Integral Transforms

Roe Goodman

Introduction to Math at Rutgers

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$L^2(G)$ – square integrable complex valued functions on G

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Solution: Use characters of G and Fourier transform

Example 1 $G = \mathbb{Z}/n\mathbb{Z}$ (additive group of integers mod n)
$L^2(G) = \{\phi : \mathbb{Z} \to \mathbb{C} \;:\; \phi(k+n) = \phi(k) \text{ for all } k \in \mathbb{Z}\}$
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- Eigenfunctions for translations $\quad T_k e_p = w^{-kp} e_p$
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- Diagonalization $\quad \psi = T_k \phi \quad \Rightarrow \quad \widehat{\psi}(p) = w^{-kp}\widehat{\phi}(p)$
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Theorem

Let $G = \mathbb{Z}/n\mathbb{Z}$. Let C be a translation invariant operator on $L^2(G)$. There is a function F on $\widehat{G} \cong \mathbb{Z}/n\mathbb{Z}$ such that

$(\star) \qquad \widehat{C\phi}(p) = F(p)\widehat{\phi}(p)$ *for all* $\phi \in L^2(G)$ *and* $p \in \mathbb{Z}$.

Conversely, every function F on $\mathbb{Z}/n\mathbb{Z}$ defines a translation invariant operator C on $L^2(G)$ by $(\star)$ ($C =$ convolution by f, where $\widehat{f} = F$).

Diagonalization of Translation Invariant Operators

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Proof.

Let $S = T_1$ (shift operator). Then S has n distinct eigenvalues $\lambda_p = w^{-p}$ for $p = 0, \ldots, n-1$ with eigenvectors e_p. Since C commutes with S, the function Ce_p is an eigenvector for S with eigenvalue w^{-p}. Hence $Ce_p = F(p)e_p$ for some scalar $F(p) \in \mathbb{C}$. The Fourier inversion formula now implies $(\star)$. □

General Version of Fourier Transform

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$\widehat{G}$ – all characters of G:

$\psi : G \to \mathbb{T} = \{z \in \mathbb{C} \ : \ |z| = 1\}$ (continuous)

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Example

$G = \mathbb{Z}/n\mathbb{Z} \quad \widehat{G} = \{e_p \ : \ p \in \mathbb{Z}/n\mathbb{Z}\} \cong G$
Choose basic character e_1. Then $e_p(k) = e_1(pk)$

Example 2 $\quad G = \mathbb{R}/\mathbb{Z}$ (additive group of real numbers modulo 1)

$L^2(\mathbb{R}/\mathbb{Z}) \quad \phi : \mathbb{R} \to \mathbb{C},\ \phi(x+1) = \phi(x)$ (periodic, measurable)

$\int_0^1 |\phi(x)|^2 \, \mathrm{d}x < \infty$ (Lebesgue integral)

Inner product $\quad (\phi, \psi) = \int_0^1 \phi(x)\overline{\psi(x)} \, \mathrm{d}x$

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$CS = SC \Rightarrow Ce_p = F(p)e_p$ with $F(p) \in \mathbb{C}$. Then C bounded $\Rightarrow$ $\|F\|_\infty := \sup_p |F(p)| < \infty$. Hence

$$C\phi = \sum_{p\in\mathbb{Z}} \widehat{\phi}(p) Ce_p = \sum_{p\in\mathbb{Z}} \widehat{\phi}(p) F(p) e_p$$

□

Fourier Analysis of $C^{\infty}(\mathbb{R}/\mathbb{Z})$

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Conversely, every such function F on $\mathbb{Z}$ defines a continuous translation invariant operator C on $C^\infty(\mathbb{R}/\mathbb{Z})$ by $(\star)$.

Example 3 $\quad G = \mathbb{R}$ (additive group of real numbers)

$L^2(\mathbb{R}) \quad \phi : \mathbb{R} \to \mathbb{C}$, (measurable) $\int_{-\infty}^{\infty} |\phi(x)|^2 \, \mathrm{d}x < \infty$

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Fourier transform For $\phi \in L^1(\mathbb{R})$ define
$\widehat{\phi}(\xi) = \int_{-\infty}^{\infty} \phi(x)e_{-\xi}(x) \,\mathrm{d}x$ (integral converges absolutely)

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- $e_\xi(x)e_\tau(x) = e_{\xi+\tau}(x)$, so $\widehat{\mathbb{R}} \cong \mathbb{R}$ under $e_\xi \leftrightarrow \xi$
- $\mathbb{R}$ not compact $\Rightarrow e_\xi \notin L^2(\mathbb{R})$ (plane wave, frequency ξ)

Fourier transform For $\phi \in L^1(\mathbb{R})$ define

$\widehat{\phi}(\xi) = \int_{-\infty}^{\infty} \phi(x) e_{-\xi}(x) \, \mathrm{d}x$ (integral converges absolutely)

- Fourier transform extends to isometry $L^2(\mathbb{R}) \to L^2(\widehat{\mathbb{R}})$
- Plancherel formula $\quad (\phi, \psi) = \int_{-\infty}^{\infty} \widehat{\phi}(\xi)\overline{\widehat{\psi}(\xi)} \, \mathrm{d}\xi$
- Bounded translation invariant operator C on $L^2(\mathbb{R}) \longleftrightarrow$ multiplication by bounded measurable function F on $\widehat{\mathbb{R}}$

