $n \times n$ matrix $I_n - 2\boldsymbol{\alpha}\boldsymbol{\alpha}'$, where I_n is the $n \times n$ identity matrix.

Examples of Reflection Matrices. In $\mathbb{R}^2$, $\boldsymbol{\alpha} = [\ 0 \quad -1\ ]'$ is a root vector for the mirror $x_2 = 0$ along the x_1 axis, and

$$R_{\boldsymbol{\alpha}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 2 \begin{bmatrix} 0 \\ -1 \end{bmatrix} [\ 0 \quad -1\ ] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

[3] Hessel in 1830 and Bravais in 1849 had already determined the finite groups of three-dimensional rotations in connection with crystallography.

[4] See [6] for some ways of visualizing reflections and regular solids in four-dimensional space using quaternions (pairs of complex numbers).

(which is obvious). The vector $\boldsymbol{\beta} = [-\sin\theta, \cos\theta]'$ is a root vector for the mirror at an angle θ with the x_1 axis, and

$$\begin{aligned} R_{\boldsymbol{\beta}} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 2 \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} [-\sin\theta \ \ \cos\theta] = \begin{bmatrix} 1 - 2\sin^2\theta & 2\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & 1 - 2\cos^2\theta \end{bmatrix} \\ &= \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}. \end{aligned}$$

From this calculation we see that matrix $R_{\boldsymbol{\alpha}} R_{\boldsymbol{\beta}}$ is rotation by an angle 2θ.

Properties of Reflections.

- $R(Rv) = v$, so R^2 is the *identity transformation.*
- $(R\mathbf{v}) \cdot (R\mathbf{v}) = \mathbf{v} \cdot \mathbf{v}$, so R is an *orthogonal* transformation ($RR' = I_n$).
- R reverses orientation ($\det R = -1$).

Remark. Reflection matrices are also called *Householder matrices* [9, §2.2.4]. Every orthogonal matrix is a product of reflection matrices, and this provides an extremely efficient and accurate tool in many algorithms of numerical linear algebra. In fact, the familiar Gram-Schmidt algorithm taught in elementary linear algebra courses is too unstable and inefficient for numerical calculation, and it is replaced by an algorithm using reflection matrices.

3.2. **Dihedral Kaleidoscopes.**

Theorem 1. *Take two mirrors in $\mathbb{R}^2$ that pass through 0 and have root vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. Let $\theta \le \pi/2$ be the dihedral angle between the mirrors and let C be the closed cone between the mirrors. Assume that the interior of C does not contain any virtual mirror generated by multiple reflections in the two mirrors.*

(i) *The group G of matrices*

$$I_2, R_{\boldsymbol{\alpha}},\ R_{\boldsymbol{\beta}},\ R_{\boldsymbol{\alpha}}R_{\boldsymbol{\beta}},\ R_{\boldsymbol{\beta}}R_{\boldsymbol{\alpha}},\ R_{\boldsymbol{\alpha}}R_{\boldsymbol{\beta}}R_{\boldsymbol{\alpha}},\ \cdots \tag{1}$$

generated by $R_{\boldsymbol{\alpha}}, R_{\boldsymbol{\beta}}$ is finite if and only if $\theta = \pi/m$ for some integer $m \ge 2$. In this case G is the dihedral group $I_2(m)$ *of symmetries of the regular $2m$-gon.*

(ii) *Assume $\theta = \pi/m$. The images $g \cdot C$ for $g \in G$ (the* fundamental chamber *for $g = 1$ and the* virtual chambers *for $g \ne 1$) have disjoint interiors and fill up $\mathbb{R}^2$. Furthermore, if $gC = C$ then $g = I$. Hence the chambers (fundamental and virtual) correspond uniquely to the elements of G. In particular, $|G| = 2m$.*

(iii) *Assume $\theta = \pi/m$. As an abstract group G is generated by $a = R_{\boldsymbol{\alpha}}$ and $b = R_{\boldsymbol{\beta}}$ with all relations generated by the three relations*

$$a^2 = 1, \quad b^2 = 1, \quad (ab)^m = 1.$$

Proof: Set $g = R_{\boldsymbol{\alpha}} R_{\boldsymbol{\beta}}$. Then g is rotation by the angle 2θ, hence it is of finite order if and only if θ is a rational multiple of π.

Assume $\theta = n\pi/m$, with m, n relatively prime positive integers. We claim that $n = 1$. To prove this, note that the action of g on the given mirrors produces two virtual mirrors that make angles of 2θ and 3θ with the first mirror. Hence for every integer k there is a virtual mirror that makes an angle of $k\theta$ with the first mirror. Since m, n are relatively prime, there is an integer k such that $kn \equiv 1 \mod (m)$ (by the Euclidean algorithm). Hence if $n > 1$ there is virtual mirror that makes an angle of π/m with the first mirror, contradicting the assumption that there are no virtual mirrors between the two given mirrors.

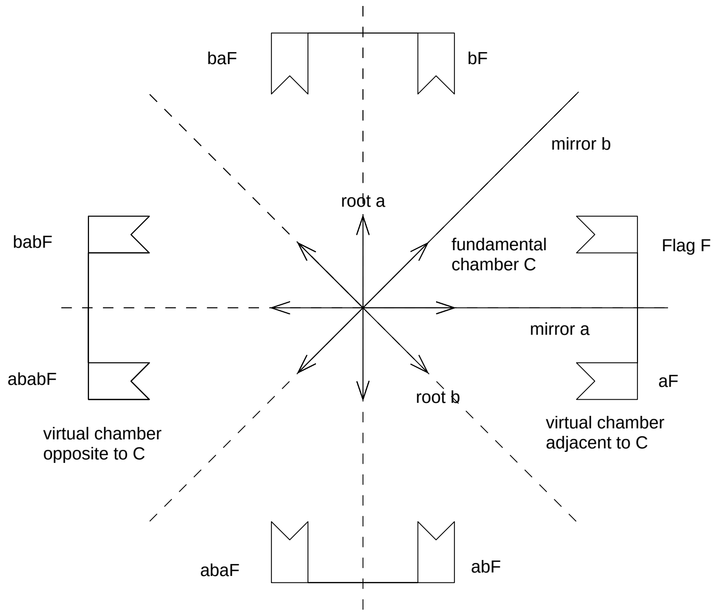

FIGURE 3. Type B_2 Kaleidoscope ($45°$ dihedral angle)

Now assume $\theta = \pi/m$. From the relations

$$R_{\boldsymbol{\alpha}}^2 = R_{\boldsymbol{\beta}}^2 = (R_{\boldsymbol{\alpha}} R_{\boldsymbol{\beta}})^m = I$$

one checks that there are at most $2m$ distinct matrices in the set (1) (the cases m even and m odd need separate consideration). This proves (i). Part (ii) follows from elementary geometry.

