Multilinear Algebra and Tensor Symmetries

Roe Goodman

Introduction to Math at Rutgers

August 28, 2011

$\mathbb{F}$ = field: $\mathbb{R}$, $\mathbb{C}$, ...
V = finite-dimensional vector space over $\mathbb{F}$ ($\dim V = n$)
$V \longleftrightarrow \mathbb{F}^n = n \times 1$ column vectors when basis $\{\mathbf{v}_i\}$ fixed
$\mathrm{GL}(V)$ = group of invertible linear transformations $g : V \to V$
$\longleftrightarrow \mathrm{GL}(n, \mathbb{F})$ ($n \times n$ matrices with $\det \neq 0$)
V^* = dual space of linear functions $\mathbf{v}^* : V \to \mathbb{F}$ covectors

Duality pairing $V^* \times V \to \mathbb{F}$ (bilinear): $\langle \mathbf{v}^*, \mathbf{u} \rangle \stackrel{def}{=} \mathbf{v}^*(\mathbf{u})$
$V^* \longleftrightarrow 1 \times n$ row vectors using dual basis: $\langle \mathbf{v}_j^*, \mathbf{v}_i \rangle = \delta_{ij}$
When basis/dual basis fixed, then $\langle \mathbf{v}^*, \mathbf{u} \rangle = \mathbf{v}^*\mathbf{u}$.
(row vector $\times$ column vector = scalar)
If $g \in \mathrm{GL}(V)$, then transpose ${}^t g \in \mathrm{GL}(V^*)$: $\langle {}^t g\mathbf{v}^*, \mathbf{u} \rangle \stackrel{def}{=} \langle \mathbf{v}^*, g\mathbf{u} \rangle$.
Calculate as $\mathbf{v}^* g$ (matrix product) when $\mathbf{v}^*$ = row vector.
Same as: Use (transposed matrix) $\times$ (column vector)

Change of Basis vs. Moving a Vector

- Express $\mathbf{u}$ in terms of basis $\{\mathbf{v}_i\}$ or basis $\{g\mathbf{v}_i\}$ ($g \in \mathrm{GL}(V)$).
- Change vector $\mathbf{u}$ to $g\mathbf{u}$ (orbit of $\mathbf{u}$ under action of $\mathrm{GL}(V)$).

Expansion Formulas: $\mathbf{u} = \sum_i \langle \mathbf{v}_i^*, \mathbf{u} \rangle \mathbf{v}_i$, $\mathbf{v}^* = \sum_i \langle \mathbf{v}^*, \mathbf{v}_i \rangle \mathbf{v}_i^*$
for $\mathbf{u} \in V$ and $\mathbf{v}^* \in V^*$

Components of $\mathbf{u}$ relative to basis $\{\mathbf{v}_i\}$ are $x_i \overset{def}{=} \langle \mathbf{v}_i^*, \mathbf{u} \rangle$.

Components of $\mathbf{v}^*$ relative to basis $\{\mathbf{v}_i^*\}$ are $y^i \overset{def}{=} \langle \mathbf{v}^*, \mathbf{v}_i \rangle$.

Key Property: $\langle {}^t g^{-1} \mathbf{v}^*,\ g\mathbf{u} \rangle = \langle \mathbf{v}^*,\ g^{-1} g\mathbf{u} \rangle = \langle \mathbf{v}^*, \mathbf{u} \rangle$
Hence $f(\mathbf{v}^*, \mathbf{u}) \overset{def}{=} \langle \mathbf{v}^*, \mathbf{u} \rangle$ is a $\mathrm{GL}(V)$-invariant function on $V^* \times V$.

Consequences:

- $\{{}^t g^{-1} \mathbf{v}_i^*\}$ is the dual basis to $\{g\mathbf{v}_i\}$
- Components of $g\mathbf{u}$ relative to basis $\{g\mathbf{v}_i\}$ are the same $\{x_i\}$.
- Components of ${}^t g^{-1} \mathbf{v}^*$ relative to basis $\{{}^t g^{-1} \mathbf{v}_i^*\}$ are the same $\{y^i\}$.
- $\langle \mathbf{v}^*, \mathbf{u} \rangle = \sum_i y_i x^i$ (contraction of covector and vector)

Question: How do we multiply vector spaces?

