Alice through Looking Glass after Looking Glass: The Mathematics of Mirrors and Kaleidoscopes

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Let us imagine that Lewis Carroll stopped condensing determinants long enough to write a third Alice book called *Alice Through Looking Glass After Looking Glass*. The book opens with Alice in her chamber in front of several looking glasses. She enters one of them and discovers that she is in a new *virtual chamber* that looks almost like her own. On closer examination she discovers that she is now left-handed and her books are all written backwards. There are also *virtual mirrors* in this chamber. Stepping through one of them, she continues her trip through many virtual chambers until, to her great relief, she suddenly finds herself back in her own real chamber. Eager to have new adventures, Alice wonders how many different ways the mirrors could be arranged so that she could have other trips through the looking glasses.

Alice's Kaleidoscope Problem:

• *Kaleidoscope:* Arrangement of n mirror hyperplanes in $\mathbf{R}^n$

• *Kaleidoscope Condition:* Reflections in the mirrors generate a finite group of orthogonal matrices (finite number of images)

Solution (Classification of Finite Reflection Groups):

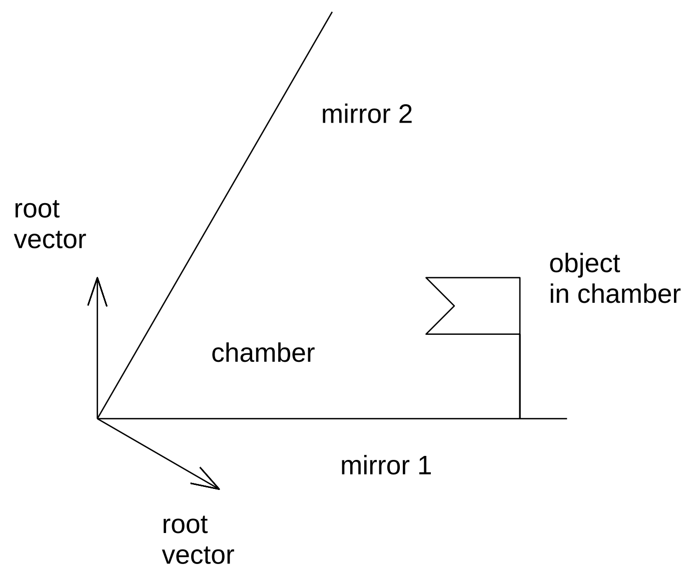
• Infinite number of 2-dimensional kaleidoscopes ($\longleftrightarrow$ Regular Polygons)

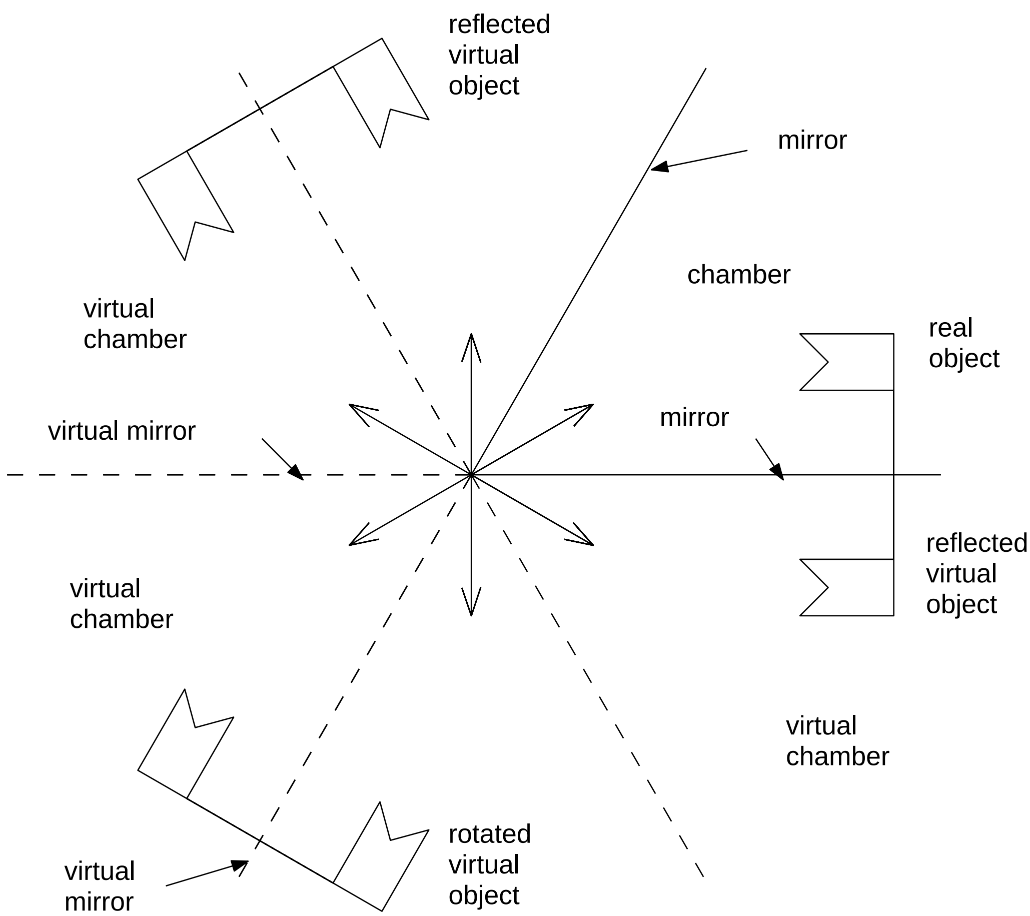
• Three types of 3-dimensional kaleidoscopes ($\longleftrightarrow$ Platonic solids)