$\mathcal{S}(\mathbb{R}) =$ rapidly decreasing C^∞ functions on $\mathbb{R}$:

$$\sup_{x\in\mathbb{R}} \left| x^m \left(\frac{d}{dx}\right)^k \phi(x) \right| < \infty \text{ for all positive integers } m, k$$

Example $\quad \phi(x) = p(x)e^{-\pi x^2}$ with $p(x)$ a polynomial

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Theorem

Let C be a continuous translation invariant operator on $\mathcal{S}(\mathbb{R})$. Then there is a C^∞ function F on $\mathbb{R}$ with all derivatives of polynomial growth at ∞ such that

$(\star)$ $\qquad \widehat{C\phi}(\xi) = F(\xi)\widehat{\phi}(\xi)$ *for all* $\phi \in \mathcal{S}(\mathbb{R})$.

Conversely, every such function F on $\mathbb{R}$ defines a continuous translation invariant operator C on $\mathcal{S}(\mathbb{R})$ by $(\star)$.

$\mathbb{R}^\times = \mathbb{R} \setminus \{0\}$ – locally compact group under multiplication

- Invariant integral $\displaystyle\int_{-\infty}^{\infty} f(x)\,\frac{\mathrm{d}x}{|x|}$
- Characters $e_{\tau,\epsilon}(x) = \mathrm{sgn}(x)^{\epsilon}|x|^{\mathrm{i}\tau}$ with $\tau \in \mathbb{R}$ and $\epsilon = \pm 1$
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  for $f \in L^1(\mathbb{R}, \frac{\mathrm{d}x}{|x|})$
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  $$\int_{-\infty}^{\infty} f(x)\overline{g(x)}\,\frac{\mathrm{d}x}{|x|} = \sum_{\epsilon=\pm 1}\int_{-\infty}^{\infty} \widehat{f}(\tau,\epsilon)\overline{\widehat{g}(\tau,\epsilon)}\,\mathrm{d}\tau$$
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Log Trick: Use group homomorphism $x \mapsto (\log|x|, \mathrm{sgn}(x))$ to turn Fourier-Mellin transform into Fourier transform on $\mathbb{R} \times (\mathbb{Z}/2\mathbb{Z})$.

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$\widehat{\{p^k\}_{k \in \mathbb{Z}}} \cong \mathbb{R}/\mathbb{Z}$ (compact) $\quad \widehat{A} \cong \lim\limits_{\overleftarrow{k \to \infty}} \mathbb{Z}/(p^k\mathbb{Z})$ (countable)

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Radon Transform Method:

Use integral transform that turns Δ into $(\partial/\partial p)^2$ on even functions of $p \in \mathbb{R}$ with parameter $\omega \in \mathbb{S}^{n-1}$ (no singularity). Then diagonalize by one-dimensional Fourier transform.

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- Radon transform of $\Delta f(x)$ is $(\partial/\partial p)^2 F(\omega, p)$

For $x \in \mathbb{R}^n$

$$K(x) = \text{all hyperplanes } \xi \text{ containing } x$$
$$= \{(\omega, p) \ : \ x \cdot \omega = p\} \cong \mathbb{S}^{n-1}/\pm 1$$

Let $\mathrm{d}\mu =$ invariant measure on $K(x)$ (total mass 1)

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$$
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For $F \in \mathcal{S}(\mathbb{P}^n)$ define dual Radon transform

$$
\widetilde{F}(x) = \int_{\xi \in K(x)} F(\xi)\, \mathrm{d}\mu(\xi) = \int_{\omega \in \mathbb{S}^{n-1}} F(\omega, \omega \cdot x)\, \mathrm{d}\omega
$$

For $x \in \mathbb{R}^n$

$$K(x) = \text{all hyperplanes } \xi \text{ containing } x$$
$$= \{(\omega, p) \ : \ x \cdot \omega = p\} \cong \mathbb{S}^{n-1}/\pm 1$$

Let $\mathrm{d}\mu =$ invariant measure on $K(x)$ (total mass 1)

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Radon Inversion Formula

If $f \in \mathcal{S}(\mathbb{R}^n)$ and $F =$ Radon transform of f, then

$$f(x) = c\,(-\Delta)^{(n-1)/2}\,\widetilde{F}(x) \qquad (c = \text{normalizing constant})$$

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odd dimensions: Inversion formula is local - differential operator applied to $\widetilde{F}(x)$

even dimensions: Inversion formula is non-local - square root of differential operator (Hilbert transform) applied to $\widetilde{F}(x)$

Further Reading

The Wikipedia articles on Fourier Analysis, p-adic Numbers, and Radon Transform are good starting points. Here are some books:

- Fourier analysis on locally compact abelian groups:
  W. Rudin, **Fourier Analysis on Groups**, Wiley (1962)
  G. Folland, **A Course in Abstract Harmonic Analysis**, CRC Press (1995)
- Finite Fourier transform:
  A. Terras, **Fourier Analysis on Finite Groups and applications**, Cambridge (1999)
- Fourier analysis on $\mathbb{R}/\mathbb{Z}$ and $\mathbb{R}$:
  G. Folland, **Real Analysis: Modern Techniques and Their Applications**, Wiley (1999)
- Radon Transform:
  S. Helgason, **Groups and Geometric Analysis**, Academic Press (1984)