Let U, V be finite-dimensional vector spaces. Fix bases $\{\mathbf{u}_i\}$, $\{\mathbf{v}_j\}$.
Define tensor product $U \otimes V =$ vector space with basis $\{\mathbf{u}_i \otimes \mathbf{v}_j\}$

- $\mathbf{x} = \sum_{ij} x_{ij}\, \mathbf{u}_i \otimes \mathbf{v}_j \in U \otimes V$ has components $\{x_{ij}\}$ relative to basis $\{\mathbf{u}_i \otimes \mathbf{v}_j\}$.
- $\dim U \otimes V = \dim U \dim V$.
- Use dual bases to get bilinear map $\tau : U \times V \to U \otimes V$
  $\tau(\mathbf{u}, \mathbf{v}) = \sum_{i,j} x_i y_j\, \mathbf{u}_i \otimes \mathbf{v}_j \stackrel{def}{=} \mathbf{u} \otimes \mathbf{v}$ Kronecker product
  Here $\{x_i\} =$ components of $\mathbf{u}$, $\{y_j\} =$ components of $\mathbf{v}$

Universal Linearization Property: Let W be any vector space, and $\beta : U \times V \to W$ a bilinear map (linear in each variable)
Set $B\big(\sum_{i,j} x_{ij}\, \mathbf{u}_i \otimes \mathbf{v}_j\big) = \sum_{ij} x_{ij}\, \beta(\mathbf{u}_i, \mathbf{v}_j)$

- $B : U \otimes V \to W$ is linear
- $B(u \otimes v) = \beta(u, v)$

Let $\mathrm{Hom}(U, V)$ = all linear maps $T : U \to V$

Theorem

If X, Y are vector spaces and $S \in \mathrm{Hom}(U, X)$, $T \in \mathrm{Hom}(V, Y)$, then there exists a unique $S \otimes T \in \mathrm{Hom}(U \otimes V, X \otimes Y)$ such that

$$(S \otimes T)(\mathbf{u} \otimes \mathbf{v}) = (S\mathbf{u}) \otimes (T\mathbf{v}) \quad \text{for } \mathbf{u} \in U, \mathbf{v} \in V.$$

This sets up a linear isomorphism

$(\star)$ $\quad \mathrm{Hom}(U, X) \otimes \mathrm{Hom}(V, Y) \cong \mathrm{Hom}(U \otimes V, X \otimes Y)$

(Each side has dimension $= \dim U \dim X \dim V \dim Y$*.)*

Special Cases of $(\star)$

- $X = Y = \mathbb{F} \cong \mathbb{F} \otimes \mathbb{F}$, so $U^* \otimes V^* \cong (U \otimes V)^*$
  Basis $\{\mathbf{u}_i^* \otimes \mathbf{v}_j^*\}$ for $U^* \otimes V^*$ dual to basis $\{\mathbf{u}_i \otimes \mathbf{v}_j\}$ for $U \otimes V$
  Components of $\mathbf{x} \in U \otimes V$ are $x_{ij} = \langle \mathbf{u}_i^* \otimes \mathbf{v}_j^*, \mathbf{x} \rangle$
- $U \otimes V^* \cong \mathrm{Hom}(V, U)$:
  $\mathbf{u} \otimes \mathbf{v}^*$ gives transformation $T_{\mathbf{u},\mathbf{v}^*} : \mathbf{x} \mapsto \langle \mathbf{v}^*, \mathbf{x} \rangle \mathbf{u}$
  $U = \mathbb{F}^m$, $V = \mathbb{F}^n$: $\mathrm{Hom}(V, U) = m \times n$ matrices
  $T_{\mathbf{u},\mathbf{v}^*} = \mathbf{u}\mathbf{v}^*$ (column $\times$ row) rank one matrix

Iterated Tensor Products: Linearizing Multilinear Maps

Associativity of Tensor Product: U, V, W vector spaces
Define bilinear map $\tau : (U \otimes V) \times W \to U \otimes (V \otimes W)$ by

$$\tau(\mathbf{u} \otimes \mathbf{v}, \mathbf{w}) = \mathbf{u} \otimes (\mathbf{v} \otimes \mathbf{w}).$$