• Finite number of n-dimensional kaleidoscopes when $n > 3$ ($\longleftrightarrow$ root systems in $\mathbf{R}^n$)

Dihedral Kaleidoscopes (Brewster, 1819):

Two mirrors, chamber, and object





Reflections in the two mirrors generate *virtual* mirrors, chambers, and objects

Mirror in Euclidean space $\mathbf{R}^n$:

$(n-1)$-dimensional subspace $M \subset \mathbf{R}^n$

unit vector $\alpha \perp M$ (*root vector* for Mirror)

Reflection matrix: $R_\alpha = I - 2\alpha\,\alpha'$

$$R_\alpha \mathbf{v} = \begin{cases} -\mathbf{v} & \text{if } \mathbf{v} \perp M \\ \mathbf{v} & \text{if } \mathbf{v} \in M \end{cases}$$

$R_\alpha^2 = I$, $\quad \det R_\alpha = -1$ (reverses orientation)

$R_\alpha' = R_\alpha$ (symmetric and orthogonal matrix)

Examples:

$$\alpha = [1\ \ 0]', \quad \beta = [\cos\theta\ \ \sin\theta]'$$

$$R_\alpha = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad R_\beta = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

$$R_\alpha R_\beta = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix} \text{(Rotation by } 2\theta)$$

Theorem 1 *Take two mirrors in* $\mathbf{R}^2$ *that pass through* 0 *and have root vectors* α *and* β. *Let* $\theta \leq \pi/2$ *be the dihedral angle between the mirrors and* C *the closed acute cone between the mirrors (the* fundamental chamber*).*

(i) *The group* G *of matrices*

$$I_2, R_\alpha, R_\beta, R_\alpha R_\beta, R_\beta R_\alpha, R_\alpha R_\beta R_\alpha, \cdots$$

is finite $\iff \theta = \pi/m$ *for some integer* $m \geq 2$. *In this case* G *is the* dihedral group $I_2(m)$.

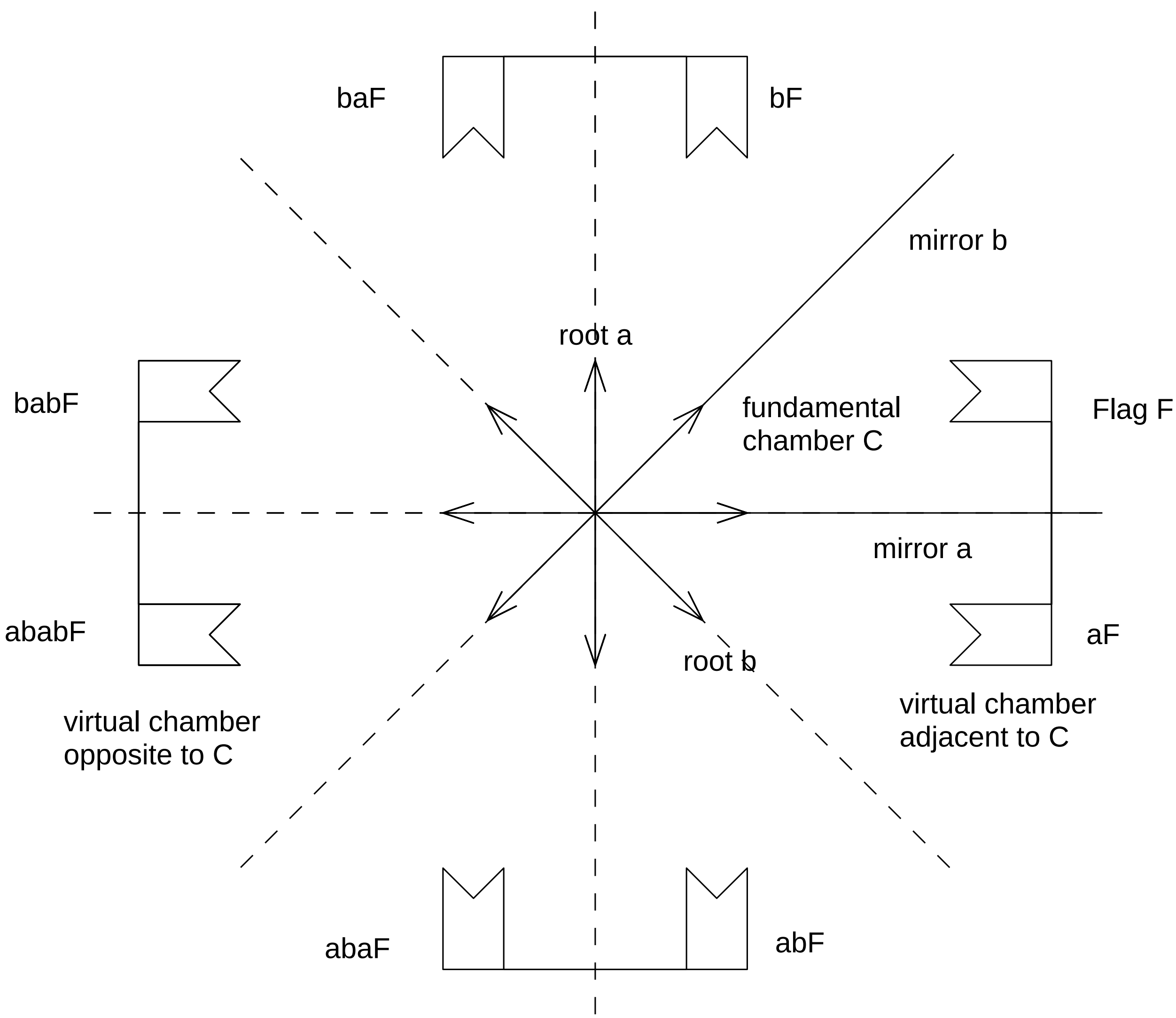
(ii) *The images* $g \cdot C$ *for* $g \in G$ *(the* virtual chambers*) have disjoint interiors and fill up* $\mathbf{R}^2$. *Furthermore, if* $gC = C$ *then* $g = I$. *Hence the chambers (fundamental and virtual) correspond uniquely to the elements of* G *and* $|G| = 2m$.