To present $I_2(m)$ as an abstract group in terms of generators and relations, use the relations, as in (i), to see that at most $2m$ distinct words

$$1, a, b, ab, ba, aba, bab, abab, \ldots$$

can be formed. Since $I_2(m)$ has $2m$ elements, this proves (iii) (see [11, §1.3] for more details). ■

Example: Type B_2 Mirror System. The case $\theta = 45°$ is shown in Figure 3. The fundamental and virtual chambers are labeled by the associated elements of the dihedral group $I_2(4)$, with $a = R_{\boldsymbol{\alpha}}$ and $b = R_{\boldsymbol{\beta}}$. The labeling is determined by the group element g needed to move the flag F in the fundamental chamber to the flag gF in the virtual chamber. In this case the relations $a^2 = b^2 = (ab)^4 = 1$ can be used to reduce any word in a, b to one of the eight words

$$1, a, b, ab, ba, aba, bab, abab$$

that correspond to the eight chambers in Figure 3. For example, $baba = abab$ because $(abab)^2 = 1$. This is clear geometrically from Figure 3 by reflecting the virtual image $abaF$ through mirror b to get the virtual image $ababF$. Note that the composition of an *even* number of reflections *preserves* orientation, while the composition of an *odd* number of reflections reverses orientation.

4. Mirror Systems and Kaleidoscopes in Three Dimensions

Now consider a configuration of three mirrors in three dimensions and the group G generated by reflections in these mirrors. We assume the three mirror planes each contain 0. The mirror planes divide $\mathbb{R}^3$ into several *solid cones* whose walls are the mirrors. Fix one cone C, which we will call the *fundamental chamber*. Then C has three walls, which are two-dimensional wedges extending to infinity (see Figure 4).

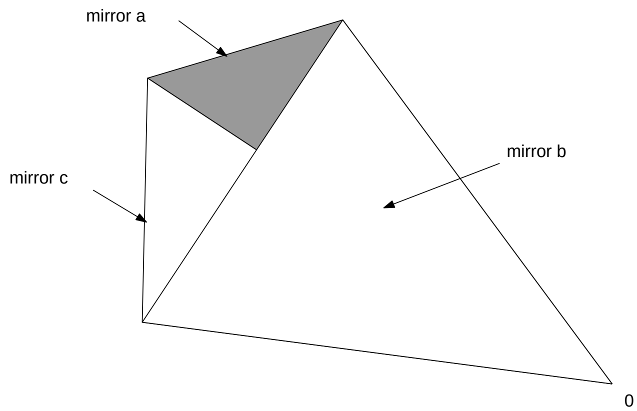

Figure 4. Fundamental Chamber for a Kaleidoscope in $\mathbb{R}^3$

Let π/p, π/q, π/r be the interior dihedral angles between the walls of C (there are three pairs of walls, and each pair has a dihedral angle). The *interior* walls of C are mirrored; in order to get multiple reflections of one wall in another, we can assume that

$$2 \leq p \leq q \leq r.$$

If $p = q = 2$ then one of the mirrors is perpendicular to both of the other mirrors, and we are in the situation of two mirrors in two dimensions and one mirror in the remaining dimension. The group G in this case is simply the *product* of the dihedral group for the two mirrors and the two-element group for one mirror. So now we assume that $q > 2$.

Theorem 2. *Let G be the group of orthogonal matrices generated by reflections in the three walls of the chamber C. Assume that the interior of C does not contain any virtual mirror generated by multiple reflections in the three walls of C.*

(i) *Suppose that the orbit $G \cdot \mathbf{x}$ is finite for some point $\mathbf{x}$ inside C. Then p, q, r are positive integers that satisfy*

$$2 \leq p \leq q \leq r, \quad \frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1. \tag{2}$$

(ii) *The integer solutions to* (2) *with $q > 2$ are* $(2, 3, 3)$, $(2, 3, 4)$, *and* $(2, 3, 5)$.

(iii) *Let (p, q, r) be one of the triples in* (ii) *and let C be a chamber (3-sided cone) in $\mathbb{R}^3$ with the corresponding dihedral angles. The images $g \cdot C$ for $g \in G$ (the* virtual chambers*) have disjoint interiors and fill up $\mathbb{R}^3$. Furthermore, if $gC = C$ then $g = I$. Hence the chambers (fundamental and virtual) correspond uniquely to the elements of G and G is finite.*

Proof. (i) Take a pair of mirrors and consider the group $H \subset G$ generated by reflections in these two mirrors. Then $H \cdot \mathbf{x}$ must be finite, so Theorem 1 implies that the dihedral angle between the mirrors must be an integral submultiple of π. Hence p, q, and r are integers.

To obtain a relation among p, q, and r, consider a triangular cross-section of the cone C (see Figure 4). The angles of this triangle are no bigger than the corresponding dihedral angles of the mirrors, and at least one of the angles is strictly smaller than the dihedral angle. Since the sum of the interior angles of the triangle is π, condition (2) follows.

(ii) The integer solutions to (2) must satisfy $p = 2$, $q = 3$, and $r \leq 5$ because

$$\frac{1}{3} + \frac{1}{3} + \frac{1}{3} = \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = \frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1$$

(iii) For each admissible triple (p, q, r) we can construct

- Regular polyhedron (tetrahedron, cube, icosahedron) centered at 0.
- Triangulation of the faces of the polyhedron by congruent triangles
- Cone C from 0 through one of the triangles with dihedral angles π/p, π/q, π/r.

This is illustrated in Figure 5 for the triple $(2, 3, 4)$ (the regular polyhedron is a cube in this case). We verify that each reflection in a wall of C permutes the vertices of this polyhedron. Hence G is finite since there are three linearly independent vertices (see the illustrations in [7, Chapter 3] and [1, Chapter 5, page 157]). Also, G permutes the triangles, so G permutes the chambers.

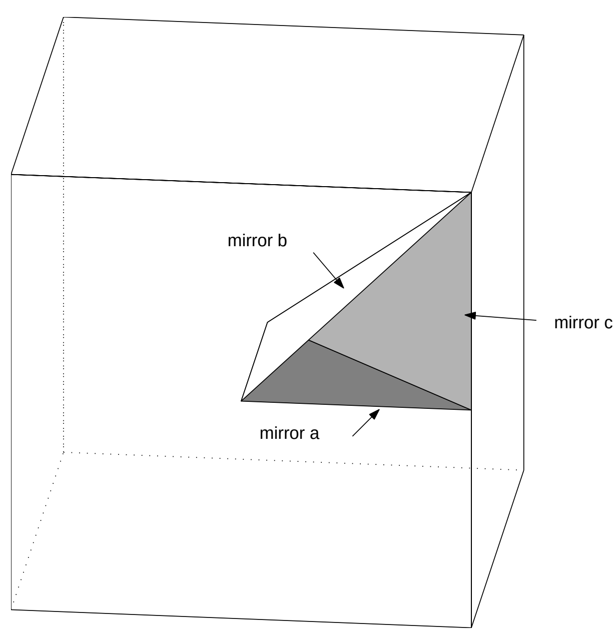

FIGURE 5. Fundamental Chamber in a Cube

To establish the correspondence between $g \in G$ and the (virtual) chamber $g \cdot C$ through one of the triangular faces of the polyhedron, write

$$g = R_1 R_2 \cdots R_k$$

where each R_i is reflection in some wall of C. Such a product is called a *reduced word* if k is as small as possible, and we define $\mathsf{length}(g) = k$ in this case (the identity element has length zero). One can show that $\mathsf{length}(g)$ is the minimal number of mirrors (real and virtual) that must be crossed in order to go from the fundamental chamber C to the virtual chamber $g \cdot C$ [7, Chapter 6] (this is obvious when $\mathsf{length}(g) = 1$, since the virtual chamber $g \cdot C$ shares a wall with the fundamental chamber C in that case). This property of the length function immediately shows that 1 is the only element of G that fixes C (as a set). So if $g, h \in G$ and $g \cdot C = h \cdot C$, then $g^{-1}h$ fixes C and must be the identity. This proves that g is uniquely determined by the chamber $g \cdot C$. ■

Table 1 summarizes the situation in three dimensions. In the table S_n denotes the symmetric group on n letters, and the three groups are designated as types A_3, B_3, H_3 (to be consistent with the classification in higher dimensions). The *tetrahedral group* of type A_3 is S_4, which acts by permuting the four vertices of the tetrahedron. If the vertices are numbered 1 to 4, then the transposition $1 \leftrightarrow 2$ acts by the reflection through the plane containing vertices $3, 4$ and the midpoint of the edge joining vertices 1 and 2. Since S_4 is generated by transpositions, every permutation of the vertices of the tetrahedron can be obtained as a product of reflections.

Dihedral Angles	Regular Polyhedron	Group	# Mirrors	# Chambers
$\pi/2$ - $\pi/3$ - $\pi/3$	Tetrahedron	$A_3 = S_4$	6	24
$\pi/2$ - $\pi/3$ - $\pi/4$	Cube Octahedron	$B_3 = S_3 \ltimes \{[\pm 1, \pm 1, \pm 1]\}$	9	48
$\pi/2$ - $\pi/3$ - $\pi/5$	Icosahedron Dodecahedron	$H_3 = \mathrm{Alt}_5 \times \{\pm 1\}$	15	120

TABLE 1. Finite Reflection Groups in Three Dimensions

The symmetry group B_3 of the cube or octahedron consists of the *signed permutations* of three objects (which we can take as the three basis vectors in three dimensions that define the cube). It is the *semidirect product* of the group S_3, realized as 3×3 permutation matrices, and the normal abelian subgroup of 3×3 diagonal matrices with entries ± 1 (these diagonal matrices give the reflections in the three coordinate planes).

The group of rotational symmetries (orientation-preserving) of the icosahedron is isomorphic to the alternating group Alt_5 of order 60 (the *even* permutations in S_5). The full symmetry group of the icosahedron is $H_3 \cong \mathrm{Alt}_5 \times \{\pm 1\}$ of order 120. Here -1 acts by the transformation $\mathbf{x} \mapsto -\mathbf{x}$ of $\mathbb{R}^3$ that commutes with all rotations, so H_3 is *not* isomorphic to S_5, whose center is $\{1\}$ (see [7, §2.4] and [3] for more details and the recent appearance of icosahedral symmetry in chemistry).

The mirror count in Table 1 includes the virtual mirrors (reflecting planes) together with the three mirrors that bound the fundamental chamber. The number of chambers is the same as the order of the symmetry group G, by Theorem 2. For the dihedral groups in two dimensions, the number of chambers is always twice the number of mirrors. But in three dimensions, the ratio #(chambers)/#(mirrors) is 4, 16/3, or 8, in the three cases.

5. Kaleidoscopes in n Dimensions

5.1. **Mirror Systems and Root Systems.** A finite arrangement of mirrors (reflecting hyperplanes) in $\mathbb{R}^n$ can be specified by giving a pair of root vectors $\pm\boldsymbol{\alpha}$ for each mirror (we take the pair $\pm\boldsymbol{\alpha}$ to avoid choosing an orientation). We normalize $\boldsymbol{\alpha}$ to have length one and let Φ be the resulting collection of unit vectors. We are interested in the multiple reflections in the mirrors, so we assume that the arrangement of mirrors is invariant under the reflection $R_{\boldsymbol{\alpha}}$ for every $\boldsymbol{\alpha} \in \Phi$. This condition can be stated in terms of the root vectors as follows.

Kaleidoscope Condition: For every $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \Phi$, the reflected vector $R_{\boldsymbol{\alpha}}\boldsymbol{\beta} \in \Phi$.

A finite set Φ of unit vectors in $\mathbb{R}^n$ satisfying the kaleidoscope condition is called a *root system*[5]. We may assume that Φ spans $\mathbb{R}^n$, since any vector that is perpendicular to all mirrors will be fixed by all the reflections. By the kaleidoscope condition the group of orthogonal matrices generated by the reflections $R_{\boldsymbol{\alpha}}$ for $\boldsymbol{\alpha} \in \Phi$ is finite, since an element of this group is determined by its action on the finite set Φ.

In the cases $n = 2$ and $n = 3$ already studied Φ must contain the root vectors to the virtual mirrors in Φ to have the kaleidoscope condition satisfied (see Figure 2 with the six root vectors of the A_2 system and Figure 3 with the eight root vectors of the B_2 system).

How can we identify the n outward-pointing root vectors to the physical mirrors that form the walls of the kaleidoscope? This is answered by the next result, whose proof is an easy linear algebra argument (see [10, Theorem 1.3]).

Theorem 3. *If Φ is a root system, then it contains a subset $\Delta = \{\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_n\}$ of* simple roots *such that*

(i) *Δ is a basis for $\mathbb{R}^n$.*
(ii) *$\boldsymbol{\alpha}_i \cdot \boldsymbol{\alpha}_j \leq 0$ for $i \neq j$*
(iii) *If $\boldsymbol{\beta} \in \Phi$ then the expression of $\boldsymbol{\beta}$ in terms of the basis Δ has coefficients that are all of the same sign.*

Fix a set of simple roots Δ and write R_i for the reflection $R_{\boldsymbol{\alpha}_i}$. The *fundamental chamber* C determined by Δ is the simplicial cone in $\mathbb{R}^n$ defined by the inequalities

$$\boldsymbol{\alpha}_i \cdot \mathbf{x} \geq 0 \quad \text{for } i = 1, \dots, n.$$

If θ_{ij} is the dihedral angle between the mirrors for $\boldsymbol{\alpha}_i$ and $\boldsymbol{\alpha}_j$, then $\theta_{ij} \leq \pi/2$ since $\boldsymbol{\alpha}_i \cdot \boldsymbol{\alpha}_j \leq 0$. Part (iii) of Theorem 3 implies that if a point $\mathbf{x}$ is in the interior of C, then $\boldsymbol{\beta} \cdot \mathbf{x} \neq 0$ for every $\boldsymbol{\beta} \in \Phi$. Hence none of the reflecting hyperplanes determined by the root vectors pass through the interior of C, so from the two-dimensional case (Theorem 1) we conclude that

$$\theta_{ij} = \frac{\pi}{p_{ij}},$$

where p_{ij} is an integer. Furthermore, R_i and R_j satisfy the relations

$$R_i^2 = (R_i R_j)^{p_{ij}} = 1. \tag{3}$$

[5]The term *root system* has a slightly different meaning in connection with Lie algebras (see [10])

5.2. **Coxeter Graphs.** We have now translated the study of kaleidoscopes in n dimensions into the study of root systems. The n (actual) mirrors of the kaleidoscope correspond to the set Δ of n simple roots, and the interior of the kaleidoscope is the fundamental chamber C. We encode this information in the *Coxeter graph* of the root system. This is a *labeled graph* with n vertices, with the ith vertex corresponding to the ith mirror. We draw an edge between vertex i and vertex j if $p_{ij} > 2$, and we label the edge with the integer p_{ij} if $p_{ij} > 3$. Figure 6 gives the Coxeter graphs for the three-dimensional mirror systems we have studied.

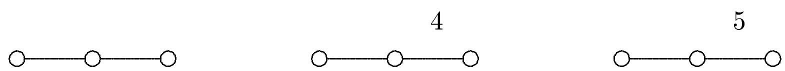

FIGURE 6. Coxeter Graphs for Three-Dimensional Kaleidoscopes

We define the *Coxeter matrix* of the root system to be the $n \times n$ symmetric matrix $A = [\boldsymbol{\alpha}_i \cdot \boldsymbol{\alpha}_j]$ (matrix of inner products). Since

$$a_{ii} = 1, \quad a_{ij} = -\cos(\pi/p_{ij}) \quad \text{for } i \neq j,$$

we see that A is completely determined by the Coxeter graph, without reference to the root system.

Since the Coxeter matrix of a root system is the matrix of inner products for a basis of $\mathbb{R}^n$, we obtain the following *necessary* condition for a labeled graph to be the Coxeter graph of a root system.

Theorem 4. *The Coxeter matrix of a root system is positive definite.*

To appreciate the power of this result, we use it to show again that a kaleidoscope with dihedral angles $\pi/2$, $\pi/3$, and π/p can only exist if $p \leq 5$. The Coxeter graph would be as in Figure 6 (with 4 or 5 replaced by p), and the Coxeter matrix would be

$$\begin{bmatrix} 1 & -1/2 & 0 \\ -1/2 & 1 & -c \\ 0 & -c & 1 \end{bmatrix}, \quad c = \cos(\pi/p).$$

Recall that a real symmetric matrix is positive definite if and only if all the principal minors are positive. In particular, the determinant of this matrix is $\frac{3}{4} - c^2$, which is positive if and only if $p < 6$. Since p must be an integer, this forces $p = 3$, 4, or 5.

Coxeter [4] found all the graphs that satisfy the positive definiteness condition. The key observation is the *Monotonicity Property:* Every subgraph (with possibly smaller labels) of a positive-definite graph is also positive definite.

To use the monotonicity property, we need a supply of Coxeter graphs that are *not* positive definite. Some examples of Coxeter graphs whose Coxeter matrix is positive semidefinite are shown in Figure 7, and the complete set is shown in [10, Figure 2.2].[6] This collection of Coxeter graphs and the monotonicity property imply that a connected positive-definite Coxeter graph with three or more vertices cannot have the following features:

- Circuits
- More than one branch point
- A branch point with four or more edges

[6]These diagrams are associated with so-called *affine root systems* and kaleidoscopes in $\mathbb{R}^n$ with $n+1$ mirrors, such as the familiar three-mirror cylindrical kaleidoscope whose images are in $\mathbb{R}^2$.

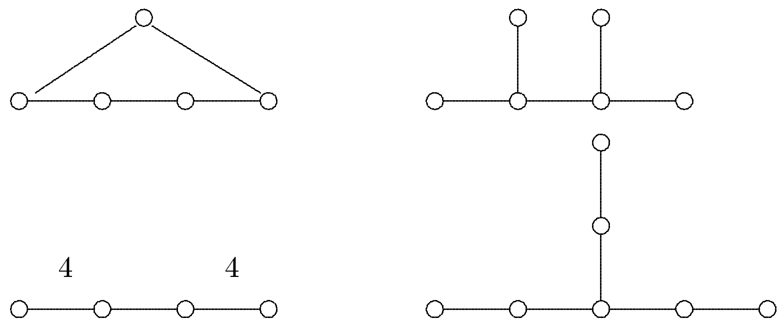

FIGURE 7. Some Semi-Definite Coxeter Graphs

- Two or more labels larger than 3
- Two or more vertices on each each of the three edges at a branch point

From constraints of this sort, one finds by a process of elimination that in dimension four there are exactly five different connected positive-definite Coxeter graphs, shown in Figure 8. In all dimensions, the positive-definite graphs can be classified by the same method, and root systems with these graphs can be constructed. The reflection groups associated with the root systems are finite, and all the relations in the groups are generated by the *Coxeter Relations* (3). See [10, Chapter 2] for details.

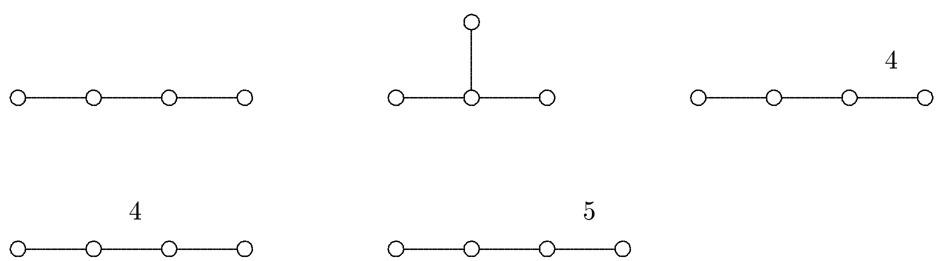

FIGURE 8. Coxeter Graphs for Four-Dimensional Kaleidoscopes

If we assume that the Coxeter graph is *connected* (this means we do not have two sets of mirrors, with every mirror in the first set orthogonal to every mirror in the second set), then the number of finite reflection groups in each dimension greater than three is very small. The list of connected root systems and their reflection groups is given in Table 2.

In each dimension, the first three groups listed in the table are the *classical* finite reflection groups: the *symmetric group* S_{n+1}, the group D_n of *evenly signed* permutations of n letters, and the *hyperoctahedral group* B_n of *signed* permutations of n letters. The remaining groups in the table (which only occur for dimensions 4, 6, 7 and 8) are the *exceptional* finite reflection groups of types F_4, H_4 in dimension 4 and types E_6, E_7, E_8 in dimensions 6, 7, 8. Except for H_4, these groups are *crystallographic*: they can be represented by integer matrices relative to a suitable basis[7].

The exceptional reflection group H_4 is the symmetry group of a regular solid in four dimensional Euclidean space, just as H_3 is the symmetry group of the icosahedron. The

[7]These are the *Weyl groups* of five of the six *exceptional* finite-dimensional simple Lie algebras, which were discovered by W. Killing in 1888 [8]. In two dimensions, the dihedral groups $I_2(m)$ are crystallographic only for $m = 2, 3, 6$, and $I_2(6)$ is the Weyl group of the remaining exceptional Lie algebra G_2.

Dimension	# Groups	# Mirrors	# Chambers
4	5	10, 12, 16 24 60	$5 \cdot 4!$, $2^3 \cdot 4!$, $2^4 \cdot 4!$ $2 \cdot 6 \cdot 8 \cdot 12$ $2 \cdot 12 \cdot 20 \cdot 30$
5	3	15, 20, 25	$6 \cdot 5!$, $2^4 \cdot 5!$, $2^5 \cdot 5!$
6	4	21, 30, 36 36	$7 \cdot 6!$, $2^5 \cdot 6!$, $2^6 \cdot 6!$ $2 \cdot 5 \cdot 6 \cdot 8 \cdot 9 \cdot 12$
7	4	28, 42, 49 63	$8 \cdot 7!$, $2^7 \cdot 7!$, $2^6 \cdot 7!$ $2 \cdot 6 \cdot 8 \cdot 10 \cdot 12 \cdot 14 \cdot 18$
8	4	36, 56, 64 120	$9 \cdot 8!$, $2^7 \cdot 8!$, $2^8 \cdot 8!$ $2 \cdot 8 \cdot 12 \cdot 14 \cdot 18 \cdot 20 \cdot 24 \cdot 30$
$n > 8$	3	$n(n+1)/2$, $n(n-1)$, n^2	$(n+1) \cdot n!$, $2^{n-1} \cdot n!$, $2^n \cdot n!$

TABLE 2. Finite Reflection Groups in Higher Dimensions

root systems of type H_3 and H_4 can be constructed quite directly as finite subgroups of the quaternions [10, §2.13]. We note that the exceptional groups are *much* larger than the classical groups in the same dimension (E_8 has order 1,920 times the order of S_9).

The number of mirrors listed in the table counts the n mirrors that are the walls of the fundamental chamber, together with all the virtual mirrors that are the reflections of these walls; it is one-half the number of roots (recall that each root occurs with its negative). Each chamber is associated with a group element, just as in the two- and three-dimensional cases, so the number of chambers is the same the order of the group G. This number is the product of the degrees of n so-called *basic invariants* (a set of n algebraically independent homogeneous polynomials in n variables that generate all G-invariant polynomials), and the table shows this factorization; for E_8 the degrees are $2, 8, 12, 14, 18, 20, 24, 30$. The degrees of the basic invariants are important for many reasons (for example, they determine the Betti numbers of the associated Lie groups). For the classical groups they are well-known (in the case of S_{n+1} one may take the elementary symmetric functions as basic invariants). For the exceptional groups they were determined by C. Chevalley [2].

We see from the table that the number of chambers is enormously larger than the number of mirrors, and that the exceptional systems are also exceptional in this regard. For the E_8 root system the ratio is almost six million, whereas for the three classical root systems in $\mathbb{R}^8$ it is in the range 10,000 to 80,000.

6. Construction of Three-Dimensional Kaleidoscopes

In the introduction to [4], Coxeter recalls the cases of finite reflection groups in two- and three-dimensions and writes

> 'These groups can be made vividly comprehensible by using actual mirrors for the generating reflections. It is found that a candle makes an excellent

> object to reflect. By hinging two vertical mirrors at an angle π/k we easily see $2k$ candle flames, in accordance with the group $[k]$. To illustrate the groups $[k_1, k_2]$, we hold a third mirror in the appropriate positions.' [8]

The actual construction of 3-dimensional kaleidoscopes ('holding a third mirror in the appropriate position') is not easy, however, compared to making a traditional dihedral kaleidoscope. In [6, Remark 3.5] Coxeter mentions that 'a very accurate icosahedral kaleidoscope was made by in Minneapolis (by Litton Industries) for a film project that was never completed because the expected financial support was withdrawn.'[9] Several recent United States Patents have been granted for 3-dimensional kaleidoscopes.[10]

Insert photo.jpg here

FIGURE 9. Kaleidoscopes of Type A_3, B_3 and H_3

In this section we give mirror dimensions for 3-dimensional kaleidoscopes of symmetry types A_3, B_3 and H_3 (see also [6, Chapter 3]). Our method of generating an image in the kaleidoscope is to truncate the fundamental chamber and place a circular disc over the opening that is free to rotate. Let P be the polyhedron that is the convex hull of the Platonic solid and its dual associated with the Coxeter graph for the kaleidoscope. Reflections of the edges at the plane of truncation generate an image of P, and a graphic pattern placed on the disc appears on each face of P by the multiple reflections in the mirrors (see the photograph in Figure 9, in which the kaleidoscope on the right shows the icosahedron/dodecahedron

[8]Here the group $[k]$ is the dihedral group $I_2(k)$ of order $2k$, while $[k_1, k_2]$ is the group in Theorem 2 associated with the regular polyhedron having faces with k_1 edges and vertices with k_2 edges.

[9]Presumably this was to be a sequel to Coxeter's 1966 film *Dihedral Kaleidoscopes*, distributed by International Film Bureau, Chicago.

[10]Some interesting examples are Patent #5,475,532 granted to J. Sandoval and J. Bracho, December 12, 1995 and Patent #5,651,679 granted to F. Altman, July 29, 1997. The U.S. Patent Office web site `http://www.uspto.gov/patft` gives details. I do not know if any of these designs have been manufactured; in my own experience of building three-dimensional kaleidoscopes by hand I found that large front-silvered mirrors of high optical quality are unavailable and assembling mirrors accurately is difficult.

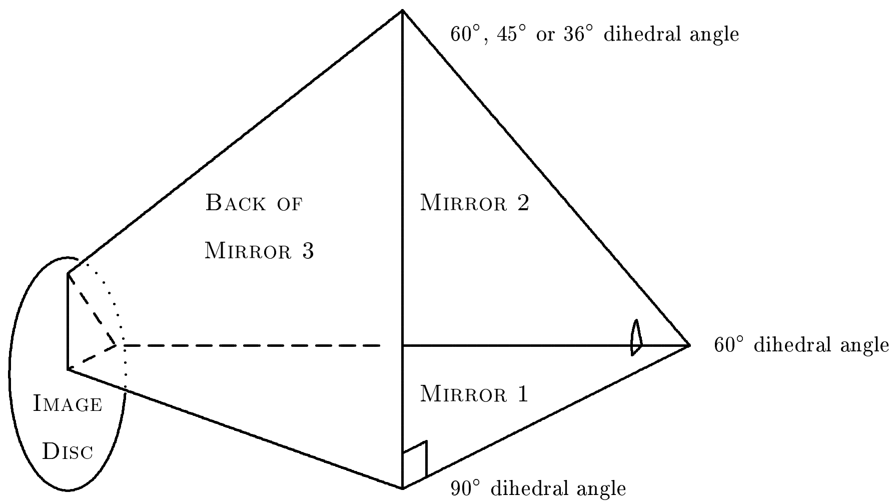

Figure 10. Three-Dimensional Kaleidoscope

polyhedron). When the disc is rotated, the patterns on the faces of P move. Since P has many faces, this generates a striking effect. When the graphic pattern is just a single line, rotation of the disc gives a continuous transition between the image of a Platonic solid and its dual.

An assembled kaleidoscope and its pattern disc is illustrated in Figure 10. The dimensions of the mirrors are given in Table 3 and at the end of the paper. The proportions of the truncated mirrored cones are chosen so that the following linear dimensions are the same for the three types:

- the radius r of the polyhedral image P appearing in the mirrors
- the length z of the longer leg of the front right triangle

The vertex angles α, β and γ of the three cones are different, with Type A the most open and Type H the most closed. Consequently the length L of the truncated cone (measured perpendicular to the truncation planes) varies considerably when the dimensions are fixed as above. If the orientation of the cone is chosen so that $\alpha \leq \gamma$, then the relations among the parameters are given in Table 3, where $\varphi = (1+\sqrt{5})/2$ is the golden mean. The table also gives y, the length of the short leg in the back right triangle. The calculations in the table are based on the property that the point with coordinates

$$\left[\frac{y}{\tan\alpha},\ 0,\ \frac{y\tan\gamma}{\tan\alpha}\right]$$

is a vertex of the polyhedral image at distance r from 0, and the relation

$$z = \left(L + \frac{y}{\tan\alpha}\right)\tan\gamma.$$

The case-by-case calculation of the angles α, β, γ is given at the end of this section.

We found experimentally (by building several kaleidoscopes) that the proportion $L = 8y$ for the B_3 cone gives an easily-viewed image. This corresponds to the proportion

$$z = \left(4 + \frac{1}{\sqrt{2}}\right) r.$$

With this ratio fixed, we calculated the dimensions for the two other types from the table to obtain Figures 11, 12, and 13 at the end of the paper (these figures are to scale).

Type A_3. For this case it is convenient to use the subspace $\mathbb{E}^3 \subset \mathbb{R}^4$ consisting of vectors orthogonal to [1, 1, 1, 1] and to normalize the root vectors to have length $\sqrt{2}$. The root system consists of the twelve vectors in $\mathbb{E}^3$ whose coordinates are $1, -1, 0, 0$ (in any order). This set of roots is invariant under the natural action of S_4 on $\mathbb{R}^4$ as permutations of coordinates. A set Δ of simple roots is

$$\boldsymbol{\alpha}_1 = [1,\ -1,\ 0,\ 0], \quad \boldsymbol{\alpha}_2 = [0,\ 1,\ -1,\ 0,], \quad \boldsymbol{\alpha}_3 = [0,\ 0,\ 1,\ -1].$$

The kaleidoscope mirrors are the planes with normal vectors $\boldsymbol{\alpha}_i$. Define

$$\boldsymbol{\lambda}_1 = \left[\frac{3}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}\right], \quad \boldsymbol{\lambda}_2 = \left[\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right], \quad \boldsymbol{\lambda}_3 = \left[\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, -\frac{3}{4}\right].$$

Then $\boldsymbol{\lambda}_i \in \mathbb{E}^3$, $\boldsymbol{\lambda}_i \cdot \boldsymbol{\alpha}_i = 1$, and $\boldsymbol{\lambda}_i \cdot \boldsymbol{\alpha}_j = 0$ for $i \neq j$. Hence the vector $\boldsymbol{\lambda}_i$ lies on the edge of the kaleidoscope formed by mirrors j, k (for i, j, k all different). Let α, β and γ be the vertex angles of the kaleidoscope, with α the angle between $\boldsymbol{\lambda}_2$ and $\boldsymbol{\lambda}_3$, and so forth.

Taking dot products we calculate that

$$\tan\alpha = \sqrt{2}, \quad \tan\beta = 2\sqrt{2}, \quad \tan\gamma = \sqrt{2}$$

Hence $\alpha = \gamma \doteq 54.74°$ and $\beta \doteq 70.53°$. From the identity

$$\tan(\alpha + \beta + \gamma) = \frac{a + b + c - abc}{1 - ab - ac - bc} \qquad (a = \tan\alpha,\ b = \tan\beta,\ c = \tan\gamma)$$

we find that $\alpha + \beta + \gamma = 180°$. The pattern for the kaleidoscope is shown in Figure 11. The dihedral angles at edges a and b are 60° and the dihedral angle at edge c is 90°.

Type B_3. In this case the root system consists of the eighteen vectors in $\mathbb{R}^3$ whose coordinates (in any order) are either $\pm 1, \pm 1, 0$ or $\pm\sqrt{2}, 0, 0$ (again it is convenient to normalize the roots to have length $\sqrt{2}$). A set Δ of simple roots is

$$\boldsymbol{\alpha}_1 = [1,\ -1,\ 0], \quad \boldsymbol{\alpha}_2 = [0,\ 1,\ -1], \quad \boldsymbol{\alpha}_3 = [0,\ 0,\ \sqrt{2}].$$

The kaleidoscope mirrors are the planes with normal vectors $\boldsymbol{\alpha}_i$. Define

$$\boldsymbol{\lambda}_1 = [1,\ 0,\ 0], \quad \boldsymbol{\lambda}_2 = [1,\ 1,\ 0] \quad \boldsymbol{\lambda}_3 = \left[\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right]$$

Then $\boldsymbol{\lambda}_i \cdot \boldsymbol{\alpha}_i = 1$, and $\boldsymbol{\lambda}_i \cdot \boldsymbol{\alpha}_j = 0$ for $i \neq j$. Hence the vector $\boldsymbol{\lambda}_i$ lies on the edge of the kaleidoscope formed by mirrors j, k (for i, j, k all different). Let α, β and γ be the vertex

Type	A_3	B_3	H_3
$\alpha + \beta + \gamma$	180°	135°	90°
L	$\frac{z}{\sqrt{2}} - \frac{r}{\sqrt{3}}$	$z - \frac{r}{\sqrt{2}}$	$z\varphi - \frac{r\varphi}{\sqrt{\varphi+2}}$
y	$r\sqrt{\frac{2}{3}}$	$\frac{r}{2}$	$\frac{r}{\varphi\sqrt{\varphi+2}}$

TABLE 3. Mirror Dimensions for 3-Dimensional Kaleidoscopes

angles of the kaleidoscope, with α the angle between $\boldsymbol{\lambda}_2$ and $\boldsymbol{\lambda}_3$, and so forth. As in case A_3, we calculate that

$$\tan\alpha = \frac{1}{\sqrt{2}}, \quad \tan\beta = \sqrt{2}, \quad \tan\gamma = 1$$

and hence $\alpha \doteq 35.26°$, $\beta \doteq 54.74°$ and $\gamma = 45°$. The addition formula for the tangent shows that $\alpha+\beta+\gamma = 135°$. The pattern for the kaleidoscope is shown in Figure 12. The dihedral angle at edge a is $60°$, the dihedral angle at edge b is $45°$ and the dihedral angle at edge c is $90°$.

Type H_3. In this case the alcove is determined by the vertices of an icosahedron, and its coordinates are expressed in terms of the golden mean $\varphi = (1+\sqrt{5})/2$ (see [7, §4.2]). Let α, β and γ be the vertex angles of the alcove. One calculates that

$$\tan\alpha = \frac{1}{\varphi+1}, \quad \tan\beta = \frac{2}{\varphi+1}, \quad \tan\gamma = \frac{1}{\varphi}$$

and hence $\alpha \doteq 20.90°$, $\beta \doteq 37.38°$ and $\gamma \doteq 31.72°$. From the addition formula for the tangent and the relation $\varphi^2 = \varphi + 1$ we find that $\alpha+\beta+\gamma = 90°$, as indicated by the numerical approximations for these angles. The pattern for the kaleidoscope is shown in Figure 13. The dihedral angle at edge a is $60°$, the dihedral angle at edge b is $36°$ and the dihedral angle at edge c is $90°$.

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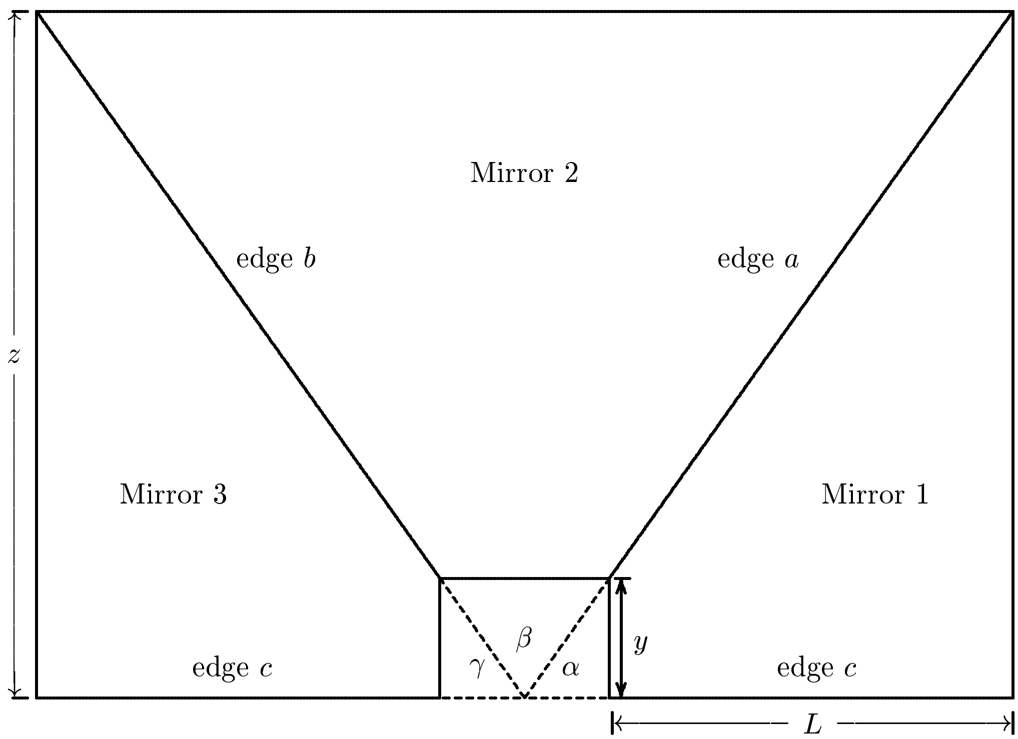

FIGURE 11. Mirrors for Tetrahedral (Type A_3) Kaleidoscope

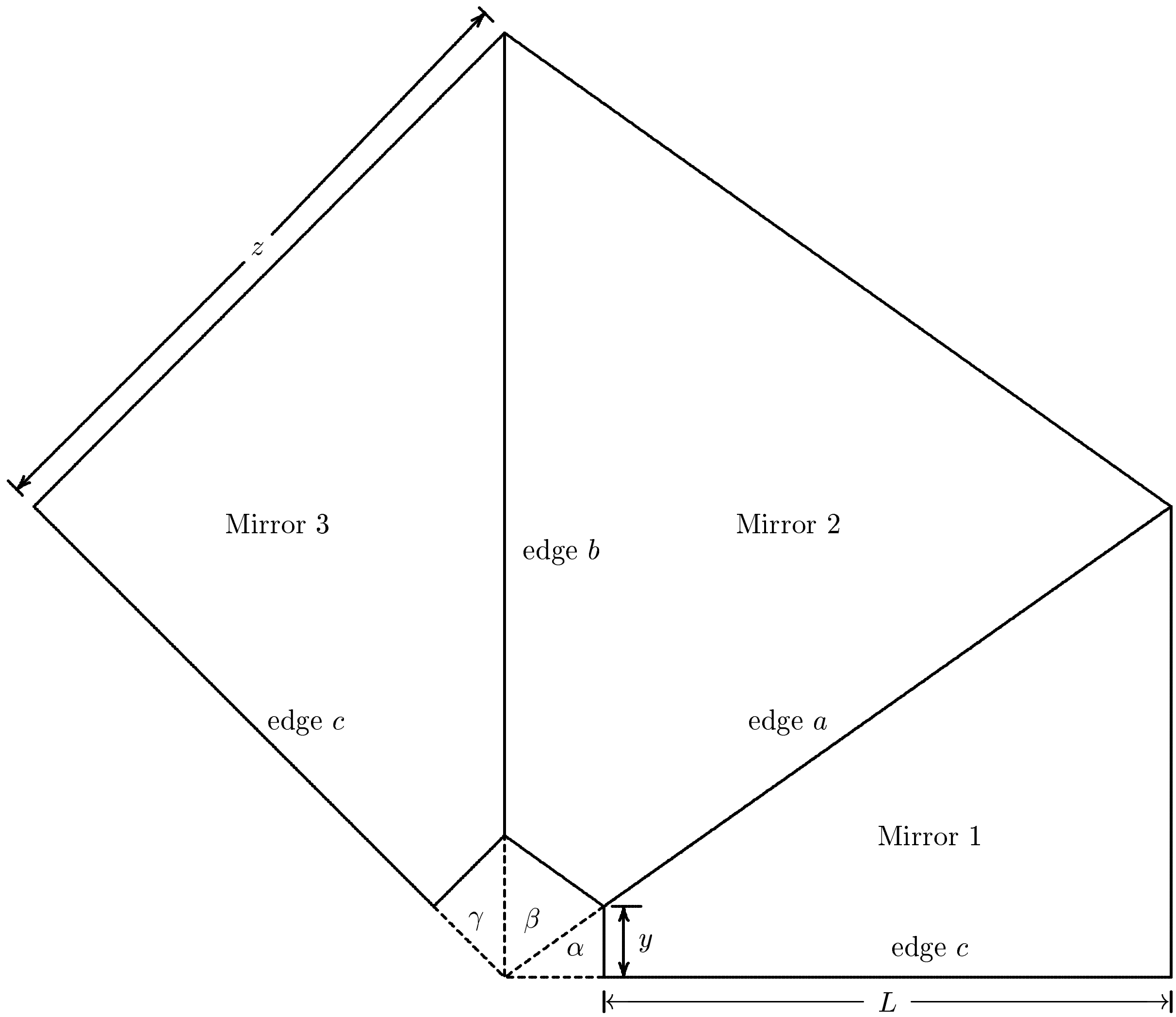

FIGURE 12. Mirrors for Octahedral (Type B_3) Kaleidoscope

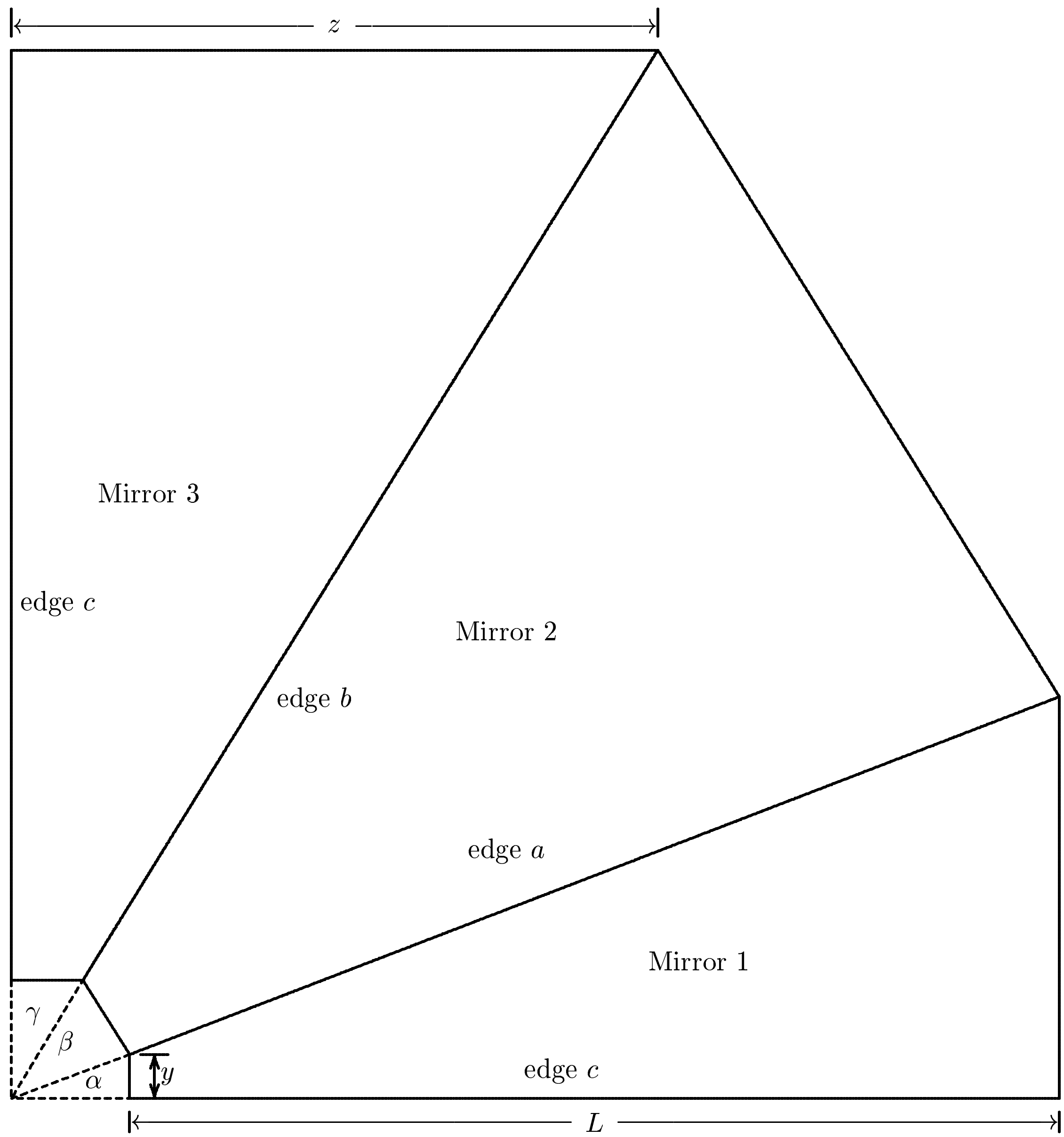

FIGURE 13. Mirrors for Icosahedral (Type H_3) Kaleidoscope