Construction of tensor product gives isomorphism

$$T : (U \otimes V) \otimes W \xrightarrow{\cong} U \otimes (V \otimes W)$$

On basis: $T(\mathbf{u}_i \otimes \mathbf{v}_j) \otimes \mathbf{w}_k = \mathbf{u}_i \otimes (\mathbf{v}_j \otimes \mathbf{w}_k)$
Write $(U \otimes V) \otimes W = U \otimes (V \otimes W) = U \otimes V \otimes W$ (note order)
$\mathbf{x} = \sum_{i,j,k} x_{ijk} \mathbf{u}_i \otimes \mathbf{v}_j \otimes \mathbf{w}_k$ components $x_{ijk} = \langle \mathbf{u}_i^* \otimes \mathbf{v}_j^* \otimes \mathbf{w}_k^*, \mathbf{x} \rangle$
Linearization Property: If $f : U \times V \times W \to Z$ is a trilinear map, then there exists unique linear map $F : U \otimes V \otimes W \to Z$ with

$$f(\mathbf{u}, \mathbf{v}, \mathbf{w}) = F(\mathbf{u} \otimes \mathbf{v} \otimes \mathbf{w}).$$

General Case: For vector spaces $V_1, \ldots, V_p, Z$ the tensor product $V_1 \otimes \cdots \otimes V_p$ has basis $\{\mathbf{v}_{i_1} \otimes \cdots \otimes \mathbf{v}_{i_p}\}$ and linearizes p-multilinear maps $f : V_1 \times \cdots \times V_p \to Z$.

Notation: $V^{\otimes p} = V \otimes \cdots \otimes V$ (p factors)

$V^{\otimes(p,q)} = V^{\otimes p} \otimes (V^*)^{\otimes q}$ mixed tensors of type (p, q)

Basis for $V^{\otimes(p,q)}$ from basis/dual basis for V and V^*:

$$\{\mathbf{v}_{i_1} \otimes \cdots \otimes \mathbf{v}_{i_p} \otimes \mathbf{v}^*_{k_1} \otimes \cdots \otimes \mathbf{v}^*_{k_q}\} \quad (i_j, k_j = 1, \ldots, n = \dim V)$$

Classic Tensor Notation: Components of $\mathbf{x} \in V^{\otimes(p,q)}$ are written

$$\mathbf{x}^{k_1 \cdots k_q}_{i_1 \cdots i_p} \stackrel{def}{=} \langle \mathbf{v}^*_{i_1} \otimes \cdots \otimes \mathbf{v}^*_{i_p} \otimes \mathbf{v}_{k_1} \otimes \cdots \otimes \mathbf{v}_{k_q}, \mathbf{x} \rangle$$

Identify $(V^*)^* = V$ as usual; $\{\mathbf{v}_i\}$ is dual basis to $\{\mathbf{v}^*_i\}$.

Call $i_1, \ldots, i_p$ the covariant indices of $\mathbf{x}$ and

$k_1, \ldots, k_q$ the contravariant indices of $\mathbf{x}$.

Representation of $\mathrm{GL}(V)$:

Define a group homomorphism $\rho : \mathrm{GL}(V) \to \mathrm{GL}(V^{\otimes(p,q)})$ by

$$\rho(g)(\mathbf{x}_1 \otimes \cdots \otimes \mathbf{x}_p \otimes \mathbf{y}^*_1 \otimes \cdots \otimes \mathbf{y}^*_q) = g\mathbf{x}_1 \otimes \cdots \otimes g\mathbf{x}_p \otimes {}^t g^{-1}\mathbf{y}^*_1 \otimes \cdots \otimes {}^t g^{-1}\mathbf{y}^*_q$$

(for any $\mathbf{x}_i \in V$ and $\mathbf{y}^*_j \in V^*$).

Need ${}^t g^{-1}$ on $\mathbf{y}^*_i$ to have $\rho(gh) = \rho(g)\rho(h)$ for $g, h \in \mathrm{GL}(V)$.

For each $1 \leq r \leq p$ and $1 \leq s \leq q$, define a linear map

$C^s_r : V^{\otimes(p,q)} \longrightarrow V^{\otimes(p-1,q-1)}$ (r, s) contraction

by taking components of $C^s_r \mathbf{x}$ as

$$\sum_{1\leq j\leq n} \mathbf{x}^{k_1 \cdots k_{s-1}\, j\, k_{s+1} \cdots k_{q-1}}_{i_1 \cdots i_{r-1}\, j\, i_{r+1} \cdots i_{p-1}} \qquad \text{(set } i_r = k_s = j \text{ and sum on } j\text{).}$$

Theorem

Contractions commute with the action of $\mathrm{GL}(V)$ *on mixed tensors.*

Examples: 1) $V^{\otimes(1,1)} = V \otimes V^* \cong \mathrm{End}(V)$, and
$C^1_1 : V^{\otimes(1,1)} \longrightarrow V^{\otimes(0,0)} = \mathbb{F}$ by $C^1_1 \mathbf{x} = \sum_j x^j_j = \mathrm{tr}(\mathbf{x})$ (trace of $\mathbf{x}$)
If $\mathbf{x} = \mathbf{u} \otimes \mathbf{u}^*$ then $C^1_1(\mathbf{x}) = \langle \mathbf{u}^*, \mathbf{u} \rangle$.