(iii) *As an abstract group* G *is generated by* $a = R_\alpha$ *and* $b = R_\beta$ *with all relations generated by* $a^2 = b^2 = (ab)^m = 1$.

Example: $\theta = \pi/4,\ a = R_\alpha,\ b = R_\beta$

Relations $a^2 = b^2 = (ab)^4 = 1, \quad |I_2(4)| = 8$

Chambers $\longleftrightarrow$ $1, a, b, ab, ba, aba, bab, abab$



Comments on proof of Theorem 1

(i) because $R_\alpha R_\beta$ is rotation by $2\pi/m$.

(ii) by geometry

(iii) At most $2m$ distinct words

$$1, a, b, ab, ba, aba, bab, abab, ...$$

can be formed. All correspond to distinct orthogonal matrices.

$m = 4$: $baba = abab$ because $(abab)^2 = 1$.

Reflect $abaF$ through mirror b to get $ababF$.

even/odd number of reflections *preserves/reverses* orientation

Longest word $abab \longleftrightarrow -C$ opposite fundamental chamber.

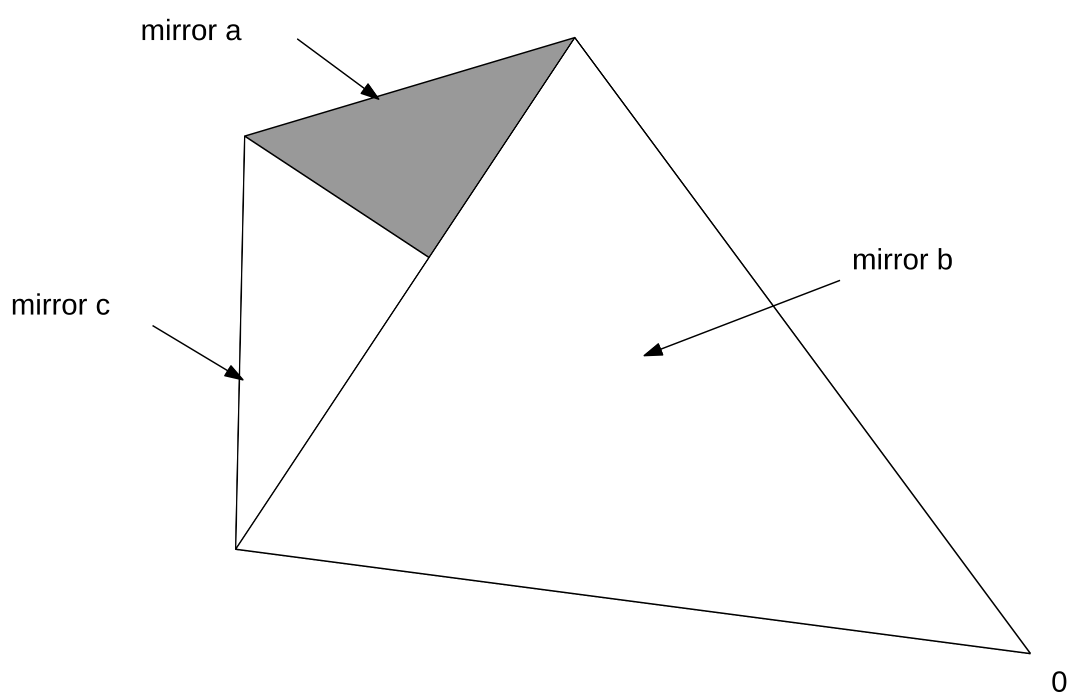
Kaleidoscopes in Three Dimensions

Three mirrors in $\mathbf{R}^3$ through 0

$C =$ fundamental chamber (acute dihedral angles π/p, π/q, π/r at walls)

Can assume $2 \leq p \leq q \leq r$ and $q > 2$

$G =$ group generated by reflections in walls C.



If $p = q = 2$, then one mirror is perpendicular to the other two mirrors

Same as two mirrors in two dimensions and one mirror in the remaining dimension.

G is *product* of the dihedral group for the two mirrors and ± 1 for one mirror.

So can assume $q > 2$.

Theorem 2 *Let G be the group generated by reflections in the walls of C.*

(i) *Suppose that the orbit $G \cdot x$ is finite for some point x inside C. Then p, q, r are positive integers that satisfy*

$$2 \leq p \leq q \leq r, \quad \frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1. \qquad (1)$$

(ii) *The integer solutions to* (1) *with $q > 2$ are* (2, 3, 3), (2, 3, 4), *and* (2, 3, 5).

(iii) *Let* (p, q, r) *be one of the triples in* (ii) *and let C be a chamber (3-sided cone) in $\mathbf{R}^3$ with the corresponding dihedral angles. The images $g \cdot C$ for $g \in G$ (the* virtual chambers*) have disjoint interiors and fill up $\mathbf{R}^3$. Furthermore, if $gC = C$ then $g = I$. Hence the chambers (fundamental and virtual) correspond uniquely to the elements of G and G is finite.*

Proof of (i): Take the group $H \subset G$ generated by reflections in a pair of walls of C.

$|H \cdot \mathbf{x}| < \infty \implies$ dihedral angle $= \pi/m$

Hence p, q, and r are integers.

Now take a triangular cross-section T of C.

Each Angle of $T \leq$ Dihedral Angle of C

(at least one $<$)

Sum(Angles of T) $<$ Sum(Dihedral Angles)

Solutions to (1): Must have $p = 2$, $q = 3$, and $r \leq 5$ because

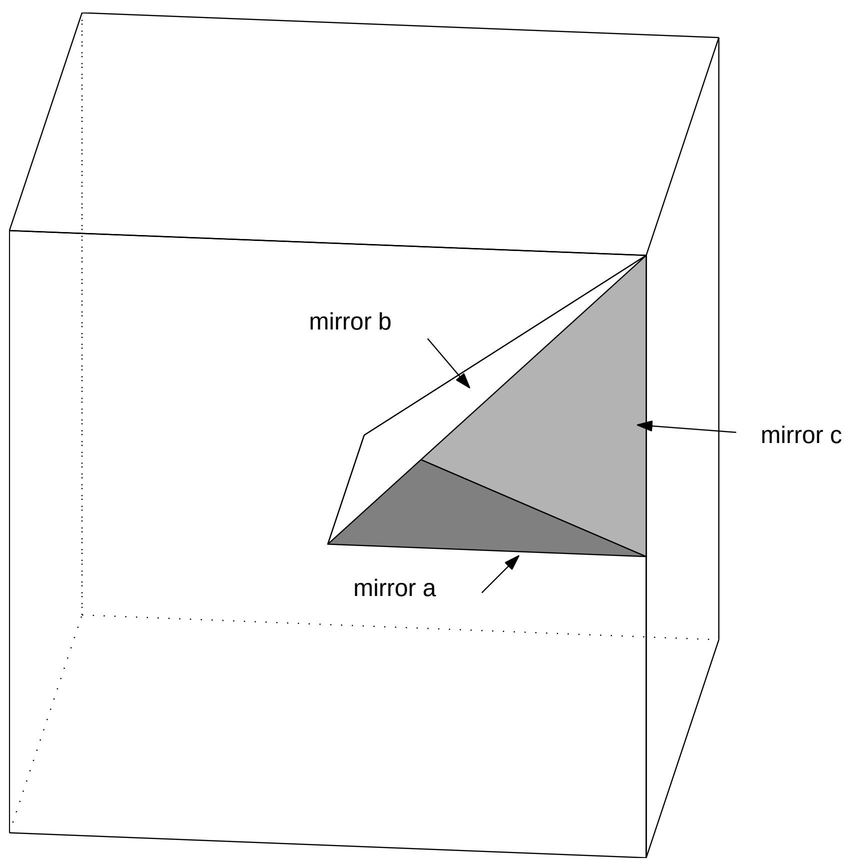
$$\frac{1}{3} + \frac{1}{3} + \frac{1}{3} = \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = \frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1$$

For each admissible triple (p, q, r) construct

• Regular polyhedron (tetrahedron, cube, icosahedron) centered at 0.

• Triangulation of the faces of the polyhedron by congruent triangles

• Cone C from 0 through one of the triangles with dihedral angles π/p, π/q, π/r.



Verify that each reflection in a wall of C permutes the vertices of this polyhedron.

Hence G permutes the vertices of the polyhedron, so G is finite since there are three linearly independent vertices

Also, G permutes the triangles, so G permutes the chambers.

Correspondence between $g \in G$ and (virtual) chamber $g \cdot C$:

Write $g = R_1 R_2 \cdots R_k$ $\quad$ (R_i reflection in some wall of the fundamental chamber.

reduced word $\iff k$ minimum.

Set $\mathsf{length}(g) = k$.

Geometric Meaning: $\mathsf{length}(g)$ is the minimal number of mirrors (real and virtual) that must be crossed in order to go from the fundamental chamber C to the virtual chamber $g \cdot C$.

Example: $\mathsf{length}(g) = 1 \iff g$ is reflection in wall of $C \iff g \cdot C$ shares a wall with C.

$\{g \cdot C = C\} \iff \{\mathsf{length}(g) = 0\} \iff \{g = 1\}$

Hence g is uniquely determined by the chamber $g \cdot C$.