2) Take $\mathbb{F} = \mathbb{R}$ and $V =$ tangent space at a point of a Riemannian manifold. Then the curvature tensor $R \in V^{\otimes(2,2)}$.

contraction of R gives Ricci curvature $\mathrm{Ric} \in V^{\otimes(1,1)}$

contraction (trace) of Ric gives scalar curvature in $V^{\otimes(0,0)} = \mathbb{R}$

"Contraction is an operation of almost magical efficiency"

(*Tensor Analysis*, Encyclopedia Britannica, 14th ed.)

$\mathfrak{S}_k$ = symmetric group (permutations of $\{1, \ldots, k\}$
Permutation $s \in \mathfrak{S}_k$ acts on $V^{\otimes k}$ by moving the vectors:

$$\sigma(s)(\mathbf{u}_1 \otimes \cdots \otimes \mathbf{u}_k) = \mathbf{u}_{s^{-1}(1)} \otimes \cdots \otimes \mathbf{u}_{s^{-1}(k)}$$

(vector in position $i \longrightarrow$ position $s(i)$)

- $\sigma : \mathfrak{S}_k \longrightarrow \mathrm{GL}(V^{\otimes k})$ is a representation:
  $\sigma(st) = \sigma(s)\sigma(t)$ and $\sigma_k(1) = I$.
- The transformation $\sigma(s)$ commutes with the transformation $\rho(g)$ for all $s \in \mathfrak{S}_k$ and $g \in \mathrm{GL}(V)$.

Theorem (Schur duality)

(1) Any linear transformation on $V^{\otimes k}$ *that commutes with* $\rho(g)$ *for all* $g \in \mathrm{GL}(V)$ *is a linear combination of* $\{\sigma(s) : s \in \mathfrak{S}_k\}$.
(2) Any linear transformation on $V^{\otimes k}$ *that commutes with* $\sigma(s)$ *for all* $s \in \mathfrak{S}_k$ *is a linear combination of* $\{\rho(g) : g \in \mathrm{GL}(V)\}$.

Symmetrizer operator $\mathbf{Sym} : V^{\otimes k} \to V^{\otimes k}$ (assume $\mathrm{char}(\mathbb{F}) = 0$)

$$\mathbf{Sym}(\mathbf{u}_1 \otimes \cdots \otimes \mathbf{u}_k) = \frac{1}{k!} \sum_{s \in \mathfrak{S}_k} \sigma(s)(\mathbf{u}_1 \otimes \cdots \otimes \mathbf{u}_k)$$

- $\mathbf{Sym}^2 = \mathbf{Sym}$ (projection operator)

  Proof:

  $$\mathbf{Sym}^2 = \frac{1}{(k!)^2} \sum_{s,t \in \mathfrak{S}_k} \sigma(s)\sigma(t) = \frac{1}{(k!)^2} \sum_{s,t \in \mathfrak{S}_k} \sigma(st) = \frac{1}{k!} \sum_{r \in \mathfrak{S}_k} \sigma(r) = \mathbf{Sym}$$

- $\sigma(t)\,\mathbf{Sym} = \mathbf{Sym}$ for all $t \in \mathfrak{S}_k$

  Proof:

  $$\sigma(t)\,\mathbf{Sym} = \frac{1}{k!} \sum_{s \in \mathfrak{S}_k} \sigma(t)\sigma(s) = \frac{1}{k!} \sum_{r \in \mathfrak{S}_k} \sigma(r) = \mathbf{Sym}$$

Define $S^k(V) = \mathbf{Sym}(V^{\otimes k})$ (*symmetric k-tensors*)
If $\mathbf{x} \in S^k(V)$ then the components $x_{i_1 \ldots i_k}$ are symmetric in the indices (unchanged under any transposition of indices), and conversely.

Theorem

Let $f : V \times \cdots \times V \to W$ (k factors) be any k-multilinear map that is symmetric in its arguments.
There is a unique linear map $F : S^k(V) \to W$ such that

$$F(\mathbf{Sym}(\mathbf{u}_1 \otimes \cdots \otimes \mathbf{u}_k)) = f(\mathbf{u}_1, \ldots, \mathbf{u}_k).$$

Proof: There exists linear map $F : V^{\otimes k} \to W$ with

$$F(\mathbf{u}_1 \otimes \cdots \otimes \mathbf{u}_k) = f(\mathbf{u}_1, \ldots, \mathbf{u}_k)$$

Then $F \circ \mathbf{Sym} = F$ since f is symmetric, so $F \circ (I - \mathbf{Sym}) = 0$.

Example

$(V^*)^{\otimes 2} \longleftrightarrow$ bilinear forms on V

$$\mathbf{b} = \sum_{i,j} b^{ij} \mathbf{v}_i^* \otimes \mathbf{v}_j^* \longleftrightarrow B(\mathbf{x}, \mathbf{y}) = \sum_{ij} b^{ij} x_i y_j$$

(written as $b^{ij} x_i y_j$ in *Einstein summation notation*)

Then $\mathbf{b} \in S^2(V^*) \iff b^{ij} = b^{ji} \iff B(\mathbf{x}, \mathbf{y}) = B(\mathbf{y}, \mathbf{x})$

For $s \in \mathfrak{S}_k$ define $\operatorname{sgn}(s) = (-1)^m$ if s is a product of m transpositions. Then $\operatorname{sgn}(s)\operatorname{sgn}(t) = \operatorname{sgn}(st)$, $\operatorname{sgn}(1) = 1$.

Alternation operator $\mathbf{Alt} : V^{\otimes k} \to V^{\otimes k}$

$$\mathbf{Alt}(\mathbf{u}_1 \otimes \cdots \otimes \mathbf{u}_k) = \frac{1}{k!} \sum_{s \in \mathfrak{S}_k} \operatorname{sgn}(s)\sigma(s)(\mathbf{u}_1 \otimes \cdots \otimes \mathbf{u}_k)$$

Properties:

- $\mathbf{Alt}^2 = \mathbf{Alt}$ (projection operator)

Proof:

$$\begin{aligned}\mathbf{Alt}^2 &= \frac{1}{(k!)^2} \sum_{s,t \in \mathfrak{S}_k} \operatorname{sgn}(s)\operatorname{sgn}(t)\sigma(s)\sigma(t) \\ &= \frac{1}{(k!)^2} \sum_{s,t \in \mathfrak{S}_k} \operatorname{sgn}(st)\sigma(st) = \frac{1}{k!} \sum_{r \in \mathfrak{S}_k} \operatorname{sgn}(r)\sigma(r) \\ &= \mathbf{Alt}\end{aligned}$$

- $\sigma(t)\,\mathbf{Alt} = \operatorname{sgn}(t)\,\mathbf{Alt}$ for all $t \in \mathfrak{S}_k$

Proof:

$$\begin{aligned}\sigma(t)\,\mathbf{Alt} &= \frac{1}{k!} \sum_{s \in \mathfrak{S}_k} \operatorname{sgn}(s)\sigma(t)\sigma(s) = \frac{1}{k!} \sum_{r \in \mathfrak{S}_k} \operatorname{sgn}(t)^{-1}\sigma(r) \\ &= \operatorname{sgn}(t)\,\mathbf{Alt}\end{aligned}$$

Universal Linearization Property of $\bigwedge^k(V)$

Define $\bigwedge^k(V) = \mathbf{Alt}(V^{\otimes k})$ (*alternating k-tensors*)

If $\mathbf{x} \in \bigwedge^k(V)$ then the components $x_{i_1 \dots i_k}$ are skew-symmetric in the indices (change sign under any transposition), and conversely.

Theorem

Let $f : V \times \cdots \times V \to W$ (k factors) be any k-multilinear map that is alternating *in its arguments (changes sign when two arguments are permuted).*

There is a unique linear *map $F : \bigwedge^k(V) \to W$ such that*

$$F(\mathbf{Alt}(\mathbf{u}_1 \otimes \cdots \otimes \mathbf{u}_k)) = f(\mathbf{u}_1, \dots, \mathbf{u}_k).$$

Proof: There exists linear map $F : V^{\otimes k} \to W$ with

$$F(\mathbf{u}_1 \otimes \cdots \otimes \mathbf{u}_k) = f(\mathbf{u}_1, \dots, \mathbf{u}_k)$$

Then $F \circ \mathbf{Alt} = F$ since f is alternating, so $F \circ (I - \mathbf{Alt}) = 0$.

Notation: $\mathbf{Alt}(\mathbf{u}_1 \otimes \cdots \otimes \mathbf{u}_k) = \mathbf{u}_1 \wedge \cdots \wedge \mathbf{u}_k$ (*exterior product* of vectors)

Comparisons between Symmetric and Alternating Tensors

$S^k(V)$ $(\dim V = n)$

- basis $\{\mathbf{Sym}(\mathbf{v}_{i_1} \otimes \cdots \otimes \mathbf{v}_{i_k}\}$ labelled by indices $1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq n$
- $\dim S^k(V) = \binom{n+k-1}{k} = \#\{k \text{ balls in } n \text{ boxes}\}$
- $S^k(V) \cong \mathcal{P}^k(V^*) =$ homogeneous polynomials of degree k on V^* (nonzero space for all k):
  $\mathbf{Sym}(\mathbf{u}_1 \otimes \cdots \otimes \mathbf{u}_k) \longleftrightarrow$ monomial $\langle \mathbf{u}^*, \mathbf{u}_1 \rangle \cdots \langle \mathbf{u}^*, \mathbf{u}_k \rangle$
- $\mathfrak{S}_k$ acts by identity on $S^k(V)$
- $S^k(V)$ is invariant under $\mathrm{GL}(V)$, and contains no proper subspace that is invariant under $\mathrm{GL}(V)$ (irreducible representation of $\mathrm{GL}(V)$)
- Quantum mechanics: $S^k(V)$ describes systems of k *bosons*

$\bigwedge^k(V)$ $\quad(\dim V = n)$

- basis $\{\mathbf{v}_{i_1} \wedge \cdots \wedge \mathbf{v}_{i_k}\}$ labelled by indices $1 \le i_1 < i_2 < \cdots < i_k \le n$
- $\dim \bigwedge^k(V) = \binom{n}{k} = \#\{\text{subsets of size } k \text{ in } n\text{-element set}\}$
- $\bigwedge^k(V) \cong$ homogeneous alternating functions of degree k on V^* (zero if $k > n$)
- $\mathfrak{S}_k$ acts by sgn on $\bigwedge^k(V)$
- $\bigwedge^k(V)$ is invariant under $\mathrm{GL}(V)$, and contains no proper subspace that is invariant under $\mathrm{GL}(V)$ (irreducible representation of $\mathrm{GL}(V)$)
- $\dim \bigwedge^n V = 1$. Fix basis vector $\mathbf{u}$. Then $\rho(g)\mathbf{u} = \det(g)\mathbf{u}$ for $g \in \mathrm{GL}(V)$.
- Geometry: $\mathbf{u}_1 \wedge \cdots \wedge \mathbf{u}_k \neq 0 \iff \{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ are linearly independent (span a k-plane).
- Quantum Mechanics: $\bigwedge^k(V)$ describes systems of k *fermions* (Pauli exclusion principle)

The group $\mathrm{GL}(V) \times \mathfrak{S}_k$ acts on $V^{\otimes k}$.
Program: Decompose $V^{\otimes k}$ into a direct sum of subspaces that are irreducible under the action of $\mathrm{GL}(V) \times \mathfrak{S}_k$
We already have two such spaces, namely $S^k(V)$ and $\bigwedge^k(V)$.

Example When $k = 2$, then $V^{\otimes 2} = S^2(V) \oplus \bigwedge^2(V)$
(Every bilinear form is the sum of a symmetric form and a skew-symmetric form)

For $k > 2$? $\dim V^{\otimes k} = n^k >> \dim S^2(V) + \dim \bigwedge^2(V)$

As a module for $\mathrm{GL}(V) \times \mathfrak{S}_k$, $\quad V^{\otimes k} \cong \bigoplus_\lambda E_\lambda \otimes F_\lambda$

- λ runs over all *partitions* of k with at most $\dim V$ parts
- E_λ is an irreducible representation of $\mathrm{GL}(V)$, and only occurs in the decomposition paired with F_λ
- F_λ is an irreducible representation of $\mathfrak{S}_k$, and only occurs in the decomposition paired with E_λ

  trivial representation of $\mathfrak{S}_k \longleftrightarrow S^k(V)$

  sgn representation of $\mathfrak{S}_k \longleftrightarrow \bigwedge^k(V)$
- $\mathrm{GL}(V)$ acts on the first tensor factor in $E_\lambda \otimes F_\lambda$
- $\mathfrak{S}_k$ acts on the second tensor factor in $E_\lambda \otimes F_\lambda$
- There are explicit operators (*Young symmetrizers*) that project on the subspaces E_λ and $E_\lambda \otimes F_\lambda$

Lots of interesting algebra, analysis, and combinatorics!