Finite Reflection Groups in Three Dimensions:

ANGLES	POLYHEDRON	GROUP	# MIRRORS	# CHAMBERS
$\pi/2$ - $\pi/3$ - $\pi/3$	TETRAHEDRON	S_4	6	24
$\pi/2$ - $\pi/3$ - $\pi/4$	CUBE	$S_3 \times \{\pm 1, \pm 1, \pm 1\}$	9	48
$\pi/2$ - $\pi/3$ - $\pi/5$	ICOSAHEDRON	$A_5 \times \{\pm 1\}$	15	120

Mirrors:

{ Virtual Mirrors } $\cup$ { Real Mirror Walls of C}

Chambers = $|G|$

In 2 dimensions:

$$\frac{\#(\text{chambers})}{\#(\text{mirrors})} = 2$$

In 3 dimensions:

$$\frac{\#(\text{chambers})}{\#(\text{mirrors})} = \begin{cases} 4 & \text{tetrahedron} \\ 16/3 & \text{cube} \\ 8 & \text{icosahedron} \end{cases}$$

Kaleidoscopes in $\mathbf{R}^n$

Mirror $\longleftrightarrow$ pair of unit root vectors $\pm\alpha$

$\Phi =$ root vectors for finite set of Mirrors

Call Φ a *Root System* if it satisfies

Kaleidoscope Condition: For every $\alpha, \beta \in \Phi$, the reflected vector $R_\alpha \beta \in \Phi$.

Examples: $\Phi =$ all roots (for real and virtual mirrors) of a Kaleidoscope in $\mathbf{R}^2$ or $\mathbf{R}^3$.

$\#(\text{roots}) = 2 \cdot \#(\text{real and virtual mirrors})$

May assume: Φ spans $\mathbf{R}^n$ ($\mathbf{v} \perp \Phi$ is fixed by all the reflections).

Theorem 3 *If Φ is a root system, then it contains a subset $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ of* **simple roots** *such that*

(i) Δ *is a basis for $\mathbf{R}^n$.*

(ii) $\alpha_i \cdot \alpha_j \leq 0$ *for $i \neq j$*

(iii) *If $\beta \in \Phi$ then the expression of β in terms of the basis Δ has coefficients that are all of the same sign.*

Fundamental Chamber (*simplicial cone*):

$C = \{\mathbf{v} \in \mathbf{R}^n \,|\, \alpha_i \cdot \mathbf{v} \geq 0 \text{ for } i = 1, \ldots, n\}$

$M_i =$ mirror for α_i, $\quad R_i =$ reflection in M_i

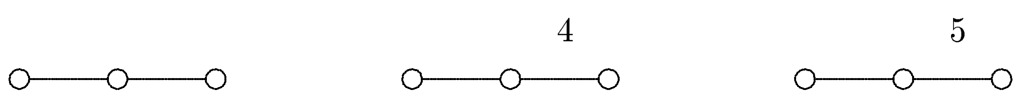
$\theta_{ij} =$ dihedral angle$(M_i, M_j) = \dfrac{\pi}{p_{ij}} \quad (p_{ij} \geq 2)$

Kaleidoscope Condition (pairs of mirrors) $\Longrightarrow$
$p_{ij} \in \{2, 3, 4, \ldots\}$ and $(R_i R_j)^{p_{ij}} = 1$

Coxeter Graph of a Root System

- vertices $\longleftrightarrow$ simple roots $\alpha_1, \ldots, \alpha_n$
- edge between vertex i and vertex j if $p_{ij} > 2$
- label the edge with p_{ij} if $p_{ij} > 3$

Coxeter graphs for Kaleidoscopes in $\mathbf{R}^3$:



Define the *Coxeter Matrix* of the root system:

$A = [\alpha_i \cdot \alpha_j]$ (matrix of inner products). Since

$$A_{ii} = 1, \quad A_{ij} = -\cos(\pi/p_{ij}) \quad (i \neq j),$$

A is determined by the Coxeter graph, without reference to the root system.

Theorem 4 *The Coxeter matrix of a root system is positive definite.*

Proof: It is the matrix of inner products of a basis for $\mathbf{R}^n$

Example. Show that a kaleidoscope with dihedral angles $\pi/2$, $\pi/3$, and π/p can only exist if $p \leq 5$. The Coxeter matrix of the mirror configuration is

$$A = \begin{bmatrix} 1 & -1/2 & 0 \\ -1/2 & 1 & -c \\ 0 & -c & 1 \end{bmatrix}, \quad c = \cos(\pi/p).$$

$\{A \text{ positive def.}\} \iff \{\text{principal minors} > 0\}$

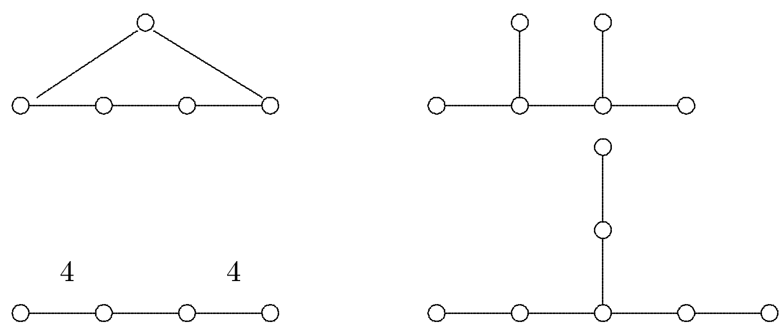
$$\det \begin{bmatrix} 1 & -1/2 \\ -1/2 & 1 \end{bmatrix} = \frac{3}{4}$$

$\det(A) = \frac{3}{4} - c^2 \quad (\text{positive} \iff p < 6)$

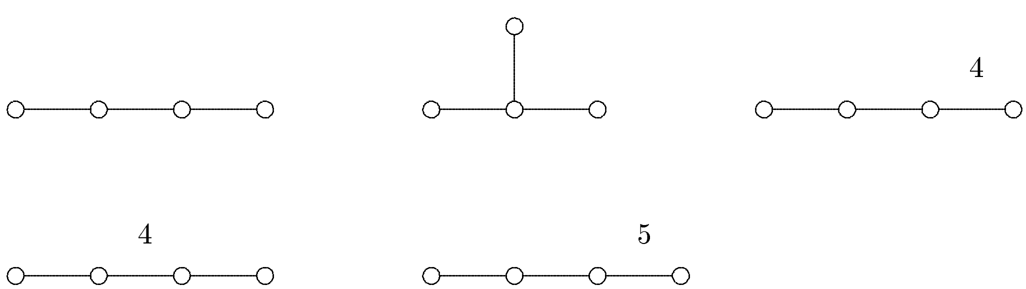
Classification Problem: Determine all Coxeter graphs whose matrix is positive definite.

Monotonicity Property: Every subgraph (with smaller labels) has a positive definite matrix.

Strategy: Construct Coxeter graphs that are *semi-definite* by adding mirror across face of C. These graphs can't occur as subgraphs:



Solution to Classification Problem ($n = 4$):



Existence Problem: Construct a root system for each positive-definite graph.

G = Reflection group for root system is finite

All relations in G generated by

$$R_i^2 = (R_i R_j)^{p_{ij}} = 1$$

(Proved by H.S.M. Coxeter - 1934)

3 Families of Classical Groups: all (signed, evenly signed) permutations for every $N \geq 4$

5 Exceptional Groups: F_4, H_4, E_6, E_7, E_8

Finite Reflection Groups in $\mathbf{R}^N$ for $N \geq 4$ with connected Coxeter Graph:

N	# GROUPS	# MIRRORS	# CHAMBERS
4	5	10, 12, 16 24 60	$5 \cdot 4!, \quad 2^3 \cdot 4!, \quad 2^4 \cdot 4!$ $2 \cdot 6 \cdot 8 \cdot 12$ $2 \cdot 12 \cdot 20 \cdot 30$
5	3	15, 20, 25	$6 \cdot 5!, \quad 2^4 \cdot 5!, \quad 2^5 \cdot 5!$
6	4	21, 30, 36 36	$7 \cdot 6!, \quad 2^5 \cdot 6!, \quad 2^6 \cdot 6!$ $2 \cdot 5 \cdot 6 \cdot 8 \cdot 9 \cdot 12$
7	4	28, 42, 49 63	$8 \cdot 7!, \quad 2^7 \cdot 7!, \quad 2^6 \cdot 7!$ $2 \cdot 6 \cdot 8 \cdot 10 \cdot 12 \cdot 14 \cdot 18$
8	4	36, 56, 64 120	$9 \cdot 8!, \quad 2^7 \cdot 8!, \quad 2^8 \cdot 8!$ $2 \cdot 8 \cdot 12 \cdot 14 \cdot 18 \cdot 20 \cdot 24 \cdot 30$
$N > 8$	3	$N(N+1)/2$ $N(N-1)$ N^2	$(N+1) \cdot N!$ $2^{N-1} \cdot N!$ $2^N \cdot N!$

The Fourth Dimension is the most interesting!

Number of mirrors: N walls of C + virtual mirrors (reflections of these walls)

Chamber $g \cdot C \longleftrightarrow g$

Number of chambers = order of G

(# Chambers)/(# Mirrors) for A_8, D_8, B_8:

10,080, 46,080, and 80,640.

Ratio for E_8 almost six million.

Order(G) = product of degrees of the basic G-invariant polynomials in N variables (elementary symmetric functions when $G = S_{N+1}$)

Coxeter element: $R_1 R_2 \cdots R_N$ (product of simple reflections)

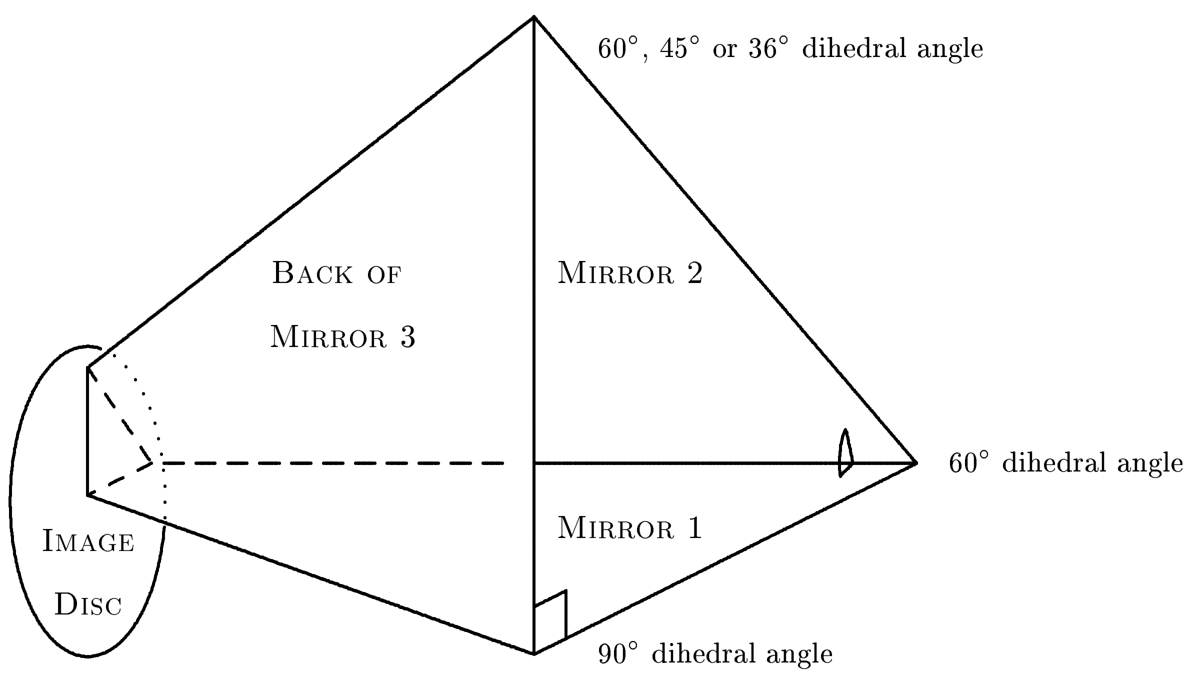
Longest cycle when $G = S_{N+1}$.

Order is the *Coxeter number*.

Eigenvalues give degrees of basic invariants

Construction of 3-D Kaleidoscopes

"These groups can be made vividly comprehensible by using actual mirrors for the generating reflections. It is found that a candle makes an excellent object to reflect. By hinging two vertical mirrors at an angle π/k we easily see $2k$ candle flames, in accordance with the group $[k]$. To illustrate the groups $[k_1, k_2]$, we hold a third mirror in the appropriate positions." (H.S.M. Coxeter)



Notation: $[k]$ - dihedral group $I_2(k)$ of order $2k$

$[k_1, k_2]$ - group for regular polyhedron with faces of k_1 edges, vertices of k_2 edges.

The actual construction of 3-dimensional kaleidoscopes ('holding a third mirror in the appropriate position') is not easy, however, compared to making a traditional cylindrical kaleidoscope.

Coxeter: 'a very accurate icosahedral kaleidoscope was made by in Minneapolis (by Litton Industries) for a film project that was never completed because the expected financial support was withdrawn.' (sequel to Coxeter's 1966 film *Dihedral Kaleidoscopes*?)

Recent U.S. Patents for 3-dim. kaleidoscopes: J. Sandoval and J. Bracho (1995), F. Altman (1997) None manufactured?

Kaleidoscope for Platonic Solid P:

- Truncate the cone C near vertex

- Reflected Image of truncation triangle is

$$S = \text{ convex hull}(P \cup P')$$

($P' =$ dual solid)

- Circular pattern disc over triangle

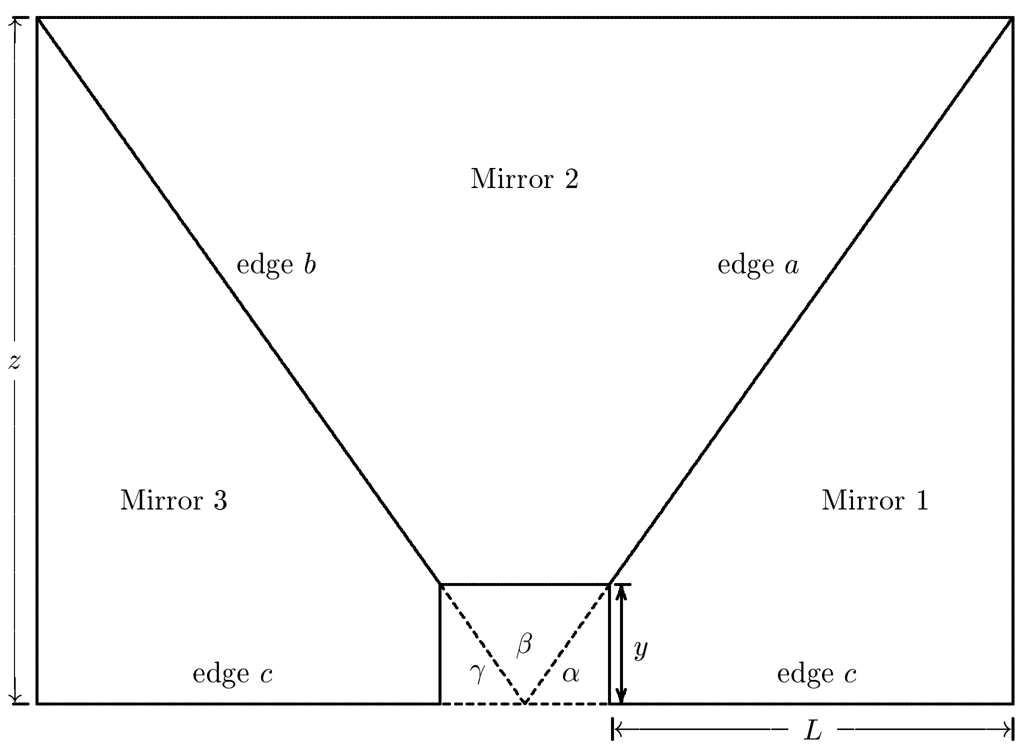
- Pattern on the disc appears on each face of S by the multiple reflections in the mirrors, and moves when disc rotates.

- Shows a continuous transition between P and P'.

Mirror Dimensions for 3-Dimensional Kaleidoscopes (r, z – scaling parameters)

Type	A_3	B_3	H_3
$\alpha+\beta+\gamma$	$180°$	$135°$	$90°$
L	$\frac{z}{\sqrt{2}} - \frac{r}{\sqrt{3}}$	$z - \frac{r}{\sqrt{2}}$	$z\phi - \frac{r\phi}{\sqrt{\phi+2}}$
y	$r\sqrt{\frac{2}{3}}$	$\frac{r}{2}$	$\frac{r}{\phi\sqrt{\phi+2}}$

Mirrors for Tetrahedral (Type A_3) Kaleidoscope



$r =$ radius of the image S in kaleidoscope (free parameter)

$z =$ length of longer leg of front right triangle

α, β, γ - vertex angles of the three cones

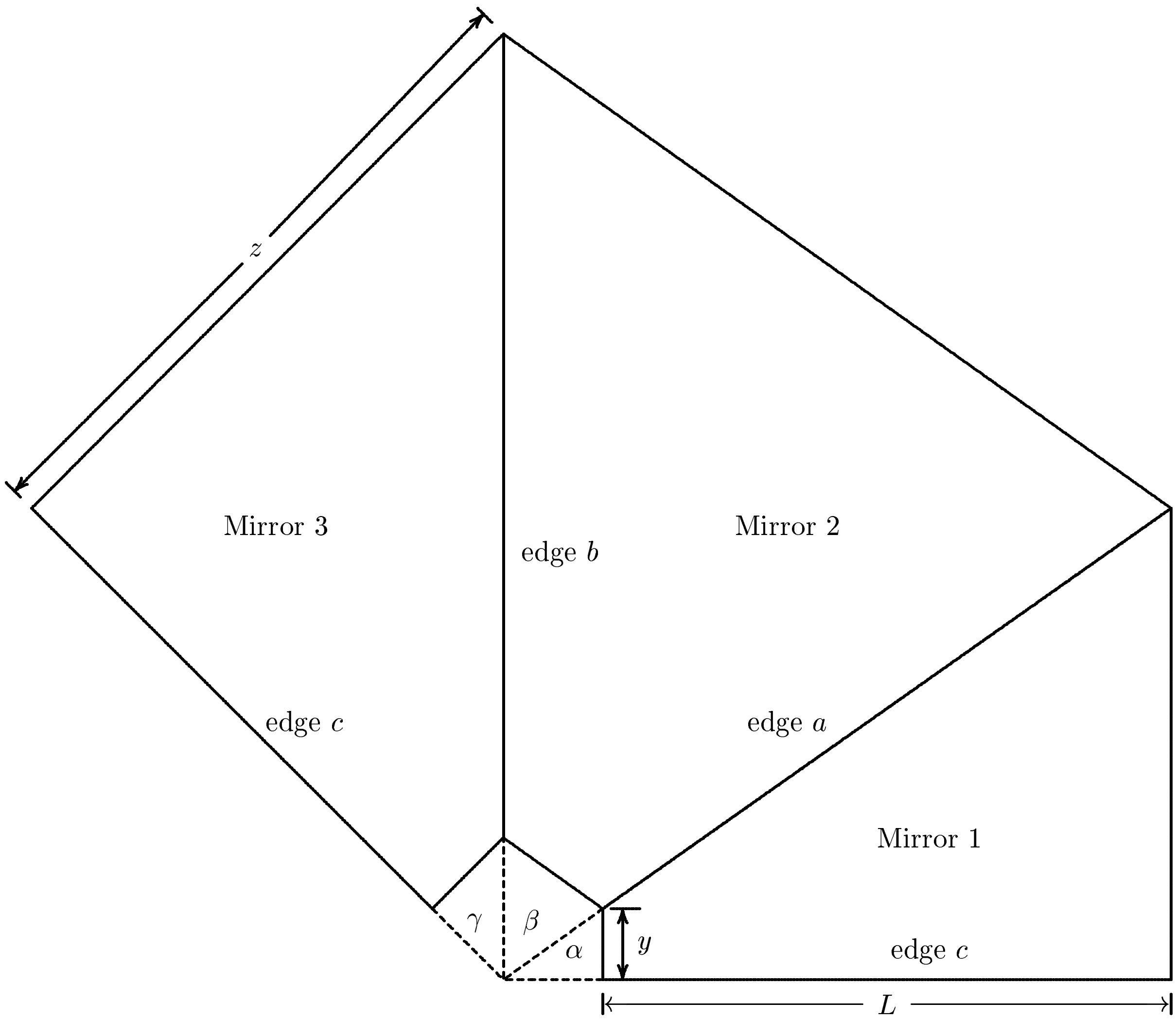
$\phi = (1 + \sqrt{5})/2$ - the golden mean

$y =$ the length of short leg of back right triangle

$z = \left(L + \frac{y}{\tan \alpha}\right) \tan \gamma$

Choose $L = 8y$. Then $z = \left(4 + \frac{1}{\sqrt{2}}\right) r$

Mirrors for Octahedral (Type B_3) Kaleidoscope:



Mirrors for Icosahedral (Type H_3) Kaleidoscope: