Classical Groups, Representations, and Invariants *

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Part 1: Linear Algebraic Groups

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Part 2: Stucture of Classical Groups

Part 3: Homogeneous Spaces

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Part 1: Linear Algebraic Groups

Lecture 1. Classical Groups and Linear Algebraic Groups

Definition of a Linear Algebraic Group

Let $\mathrm{GL}(n, \mathbb{C})$ be the group of invertible $n \times n$ complex matrices, and let $M_n(\mathbb{C})$ be the algebra of all $n \times n$ complex matrices. For $y \in M_n(\mathbb{C})$ and $1 \le i, j \le n$ we write $x_{ij}(y)$ for the i, j entry in y. A complex-valued function f on $M_n(\mathbb{C})$ is a *polynomial function* if

$$f(y) = p(x_{11}(y), x_{12}(y), \ldots, x_{nn}(y))$$

where $p \in \mathbb{C}[x_{11}, x_{12}, \ldots, x_{nn}]$.

Definition: A subgroup $G \subset \mathrm{GL}(n, \mathbb{C})$ is a *linear algebraic group* if there is a set A of polynomial functions on $M_n(\mathbb{C})$ so that

$$G = \{g \in \mathrm{GL}(n, \mathbb{C}) \,:\, f(g) = 0 \text{ for all } f \in A\}.$$

General and Special Linear Groups

The *general linear group* $\mathrm{GL}(n, \mathbb{C})$ is a linear algebraic group. The *special linear group* $\mathrm{SL}(n, \mathbb{C})$ consists of all matrices $g \in \mathrm{GL}(n, \mathbb{C})$ with $\det(g) = 1$. We shall call $\mathrm{SL}(n, \mathbb{C})$ a group of *Type* A_l, where $l = n - 1$.

Orthogonal Groups

Let B be a nondegenerate symmetric bilinear form on $\mathbb{C}^n$. The *orthogonal group* relative to B is

$$\mathrm{O}(\mathbb{C}^n, B) = \{g \in \mathrm{GL}(n, \mathbb{C}) \,:\, B(gx, gy) = B(x, y) \text{ for } x, y \in \mathbb{C}^n\}.$$

Let S be the matrix of the bilinear form: $B(x, y) = x^t S y$. Then S is a symmetric, invertible matrix and

$$g \in \mathrm{O}(\mathbb{C}^n, B) \iff g^t S g = S. \tag{1.1}$$

Proposition 1.1 *Let B, B' be nondegenerate symmetric bilinear forms on $\mathbb{C}^n$. Then there exists $\gamma \in \mathrm{GL}(n, \mathbb{C})$ such that $\mathrm{O}(\mathbb{C}^n, B') = \gamma \mathrm{O}(\mathbb{C}^n, B)\gamma^{-1}$.*

We call $\mathrm{SO}(\mathbb{C}^{2l}, B)$ a group of *type* D_l and $\mathrm{SO}(\mathbb{C}^{2l+1}, B)$ a group of *type* B_l.

Symplectic Groups

Let Ω be a nondegenerate skew symmetric bilinear form on $\mathbb{C}^n$. Then $n = 2l$ must be even. We define the *symplectic group* relative to Ω as

$$\mathrm{Sp}(\mathbb{C}^{2l}, \Omega) = \{g \in \mathrm{GL}(2l, \mathbb{C}) \,:\, \Omega(gx, gy) = \Omega(x, y) \text{ for } x, y \in \mathbb{C}^{2l}\}.$$

Let R be the matrix of the bilinear form: $\Omega(x, y) = x^t R y$. Then R is a skew-symmetric, invertible matrix and

$$g \in \mathrm{Sp}(\mathbb{C}^{2l}, \Omega) \iff g^t R g = R. \tag{1.2}$$

Proposition 1.2 *Let Ω and Ω' be nondegenerate skew symmetric bilinear forms on $\mathbb{C}^{2l}$. Then there exists $\gamma \in \mathrm{GL}(2l, \mathbb{C})$ such that $\mathrm{Sp}(\mathbb{C}^{2l}, \Omega') = \gamma \mathrm{Sp}(\mathbb{C}^{2l}, \Omega)\gamma^{-1}$.*

We call $\mathrm{Sp}(\mathbb{C}^{2l}, \Omega)$ a group of *type* C_l.

The groups $\mathrm{GL}(n, \mathbb{C})$, $\mathrm{SL}(n, \mathbb{C})$, $\mathrm{O}(n, \mathbb{C})$, $\mathrm{SO}(n, \mathbb{C})$ and $\mathrm{Sp}(l, \mathbb{C})$ are called the *classical groups.*

Regular Functions on Linear Algebraic Groups

The group $\mathrm{GL}(V)$ is the principal open set $\{g \in M_n(\mathbb{C}) : \det(g) \neq 0\}$ in the vector space $M_n(\mathbb{C})$. Thus

$$\mathrm{Aff}(\mathrm{GL}(V)) = \mathbb{C}[x_{11}, x_{12}, \ldots, x_{nn}, (\det)^{-1}],$$

where $\{x_{ij}\}$ are the matrix coordinates relative to a basis for V.

Proposition 1.3 *A subgroup $G \subset \mathrm{GL}(V)$ is a linear algebraic group if and only if G is a closed subset of $\mathrm{GL}(V)$, relative to the Zariski topology.*

A complex-valued function f on G is called *regular* if it is the restriction to G of a regular function on $\mathrm{GL}(V)$. The set $\mathrm{Aff}(G)$ of regular functions on G is a commutative algebra over $\mathbb{C}$ under pointwise multiplication. Define

$$\mathcal{I}_G = \{f \in \mathrm{Aff}(\mathrm{GL}(V)) \,:\, f(G) = 0\}.$$

The map $f \mapsto f|_G$ gives an algebra isomorphism

$$\mathrm{Aff}(G) \cong \mathrm{Aff}(\mathrm{GL}(V))/\mathcal{I}_G. \tag{1.3}$$

If G, H are linear algebraic groups, then an (abstract) group homomorphism $\phi : G \to H$ is *regular* if $\phi^*(\mathrm{Aff}(H)) \subset \mathrm{Aff}(G)$. We say that G and H are *isomorphic as algebraic groups* if there exists a regular homomorphism $\phi : G \to H$ which has a regular inverse.
The set $G \times G$ carries the structure of an affine algebraic set, with the algebra of regular functions

$$\mathrm{Aff}(G \times G) \cong \mathrm{Aff}(G) \otimes \mathrm{Aff}(G).$$

In this isomorphism, $f' \otimes f'' \in \mathrm{Aff}(G) \otimes \mathrm{Aff}(G)$ is identified with the function $(g, h) \mapsto f'(g)f''(h)$ on $G \times G$.

Proposition 1.4 *The maps $\mu : G \times G \to G$ and $\iota : G \to G$ given by multiplication and inversion are regular. If $f \in \mathrm{Aff}(G)$ then there exists an integer p and f_i', $f_i'' \in \mathrm{Aff}(G)$ for $i = 1, \ldots, p$, such that*

$$f(gh) = \sum_{i=1}^{p} f_i'(g)\, f_i''(h) \text{ for } g, h \in G. \tag{1.4}$$

Furthermore, for fixed $g \in G$ the maps $x \mapsto L_g(x) = gx$ and $x \mapsto R_g(x) = xg$ from $G \to G$ are regular.

If $G \subset \mathrm{GL}(V)$, $H \subset \mathrm{GL}(W)$ are linear algebraic groups, then we make the group-theoretic direct product $K = G \times H$ into an algebraic group by the natural block diagonal embedding into $\mathrm{GL}(V \oplus W)$ as the elements

$$k = \begin{bmatrix} g & 0 \\ 0 & h \end{bmatrix} \quad g \in G,\, h \in H.$$

This embedding defines an isomorphism

$$\mathrm{Aff}(K) \cong \mathrm{Aff}(G) \otimes \mathrm{Aff}(H).$$

Appendix: Algebraic Geometry for Lecture 1.

Affine Algebraic Sets and Regular Functions

Let V be a finite-dimensional complex vector space. Let $\mathcal{P}(V)$ be the commutative algebra of polynomial functions on V. A subset $X \subset V$ is an *affine algebraic set* if there exist $f_1, \ldots, f_m \in \mathcal{P}(V)$ such that

$$X = \{v \in V \,:\, f_i(v) = 0 \text{ for } i = 1, \ldots, m\}.$$

We define the *affine ring* of X to be the functions on X that are restrictions of polynomials on V:

$$\mathrm{Aff}(X) = \{f|_X \,:\, f \in \mathcal{P}(V)\}.$$

We call these functions the *regular functions* on X. Define

$$\mathcal{I}_X = \{f \in \mathcal{P}(V) \,:\, f|_X = 0\}.$$

Then $\mathcal{I}_X$ is an ideal in $\mathcal{P}(V)$, and $\mathrm{Aff}(X) \cong \mathcal{P}(V)/\mathcal{I}_X$.

Theorem 1.5 (Hilbert basis theorem) *Let $\mathcal{I} \subset \mathcal{P}(V)$ be an ideal. Then $\mathcal{I}$ is finitely generated: there is a finite set of polynomials $f_1, \ldots, f_d$ in $\mathcal{I}$ so that every $g \in \mathcal{I}$ can be written as*

$$g = g_1 f_1 + \cdots + g_d f_d$$

for some choice of $g_1, \ldots, g_d \in \mathcal{P}(V)$.

Let $a \in X$. Then

$$\mathfrak{m}_a = \{f \in \mathrm{Aff}(X) \,:\, f(a) = 0\}$$

is a maximal ideal in $\mathrm{Aff}(X)$, since $f - f(a) \in \mathfrak{m}_a$ for all $f \in \mathrm{Aff}(X)$.

Theorem 1.6 (Hilbert Nullstellensatz) *Let X be an affine algebraic set. If $\mathfrak{m}$ is a maximal ideal in $\mathrm{Aff}(X)$ then there is a unique point $a \in X$ such that $\mathfrak{m} = \mathfrak{m}_a$.*

If A is an algebra with 1 over $\mathbb{C}$, then $\mathrm{Hom}(A, \mathbb{C})$ is the set of all linear maps $\phi : A \to \mathbb{C}$ such that $\phi(1) = 1$ and $\phi(a'a'') = \phi(a')\phi(a'')$ for all $a', a'' \in A$ (the *multiplicative linear functionals* on A). When X is an affine algebraic set and $A = \mathrm{Aff}(X)$, then every $x \in X$ defines a homomorphism ϕ_x by evaluation: $\phi_x(f) = f(x)$ for $f \in A$.

Corollary 1.7 *Let X be an affine algebraic set, and let $A = \mathrm{Aff}(X)$. The map $x \mapsto \phi_x$ is a bijection between X and $\mathrm{Hom}(A, \mathbb{C})$.*

Let $X \subset V$ be an algebraic subset. If $Y \subset X$, then we say that Y is *Zariski closed* in X if Y is an algebraic subset of V. Given $0 \neq f \in \mathrm{Aff}(X)$, the *principal open subset* of X defined by f is

$$X^f = \{x \in X \,:\, f(x) \neq 0\}.$$

Lemma 1.8 *The Zariski closed sets of X give X the structure of a topological space. The finite unions of principal open sets X^f, for $0 \neq f \in \mathrm{Aff}(X)$, are the non-empty open sets in this topology (the* Zariski *topology).*

Let V and W be finite-dimensional complex vector spaces. Suppose $X \subset V$ and $Y \subset W$ are algebraic sets and $f : X \to Y$. If g is a complex-valued function on Y define $f^*(g)$ to be the function

$$f^*(g)(x) = g(f(x)) \quad \text{for } x \in X.$$

We say that f is a *regular map* if $f^*(g)$ is in $\mathrm{Aff}(X)$ for all $g \in \mathrm{Aff}(Y)$.
Let X be an affine algebraic subset of V, and let $f \in \mathrm{Aff}(X)$, with $f \not\equiv 0$. We make the principal open set X^f into an affine algebraic set as follows: Define a map $\psi : X^f \to V \times \mathbb{C}$ by

$$\psi(x) = (x, f(x)^{-1}).$$

This map is injective, and we use it to define the structure of an affine algebraic set on X^f by

$$\mathrm{Aff}(X^f) = \{g \circ \psi \, : \, g \in \mathcal{P}(V \times \mathbb{C})\}.$$

Thus the regular functions on X^f are the restrictions to X^f of the functions

$$p(x_1, \ldots, x_n, f^{-1}), \quad \text{where } p \in \mathbb{C}[t_1, \ldots, t_{n+1}].$$

Here $x_1, \ldots, x_n$ are linear coordinate functions on V.

Exercises for Lecture 1.

1. Show that the homomorphism $\mathbb{C}^\times \times \mathrm{SL}(n, \mathbb{C}) \to \mathrm{GL}(n, \mathbb{C})$ given by $(\lambda, g) \mapsto \lambda g$ is surjective. What is its kernel?

2. Consider the bilinear form $\Omega(v, w) = \det[vw]$ for $v, w \in \mathbb{C}^2$.

   (a) Show that Ω is skew-symmetric and nondegenerate.

   (b) Show that $g \in \mathrm{GL}(2, \mathbb{C})$ preserves Ω if and only if $\det(g) = 1$.

   Hence $\mathrm{SL}(2, \mathbb{C}) = \mathrm{Sp}(\mathbb{C}^2, \Omega)$.

3. Let A be in $M_n(\mathbb{C})$. Define $G_A = \{g \in \mathrm{GL}(n, \mathbb{C}) : gAg^t = A\}$. Set $A_{\text{symm}} = \frac{1}{2}(A + A^t)$, $A_{\text{skew}} = \frac{1}{2}(A - A^t)$. Show that $G_A = G_{A_{\text{symm}}} \cap G_{A_{\text{skew}}}$.

4. Let $\mathcal{A}$ be a finite-dimensional algebra over $\mathbb{C}$. This means that there is a *multiplication map* $\mu : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ which is bilinear (it is not assumed to be associative). Define the *automorphism group* of $\mathcal{A}$ to be

   $$\mathrm{Aut}(\mathcal{A}) = \{g \in \mathrm{GL}(\mathcal{A}) \, : \, g\mu(X, Y) = \mu(gX, gY), \text{ for } X, Y \in \mathcal{A}\}.$$

   Show that $\mathrm{Aut}(\mathcal{A})$ is an algebraic subgroup of $\mathrm{GL}(\mathcal{A})$.

5. Let Ω be a nondegenerate skew-symmetric bilinear form on a finite-dimensional vector space V. Define $\mathrm{GSp}(V,\Omega)$ to be all $g \in \mathrm{GL}(V)$ for which there is a $\lambda \in \mathbb{C}^\times$ (depending on g) so that

$$\Omega(gx, gy) = \lambda\Omega(x, y) \text{ for all } x, y \in V.$$

(a) Show that the homomorphism $\mathbb{C}^\times \times \mathrm{Sp}(V,\Omega) \to \mathrm{GSp}(V,\Omega)$ given by $(\lambda, g) \mapsto \lambda g$ is surjective. What is its kernel?

(b) Show that $\mathrm{GSp}(V,\Omega)$ is Zariski-closed in $\mathrm{GL}(V)$ and is thus a linear algebraic group.

Lecture 2. Representations, Connected Groups

Let G be a linear algebraic group. A *representation* of G is a pair (ρ, V), where V is a complex vector space (not necessarily finite-dimensional), and $\rho : G \to \mathrm{GL}(V)$ is a group homomorphism. We say that the representation is *regular* if $\dim V < \infty$ and the functions on G

$$g \mapsto \langle \rho(g)v, v^* \rangle, \tag{2.1}$$

which we call *matrix coefficients* of ρ, are regular, for all $v \in V$ and $v^* \in V^*$.
For $B \in \mathrm{End}(V)$ define the function f_B^ρ on G by

$$f_B^\rho(g) = \mathrm{tr}_V(\rho(g)B)$$

Then (ρ, V) is regular if and only if f_B^ρ is a regular function on G, for all $B \in \mathrm{End}(V)$. We set

$$E^\rho = \{f_B^\rho \,:\, B \in \mathrm{End}(V)\}.$$

(the space of *representative functions* associated with ρ).
If (ρ, V) is a regular representation and $W \subset V$ is a linear subspace, then we say that W is G-*invariant* if $\rho(g)w \in W$ for all $g \in G$ and $w \in W$. In this case we obtain a representation σ of G on W by restriction of $\rho(g)$. We also obtain a representation τ of G on the quotient space V/W by setting $\tau(g)(v + W) = \rho(g)v + W$.
If (ρ, V) and (τ, W) are representations of G, then we say that they are *equivalent* if there is a linear bijection $T : V \to W$ so that

$$T\rho(g)T^{-1} = \tau(g) \quad \text{for all } g \in G.$$

In this case we write $\rho \cong \tau$.
We say that a representation (ρ, V) with $V \neq \{0\}$ is *reducible* if there is a G-invariant subspace $W \subset V$ such that $W \neq \{0\}$ and $W \neq V$. If not such W exists, we call the representation *irreducible.*

Examples

1. Let $G \subset \mathrm{GL}(V)$ be a linear algebraic group. By definition of $\mathrm{Aff}(G)$, the representation $\rho(g) = g$ on V is regular. We call ρ the *defining* representation of G.

2. Let (ρ, V) be a regular representation. Define the *contragredient* (or *dual*) representation (ρ^*, V^*) by $\rho^*(g)v^* = v^* \circ \rho(g^{-1})$. Then

$$E^{\rho^*} = \iota^* E^\rho$$

where $(\iota^* f)(x) = f(x^{-1})$ for $f \in \mathrm{Aff}(G)$.

3. Let (ρ, V) and (σ, W) be regular representations of G. Define the *direct sum* representation $\rho \oplus \sigma$ on $V \oplus W$ by

$$(\rho \oplus \sigma)(g)(v \oplus w) = \rho(g)v \oplus \sigma(g)w$$

for $g \in G$, $v \in V$ and $w \in W$. Then

$$E^{\rho \oplus \sigma} = E^\rho + E^\sigma.$$

4. Let (ρ, V) and (σ, W) be regular representations of G. Define the *tensor product* representation $\rho \otimes \sigma$ on $V \otimes W$ by

$$(\rho \otimes \sigma)(g)(v \otimes w) = \rho(g)v \otimes \sigma(g)w$$

for $g \in G$, $v \in V$ and $w \in W$. Then

$$E^{\rho\otimes\sigma} = \mathrm{Span}(E^{\rho} \cdot E^{\sigma})$$

5. Consider the representations L and R of G on $\mathrm{Aff}(G)$ given by *left* and *right translations*:

$$L(x)f(y) = f(x^{-1}y), \quad R(x)f(y) = f(yx) \text{ for } f \in \mathrm{Aff}(G).$$

These representations are *locally regular*: for any regular function f on G,

$$V(f) = \mathrm{Span}\{L(x)R(y)f \,:\, x, y \in G\}$$

is a finite-dimensional subspace of $\mathrm{Aff}(G)$ which is invariant under $R(G)$ and $L(G)$.

Proposition 2.1 *Suppose that G and H are algebraic subgroups of $\mathrm{GL}(n,\mathbb{C})$, and $H \subset G$. Then*

$$H = \{g \in G \,:\, R(g)\mathcal{I}_H \subset \mathcal{I}_H\}.$$

Connected Groups

Theorem 2.2 *Let G be a linear algebraic group. Then G contains a unique subgroup G° which is closed, irreducible, and of finite index in G. Furthermore, G° is a normal subgroup and its cosets in G are both the irreducible components and the connected components of G.*

Corollary 2.3 *A linear algebraic group is (Zariski) connected if and only if it is irreducible.*

Appendix: Algebraic Geometry for Lecture 2.

Irreducible Components of an Algebraic Set

Let V be a finite-dimensional complex vector spaces. Let $X \subset V$ be a nonempty algebraic set. We say that X is *reducible* if there are nonempty closed subsets $X_i \neq X$, $i = 1, 2$ such that $X = X_1 \cup X_2$. We say that X is *irreducible* if it is not reducible.

Lemma 2.4 *An algebraic set X is irreducible if and only if $\mathcal{I}_X$ is a prime ideal ($\mathrm{Aff}(X)$ has no zero divisors).*

Lemma 2.5 *Let X be an irreducible algebraic set. Every nonempty open subset of X is dense in X. Furthermore, if $Y \subset X$ and $Z \subset X$ are nonempty open subsets, then $Y \cap Z$ is nonempty.*

Lemma 2.6 *If X is an irreducible algebraic set then so is X^f, for any $0 \neq f \in \mathrm{Aff}(X)$.*

Lemma 2.7 *Let V and W be finite-dimensional vector spaces. Suppose $X \subset V$ and $Y \subset W$ are irreducible algebraic sets. Then $X \times Y$ is an irreducible algebraic set in $V \oplus W$.*

Lemma 2.8 *Suppose $f : X \to Y$ is a regular map between affine algebraic sets. Suppose X is irreducible. Then $\overline{f(X)}$ is irreducible.*

Lemma 2.9 *If X is any algebraic set, then there exists a finite collection of irreducible closed sets X_i such that*

$$X = X_1 \cup \cdots \cup X_r \text{ and } X_i \not\subset X_j \text{ for } i \neq j. \tag{2.2}$$

Furthermore, such a decomposition (2.2) *is unique up to a permutation of the indices, and is called an* incontractible decomposition *of X. The sets X_i are called the* irreducible components *of X.*

Exercises for Lecture 2.

1. Let ω be a nondegenerate skew-symmetric bilinear form on $\mathbb{C}^{2l}$. Show that $\det(g) = 1$ for all $g \in \mathrm{Sp}(\mathbb{C}^{2l}, \omega)$. (*Hint:* Consider ω to be an element of $\bigwedge^2(\mathbb{C}^{2l})^*$ and let Ω be the l-fold wedge power of ω. Show that $\Omega \neq 0$, and hence $\mathbb{C}\Omega = \bigwedge^{2l}(\mathbb{C}^{2l})^*$.)

2. Let $G = \mathrm{GL}(n, \mathbb{C})$ and let ρ be the defining representation of G on $V = \mathbb{C}^n$.

   (a) Define a representation π of G on $M_n(\mathbb{C})$ by $\pi(g)B = gBg^t$ for $g \in G$ and $B \in M_n(\mathbb{C})$. Show that $(\pi, M_n(\mathbb{C}))$ is equivalent to $(\rho \otimes \rho, V \otimes V)$. (*Hint:* Let $B = [b_{ij}] \in M_n(\mathbb{C})$ be an $n \times n$ matrix. Set $T(B) = \sum_{i,j=1}^n b_{ij}\, e_i \otimes e_j$, where $\{e_i\}$ is the standard basis for $\mathbb{C}^n$. Show that $\rho^{\otimes 2}(g)T(B) = T(gBg^t)$.)

   (b) Describe the action of G on the symmetric and the skew-symmetric two-tensors in terms of matrices as in part (a).

3. Let (ρ, V) be a regular representation of the linear algebraic group G.

   (a) Prove that (ρ, V) is irreducible if and only if the dual representation (ρ^*, V^*) is irreducible. (*Hint:* Let $E \subset V$ be a linear subspace. Show that E is G-invariant if and only if $E^\perp \subset V^*$ is G-invariant.)

   (b) Assume that (ρ, V) is irreducible. Fix $v^* \in V^*$ with $v^* \neq 0$. For $v \in V$ let $\varphi_v \in \mathrm{Aff}(G)$ be the representative function $\varphi_v(g) = \langle v^*, \rho(g)v \rangle$. Let $E = \{\varphi_v : v \in V\}$ and let $T : V \to E$ be the map $Tv = \varphi_v$. Prove that T is a bijective linear map and that $T\rho(g) = R(g)T$ for all $g \in G$, where $R(g)f(x) = f(xg)$ for $f \in \mathrm{Aff}(G)$. (*Hint:* To prove that T is injective, use (a) to show that $\rho^*(G)v^*$ spans V^*.)

   Thus every irreducible regular representation of G is equivalent to a subrepresentation of $(R, \mathrm{Aff}(G))$.

4. Let $\mathcal{A}$ be a finite-dimensional associative algebra with unit 1. Let G be the set of all $g \in \mathcal{A}$ such that g is invertible in $\mathcal{A}$.

   (a) Let $f : \mathcal{A} \to \mathbb{C}$ be given by $f(a) = \det(L_a)$, where $L_a \in \mathrm{End}(\mathcal{A})$ is the operator of left multiplication by a. Show that G is the principal open set $\mathcal{A}^f$.

   (b) Define $\Phi : G \to \mathrm{GL}(\mathcal{A})$ by $\Phi(g) = L_g$. Show that $\Phi(G)$ is a closed linear algebraic subgroup in $\mathrm{GL}(\mathcal{A})$ and that $\Phi(G)$ is isomorphic with $\mathcal{A}^f$ as an algebraic subset. (*Hint*: To show that $\Phi(G)$ is closed, prove that $T \in \mathrm{End}(\mathcal{A})$ commutes with all the operators of right multiplication by elements of $\mathcal{A}$ if and only if $T = L_a$ for some $a \in \mathcal{A}$.)

Lecture 3. Subgroups and Homomorphisms
Group Structures on Affine Varieties

Subgroups of Algebraic Groups

Let $G \subset \mathrm{GL}(V)$ be a linear algebraic group.

Lemma 3.1 *Let K be a subgroup of G. Then the closure (in the Zariski topology) $\overline{K}$ of K is a subgroup, and hence an algebraic subgroup of G. Furthermore, if K contains a non-empty open subset of $\overline{K}$ then K is closed.*

Regular Homomorphisms of Algebraic Groups

Theorem 3.2 *Let $\phi : G \to H$ be a regular homomorphism of linear algebraic groups. Then $F = \mathrm{Ker}(\phi)$ is a closed subgroup of G, and $\phi(G)$ is a closed subgroup of H. Hence $\phi(G)$ is an algebraic group. Furthermore, $\phi(G^\circ) = \phi(G)^\circ$.*

Corollary 3.3 *Let $\phi : G \to H$ be a regular homomorphism of linear algebraic groups. Set $K = \phi(G)$. Let $\iota : K \to H$ be the inclusion map and let $\psi : G \to K$ be the homomorphism ϕ, viewed as having image K. Then ι is regular and injective, ψ is regular and surjective, and ϕ factors as $\phi = \iota \circ \psi$.*

Group Structures on Affine Algebraic Sets

Theorem 3.4 *Let X be an affine algebraic set. Assume that X has a group structure such that $x, y \mapsto xy$ and $x \mapsto x^{-1}$ are regular mappings. Then there exists a linear algebraic group G and a group isomorphism $\Phi : X \to G$ such that Φ also an isomorphism of affine algebraic sets.*

Theorem 3.5 *Let G and H be linear algebraic groups. Suppose $\sigma : G \to H$ is a bijective regular homomorphism. Then $\sigma^{-1} : H \to G$ is regular, and hence $G \cong H$ as algebraic groups.*

Appendix: Algebraic Geometry for Lecture 3.

Dominant Regular Maps of Algebraic Sets

Let X, Y be affine algebraic sets. A map $f : X \to Y$ is called *dominant* if it is regular and $f(X)$ is dense in Y. This is equivalent to the injectivity of $f^* : \mathrm{Aff}(Y) \to \mathrm{Aff}(X)$.

Theorem 3.6 *Assume that X, Y are irreducible affine algebraic sets and $f : X \to Y$ is a dominant map. Let $M \subset X$ be a nonempty open set. Then $f(M)$ contains a nonempty open subset of Y.*

This is proved using the following result on extensions of homomorphisms. Let A be an algebra with 1 over $\mathbb{C}$. Given $0 \neq a \in A$, we set

$$\mathrm{Hom}(A, \mathbb{C})^a = \{\phi \in \mathrm{Hom}(A, \mathbb{C}) \,:\, \phi(a) \neq 0\}.$$

Theorem 3.7 *Let B be a commutative algebra over $\mathbb{C}$. Assume $1 \in B$ and B has no zero divisors. Suppose that $A \subset B$ is a subalgebra such that $B = A[b_1, \ldots, b_n]$ for some elements $b_i \in B$. Then given $0 \neq b \in B$, there exists $0 \neq a \in A$ such that every $\phi \in \mathrm{Hom}(A, \mathbb{C})^a$ extends to $\psi \in \mathrm{Hom}(B, \mathbb{C})^b$.*

Corollary 3.8 *Let B be a finitely generated commutative algebra over $\mathbb{C}$ having no zero divisors. Given $0 \neq b \in B$, there exists $\psi \in \mathrm{Hom}(B, \mathbb{C})$ such that $\psi(b) \neq 0$.*

Theorem 3.9 *Let $f : X \to Y$ be a regular map between affine algebraic sets. Then $f(X)$ contains an open subset of $\overline{f(X)}$.*

Rational Maps

Let A be a commutative ring with 1 and without zero divisors. Then A is embedded in its *quotient field* $\mathrm{Quot}(A)$. The elements of this field are the formal expressions $f = g/h$, where $g, h \in A$ and $h \neq 0$, with the usual algebraic operations on fractions. Let X be an irreducible algebraic set. The algebra $A = \mathrm{Aff}(X)$ has no zero divisors, so it has a quotient field. We denote this field by $\mathrm{Rat}(X)$ and call it the field of *rational functions* on X.

We may view the elements of $\mathrm{Rat}(X)$ as functions, as follows. If $f \in \mathrm{Rat}(X)$, then we say that f is *defined at a point* $x \in X$ if there exist $g, h \in \mathrm{Aff}(X)$ with $f = g/h$ and $h(x) \neq 0$. In this case we set $f(x) = g(x)/h(x)$. The *domain* $\mathcal{D}_f$ of f is the subset of X at which f is defined. It is a dense open subset of X, since it contains the principal open set X^h.

A map f from X to an algebraic set Y is called *rational* if $\phi \circ f$ is a rational function on X for all $\phi \in \mathrm{Aff}(Y)$. Suppose $Y \subset \mathbb{C}^n$ and y_i is the restriction to Y of the ith linear coordinate function. Set $f_i = y_i \circ f$. Then f is rational if and only if $f_i \in \mathrm{Rat}(X)$ for $i = 1, \ldots, n$. The domain of a rational map f is defined as

$$\mathcal{D}_f = \bigcap_{\phi \in \mathrm{Aff}(Y)} \mathcal{D}_{\phi \circ f}.$$

By Lemma 2.5 $\mathcal{D}_f = \bigcap_{i=1}^n \mathcal{D}_{y_i \circ f}$ is a dense open subset of X.

Lemma 3.10 *Suppose X is irreducible and $f : X \to Y$ is a rational map. If $\mathcal{D}_f = X$ then f is a regular map.*

Let $A \subset B$ be a subalgebra, and identify $\mathrm{Quot}(A)$ with the subfield of $\mathrm{Quot}(B)$ generated by A. If $A = \mathrm{Aff}(X)$ for an irreducible variety X, and $B = \mathrm{Aff}(X^f)$ for some non-zero $f \in A$, then $B = A[b] \subset \mathrm{Quot}(A)$, where $b = 1/f$. In this example, every $\psi \in \mathrm{Hom}(B, \mathbb{C})$ such that $\psi(b) \neq 0$ is given by evaluation at a point $x \in X^f$, and hence ψ is uniquely determined by its restriction to A.

Theorem 3.11 *Let B be a finitely generated algebra over $\mathbb{C}$ with no zero divisors. Let $A \subset B$ be a finitely generated subalgebra. Assume that there exists a nonzero element $b \in B$ so that every element of $\mathrm{Hom}(B, \mathbb{C})^b$ is uniquely determined by its restriction to A. Then $B \subset \mathrm{Quot}(A)$.*

Suppose maps f, g and h satisfy the commutative diagram

$$\begin{tikzcd} M \arrow[r, "h"] \arrow[d, "f"'] & P \\ N \arrow[ur, "g"'] & \end{tikzcd}$$

Then h is constant on the fibers of f, since $f(m) = f(m')$ implies $h(m) = g(f(m)) = h(m')$. Furthermore, if f is surjective, then g is uniquely determined by f and h. Conversely, given f and h satisfying these conditions, we can ask for the regularity properties of the map g such that $h = g \circ f$. We weaken the fiber and surjectivity conditions with the aim of obtaining a rational map g.

Theorem 3.12 *Let M, N and P be irreducible affine varieties, and let $f : M \to N$ and $h : M \to P$ be dominant regular maps. Assume that there is a non-empty open subset U of M so that $f(m) = f(m')$ implies $h(m) = h(m')$ for $m, m' \in U$. Then there exists a rational map $g : N \to P$ such that $h = g \circ f$.*

Exercises for Lecture 3.

1. Let N be the group of matrices
$$\begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix}, \quad z \in \mathbb{C}$$
and let Γ be the subgroup of N consisting of the matrices with $z \in \mathbb{Z}$ an integer. Prove that Γ is Zariski-dense in N.

2. Define a multiplication μ on $\mathbb{C}^3$ by
$$\mu([x_1, x_2, x_3],\, [y_1, y_2, y_3]) = [x_1 + y_1,\, x_2 + y_2,\, x_3 + y_3 + x_1 y_2]$$
(a) Prove that μ satisfies the group axioms and that the inversion map is regular.

(b) Let $N = (\mathbb{C}^3, \mu)$ be the linear algebraic group with regular functions $\mathbb{C}[x_1, x_2, x_3]$ and multiplication μ. Let $R(y)f(x) = f(\mu(x, y))$ be the right translation representation of N on $\mathrm{Aff}(N)$. Let $V \subset \mathbb{C}[x_1, x_2, x_3]$ be the space spanned by 1, x_1, x_2, and x_3. Show that V is invariant under $R(y)$, for $y \in N$.

(c) Let $\rho(y) = R(y)|_V$ for $y \in N$. Calculate the matrix of $\rho(y)$ relative to the basis $\{1, x_1, x_2, x_3\}$ of V. Prove that $\rho : N \to \mathrm{GL}(4, \mathbb{C})$ is injective, and that $N \cong \rho(N)$ as algebraic groups.

3. Define a multiplication μ on $\mathbb{C}^\times \times \mathbb{C}$ by
$$\mu([x_1, x_2],\, [y_1, y_2]) = [x_1 y_1,\, x_2 + x_1 y_2]$$
(a) Prove that μ satisfies the group axioms and that the inversion map is regular.

(b) Let $S = (\mathbb{C}^\times \times \mathbb{C}, \mu)$ be the linear algebraic group with regular functions $\mathbb{C}[x_1, x_1^{-1}, x_2]$ and multiplication μ. Let $R(y)f(x) = f(\mu(x, y))$ be the right translation representation of S on $\mathrm{Aff}(S)$. Let $V \subset \mathrm{Aff}(S)$ be the space spanned by the functions x_1 and x_2. Show that V is invariant under $R(y)$, for $y \in S$.

(c) Let $\rho(y) = R(y)|_V$ for $y \in S$. Calculate the matrix of $\rho(y)$ relative to the basis $\{x_1, x_2\}$ of V. Prove that $\rho : S \to \mathrm{GL}(2, \mathbb{C})$ is injective, and that $S \cong \rho(S)$ as an algebraic group.

Lecture 4. Lie Algebra of an Algebraic Group

Left-invariant Vector Fields

Let $G = \mathrm{GL}(V)$. For any $A \in \mathrm{End}(V)$, $f \in \mathrm{Aff}(G)$ and $x \in G$, define a linear transformation X_A on $\mathrm{Aff}(G)$ by

$$X_A f(x) = \frac{d}{dt} f(x(I + tA))|_{t=0}, \quad \text{for } f \in \mathrm{Aff}(G),\, x \in G.$$

Fix a basis $\{e_1, \ldots, e_n\}$ for V, let E_{ij} be the corresponding elementary matrices and let $\{x_{ij}\}$ be the matrix coordinates. Define $\partial/\partial x_{ij}$ to be the vector field

$$\frac{\partial}{\partial x_{ij}} f(x) = \frac{d}{dt} f(x + tE_{ij})|_{t=0}$$

on $M_n(\mathbb{C})$. Then

$$X_{E_{ij}} f(x) = \frac{d}{dt} f(x + txE_{ij})|_{t=0} = \sum_{r=1}^{n} x_{ri} \frac{\partial}{\partial x_{rj}} f(x) \tag{4.1}$$

If $A = \sum_{i,j} a_{ij} E_{ij}$ with $a_{ij} \in \mathbb{C}$, then X_A is the vector field

$$X_A = \sum_{i,j} a_{ij} X_{E_{ij}}.$$

The operator X_A has the following properties:

$$X_A(f_1 f_2) = (X_A f_1) f_2 + f_1 (X_A f_2) \quad \text{for } f_1, f_2 \in \mathrm{Aff}(G)$$

(the product rule for differentiation) and

$$X_A(L(g)f) = L(g)(X_A f) \quad \text{for } f \in \mathrm{Aff}(G),\, g \in G,$$

where $L(g)f(y) = f(g^{-1}y)$ is the left representation of G on $\mathrm{Aff}(G)$. These two properties say that X_A is a *left-invariant vector field* on G.

Lemma 4.1 *Let $G = \mathrm{GL}(V)$. If $A, B \in \mathrm{End}(V)$ then*

$$[X_A, X_B] = X_{[A,B]}.$$

Furthermore, every left-invariant vector field Y on G is of the form X_A for a unique $A \in \mathrm{End}(V)$.

For $C \in \mathrm{End}(V)$ we have define a function f_C on G by

$$f_C(g) = \mathrm{tr}(gC), \quad \text{for } g \in \mathrm{GL}(V).$$

The functions f_C together with $(\det)^{-1}$ generate the algebra $\mathrm{Aff}(G)$, as C ranges over $\mathrm{End}(V)$. If Y is a vector field on G, then

$$(Y \det{}^{-1})(g) = -\det(g)^{-2}(Y \det)(g).$$

Since $\det(g)$ is a polynomial in the linear functions $\{f_C : C \in \mathrm{End}(V)\}$, it follows from the product rule for derivations that Y is completely determined by its action on the functions f_C.
We define $\mathrm{Lie}(\mathrm{GL}(V)) = \mathrm{End}(V)$, viewed as a Lie algebra with Lie bracket $[A, B] = AB - BA$ as above. If $G \subset \mathrm{GL}(V)$ is an algebraic subgroup, we define

$$\mathrm{Lie}(G) = \{A \in \mathrm{End}(V) : X_A f \in \mathcal{I}_G \quad \text{for all } f \in \mathcal{I}_G\}.$$

Proposition 4.2 *Let G be an algebraic subgroup of* $\mathrm{GL}(V)$. *If $A, B \in \mathrm{Lie}(G)$ and $\lambda \in \mathbb{C}$, then $A + \lambda B$ and $[A, B] \in \mathrm{Lie}(G)$.*

Theorem 4.3 *Let G be a linear algebraic group. For every $g \in G$ the map $A \mapsto (X_A)_g$ is a linear isomorphism from* $\mathrm{Lie}(G)$ *onto $T(G)_g$. Hence G is a smooth algebraic set and* $\dim \mathrm{Lie}(G) = \dim G$.

Lie Algebras of the Classical Groups

Lemma 4.4 *Suppose $G \subset \mathrm{GL}(n, \mathbb{C})$ is a linear algebraic group. Let $z \mapsto \phi(z)$ be a rational map from $\mathbb{C}$ to $M_n(\mathbb{C})$ such that $\phi(0) = I$ and $\phi(z) \in G$ for all $z \in \mathbb{C}$ except possibly for a finite set of nonzero complex numbers. Then the matrix $A = (d/dz)\phi(z)|_{z=0}$ is in* $\mathrm{Lie}(G)$.

Special Linear Group

Let $G = \mathrm{SL}(n, \mathbb{C})$. Then

$$\mathrm{Lie}(G) = \mathfrak{sl}(n, \mathbb{C}) = \{A \in M_n(\mathbb{C}) \, : \, \mathrm{tr}(A) = 0\}.$$

Orthogonal and Symplectic Groups

Let $\Gamma \in M_n(\mathbb{C})$ be nonsingular. Let

$$G_\Gamma = \{g \in \mathrm{GL}(n, \mathbb{C}) \, : \, \Gamma^{-1} g^t \Gamma g = I\}.$$

be the subgroup of $\mathrm{GL}(n, \mathbb{C})$ which preserves the nondegenerate bilinear form $x^t \Gamma y$ on $\mathbb{C}^n$.

Lemma 4.5 *Suppose $A \in M_n(\mathbb{C})$ and* $\det(I - A) \neq 0$. *Then $c(A) \in G_\Gamma$ if and only if $A^t\Gamma + \Gamma A = 0$.*

Theorem 4.6 *The Lie algebra $\mathfrak{g}_\Gamma = \mathrm{Lie}(G_\Gamma)$ consists of all $A \in M_n(\mathbb{C})$ such that*

$$A^t \Gamma + \Gamma A = 0. \tag{4.2}$$

Suppose $n = 2l$ is even. We denote by s_0 the $l \times l$ matrix

$$s_0 = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

with 1 on the skew diagonal and 0 elsewhere. Set

$$J_+ = \begin{bmatrix} 0 & s_0 \\ s_0 & 0 \end{bmatrix}, \qquad J_- = \begin{bmatrix} 0 & s_0 \\ -s_0 & 0 \end{bmatrix},$$

and define the bilinear forms

$$B(x, y) = (x, J_+ y), \qquad \Omega(x, y) = (x, J_- y) \quad \text{for } x, y \in \mathbb{C}^n. \tag{4.3}$$

The form B is nondegenerate and *symmetric*, and the form Ω is nondegenerate and *skew symmetric.*

Corollary 4.7 *The Lie algebra* $\mathfrak{so}(\mathbb{C}^{2l}, B)$ *of* $\mathrm{SO}(\mathbb{C}^{2l}, B)$ *consists of all matrices*

$$A = \begin{bmatrix} a & b \\ c & -s_0 a^t s_0 \end{bmatrix}$$

where $a \in \mathfrak{gl}(l, \mathbb{C})$*, and* b, c *are* $l \times l$ *matrices such that*

$$b^t = -s_0 b s_0, \quad c^t = -s_0 c s_0$$

(b *and* c *are skew symmetric around the skew diagonal).*

Corollary 4.8 *The Lie algebra* $\mathfrak{sp}(\mathbb{C}^{2l}, \Omega)$ *of* $\mathrm{Sp}(\mathbb{C}^{2l}, \Omega)$ *consists of all matrices*

$$A = \begin{bmatrix} a & b \\ c & -s_0 a^t s_0 \end{bmatrix}$$

where $a \in \mathfrak{gl}(l, \mathbb{C})$*, and* b, c *are* $l \times l$ *matrices such that*

$$b^t = s_0 b s_0, \quad c^t = s_0 c s_0$$

(b *and* c *are symmetric around the skew diagonal).*

Corollary 4.9 *The Lie algebra* $\mathfrak{so}(\mathbb{C}^{2l+1}, B)$ *of* $\mathrm{SO}(\mathbb{C}^{2l+1}, B)$ *consists of all matrices*

$$A = \begin{bmatrix} a & w & b \\ u & 0 & -w^t s_0 \\ c & -s_0 u^t & -s_0 a^t s_0 \end{bmatrix}$$

where $a \in \mathfrak{gl}(l, \mathbb{C})$*,* b, c *are* $l \times l$ *matrices such that*

$$b^t = -s_0 b s_0, \quad c^t = -s_0 c s_0$$

(b *and* c *are skew symmetric around the skew diagonal),* w *is a* $l \times 1$ *matrix (column vector), and* u *is an* $1 \times l$ *matrix (row vector).*

Appendix: Algebraic Geometry for Lecture 4.

Tangent Spaces

Suppose $X \subset \mathbb{C}^n$ is an algebraic set. If $x \in X$, then a *tangent vector* to X at x is a linear map $v : \mathrm{Aff}(X) \to \mathbb{C}$ such that

$$v(fg) = v(f)g(x) + f(x)v(g) \tag{4.4}$$

for all $f, g \in \mathrm{Aff}(X)$. We call the set of all tangent vectors at x the *tangent space* of X at x. Let $\mathfrak{m}_x \subset \mathrm{Aff}(X)$ be the maximal ideal of all functions which vanish at x. Then $f - f(x) \in \mathfrak{m}_x$ for any $f \in \mathrm{Aff}(X)$, and $v(f) = v(f - f(x))$. Hence v is determined by its restriction to $\mathfrak{m}_x$. On the other hand, by (4.4) we see that $v(\mathfrak{m}_x^2) = 0$, so v naturally defines an element $\tilde{v} \in (\mathfrak{m}_x/\mathfrak{m}_x^2)^*$. This gives a natural isomorphism

$$T(X)_x \cong (\mathfrak{m}_x/\mathfrak{m}_x^2)^*. \tag{4.5}$$

Vector Fields

A *Lie algebra* is a vector space $\mathfrak{g}$ with a bilinear multiplication (called the *Lie bracket* or *commutator*)

$$\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}, \quad x, y \mapsto [x, y],$$

such that $[x, y] = -[y, x]$ (skew-symmetry) and

$$[x, [y, z]] = [[x, y], z] + [y, [x, z]] \quad \text{(Jacobi identity)}$$

for all $x, y, z \in \mathfrak{g}$. A *derivation* of an algebra A is a linear map $D : A \to A$ such that $D(ab) = D(a)b + aD(b)$. If A is commutative and D, D' are derivations of A, then any linear combination of D, D' with coefficients in A is a derivation, and the *commutator* $[D, D'] = DD' - D'D$ is a derivation. Thus the derivations of A form a Lie algebra $\mathrm{Der}(A)$ and an A-module. When $A = \mathrm{Aff}(X)$ where X is an algebraic set, a derivation of A is called a *vector field*. We denote by $\mathrm{Vect}(X)$ the Lie algebra of all vector fields on X.
Given $L \in \mathrm{Vect}(X)$ and $x \in X$, we define $L_x f = (Lf)(x)$ for $f \in \mathrm{Aff}(X)$. Then $L_x \in T(X)_x$, by the definition of tangent vector. Conversely, if we have a correspondence $x \mapsto L_x \in T(X)_x$ such that the functions $x \mapsto L_x(f)$ are regular for every $f \in \mathrm{Aff}(X)$, then L is a vector field on X.

Dimension and Smoothness of an Affine Algebraic Set

Let X be an irreducible affine algebraic set. The algebra $\mathrm{Aff}(X)$ is finitely generated over $\mathbb{C}$ and has no zero divisors. The following result (the *Noether Normalization Lemma*) describes the structure of such algebras:

Lemma 4.10 *Let k be a field and $B = k[x_1, \ldots, x_n]$ a finitely generated commutative algebra over k without zero divisors. Then there exist $y_1, \ldots, y_r \in B$ such that*
(1) *$\{y_1, \ldots, y_r\}$ is algebraically independent over k;*
(2) *Every $b \in B$ is integral over the subring $k[y_1, \ldots, y_r]$.*
The integer r is uniquely determined by properties (1) *and* (2), *and is called the* transcendence degree *of B over k. A set $\{y_1, \ldots, y_r\}$ with properties* (1) *and* (2) *is called a* transcendence basis *for B over k.*

Let $X \subset \mathbb{C}^n$ be an algebraic set. We define its *dimension* $\dim X$ as follows: When X is irreducible, we let $\dim X$ be the transcendence degree of the algebra $\mathrm{Aff}(X)$. If X is reducible, we let $\dim X$ be the maximum of the dimensions of the irreducible components of X. Let $a \in X$. Then

$$T(X)_a = \{\tilde{v} \in T(\mathbb{C}^n)_a \,:\, \tilde{v}(\mathcal{I}_X) = 0\}.$$

Let $\{f_1, \ldots, f_r\}$ be a generating set of polynomials for the ideal $\mathcal{I}_X$ and set $u_j = \tilde{v}(x_j - a_j)$. Then $\tilde{v} \in T(X)_a$ if and only if

$$\sum_{j=1}^{n} u_j \frac{\partial f_i(a)}{\partial x_j} = 0 \text{ for } i = 1, \ldots, r. \tag{4.6}$$

Hence $\dim T(X)_a = n - \mathrm{rank}(J(a))$, where $J(a)$ is the $r \times n$ Jacobian matrix $[\partial f_i(a)/\partial x_j]$.
If X is irreducible, we define

$$m(X) = \min_{x \in X} \dim T(X)_x.$$

Let $X_0 = \{x \in X \,:\, \dim T(X)_x = m(X)\}$. The points of X_0 are called *smooth.* Since these are the points at which the matrix J defined above has maximum rank $d = n - m(X)$, X_0 is Zariski dense in X. If $X_0 = X$ then X is said to be *smooth.*
If X is a reducible algebraic set with irreducible components X_i, then we say that X is smooth if each X_i is smooth. We define $m(X) = \max_i m(X_i)$ in this case.

Theorem 4.11 *Let X be an algebraic set. Then $m(X) = \dim X$.*

Exercises for Lecture 4.

1. Show that the homomorphism $\mathbb{C}^\times \times \mathrm{SL}(n, \mathbb{C}) \to \mathrm{GL}(n, \mathbb{C})$ given by $(\lambda, g) \mapsto \lambda g$ is surjective. What is its kernel?

2. Consider the bilinear form $\Omega(v, w) = \det[vw]$ for $v, w \in \mathbb{C}^2$.

   (a) Show that Ω is skew-symmetric and nondegenerate.

   (b) Show that $g \in \mathrm{GL}(2, \mathbb{C})$ preserves Ω if and only if $\det(g) = 1$.

   Hence $\mathrm{SL}(2, \mathbb{C}) = \mathrm{Sp}(\mathbb{C}^2, \Omega)$.

3. Let A be in $M_n(\mathbb{C})$. Define $G_A = \{g \in \mathrm{GL}(n, \mathbb{C}) \,:\, gAg^t = A\}$. Set $A_{\mathrm{symm}} = \frac{1}{2}(A + A^t)$, $A_{\mathrm{skew}} = \frac{1}{2}(A - A^t)$. Show that $G_A = G_{A_{\mathrm{symm}}} \cap G_{A_{\mathrm{skew}}}$.

4. Let $\mathcal{A}$ be a finite-dimensional algebra over $\mathbb{C}$. This means that there is a *multiplication map* $\mu : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ which is bilinear (it is not assumed to be associative). Define the *automorphism group* of $\mathcal{A}$ to be

   $$\mathrm{Aut}(\mathcal{A}) = \{g \in \mathrm{GL}(\mathcal{A}) \,:\, g\mu(X, Y) = \mu(gX, gY),\ \text{for } X, Y \in \mathcal{A}\}.$$

   Show that $\mathrm{Aut}(\mathcal{A})$ is an algebraic subgroup of $\mathrm{GL}(\mathcal{A})$.

5. Let Ω be a nondegenerate skew-symmetric bilinear form on a finite-dimensional vector space V. Define $\mathrm{GSp}(V, \Omega)$ to be all $g \in \mathrm{GL}(V)$ for which there is a $\lambda \in \mathbb{C}^\times$ (depending on g) so that

$$\Omega(gx, gy) = \lambda \Omega(x, y) \text{ for all } x, y \in V.$$

(a) Show that the homomorphism $\mathbb{C}^\times \times \mathrm{Sp}(V, \Omega) \to \mathrm{GSp}(V, \Omega)$ given by $(\lambda, g) \mapsto \lambda g$ is surjective. What is its kernel?

(b) Show that $\mathrm{GSp}(V, \Omega)$ is Zariski-closed in $\mathrm{GL}(V)$ and is thus a linear algebraic group.

Lecture 5. Lie Algebra Representations
Adjoint Representation

Differential of a Regular Representation

Theorem 5.1 *Let G be a linear algebraic group, and let (π, V) be a regular representation of G. There is a unique linear map $d\pi : \mathfrak{g} \to \mathrm{End}(V)$ such that*

$$X_A(f_C \circ \pi)(I) = f_{d\pi(A)C}(I) \quad \textit{for all } A \in \mathfrak{g},\, C \in \mathrm{End}(V). \tag{5.1}$$

This map is a Lie algebra homomorphism:

$$d\pi([A, B]) = [d\pi(A), d\pi(B)] \quad \textit{for } A, B \in \mathfrak{g}.$$

Furthermore, for $f \in \mathrm{Aff}(\mathrm{GL}(V))$ and $A \in \mathrm{Lie}(G)$,

$$X_A(f \circ \pi) = (X_{d\pi(A)} f) \circ \pi. \tag{5.2}$$

We call $d\pi$ the *differential* of the representation π.

Examples

1. Let π be the defining representation of $G \subset \mathrm{GL}(n, \mathbb{C})$. Then $d\pi(A) = A$, for $A \in \mathfrak{g}$.

2. Let (π, V) be a regular representation of G. For dual representation (π^*, V^*) we have

$$d\pi^*(A) = -(d\pi(A))^t \quad \text{for } A \in \mathfrak{g}. \tag{5.3}$$

3. Let (π_1, V_1) and (π_2, V_2) be regular representations of G. Let $\pi = \pi_1 \oplus \pi_2$ be the *direct sum* representation on $V = V_1 \oplus V_2$. Then

$$d\pi(X) = d\pi_1(X) \oplus d\pi_2(X).$$

4. Let (π_1, V_1) and (π_2, V_2) be regular representations of G and let $\pi = \pi_1 \otimes \pi_2$ be the *tensor product* of the representations on $V = V_1 \otimes V_2$. Then

$$d\pi(X) = d\pi_1(X) \otimes I + I \otimes d\pi_2(X). \tag{5.4}$$

Theorem 5.2 *Suppose G is a linear algebraic group with Lie algebra $\mathfrak{g}$. Let (π, V) be a regular representation of G.*
(1) *Suppose $W \subset V$ is a linear subspace such that $\pi(g)W \subset W$ for all $g \in G$. Then $d\pi(A)W \subset W$ for all $A \in \mathfrak{g}$.*
(2) *Assume that G is connected. If $W \subset V$ is a linear subspace such that $d\pi(X)W \subset W$ for all $X \in \mathfrak{g}$ then $\pi(g)W \subset W$ for all $g \in G$.*

Proposition 5.3 *If $\pi : G \to H$ is a regular homomorphism, then $d\pi(\mathrm{Lie}(G)) \subset \mathrm{Lie}(H)$ and $d\pi$ is a Lie algebra homomorphism. Furthermore, if K is a linear algebraic group and $\rho : H \to K$ is another regular homomorphism, then $d(\rho \circ \pi) = d\rho \circ d\pi$. In particular, if $G = K$ and $\rho \circ \pi$ is the identity map, then $d\rho \circ d\pi =$ identity, so that isomorphic linear algebraic groups have isomorphic Lie algebras.*

Corollary 5.4 *Suppose G and H are algebraic subgroups of* $\mathrm{GL}(n, \mathbb{C})$.
(1) *If $G \subset H$, then* $\mathrm{Lie}(G) \subset \mathrm{Lie}(H)$.
(2) *If $G \subset H$ and (π, V) is a regular representation of H, then the differential of $\pi|_G$ is $d\pi|_{\mathrm{Lie}(G)}$.*
(3) $\mathrm{Lie}(G \cap H) = \mathrm{Lie}(G) \cap \mathrm{Lie}(H)$.

Proposition 5.5 *Let G be a connected linear algebraic group with Lie algebra $\mathfrak{g}$. Suppose $\sigma : G \to$ $\mathrm{GL}(n, \mathbb{C})$ is a regular representation and $H \subset \mathrm{GL}(n, \mathbb{C})$ is a linear algebraic subgroup with Lie algebra $\mathfrak{h}$ such that $d\sigma(\mathfrak{g}) \subset \mathfrak{h}$. Then $\sigma(G) \subset H$. In particular, if H is connected and $d\sigma(\mathfrak{g}) = \mathfrak{h}$, then $\sigma(G) = H$.*

Differential of the Adjoint Representation

Let G be a linear algebraic group.

Lemma 5.6 *Let $A \in \mathrm{Lie}(G)$ and $g \in G$. Then $gAg^{-1} \in \mathrm{Lie}(G)$.*

Define $\mathrm{Ad}(g)A = gAg^{-1}$ for $g \in G$ and $A \in \mathrm{Lie}(G)$. Then by Lemma 5.6, $\mathrm{Ad}(g) : \mathrm{Lie}(G) \to \mathrm{Lie}(G)$. The representation $(\mathrm{Ad}, \mathrm{Lie}(G))$ is called the *adjoint representation* of G. For $A, B \in \mathrm{Lie}(G)$ we have

$$\mathrm{Ad}(g)[A, B] = [\mathrm{Ad}(g)A, \mathrm{Ad}(g)B],$$

Thus $\mathrm{Ad} : G \to \mathrm{Aut}(\mathrm{Lie}(G))$.

Theorem 5.7 *Let $\mathfrak{g} = \mathrm{Lie}(G)$. The differential of the adjoint representation of G is the representation $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ given by*

$$\mathrm{ad}(A)(B) = [A, B] \text{ for } A, B \in \mathfrak{g}. \tag{5.5}$$

Furthermore, $\mathrm{ad}(A)$ is a derivation of $\mathfrak{g}$, and hence $\mathrm{ad}(\mathfrak{g}) \subset \mathrm{Der}(\mathfrak{g})$.

Lemma 5.8 *Let G be a closed subgroup of the linear algebraic group H. Denote the adjoint representations of G and H by Ad_G and Ad_H. Then*

$$\mathrm{Ad}_H(g)X = \mathrm{Ad}_G(g)X, \text{ for } g \in G,\ X \in \mathrm{Lie}(G). \tag{5.6}$$

Appendix: Algebraic Geometry for Lecture 5.

Differential of a Regular Map

Let X, Y be algebraic sets and $\phi : X \to Y$ a regular map. Then the induced map $\phi^* : \mathrm{Aff}(Y) \to \mathrm{Aff}(X)$ is an algebra homomorphism. If $v \in T(X)_x$ then the linear functional $f \mapsto v(\phi^* f), \quad f \in \mathrm{Aff}(Y)$, is a tangent vector at $y = \phi(x)$ that we denote by $d\phi_x(v)$. At each point $x \in X$ we thus have a *linear* map

$$d\phi_x : T(X)_x \to T(Y)_{\phi(x)}$$

which we call the *differential* of ϕ at x.

Differential Criterion for Dominance of a Map

Proposition 5.9 *Let X, Y be affine algebraic sets and $\psi : X \to Y$ a regular map. Assume Y is irreducible and $\dim Y = m$. Suppose there exists an algebraically independent set $\{u_1, \ldots, u_m\} \subset \mathrm{Aff}(Y)$ such that the set*

$$\{\psi^* u_1, \ldots, \psi^* u_m\} \subset \mathrm{Aff}(X)$$

is also algebraically independent. Then $\psi(X)$ is dense in Y.

Corollary 5.10 *Let $X \subset Y$ with X, Y irreducible affine algebraic sets. Suppose X is closed in Y and $\dim X = \dim Y$. Then $X = Y$.*

Theorem 5.11 *Let X, Y be irreducible affine algebraic sets and $\psi : X \to Y$ a regular map. Suppose there exists a smooth point p of X such that $\psi(p)$ is a smooth point of Y and*

$$d\psi_p : T(X)_p \to T(Y)_{\psi(p)}$$

is bijective. Then $\psi(X)$ is dense in Y.

Lemma 5.12 *Let $X \subset \mathbb{C}^n$ be closed and irreducible and let $p \in X$ be a smooth point of X. Then there exists a open subset $U \subset X$ with $p \in U$ and regular maps $w_j : U \to \mathbb{C}^n$ for $j = 1, \ldots, m = \dim X$ such that*

$$T(X)_q = \bigoplus_{j=1}^{m} \mathbb{C} w_j(q)$$

for all $q \in U$.

Corollary 5.13 *Let X be an irreducible affine algebraic set. Let $K(X) = \mathrm{Quot}(\mathrm{Aff}(X))$ be the field of rational functions on X. Suppose $f \in K(X)$ and $Df = 0$ for all $D \in \mathrm{Der}(K(X))$. Then f is constant.*

Exercises for Lecture 5.

1. Let G and H be linear algebraic groups. Suppose $\phi : G \to H$ is a surjective regular homomorphism such that $\mathrm{Ker}(\phi)$ is finite. Prove that $d\phi : \mathrm{Lie}(G) \to \mathrm{Lie}(H)$ is an isomorphism. (*Hint:* Prove that $\dim G = \dim H$.)

2. Let Ω be a nondegenerate skew-symmetric form on $\mathbb{C}^{2l}$, and let $G = \mathrm{GSp}(\mathbb{C}^{2l}, \Omega)$ be the group introduced in the Exercises for Lecture #1. Find $\mathrm{Lie}(G)$. (*Hint:* Use the surjective homomorphism $\mathbb{C}^\times \times \mathrm{Sp}(\mathbb{C}^{2l}, \Omega) \to G$ and the previous exercise.)

3. Let G be a linear algebraic group and let $\mathfrak{g} = \mathrm{Lie}(G)$. Let (π, V) be a regular representation of G.

   (a) Let B be a G-invariant bilinear form on V. Show that B is $\mathfrak{g}$-invariant. (*Hint*: Consider the representation of G on $V^* \otimes V^*$.)

   (b) Let (σ, W) be another regular representation of G. Set

$$\begin{aligned} \mathrm{Hom}_G(V, W) &= \{T \in \mathrm{Hom}(V, W) : T\pi(g) = \sigma(g)T \text{ for all } g \in G\} \\ \mathrm{Hom}_{\mathfrak{g}}(V, W) &= \{T \in \mathrm{Hom}(V, W) : T d\pi(A) = d\sigma(A)T \text{ for all } A \in \mathfrak{g}\}. \end{aligned}$$

Show that $\mathrm{Hom}_G(V, W) \subset \mathrm{Hom}_{\mathfrak{g}}(V, W)$ and that equality holds if G is connected. (*Hint*: Consider the representation $V^* \otimes W$.)

(c) Show that (a) is a special case of (b).

4. Let G be a linear algebraic group. Let Int be the representation of G on $\mathrm{Aff}(G)$ given by $\mathrm{Int}(g)f(x) = f(g^{-1}xg)$ for $f \in \mathrm{Aff}(G)$ (thus $\mathrm{Int}(g) = L(g)R(g)$). Assume that H is a Zariski closed normal subgroup of G.

   (a) Let $f \in \mathcal{I}_H$. Prove that there is a finite-dimensional subspace $V \subset \mathcal{I}_H$ so that $f \in V$ and $\mathrm{Int}(g)V \subset V$.

   (b) Set $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathfrak{h} = \mathrm{Lie}(H)$. Prove that $\mathrm{Ad}(G)\mathfrak{h} \subset \mathfrak{h}$. (*Hint:* Use (a) to show that $R(g)X_A R(g)^{-1}\mathcal{I}_H \subset \mathcal{I}_H$ for all $A \in \mathfrak{h}$ and all $g \in G$.)

   (c) Prove that $[\mathfrak{g}, \mathfrak{h}] \subset \mathfrak{h}$, and hence $\mathfrak{h}$ is an ideal in $\mathfrak{g}$. (*Hint:* By (b), $\mathfrak{h}$ is an $\mathrm{Ad}(G)$-invariant subspace of $\mathfrak{g}$.)

Lecture 6. Chevalley-Jordan Decomposition
Quotient Groups

Nilpotent and Unipotent Matrices

A matrix $A \in M_n(\mathbb{C})$ is *nilpotent* if $A^k = 0$ for some positive integer k. A linear transformation $u \in M_n(\mathbb{C})$ is called *unipotent* if $u - I$ is nilpotent.
Let $A \in M_n(\mathbb{C})$ be nilpotent. Then $A^n = 0$ and we define

$$\exp A = \sum_{k=0}^{n-1} \frac{1}{k!} A^k = I + Y,$$

where $Y = A + \frac{1}{2!}A^2 + \cdots + \frac{1}{(n-1)!}A^{n-1}$ is also nilpotent. Hence $\exp A$ is unipotent. If $u = I+Y$ is unipotent set

$$\log u = \sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{k} Y^k.$$

The exponential function is a bijective polynomial map from the nilpotent elements in $\mathfrak{gl}(n,\mathbb{C})$ onto the unipotent elements in $\mathrm{GL}(n,\mathbb{C})$, with polynomial inverse $u \mapsto \log u$.

Lemma 6.1 (Taylor's Formula) *Suppose $A \in M_n(\mathbb{C})$ is nilpotent and f is a regular function on $\mathrm{GL}(n,\mathbb{C})$. Then there exists an integer k so that $(X_A)^k f = 0$, and*

$$f(\exp A) = \sum_{m=0}^{k-1} \frac{1}{m!} (X_A)^m f(I). \tag{6.1}$$

Theorem 6.2 *Let $G \subset \mathrm{GL}(n,\mathbb{C})$ be a linear algebraic group.*
(1) *Let $A \in M_n(\mathbb{C})$ be a nilpotent matrix. Then $A \in \mathrm{Lie}(G)$ if and only if $\exp A \in G$.*
(2) *Suppose $A \in \mathrm{Lie}(G)$ is a nilpotent matrix and (ρ, V) is a regular representation of G. Then $d\rho(A)$ is a nilpotent transformation on V, and*

$$\rho(\exp A) = \exp d\rho(A). \tag{6.2}$$

Semisimple One-Parameter Groups

Let V be a vector space and $T \in \mathrm{End}(V)$. For $\lambda \in \mathbb{C}$ let

$$V(T,\lambda) = \{v \in V \,:\, Tv = \lambda v\}.$$

We say that T is a *semisimple* transformation if $V = \bigoplus_\lambda V(T,\lambda)$.

Lemma 6.3 *Let $\phi : \mathbb{C}^\times \to \mathrm{GL}(n,\mathbb{C})$ be a regular homomorphism. For $p \in \mathbb{Z}$ let $E_p = \{v \in \mathbb{C}^n : \phi(z)v = z^p v\}$. Then*

$$\mathbb{C}^n = \bigoplus_{p\in\mathbb{Z}} E_p \tag{6.3}$$

and hence $\phi(z)$ is a semisimple transformation. Conversely, given a direct sum decomposition (6.3) *of $\mathbb{C}^n$, define $\phi(z)v = z^p v$ for $z \in \mathbb{C}^\times$, $v \in E_p$. Then ϕ is a regular homomorphism.*

Jordan-Chevalley Decomposition

Theorem 6.4 *Let $G \subset \mathrm{GL}(n, \mathbb{C})$ be a linear algebraic group and set $\mathfrak{g} = \mathrm{Lie}(G)$.*
(1) *If $A \in \mathfrak{g}$ and $A = S + N$ is its additive Jordan decomposition, then $S, N \in \mathfrak{g}$.*
(2) *If $g \in G$ and $g = su$ is its multiplicative Jordan decomposition, then $s, u \in G$.*

Theorem 6.5 *Let $G \subset \mathrm{GL}(n, \mathbb{C})$ be a linear algebraic group with Lie algebra $\mathfrak{g}$. Suppose (ρ, V) is a regular representation of G.*
(1) *If $A \in \mathfrak{g}$ and $A = S + N$ is its additive Jordan decomposition, then $d\rho(S)$ is semisimple, $d\rho(N)$ is nilpotent, and $d\rho(A) = d\rho(S) + d\rho(N)$ is the additive Jordan decomposition of $d\rho(A)$ in $\mathrm{End}(V)$.*
(2) *If $g \in G$ and $g = su$ is its multiplicative Jordan decomposition in G, then $\rho(s)$ is semisimple, $\rho(u)$ is unipotent, and $\rho(g) = \rho(s)\rho(u)$ is the multiplicative Jordan decomposition of $\rho(g)$ in $\mathrm{GL}(V)$.*

From theorems 6.4 and 6.5 we see that every element g of G has a *semisimple component* g_s and a *unipotent component* g_u which are independent of the embedding $G \subset \mathrm{GL}(V)$, such that $g = g_s g_u$. Likewise, every element $Y \in \mathfrak{g}$ has a *semisimple component* Y_s and a *nilpotent component* Y_n which are independent of the embedding $\mathfrak{g} \subset \mathfrak{gl}(V)$, such that $Y = Y_s + Y_n$.
We denote the set of all semisimple elements of G as G_s and the set of all unipotent elements as G_u. Likewise, we denote the set of all semisimple elements of $\mathfrak{g}$ as $\mathfrak{g}_s$ and the set of all nilpotent elements as $\mathfrak{g}_n$. Since $T \in M_n(\mathbb{C})$ is nilpotent if and only if $T^n = 0$, we have

$$\mathfrak{g}_u = \mathfrak{g} \cap \{T \in M_n(\mathbb{C}) \,:\, T^n = 0\}$$

$$G_u = G \cap \{g \in \mathrm{GL}(n, \mathbb{C}) \,:\, (I - g)^n = 0\}.$$

Thus $\mathfrak{g}_n$ is an algebraic subset of $\mathrm{End}(V)$ and G_u is an algebraic subset of $\mathrm{GL}(V)$. It follows from Theorem 6.2 that the map $N \mapsto \exp(N)$ from $\mathfrak{g}_u$ to G_u is an isomorphism of algebraic sets.

Normal Subgroups and Quotient Groups

Suppose G is a linear algebraic group and $H \subset G$ is a normal algebraic subgroup. The quotient G/H is an (abstract) group. To show that it has the structure of a linear algebraic group we need to construct some representations.

Theorem 6.6 *Suppose G is a linear algebraic group and $N \subset G$ is an algebraic subgroup.*
(1) *There exists a regular representation (π, V) of G and a 1-dimensional subspace $V_0 \subset V$ so that $N = \{g \in G \,:\, \pi(g)V_0 = V_0\}$.*
(2) *If N is normal, then there exists a regular representation (ϕ, W) of G so that $N = \mathrm{Ker}(\phi)$.*

Let G be a connected algebraic group, and $N \subset G$ a normal algebraic subgroup. We define an algebraic group structure on the abstract group $H = G/N$ by taking a regular representation (ϕ, W) of G such that $\mathrm{Ker}(\phi) = N$, whose existence is provided by Theorem 6.6. The group $K = \phi(G) \subset \mathrm{GL}(W)$ is algebraic, by Theorem 3.2. As an abstract group, K is isomorphic to G/N by the map μ such that $\phi = \mu \circ \pi$, where $\pi : G \to G/N$ is the quotient map.

$$\begin{tikzcd} G \arrow[r, "\phi"] \arrow[d, "\pi"'] & K \\ G/N \arrow[ur, "\mu"'] & \end{tikzcd}$$

We define $\mathrm{Aff}(G/N) = \mu^*\mathrm{Aff}(K)$. This gives G/N the structure of an algebraic group, which *a priori* might depend on the choice of the representation ϕ. To show that this structure is unique, we establish the following regularity result for homomorphisms.

Theorem 6.7 *Suppose that G, H and K are algebraic groups, with G connected. Let $\psi : G \to H$ and $\phi : G \to K$ be regular homomorphisms. Assume that ψ is surjective and $\mathrm{Ker}(\psi) \subset \mathrm{Ker}(\phi)$. Let $\mu : H \to K$ be the map such that $\phi = \mu \circ \psi$. Then μ is a regular homomorphism.*

Corollary 6.8 *Assume that G, H are connected algebraic groups and that $\psi : G \to H$ is a bijective regular homomorphism. Then ψ^{-1} is regular, and hence ψ is an isomorphism of algebraic groups.*

We now combine these results to obtain the existence and uniqueness of quotient groups as linear algebraic groups.

Theorem 6.9 *Let G be a connected algebraic group and N a normal algebraic subgroup.*
(1) *The algebraic group structure on G/N defined by a representation ϕ with $\mathrm{Ker}\phi = N$ is independent of the choice of ϕ, and the quotient map $\pi : G \to G/N$ is regular.*
(2) $\pi^*\mathrm{Aff}(G/N) = \mathrm{Aff}(G)^N$*, the right N-invariant regular functions on G.*

Appendix: Linear and Associative Algebra for Lecture 6.

Jordan Decompositions

Let $A \in M_n(\mathbb{C})$. Then there exist $S, N \in M_n(\mathbb{C})$ so that
(1) $A = S + N$
(2) S is semisimple and N is nilpotent
(3) $NS = SN$.
Properties (1), (2), (3) uniquely determine N and S. Furthermore, there is a polynomial $\phi(x)$ so that $S = \phi(A)$. We write $A_s = S$ and $A_n = N$ for the semisimple and nilpotent parts of A and call $A = S + N$ the *additive Jordan decomposition* of A.
There is a corresponding *multiplicative Jordan decomposition:* Let $g \in \mathrm{GL}(n, \mathbb{C})$. There exist $s, u \in \mathrm{GL}(n, \mathbb{C})$ so that
(1) $g = su$
(2) s is semisimple and u is unipotent
(3) $us = su$.
Properties (1), (2), (3) uniquely determine u and s. Furthermore, there is a polynomial $\phi(x)$ so that $s = \phi(g)$. We write $s = g_s$ and $u = g_u$ for the semisimple and unipotent factors in the multiplicative Jordan decomposition of g.

Exercises for Lecture 6.

1. Suppose V and W are finite-dimensional vector spaces over $\mathbb{C}$. Let $x \in \mathrm{GL}(V)$ and $y \in \mathrm{GL}(W)$ have multiplicative Jordan decompositions $x = x_s x_u$ and $y = y_s y_u$. Prove that the multiplicative Jordan decomposition of $x \otimes y$ in $\mathrm{GL}(V \otimes W)$ is $x \otimes y = (x_s \otimes y_s)(x_u \otimes y_u)$.

2. Suppose $\mathcal{A}$ is a finite-dimensional algebra over $\mathbb{C}$ (not necessarily associative). For example, $\mathcal{A}$ could be a Lie algebra. Let $g \in \mathrm{Aut}(\mathcal{A})$ have multiplicative Jordan decomposition $g = g_s g_u$ in $\mathrm{GL}(\mathcal{A})$. Show that g_s and g_u are also in $\mathrm{Aut}(\mathcal{A})$.

3. Let $G = \mathrm{SL}(2, \mathbb{C})$.

   (a) Show that $\{g \in G : \mathrm{tr}(g)^2 \neq 4\} \subset G_s$. (*Hint:* Show that the elements in this set have distinct eigenvalues.)

   (b) Let $u(t) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$ and $v(t) = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}$ for $t \in \mathbb{C}$. Show that $u(r)v(t) \in G_s$ whenever $rt(4 + rt) \neq 0$ and that $u(r)v(t)u(r) \in G_s$ whenever $rt(2 + rt) \neq 0$.

   (c) Show that G_s and G_u are not subgroups of G.

   (d) Show that every Zariski neighborhood of 1 in G contains unipotent elements, and hence G_s is not closed in G. (*Hint:* If $f \in \mathrm{Aff}(G)$ and $f(1) \neq 0$ then $f(u(t))$ is a non-vanishing polynomial in t.)

4. Let G be a connected linear algebraic group and let $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ be the adjoint representation of G. Let $N = \mathrm{Ker}(\mathrm{Ad})$. The group G/N is called the *adjoint group* of G.

   (a) Suppose $\mathfrak{g}$ is a simple Lie algebra. Prove that N is finite.

   (b) Suppose $G = \mathrm{SL}(n, \mathbb{C})$, so that $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$. Find N in this case. The group G/N is denoted by $\mathrm{PSL}(n, \mathbb{C})$ (the *projective linear group*).

Part 2: Stucture of Classical Groups

Lecture 7. Maximal Tori and Unipotent Generators for Classical Groups

Algebraic Tori

An *algebraic torus* is an algebraic group T isomorphic to $\mathbb{C}^\times \times \cdots \times \mathbb{C}^\times$ (l factors); the integer l is the *rank* of T. If G is a linear algebraic group, then a torus $H \subset G$ is *maximal* if it is not contained in any larger torus in G.
Suppose now that G is one of the classical groups $\mathrm{GL}(l, \mathbb{C})$, $\mathrm{SL}(l+1, \mathbb{C})$, $\mathrm{Sp}(\mathbb{C}^{2l}, \Omega)$, $\mathrm{SO}(\mathbb{C}^{2l}, B)$, or $\mathrm{SO}(\mathbb{C}^{2l+1}, B)$. We take as Ω and B the specific bilinear forms used in Lecture 4, Corollaries 4.7, 4.8, and 4.9. Let H be the subgroup of diagonal matrices in G.

(1) When $G = \mathrm{SL}(l+1, \mathbb{C})$ (type A_l), then

$$H = \{\mathrm{diag}[x_1, \ldots, x_l, (x_1 \cdots x_l)^{-1}] \,:\, x_i \in \mathbb{C}^\times\},$$

and

$$\mathrm{Lie}(H) = \{\mathrm{diag}[a_1, \ldots, a_{l+1}]\,;\, a_i \in \mathbb{C}, \quad \sum a_i = 0\}.$$

(2) When $G = \mathrm{Sp}(\mathbb{C}^{2l}, \Omega)$ (type C_l) or $G = \mathrm{SO}(\mathbb{C}^{2l}, B)$ (type D_l), then

$$H = \{\mathrm{diag}[x_1, \ldots, x_l, x_l^{-1}, \ldots, x_1^{-1}] \,:\, x_i \in \mathbb{C}^\times\},$$

and

$$\mathrm{Lie}(H) = \{\mathrm{diag}[a_1, \ldots, a_l, -a_l, \ldots, -a_1]\,;\, a_i \in \mathbb{C}\}.$$

(3) When $G = \mathrm{SO}(\mathbb{C}^{2l+1}, B)$ (type B_l), then

$$H = \{\mathrm{diag}[x_1, \ldots, x_l, 1, x_l^{-1}, \ldots, x_1^{-1}] \,:\, x_i \in \mathbb{C}^\times\}$$

and

$$\mathrm{Lie}(H) = \{\mathrm{diag}[a_1, \ldots, a_l, 0, -a_l, \ldots, -a_1]\,;\, a_i \in \mathbb{C}\}.$$

In all cases H is isomorphic as an algebraic group to the product of l copies of $\mathbb{C}^\times$, so it is a torus of rank l. Its Lie algebra is isomorphic to the vector space $\mathbb{C}^l$ with all Lie brackets zero. Define coordinate functions $x_1, \ldots, x_l$ on H as above. Then

$$\mathrm{Aff}(H) = \mathbb{C}[x_1, \ldots, x_l, x_1^{-1}, \ldots, x_l^{-1}].$$

For any algebraic group K, a *rational character* of K is a regular homomorphism $\chi : K \to \mathbb{C}^\times$. Denote by $\mathcal{X}(K)$ the set of rational characters of K. It has the natural structure of an abelian group under pointwise multiplication.

Lemma 7.1 *Let T be an algebraic torus of rank l. The group $\mathcal{X}(T)$ is isomorphic to $\mathbb{Z}^l$. Furthermore, $\mathcal{X}(T)$ is linearly independent, as a set of functions on H.*

Maximal Tori

Theorem 7.2 *Let G be* $\mathrm{GL}(n, \mathbb{C})$, $\mathrm{SL}(n, \mathbb{C})$, $\mathrm{SO}(\mathbb{C}^n, B)$ *or* $\mathrm{Sp}(\mathbb{C}^{2l}, \Omega)$ *in the form given above, H the diagonal subgroup in G. Suppose $g \in G$ and $gh = hg$ for all $h \in H$. Then $g \in H$.*

Corollary 7.3 *Let G and H be as in Theorem 7.2. Suppose $T \subset G$ is an abelian subgroup (not assumed to be algebraic). If $H \subset T$ then $H = T$. In particular, H is a maximal torus in G.*

Lemma 7.4 *Let T be a torus. Then there exists an element $t \in T$ so that the subgroup generated by t is Zariski dense in T.*

Theorem 7.5 (Notation as in Theorem 7.2) *Every semisimple element of G is G-conjugate to an element of H. Thus*

$$G_s = \bigcup_{\gamma \in G} \gamma H \gamma^{-1}. \tag{7.1}$$

Corollary 7.6 *Let T be any maximal torus in G. Then there exists $g \in G$ so that $gTg^{-1} = H$.*

From Corollary 7.6, we see that the integer $l = \dim H$ does not depend on the choice of a particular maximal torus in G. We call l the *rank* of G.

Unipotent Generators for Classical Groups

We begin with the basic case $G = \mathrm{SL}(2, \mathbb{C})$. Let N be the subgroup of G consisting of the unipotent matrices

$$u(z) = \begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix}, \quad z \in \mathbb{C}$$

and let $\bar{N}$ be the subgroup of G consisting of the unipotent matrices

$$v(z) = \begin{bmatrix} 1 & 0 \\ z & 1 \end{bmatrix}, \quad z \in \mathbb{C}.$$

Lemma 7.7 $\mathrm{SL}(2, \mathbb{C})$ *is generated by $N \cup \bar{N}$.*

Theorem 7.8 *Let G be one of the groups* $\mathrm{SL}(l+1, \mathbb{C})$, $\mathrm{SO}(2l+1, \mathbb{C})$, $\mathrm{Sp}(l, \mathbb{C})$, *with $l \geq 1$, or* $\mathrm{SO}(2l, \mathbb{C})$ *with $l \geq 2$. Then G is generated by its unipotent elements.*

Connectedness of Classical Groups

Theorem 7.9 *The algebraic groups* $\mathrm{GL}(n, \mathbb{C})$, $\mathrm{SL}(n, \mathbb{C})$, $\mathrm{SO}(n, \mathbb{C})$ *and* $\mathrm{Sp}(n, \mathbb{C})$ *are connected in the Zariski topology.*

Roots with respect to a Maximal Torus

Assume G is a connected classical group of rank l, and set $\mathfrak{g} = \mathrm{Lie}(G)$. Thus G is $\mathrm{GL}(l, \mathbb{C})$, $\mathrm{SL}(l+1, \mathbb{C})$, $\mathrm{Sp}(\mathbb{C}^{2l}, \Omega)$, $\mathrm{SO}(\mathbb{C}^{2l}, B)$, or $\mathrm{SO}(\mathbb{C}^{2l+1}, B)$ with B chosen so that the subgroup H of diagonal matrices in G is a maximal torus of rank l. We write $\mathrm{Lie}(H) = \mathfrak{h}$. We let $x_1, \ldots, x_l$ be the coordinate functions on H used in the proof of Theorem 7.2. The group $\mathcal{X}(H)$ of rational characters of H is isomorphic to the additive group $\mathbb{Z}^l$ (see Lemma 7.1). Here $\lambda = [\lambda_1, \ldots, \lambda_l] \in \mathbb{Z}^l$ corresponds to the character $h \mapsto h^\lambda$, where

$$h^\lambda = \prod_{k=1}^{l} x_k(h)^{\lambda_k}, \quad \text{for } h \in H. \tag{7.2}$$

We denote this character by e^λ.
Fix a basis for $\mathfrak{h}^*$ as follows:

(1) Let $G = \mathrm{GL}(l, \mathbb{C})$. Define the linear functional ε_i on $\mathfrak{h}$ by $\langle \varepsilon_i, A \rangle = a_i$ for $A = \mathrm{diag}[a_1, \ldots, a_l] \in \mathfrak{h}$. Then $\{\varepsilon_1, \ldots, \varepsilon_l\}$ is a basis for $\mathfrak{h}^*$.

(2) Let $G = \mathrm{SL}(l+1, \mathbb{C})$. In this case $\mathfrak{h}$ consists of all diagonal matrices of trace zero. Define ε_i as in (1) as a linear functional on the space of diagonal matrices for $i = 1, \ldots, l+1$. The restriction of ε_i to $\mathfrak{h}$ is then an element of $\mathfrak{h}^*$. With an abuse of notation we will continue to denote this linear functional as ε_i. The elements of $\mathfrak{h}^*$ can be written uniquely as

$$\sum_{i=1}^{l+1} \lambda_i \varepsilon_i, \qquad \text{with } \lambda_i \in \mathbb{C} \text{ and } \sum_{i=1}^{l+1} \lambda_i = 0.$$

The functionals

$$\varepsilon_i - \frac{1}{l+1}(\varepsilon_1 + \cdots + \varepsilon_{l+1}) \qquad \text{for } i = 1, \ldots, l$$

give a basis for $\mathfrak{h}^*$.

(3) Let G be $\mathrm{Sp}(\mathbb{C}^{2l}, \Omega)$ or $\mathrm{SO}(\mathbb{C}^{2l}, B)$. Define the linear functionals ε_i on $\mathfrak{h}$ by $\langle \varepsilon_i, A \rangle = a_i$ for $A = \mathrm{diag}[a_1, \ldots, a_l, -a_l, \ldots, -a_1] \in \mathfrak{h}$ and $i = 1, \ldots, l$. Then $\{\varepsilon_1, \ldots, \varepsilon_l\}$ is a basis for $\mathfrak{h}^*$.

(4) Let $G = \mathrm{SO}(\mathbb{C}^{2l+1}, B)$. Define the linear functionals ε_i on $\mathfrak{h}$ by

$$\langle \varepsilon_i, A \rangle = a_i \quad \text{for } A = \mathrm{diag}[a_1, \ldots, a_l, 0, -a_l, \ldots, -a_1] \in \mathfrak{h} \text{ and } i = 1, \ldots, l.$$

Then $\{\varepsilon_1, \ldots, \varepsilon_l\}$ is a basis for $\mathfrak{h}^*$.

We define $P(G) = \mathrm{Span}\{d\theta : \theta \in \mathcal{X}(H)\} \subset \mathfrak{h}^*$. With the functionals ε_i defined as above, we then have

$$P(G) = \bigoplus_{k=1}^{l} \mathbb{Z}\varepsilon_k. \tag{7.3}$$

For $\alpha \in \mathfrak{h}^*$ let

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : [A, X] = \langle \alpha, A \rangle X \text{ for all } A \in \mathfrak{h}\}.$$

If $\alpha \neq 0$ and $\mathfrak{g}_\alpha \neq 0$ then α is called a *root* and $\mathfrak{g}_\alpha$ is called a *root space.* If α is a root then a nonzero element of $\mathfrak{g}_\alpha$ is called a *root vector* for α. We call the set Φ of roots the *root system* of $\mathfrak{g}$.

Its definition requires fixing a choice of maximal torus, so we write $\Phi = \Phi(\mathfrak{g}, \mathfrak{h})$ when we want to make this choice explicit.

General Linear Group: Let $G = \mathrm{GL}(l, \mathbb{C})$, and let $E_{i,j}$, for $1 \le i, j \le l$, be the usual elementary matrix which takes the basis vector e_j to e_i. The roots are

$$\{\varepsilon_i - \varepsilon_j \,:\, 1 \le i, j \le l,\, i \ne j\},$$

each with multiplicity 1. The root space $\mathfrak{g}_\lambda = \mathbb{C}E_{i,j}$ for $\lambda = \varepsilon_i - \varepsilon_j$.

Type C: Let $G = \mathrm{Sp}(\mathbb{C}^{2l}, \Omega)$. Label the basis for $\mathbb{C}^{2l}$ as $e_{\pm 1}, \ldots e_{\pm l}$ with $e_{-i} = e_{2l+1-i}$. Let $E_{i,j}$ be the matrix that takes the basis vector e_j to e_i, where i and j range over $\pm 1, \ldots, \pm l$. Set $X_{\varepsilon_i - \varepsilon_j} = E_{i,j} - E_{-j,-i}$ for $1 \le i, j \le l$, $i \ne j$. Then $X_{\varepsilon_i - \varepsilon_j} \in \mathfrak{g}$ is a root vector for the root $\varepsilon_i - \varepsilon_j$. Set

$$X_{\varepsilon_i + \varepsilon_j} = E_{i,-j} + E_{j,-i}, \quad X_{-\varepsilon_i - \varepsilon_j} = E_{-j,i} + E_{-i,j}$$

for $1 \le i < j \le l$ and set $X_{2\varepsilon_i} = E_{i,-i}$ for $1 \le i \le l$. Then $X_{\pm(\varepsilon_i + \varepsilon_j)}$ is a root vector for the root $\pm(\varepsilon_i + \varepsilon_j)$ for $1 \le i \le j \le l$. This gives the complete set of roots.

Type D: Let $G = \mathrm{SO}(\mathbb{C}^{2l}, B)$. Label the basis for $\mathbb{C}^{2l}$ and define $X_{\varepsilon_i - \varepsilon_j}$ as in the case of $\mathrm{Sp}(\mathbb{C}^{2l}, \Omega)$. Then $X_{\varepsilon_i - \varepsilon_j} \in \mathfrak{g}$ is a root vector for the root $\varepsilon_i - \varepsilon_j$.

$$X_{\varepsilon_i + \varepsilon_j} = E_{i,-j} - E_{j,-i}, \quad X_{-\varepsilon_i - \varepsilon_j} = E_{-j,i} - E_{-i,j} \quad \text{for } 1 \le i < j \le l.$$

Then $X_{\pm(\varepsilon_i + \varepsilon_j)} \in \mathfrak{g}$ is a root vector for the root $\pm(\varepsilon_i + \varepsilon_j)$. The roots are

$$\pm(\varepsilon_i - \varepsilon_j) \text{ and } \pm(\varepsilon_i + \varepsilon_j) \text{ for } 1 \le i < j \le l,$$

each with multiplicity one.

Type B: Let $G = \mathrm{SO}(\mathbb{C}^{2l+1}, B)$. We label the basis for $\mathbb{C}^{2l+1}$ as

$$e_{-l}, \cdots, e_{-1}, e_0, e_1, \ldots, e_l,$$

where $e_0 = e_{l+1}$ and $e_{-i} = e_{2l+2-i}$. Let $E_{i,j}$ be the matrix that takes the basis vector e_j to e_i, where i and j range over $0, \pm 1, \ldots, \pm l$. Then

$$X_{\varepsilon_i - \varepsilon_j} = E_{i,j} - E_{-j,-i}, \quad X_{\varepsilon_j - \varepsilon_i} = E_{j,i} - E_{-i,-j}$$

$$X_{\varepsilon_i + \varepsilon_j} = E_{i,-j} - E_{j,-i}, \quad X_{-\varepsilon_i - \varepsilon_j} = E_{-j,i} - E_{-i,j}$$

are root vectors for $1 \le i < j \le l$. Define

$$X_{\varepsilon_i} = E_{i,0} - E_{0,-i}, \quad X_{-\varepsilon_i} = E_{0,i} - E_{-i,0}$$

for $1 \le i \le l$. Then $X_{\pm\varepsilon_i} \in \mathfrak{g}$ is a root vector. The roots of $\mathfrak{so}(\mathbb{C}^{2l+1}, B)$ are

$$\pm(\varepsilon_i - \varepsilon_j) \text{ and } \pm(\varepsilon_i + \varepsilon_j) \text{ for } 1 \le i < j \le l, \quad \pm\varepsilon_k \text{ for } 1 \le k \le l,$$

each with multiplicity one.

Theorem 7.10 *Let G be a classical group and let $H \subset G$ be a maximal torus. Let $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathfrak{h} = \mathrm{Lie}(H)$ and let $\Phi = \Phi(\mathfrak{g}, \mathfrak{h})$ be the set of roots of $\mathfrak{h}$ on $\mathfrak{g}$.*
(1) *If $\alpha \in \Phi$ then $\alpha \in P(G)$, $\dim \mathfrak{g}_\alpha = 1$, and*

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Phi} \mathfrak{g}_\alpha.$$

(2) *If $\alpha \in \Phi$ and $c\alpha \in \Phi$ for some $c \in \mathbb{C}$ then $c = \pm 1$.*
(3) *The symmetric bilinear form $(X, Y) = \mathrm{tr}(XY)$ on $\mathfrak{g}$ is invariant:*

$$([X, Y], Z) = -(Y, [X, Z]) \quad \text{for } X, Y, Z \in \mathfrak{g}.$$

(4) *Let $\alpha, \beta \in \Phi$ and $\alpha \neq -\beta$. Then $(\mathfrak{h}, \mathfrak{g}_\alpha) = 0$ and $(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$.*
(5) *The form (X, Y) on $\mathfrak{g}$ is non-degenerate.*

Exercises for Lecture 7.

1. (*Cayley Parameters*) Let Γ be a nonsingular $n \times n$ matrix. Assume that either $\Gamma = \Gamma^t$ or $\Gamma = -\Gamma^t$. Let $G = \{g \in \mathrm{GL}(n, \mathbb{C}) : g^t \Gamma g = \Gamma\}$ and let $\mathfrak{g} \subset M_n(\mathbb{C})$ be the Lie algebra of G. Set
$$\mathcal{V}_G = \{g \in G : \det(I + g) \neq 0\}, \qquad \mathcal{V}_{\mathfrak{g}} = \{X \in \mathfrak{g} : \det(I - X) \neq 0\}.$$
For $X \in \mathcal{V}_{\mathfrak{g}}$ define the Cayley transform $c(X) = (I + X)(I - X)^{-1}$. (Recall that $c(X) \in G$.)

   (a) Show that c is a bijection from $\mathcal{V}_{\mathfrak{g}}$ onto $\mathcal{V}_G$.

   (b) Show that $\mathcal{V}_{\mathfrak{g}}$ is invariant under the adjoint action of G on $\mathfrak{g}$, and that $gc(X)g^{-1} = c(gXg^{-1})$ for $g \in G$ and $X \in \mathcal{V}_{\mathfrak{g}}$.

   (c) Prove that $\mathcal{V}_G$ is a dense Zariski-open set in G containing I and invariant under inner automorphisms. (*Hint:* G is connected.)

2. Let (ρ, V) be a regular representation of a linear algebraic group G. Suppose $W \subset V$ is invariant under $d\rho(\mathfrak{g})$.

   (a) Let $X \in \mathfrak{g}$ be nilpotent. Show that $\rho(\exp X)W \subset W$ by considering the Taylor expansion of the polynomial $t \mapsto \langle v^*, \rho(\exp tX)v \rangle$ for $v^* \in V^*$ and $v \in V$.

   (b) Suppose G is generated by unipotent elements. Use (a) to prove that $\rho(G)W \subset W$.

3. For $0 \le k \le 4$ let $\bigwedge^k \mathbb{C}^4$ be the k^{th} exterior power of $\mathbb{C}^4$. It has basis $e_{i_1} \wedge \cdots \wedge e_{i_k}$, where $1 \le i_1 < \cdots < i_k \le 4$ and $e_1, \cdots, e_4$ is the usual basis for $\mathbb{C}^4$. In particular, $\dim \bigwedge^2 \mathbb{C}^4 = 6$ and $\dim \bigwedge^4 \mathbb{C}^4 = 1$. There is a representation of $\mathrm{SL}(4, \mathbb{C})$ on $\bigwedge^k \mathbb{C}^4$:
$$g \cdot (v_1 \wedge \cdots \wedge v_k) = gv_1 \wedge \cdots \wedge gv_k$$
for $g \in \mathrm{SL}(4, \mathbb{C})$ and $v_1, \ldots, v_k \in \mathbb{C}^4$. The differential of this representation gives the action
$$X \cdot (v_1 \wedge \cdots \wedge v_k) = Xv_1 \wedge \cdots \wedge v_k + \cdots + v_1 \wedge \cdots \wedge Xv_k$$

of $X \in \mathfrak{sl}(4, \mathbb{C})$. For $k = 2$ we denote this representation by ρ. The wedge product $a, b \mapsto a \wedge b$ defines a map $\bigwedge^2 \mathbb{C}^4 \times \bigwedge^2 \mathbb{C}^4 \to \bigwedge^4 \mathbb{C}^4$. Since $\bigwedge^4 \mathbb{C}^4 = \mathbb{C}\Omega$, where $\Omega = e_1 \wedge e_2 \wedge e_3 \wedge e_4$, there is a bilinear form B on $\bigwedge^2 \mathbb{C}^4$ so that $a \wedge b = B(a, b)\Omega$.

(a) Prove that the form B is symmetric and non-degenerate.

(b) Prove that $B(\rho(g)a, \rho(g)b) = B(a, b)$ and $B(d\rho(X)a, b) + B(a,\, d\rho(X)b) = 0$ for $g \in \mathrm{SL}(4, \mathbb{C})$, $X \in \mathfrak{sl}(4, \mathbb{C})$ and $a, b \in \bigwedge^2 \mathbb{C}^4$. (*Hint:* Show that Ω is invariant under $\mathrm{SL}(4, \mathbb{C})$ and use the definition of B in terms of the wedge product.)

(c) Use $d\rho$ to obtain a Lie algebra isomorphism $\mathfrak{sl}(4, \mathbb{C}) \cong \mathfrak{so}(\bigwedge^2 \mathbb{C}^4, B)$. (*Hint*: $\mathfrak{sl}(4, \mathbb{C})$ is a simple Lie algebra.)

(d) Explain the isomorphism in (c) in terms of the classification of simple Lie algebras by Dynkin diagrams.

(e) Show that $\rho : \mathrm{SL}(4, \mathbb{C}) \to \mathrm{SO}(\bigwedge^2 \mathbb{C}^4, B)$ is surjective, and $\mathrm{Ker}(\rho) = \{\pm I\}$. (*Hint:* For the surjectivity, use (c) and the fact that $\mathrm{SL}(4, \mathbb{C})$ and $\mathrm{SO}(\bigwedge^2 \mathbb{C}^4, B)$ are connected and of the same dimension. To determine $\mathrm{Ker}(\rho)$, use (c) to show that $\mathrm{Ad}(g) = I$ for all $g \in \mathrm{Ker}(\rho)$.)

4. Let B be the symmetric bilinear form on $\bigwedge^2 \mathbb{C}^4$ and let ρ be the representation of $\mathrm{SL}(4, \mathbb{C})$ on $\bigwedge^2 \mathbb{C}^4$ as in the previous exercise. Let

$$\omega = e_1 \wedge e_4 + e_2 \wedge e_3 \in \textstyle\bigwedge^2 \mathbb{C}^4.$$

Identify $\mathbb{C}^4$ with $(\mathbb{C}^4)^*$ by the inner product $(x, y) = x^t y$, so that ω can also be viewed as a skew-symmetric bilinear form on $\mathbb{C}^4$. Define $\mathcal{L} = \{a \in \bigwedge^2 \mathbb{C}^4 \,:\, B(a,\, \omega) = 0\}$.

(a) Prove that $\rho(g)\mathcal{L} \subset \mathcal{L}$ for all $g \in \mathrm{Sp}(\mathbb{C}^4, \omega)$, and that $\bigwedge^2 \mathbb{C}^4 = \mathbb{C}\omega \oplus \mathcal{L}$.

(b) Let β be the restriction of the bilinear form B to $\mathcal{L} \times \mathcal{L}$. Prove that β is non-degenerate.

(c) Let $\phi(g)$ be the restriction of $\rho(g)$ to the subspace $\mathcal{L}$, for $g \in \mathrm{Sp}(\mathbb{C}^4, \omega)$. Use $d\phi$ to obtain a Lie algebra isomorphism $\mathfrak{sp}(\mathbb{C}^4, \omega) \cong \mathfrak{so}(\mathbb{C}^5, \beta)$. (*Hint*: $\mathfrak{sp}(\mathbb{C}^4, \omega)$ is a simple Lie algebra.)

(d) Explain the isomorphism in (c) in terms of the classification of simple Lie algebras by Dynkin diagrams.

(e) Show that $\phi : \mathrm{Sp}(\mathbb{C}^4, \omega) \to \mathrm{SO}(\mathcal{L}, \beta)$ is surjective, and $\mathrm{Ker}(\phi) = \{\pm I\}$. (*Hint:* For the surjectivity, use (c) and the fact that $\mathrm{Sp}(\mathbb{C}^4, \omega)$ and $\mathrm{SO}(\mathcal{L}, \beta)$ are connected and of the same dimension. To determine $\mathrm{Ker}(\phi)$, use (c) to show that $\mathrm{Ad}(g) = I$ for all $g \in \mathrm{Ker}(\phi)$.)

Lecture 8. Adjoint Representation and Reductivity of Classical Groups

Representations of $\mathfrak{sl}(2,\mathbb{C})$

Let $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$. The matrices

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

are a basis for $\mathfrak{g}$ and satisfy the commutation relations

$$[h,e] = 2e, \quad [h,f] = -2f, \quad [e,f] = h. \tag{8.1}$$

Any triple $\{e,f,h\}$ of non-zero elements in a Lie algebra which satisfies (8.1) will be called a TDS (three-dimensional simple) triple.

Lemma 8.1 *Let (π, V) be a representation of $\mathfrak{g}$ (with V possibly infinite-dimensional). Set $E = \pi(e)$, $F = \pi(f)$ and $H = \pi(h)$.*
(1) *For all integers $k \geq 1$*

$$[H, F^k] = -2kF^k, \quad [E, F^k] = kF^{k-1}(H - k + 1). \tag{8.2}$$

(2) *Suppose $0 \neq v_0 \in V$ satisfies $Hv_0 = \lambda_0 v_0$ for some $\lambda_0 \in \mathbb{C}$ and $Ev_0 = 0$. Set $v_k = (1/k!)\, F^k v_0$ for $k = 0, 1, 2, \ldots$. Then*

$$Hv_k = (\lambda_0 - 2k)v_k, \quad Ev_k = (\lambda_0 - k + 1)v_{k-1}\,. \tag{8.3}$$

(3) *Let v_0 and λ_0 be as in (2). If $\lambda_0 \notin \{0, 1, 2, \ldots, k-1\}$ then the set*

$$\{v_0, v_1, \ldots, v_k\}$$

is linearly independent. Hence if $\dim V < \infty$ then $\lambda_0 = n$ for some nonnegative integer n and $v_{n+1} = 0$.

Proposition 8.2 (1) *Let (π, V) be a finite-dimensional representation of $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$. Then there exists $0 \neq v_0 \in V$ and an integer $n \geq 0$ such that*

$$\pi(h)v_0 = nv_0, \quad \pi(e)v_0 = 0. \tag{8.4}$$

Define $v_k = (1/k!)\, \pi(f)^k v_0$ for $k = 0, 1, \ldots, n$. Then $\{v_0, \ldots, v_n\}$ is linearly independent and spans an irreducible $\mathfrak{g}$-invariant subspace W of V. The action of $\mathfrak{g}$ on W is given by

$$\begin{gathered} \pi(h)v_k = (n - 2k)v_k \\ \pi(f)v_k = (k+1)v_{k+1}, \quad \pi(e)v_k = (n - k + 1)v_{k-1}, \end{gathered} \tag{8.5}$$

with the convention that $v_{-1} = 0$ and $v_{n+1} = 0$. In particular, if V is irreducible, then $\{v_0, \ldots, v_n\}$ is a basis for V and $\dim V = n + 1$.
(2) *Let n be a nonnegative integer. Let V be an $n+1$-dimensional vector space with basis $\{v_0, v_1, \ldots, v_n\}$. Then formulas (8.5) define an irreducible representation π of $\mathfrak{g}$ in which $\pi(h)$ is semisimple. The eigenvalues of $\pi(h)$ are*

$$n, n-2, \ldots, -n+2, -n$$

and each eigenvalue has multiplicity one.

Representations of $\mathrm{SL}(2,\mathbb{C})$

Proposition 8.3 *Let* $G = \mathrm{SL}(2,\mathbb{C})$*,* N *the upper-triangular unipotent matrices, and* $\bar{N}$ *the lower-triangular unipotent matrices in* G*. Let* $d(a) = \mathrm{diag}[a, a^{-1}]$ *for* $a \in \mathbb{C}^\times$*.*
For every integer $n \geq 0$ *there is a unique (up to equivalence) irreducible representation* (ρ, V) *of* G *of dimension* $n+1$*. The semisimple operator* $\rho(d(a))$ *has eigenvalues*

$$a^n, a^{n-2}, \ldots, a^{-n+2}, a^{-n}.$$

The space V^N *of* N*-fixed vectors is one-dimensional, and* $\rho(d(a))$ *acts on it by the scalar* a^n*. The space* $V^{\bar{N}}$ *of* $\bar{N}$*-fixed vectors is also one-dimensional, and* $\rho(d(a))$ *acts on it by the scalar* a^{-n}*. The differential of* ρ *is the representation* π *in Proposition 8.2. Every irreducible regular representation of* G *is equivalent to one of these representations.*

Commutation Relations of Root Spaces

Let G be a classical group and let $H \subset G$ be a maximal torus. Let $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathfrak{h} = \mathrm{Lie}(H)$ and let $\Phi = \Phi(\mathfrak{g}, \mathfrak{h})$ be the set of roots of $\mathfrak{h}$ on $\mathfrak{g}$.

Lemma 8.4 *For each* $\alpha \in \Phi$ *there exist* $e_\alpha \in \mathfrak{g}_\alpha$ *and* $f_\alpha \in \mathfrak{g}_{-\alpha}$ *such that the element* $h_\alpha = [e_\alpha, f_\alpha] \in \mathfrak{h}$ *satisfies* $\langle \alpha, h_\alpha \rangle = 2$*. Hence*

$$[h_\alpha, e_\alpha] = 2e_\alpha, \quad [h_\alpha, f_\alpha] = -2f_\alpha,$$

so that $\{e_\alpha, f_\alpha, h_\alpha\}$ *is a TDS triple.*

Type A: Let $\alpha = \varepsilon_i - \varepsilon_j$ with $1 \leq i < j \leq l+1$. Set $e_\alpha = E_{i,j}$ and $f_\alpha = E_{j,i}$. Then $h_\alpha = E_{i,i} - E_{j,j}$.

Type B: (a) For $\alpha = \varepsilon_i - \varepsilon_j$ with $1 \leq i < j \leq l$ set $e_\alpha = E_{i,j} - E_{-j,-i}$ and $f_\alpha = E_{j,i} - E_{-i,-j}$. Then $h_\alpha = E_{i,i} - E_{j,j} + E_{-j,-j} - E_{-i,-i}$.

(b) For $\alpha = \varepsilon_i + \varepsilon_j$ with $1 \leq i < j \leq l$ set $e_\alpha = E_{i,-j} - E_{j,-i}$ and $f_\alpha = E_{-j,i} - E_{-i,j}$. Then $h_\alpha = E_{i,i} + E_{j,j} - E_{-j,-j} - E_{-i,-i}$.

(c) For $\alpha = \varepsilon_i$ with $1 \leq i \leq l$ set $e_\alpha = E_{i,0} - E_{0,-i}$ and $f_\alpha = 2E_{0,i} - 2E_{-i,0}$. Then $h_\alpha = 2E_{i,i} - 2E_{-i,-i}$.

Type C: (a) For $\alpha = \varepsilon_i - \varepsilon_j$ with $1 \leq i < j \leq l$ set $e_\alpha = E_{i,j} - E_{-j,-i}$ and $f_\alpha = E_{j,i} - E_{-i,-j}$. Then $h_\alpha = E_{i,i} - E_{j,j} + E_{-j,-j} - E_{-i,-i}$.

(b) For $\alpha = \varepsilon_i + \varepsilon_j$ with $1 \leq i < j \leq l$ set $e_\alpha = E_{i,-j} + E_{j,-i}$ and $f_\alpha = E_{-j,i} - E_{-i,j}$. Then $h_\alpha = E_{i,i} + E_{j,j} - E_{-j,-j} - E_{-i,-i}$.

(c) For $\alpha = 2\varepsilon_i$ with $1 \leq i \leq l$ set $e_\alpha = E_{i,-i}$ and $f_\alpha = E_{-i,i}$. Then $h_\alpha = E_{i,i} - E_{-i,-i}$.

Type D: (a) For $\alpha = \varepsilon_i - \varepsilon_j$ with $1 \leq i < j \leq l$ set $e_\alpha = E_{i,j} - E_{-j,-i}$ and $f_\alpha = E_{j,i} - E_{-i,-j}$. Then $h_\alpha = E_{i,i} - E_{j,j} + E_{-j,-j} - E_{-i,-i}$.

(b) For $\alpha = \varepsilon_i + \varepsilon_j$ with $1 \leq i < j \leq l$ set $e_\alpha = E_{i,-j} - E_{j,-i}$ and $f_\alpha = E_{-j,i} - E_{-i,j}$. Then $h_\alpha = E_{i,i} + E_{j,j} - E_{-j,-j} - E_{-i,-i}$.

We call h_α the *coroot* to α. Since the space $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ is one-dimensional, h_α is uniquely determined by the properties

$$h_\alpha \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}], \qquad \langle \alpha, h_\alpha \rangle = 2.$$

From the calculations in the proof of Lemma 8.4 we see that

$$\langle \beta, h_\alpha \rangle \in \{0, \pm 1, \pm 2\} \qquad \text{for all } \alpha, \beta \in \Phi. \tag{8.6}$$

For $\alpha \in \Phi$ we denote by $\mathfrak{s}(\alpha)$ the algebra spanned by $\{e_\alpha, f_\alpha, h_\alpha\}$. It is isomorphic to $\mathfrak{sl}(2, \mathbb{C})$ under the map $e \mapsto e_\alpha$, $f \mapsto f_\alpha$, $h \mapsto h_\alpha$. The algebra $\mathfrak{g}$ becomes a module for $\mathfrak{s}(\alpha)$ by restricting the adjoint representation of $\mathfrak{g}$ to $\mathfrak{s}(\alpha)$.
Let

$$R(\alpha, \beta) = \{\beta + k\alpha \,:\, k \in \mathbb{Z}\} \cap \Phi,$$

which we call the α *root string* through β. The number of elements of a root string is called the *length* of the string. Define

$$V_{\alpha,\beta} = \sum_{\gamma \in R(\alpha,\beta)} \mathfrak{g}_\gamma.$$

Then $V_{\alpha,\beta}$ is a subspace of $\mathfrak{g}$ whose dimension is the length of the α root string through β.

Lemma 8.5 *For every $\alpha, \beta \in \Phi$ with $\alpha \neq \pm\beta$, the space $V_{\alpha,\beta}$ is invariant and irreducible under $\mathrm{ad}(\mathfrak{s}(\alpha))$.*

Corollary 8.6 *If $\alpha, \beta \in \Phi$ then $\beta - \langle \beta, h_\alpha \rangle \alpha \in \Phi$.*

Corollary 8.7 *If $\alpha, \beta \in \Phi$ and $\alpha + \beta \in \Phi$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.*

Structure of Classical Root Systems

We call a subset $\Delta = \{\alpha_1, \ldots, \alpha_l\} \subset \Phi$ a set of *simple roots* if every $\gamma \in \Phi$ can be written uniquely as

$$\gamma = n_1\alpha_1 + \cdots + n_l\alpha_l, \text{ with } n_1, \ldots, n_l \text{ integers all of the same sign.} \tag{8.7}$$

If Δ is a set of simple roots, then we define the *height* of a root $\beta = n_1\alpha_1 + \cdots + n_l\alpha_l$ (relative to Δ) as

$$\mathrm{ht}(\beta) = n_1 + \cdots + n_l.$$

The positive roots are then the roots β with $\mathrm{ht}(\beta) > 0$. A root β is called the *highest root* of Φ, relative to a set Δ of simple roots, if

$$\mathrm{ht}(\beta) > \mathrm{ht}(\gamma) \quad \text{for all roots } \gamma \neq \beta.$$

If such a root exits, it is clearly unique.

Type A ($G = \mathrm{SL}(l+1, \mathbb{C})$): Let $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ and $\Delta = \{\alpha_1, \ldots, \alpha_l\}$. The associated set of positive roots is

$$\Phi^+ = \{\varepsilon_i - \varepsilon_j \,:\, 1 \leq i < j \leq l+1\}$$

and the highest root is

$$\tilde{\alpha} = \varepsilon_1 - \varepsilon_{l+1} = \alpha_1 + \cdots + \alpha_l$$

with $\mathrm{ht}(\tilde{\alpha}) = l$.

Type B ($G = \mathrm{SO}(2l+1, \mathbb{C})$): Let $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $1 \le i \le l-1$ and $\alpha_l = \varepsilon_l$. Take $\Delta = \{\alpha_1, \dots, \alpha_l\}$. The associated set of positive roots is

$$\Phi^+ = \{\varepsilon_i - \varepsilon_j,\, \varepsilon_i + \varepsilon_j \,:\, 1 \le i < j \le l\} \cup \{\varepsilon_i \,:\, 1 \le i \le l\}.$$

The highest root is

$$\tilde{\alpha} = \varepsilon_1 + \varepsilon_2 = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_l$$

with $\mathrm{ht}(\tilde{\alpha}) = 2l - 1$.

Type C ($G = \mathrm{Sp}(l, \mathbb{C})$): Let $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $1 \le i \le l-1$ and $\alpha_l = 2\varepsilon_l$. Take $\Delta = \{\alpha_1, \dots, \alpha_l\}$. The associated set of positive roots is

$$\Phi^+ = \{\varepsilon_i - \varepsilon_j,\, \varepsilon_i + \varepsilon_j \,:\, 1 \le i < j \le l\} \cup \{2\varepsilon_i \,:\, 1 \le i \le l\}.$$

The highest root is

$$\tilde{\alpha} = 2\varepsilon_1 = 2\alpha_1 + \cdots + 2\alpha_{l-1} + \alpha_l$$

with $\mathrm{ht}(\tilde{\alpha}) = 2l - 1$.

Type D ($G = \mathrm{SO}(2l, \mathbb{C})$ with $l \ge 3$): Let $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $1 \le i \le l-1$ and $\alpha_l = \varepsilon_{l-1} + \varepsilon_l$. Take $\Delta = \{\alpha_1, \dots, \alpha_l\}$.

Lemma 8.8 *Let Φ be the root system for a classical Lie algebra $\mathfrak{g}$ of rank l and type A, B, C or D (in the case of type D assume that $l \ge 3$). Let the system of simple roots $\Delta \subset \Phi$ be chosen as above. Let Φ^+ be the positive roots and let $\tilde{\alpha}$ be the maximal root relative to Δ. Then the following properties hold:*
(1) *If $\alpha, \beta \in \Phi^+$ and $\alpha + \beta \in \Phi$, then $\alpha + \beta \in \Phi^+$.*
(2) *If $\beta \in \Phi^+$ and β is not a simple root, then there exist $\gamma, \delta \in \Phi^+$ so that $\beta = \gamma + \delta$.*
(3) *The highest root $\tilde{\alpha} \in \Phi$ is of the form*

$$\tilde{\alpha} = n_1\alpha_1 + \cdots + n_l\alpha_l, \text{ with } n_i \ge 1 \text{ for } i = 1, \dots, l.$$

For any $\beta \in \Phi^+$ with $\beta \ne \tilde{\alpha}$ there exists $\alpha \in \Phi^+$ so that $\alpha + \beta \in \Phi^+$.
(4) *If $\alpha \in \Phi^+$ then there exist $1 \le i_1, i_2, \dots, i_r \le l$ such that $\alpha = \tilde{\alpha} - \alpha_{i_1} - \cdots - \alpha_{i_r}$ and $\tilde{\alpha} - \alpha_{i_1} - \cdots - \alpha_{i_j} \in \Phi$ for all $1 \le j \le r$.*

Theorem 8.9 *Let $\mathfrak{g}$ be the Lie algebra of one of the groups*

$$\mathrm{SL}(l+1, \mathbb{C}), \quad \mathrm{Sp}(\mathbb{C}^{2l}, \Omega), \quad \text{or} \quad \mathrm{SO}(\mathbb{C}^{2l+1}, B)$$

with $l \ge 1$, or the Lie algebra of $\mathrm{SO}(\mathbb{C}^{2l}, B)$ with $l \ge 3$. Take the set of simple roots Δ and the positive roots Φ^+ as in Lemma 8.8. The subspaces

$$\mathfrak{n} = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha, \quad \bar{\mathfrak{n}} = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_{-\alpha}$$

are Lie subalgebras of $\mathfrak{g}$ which are invariant under $\mathrm{ad}(\mathfrak{h})$.
The subspace $\mathfrak{n} + \bar{\mathfrak{n}}$ generates $\mathfrak{g}$ as a Lie algebra. In particular, $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$. There is a vector space decomposition

$$\mathfrak{g} = \bar{\mathfrak{n}} + \mathfrak{h} + \mathfrak{n}. \tag{8.8}$$

Furthermore, $\mathfrak{n}$ is generated (as a Lie algebra) by the simple root spaces $\mathfrak{g}_{\alpha_1}, \dots, \mathfrak{g}_{\alpha_l}$ and $\bar{\mathfrak{n}}$ is generated by $\mathfrak{g}_{-\alpha_1}, \dots, \mathfrak{g}_{-\alpha_l}$.

Irreducibility of Adjoint Representation

Theorem 8.10 *Let G be one of the groups* $\mathrm{SL}(\mathbb{C}^{l+1})$, $\mathrm{Sp}(\mathbb{C}^{2l})$, $\mathrm{SO}(\mathbb{C}^{2l+1})$ *with* $l \geq 1$ *or* $\mathrm{SO}(\mathbb{C}^{2l})$ *with* $l \geq 3$. *Then the adjoint representation of G is irreducible.*

Reductive Groups

Theorem 8.11 *Let G be a finite group. Then G is reductive.*

Proposition 8.12 *Let G be a linear algebraic group. If the identity component G° is reductive, then G is reductive.*

Reductivity of Classical Groups

Theorem 8.13 *Let G be a classical group. Then G is reductive.*

This follows from the corresponding Lie algebra result:

Theorem 8.14 *Let $\mathfrak{g}$ be the Lie algebra of a classical group G, and assume that $\mathfrak{g}$ is a simple Lie algebra. Then every finite-dimensional representation (ρ, V) of $\mathfrak{g}$ is completely reducible.*

Exercises for Lecture 8.

1. Let $E_{ij} \in M_3(\mathbb{C})$ be the usual elementary matrices. Set $e = E_{13}$, $f = E_{31}$ and $h = E_{11} - E_{33}$.

   (a) Verify that $\{e, f, h\}$ is a TDS in $\mathfrak{sl}(3, \mathbb{C})$.

   (b) Let $\mathfrak{g} = \mathbb{C}e + \mathbb{C}f + \mathbb{C}h \cong \mathfrak{sl}(2, \mathbb{C})$ and let $U = M_3(\mathbb{C})$. Define a representation ρ of $\mathfrak{g}$ on U by $\rho(A)X = [A, X]$ for $A \in \mathfrak{g}$ and $X \in M_3(\mathbb{C})$. Prove (without calculation) that $\rho(h)$ is diagonalizable. Then calculate that $\rho(h)$ has eigenvalues ± 2 (multiplicity 1), ± 1 (multiplicity 2) and 0 (multiplicity 3). Find all $u \in U$ so that $\rho(h)u = \lambda u$ and $\rho(e)u = 0$, where $\lambda = 0, 1, 2$.

   (c) Let V_k denote the irreducible $(k+1)$-dimensional representation of $\mathfrak{g}$. Show that

   $$U \cong V_2 \oplus V_1 \oplus V_1 \oplus V_0 \oplus V_0$$

   as a $\mathfrak{g}$-module. (*Hint:* Use the results of (b) and Proposition 8.2 of the notes.)

2. Let $G = \mathrm{SL}(2, \mathbb{C})$. Let k be a non-negative integer and let W_k be the polynomials in $\mathbb{C}[x]$ of degree at most k. If

   $$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G$$

   and if $f \in W_k$ then set

   $$\sigma_k(g)f(x) = (-cx + d)^k f\left(\frac{ax - b}{-cx + d}\right).$$

   (a) Show that $\sigma_k(g)W_k \subset W_k$ and that (σ_k, W_k) defines a regular representation of G.

   (b) Let V_k be the space of homogeneous polynomials of degree k in x_1, x_2. Let ρ_k be the representation of G given by $\rho(g)\phi(x_1, x_2) = \phi(ax_1 + cx_2, bx_1 + dx_2)$. Find a G isomorphism between the representations (σ_k, W_k) and (ρ_k, V_k).

3. Let $V = \mathbb{C}[x]$. Define operators E and F on V by

$$E\phi(x) = -\frac{1}{2}\frac{d^2\phi(x)}{dx^2}, \qquad F\phi(x) = \frac{1}{2}x^2\phi(x).$$

Set $H = [E, F]$.

(a) Show that $H = -x\dfrac{d}{dx} - \dfrac{1}{2}$ and that $\{E, F, H\}$ is a TDS.

(b) Find the space $V^E = \{\phi \in V \,:\, E\phi = 0\}$.

(c) Let $\mathfrak{g} \subset \mathrm{End}(V)$ be the Lie algebra spanned by E, F, H. Let $V_{\mathrm{even}} \subset V$ be the space of *even* polynomials, and $V_{\mathrm{odd}} \subset V$ be the space of *odd* polynomials. Show that each of these spaces is invariant and irreducible under $\mathfrak{g}$. (*Hint:* Use (b) and Lemma 8.1 of the notes.)

(d) Show that $V = V_{\mathrm{even}} \oplus V_{\mathrm{odd}}$ and that V_{even} is not equivalent to V_{odd} as a module for $\mathfrak{g}$. (*Hint:* Show that the operator H is diagonalizable on V_{even} and V_{odd} and find its eigenvalues.)

4. Let G be a classical group. Let Φ be the root system for G, $\alpha_1, \ldots, \alpha_l$ the simple roots, and Φ^+ the positive roots. Verify the following:

(a) For G of type A_l, $\Phi^+ \setminus \Delta$ consists of the roots

$$\alpha_i + \cdots + \alpha_j \quad \text{for } 1 \le i < j \le l.$$

(b) For G of type B_l with $l \ge 2$, $\Phi^+ \setminus \Delta$ consists of the roots

$$\begin{aligned} \alpha_i + \cdots + \alpha_j &\qquad \text{for } 1 \le i < j \le l, \\ \alpha_i + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_l &\qquad \text{for } 1 \le i < j \le l. \end{aligned}$$

(c) For G of type C_l with $l \ge 2$, $\Phi^+ \setminus \Delta$ consists of the roots

$$\begin{aligned} \alpha_i + \cdots + \alpha_j &\qquad \text{for } 1 \le i < j < l, \\ \alpha_i + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_{l-1} + \alpha_l &\qquad \text{for } 1 \le i < j < l, \\ 2\alpha_i + \cdots + 2\alpha_{l-1} + \alpha_l &\qquad \text{for } 1 \le i < l. \end{aligned}$$

(d) For G of type D_l with $l \ge 3$, $\Phi^+ \setminus \Delta$ consists of the roots

$$\begin{aligned} \alpha_i + \cdots + \alpha_j &\qquad \text{for } 1 \le i < j < l, \\ \alpha_i + \cdots + \alpha_l &\qquad \text{for } 1 \le i < l - 1, \\ \alpha_i + \cdots + \alpha_{l-2} + \alpha_l &\qquad \text{for } 1 \le i < l - 1, \\ \alpha_i + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_{l-2} + \alpha_{l-1} + \alpha_l &\qquad \text{for } 1 \le i < j < l - 1. \end{aligned}$$

Part 3: Homogeneous Spaces

Lecture 9. G-spaces, Orbits, and Invariants

Algebraic Group Actions

Let M be a quasiprojective algebraic set. An *algebraic action* of a linear algebraic group G on M is a regular map $\alpha : G \times M \to M$, written as $(g, m) \mapsto g \cdot m$, such that

$$g \cdot (h \cdot m) = (gh) \cdot m, \quad 1 \cdot m = m$$

for all $g, h \in G$ and $m \in M$. (Recall that $G \times M$ is a quasiprojective algebraic set.)

Theorem 9.1 *For every $x \in M$, the stabilizer G_x of x is an algebraic subgroup of G and the orbit $G \cdot x$ is a smooth quasiprojective subset of M.*

Corollary 9.2 *There exists a point $x \in M$ so that $G \cdot x$ is closed in M.*

Homogeneous G Spaces

We have the following converse to Theorem 9.1. Let H be an algebraic subgroup of an algebraic group G. By Theorem 6.6 there is a regular representation (π, V) of G and a point $x_0 \in \mathbb{P}(V)$ so that H is the stabilizer of x_0. The map $g \mapsto g \cdot x_0$ is a bijection from the coset space G/H to the orbit $G \cdot x_0$. We view G/H as a smooth quasiprojective algebraic set by identifying it with the orbit $G \cdot x_0$.

Theorem 9.3 (1) *The quasiprojective algebraic set structure on G/H is independent of the choice of the representation π.*
(2) *The quotient map from G to G/H is regular.*
(3) *If M is any quasiprojective algebraic set on which G acts algebraically, and $x \in M$ is such that $H \subset G_x$, then the map $gH \mapsto g \cdot x$ from G/H to the orbit $G \cdot x$ is regular.*

Polynomial Invariants

Let G be a linear algebraic group. Suppose (π, V) is a regular representation of G. We define a representation ρ of G on the algebra $\mathcal{P}(V)$ of polynomial functions on V by

$$\rho(g) f(v) = f(g^{-1} v) \quad \text{for } f \in \mathcal{P}(V)$$

Here we write $\pi(g)v = gv$ for $g \in G$ and $v \in V$ when the representation π is understood from the context.
The finite-dimensional spaces

$$\mathcal{P}^k(V) = \{ f \in \mathcal{P}(V) \, : \, f(zv) = z^k f(v) \quad \text{for } z \in \mathbb{C}^\times \}$$

of homogeneous polynomials of degree k are G-invariant for $k = 0, 1, \ldots$ and the restriction ρ_k of ρ to $\mathcal{P}^k(V)$ is a regular representation of G.
We denote the space of G-invariant polynomials on V by $\mathcal{P}(V)^G$. It is a commutative subalgebra of $\mathcal{P}(V)$ which we call the *algebra of G-invariants*.

Theorem 9.4 *Suppose G is a reductive linear algebraic group acting by a regular representation on a vector space V. Then the algebra $\mathcal{P}(V)^G$ of G-invariant polynomials on V is finitely-generated as a $\mathbb{C}$-algebra.*

Let $\{f_1, \ldots, f_n\}$ be a set of generators for $\mathcal{P}(V)^G$ with n as small as possible. We call $\{f_1, \ldots, f_n\}$ a set of *basic invariants*. Theorem 9.4 asserts that when G is reductive, there always exists a finite set of basic invariants. Since $\mathcal{P}(V)$ and $\mathcal{J} = \mathcal{P}(V)^G$ are graded algebras, relative to the usual degree of a polynomial, there is a set of basic invariants with each f_i homogeneous, say of degree d_i. If we enumerate the f_i so that $d_1 \leq d_2 \leq \cdots$ then the sequence $\{d_i\}$ is uniquely determined (even though the set of basic invariants is not unique).

Algebraic Quotients

Let G be an algebraic group acting on an affine algebraic variety X. Assume that the algebra $\mathcal{J} = \mathrm{Aff}(X)^G$ of G-invariant regular functions on X is finitely generated over $\mathbb{C}$ (if G is reductive this is always true, by complete reducibility). This action partitions X into G-orbits, and every G-invariant function on X is constant on each orbit. An affine variety Y is called the *algebraic quotient* of X by G if there is a regular map $\pi : X \to Y$ which is constant on each G-orbit in X, with the following *universal property*: Given any algebraic variety Z and regular map $f : X \to Z$ that is constant on G-orbits, there exists a unique regular map $\tilde{f}$ such that $f = \tilde{f} \circ \pi$.

Theorem 9.5 (1) *An algebraic quotient exists and is unique up to isomorphism of affine algebraic sets. Denote it by $X//G$.*
(2) *If G is reductive, then the canonical map $\pi : X \to X//G$ is surjective.*

Proof. (1): Let Y be the set of maximal ideals of $\mathcal{J}$. We may identify the points of Y with the algebra homomorphisms $\mathcal{J} \to \mathbb{C}$ by Hilbert's Nullstellensatz. This identification gives a map $\pi : X \to Y$ defined by $\pi(x)(f) = f(x)$ for $f \in \mathcal{J}$. We must show that (Y, π) satisfies the universal property of an algebraic quotient of X by G.
Let Z be an affine variety and $f : X \to Z$ be a regular function that is constant on G-orbits. Then $f^*(\mathrm{Aff}(Z)) \subset \mathcal{J}$, by definition. Hence every homomorphism $\phi : \mathcal{J} \to \mathbb{C}$ determines a homomorphism $\tilde{f}(\phi) : \mathrm{Aff}(Z) \to \mathbb{C}$, where $\tilde{f}(\phi)(h) = \phi(h \circ f)$ for $h \in \mathrm{Aff}(Z)$. This defines a regular map $\tilde{f}$ such that $f = \tilde{f} \circ \pi$.)
It is clear that the universal property of a quotient variety uniquely determines it, up to isomorphism. Write $Y = X//G$ and call π the *canonical map.*
(2): Since G is reductive, there is a projection $g \mapsto g^\natural$ from $\mathrm{Aff}(X)$ onto the G-invariants $\mathrm{Aff}(X)^G$. To prove that the canonical map is surjective, let $\mathfrak{m} \subset \mathcal{J}$ be a maximal ideal. Then $\mathfrak{m}$ generates a proper ideal in $\mathrm{Aff}(X)$, since any relation $\sum_i f_i g_i = 1$ with $f_i \in \mathfrak{m}$ and $g_i \in \mathrm{Aff}(X)$ would give a relation $\sum f_i g_i^\natural \in \mathfrak{m} = 1$. By the Hilbert Nullstellensatz there exists $x \in X$ so that all the functions in $\mathfrak{m}$ vanish at x. Hence $\pi(x) = \mathfrak{m}$. $\Box$

Appendix: Algebraic Geometry for Lecture 9.

Projective and Quasiprojective Sets

Let V be a complex vector space. The *projective space* $\mathbb{P}(V)$ associated with V is the set of lines through 0 (one-dimensional subspaces) in V. For $x \in V \setminus \{0\}$, $[x] \in \mathbb{P}(V)$ will denote the line through x. The map $p : V \setminus \{0\} \to \mathbb{P}(V)$ given by $p(x) = [x]$ is surjective, and $p(x) = p(y)$ if and only if $x = \lambda y$ for some $\lambda \in \mathbb{C}^\times$. We denote $\mathbb{P}(\mathbb{C}^{n+1})$ by $\mathbb{P}^n$ and for $x = (x_0, \ldots, x_n) \in \mathbb{C}^{n+1}$ we call $\{x_i\}$ the *homogeneous coordinates* of $[x]$.
If $f(x_0, \ldots, x_n)$ is a homogeneous polynomial in $n+1$ variables and $0 \neq x \in \mathbb{C}^{n+1}$, set

$$A_f = \{[x] \in \mathbb{P}^n \,:\, f(x) = 0\}.$$

The *Zariski topology* on $\mathbb{P}^n$ is obtained by taking as closed sets the intersections

$$X = \bigcap_{f \in S} A_f,$$

where S is any set of homogeneous polynomials on $\mathbb{C}^{n+1}$. Any such set X will be called a *projective algebraic set* The set

$$p^{-1}(X) \cup \{0\} = \{x \in \mathbb{C}^{n+1} \,:\, f(x) = 0 \text{ for all } f \in S\}$$

is closed in $\mathbb{C}^{n+1}$ and is called the *cone over* X.
Every closed set in projective space is definable as the zero locus of a finite collection of homogeneous polynomials, and the descending chain condition for closed sets is satisfied. Hence every closed set is a finite union of irreducible closed sets, and any nonempty open subset of an irreducible closed set M is dense in M.
For $i = 0, \ldots, n$ let $\mathbb{U}_i^n = \{[x] \in \mathbb{P}^n \,:\, x_i \neq 0\}$. Each $\mathbb{U}_i^n$ is an open set in $\mathbb{P}^n$, and every point of $\mathbb{P}^n$ lies in $\mathbb{U}_i^n$ for some i. For $[x] \in \mathbb{U}_i^n$ define the *inhomogeneous coordinates* of $[x]$ to be $y_j = x_j/x_i$ for $j \neq i$. The map

$$\pi_i([x]) = (y_0, \ldots, \widehat{y_i}, \ldots, y_n)$$

(omit y_i) is a bijection between $\mathbb{U}_i^n$ and $\mathbb{C}^n$. It is also a topological isomorphism (where $\mathbb{U}_i^n$ has the relative Zariski topology from $\mathbb{P}^n$ and $\mathbb{C}^n$ carries the Zariski topology).
Thus we have a covering by $\mathbb{P}^n$ by the $n+1$ open sets $\mathbb{U}_i^n$, each homeomorphic to the affine space $\mathbb{C}^n$.

Lemma 9.6 *Let $X \subset \mathbb{P}^n$. Suppose that for all $i = 0, 1, \ldots, n$, $X \cap \mathbb{U}_i$ is the set of zeros of homogeneous polynomials $f_{ij}(y_0, \ldots, \widehat{y_i}, \ldots, y_n)$, where $\{y_k\}$ are the inhomogeneous coordinates on $\mathbb{U}_i$. Then X is closed in $\mathbb{P}^n$.*

A *quasiprojective algebraic set* is a subset $M \subset \mathbb{P}^n$ defined by a finite set of equalities and inequalities on the homogeneous coordinates of the form

$$f_i(x) = 0 \quad \text{for all } i = 1, \ldots, k \text{ and } g_j(x) \neq 0 \quad \text{for some } j = 1, \ldots, l$$

where f_i and g_j are homogeneous polynomials on $\mathbb{C}^{n+1}$. In topological terms, M is the intersection of the closed set

$$Y = \{[x] \in \mathbb{P}^n \,:\, f_i(x) = 0 \text{ for all } i = 1, \ldots, k\}$$

and the open set

$$Z = \{[x] \in \mathbb{P}^n \,:\, g_j(x) \neq 0 \text{ for some } j\}.$$

Products of Projective Sets

We begin with the basic case of projective spaces. Let x and y be homogeneous coordinates on $\mathbb{P}^m$ and $\mathbb{P}^n$. Denote the space of complex matrices of size $r \times s$ by $\mathbb{M}_{r\times s}$ and view $\mathbb{C}^r = \mathbb{M}_{1\times r}$ as row vectors. We map $\mathbb{C}^{m+1} \times \mathbb{C}^{n+1} \to \mathbb{M}_{(m+1)\times(n+1)}$ by $(x, y) \mapsto x^t y$, where x^t is the transpose of x. The image of $(\mathbb{C}^{m+1} \setminus \{0\}) \times (\mathbb{C}^{n+1} \setminus \{0\})$ consists of all rank one matrices, hence it is defined by the vanishing of all minors of size greater than 1. These minors are homogeneous polynomials in the matrix coordinates z_{ij} of $z \in \mathbb{M}_{(m+1)\times(n+1)}$. Passing to projective space, we have thus obtained an embedding

$$\mathbb{P}^m \times \mathbb{P}^n \hookrightarrow \mathbb{P}(\mathbb{M}_{(m+1)\times(n+1)}) = \mathbb{P}^{mn+m+n} \tag{9.1}$$

with closed image. We take this as the structure of a projective algebraic set on $\mathbb{P}^m \times \mathbb{P}^n$.
Let $X \subset \mathbb{P}^m$ and $Y \subset \mathbb{P}^n$. The image of $X \times Y$ under the map (9.1) in $\mathbb{P}^{m+n+mn}$ is closed if X and Y are closed. Also, the image of $X \times Y$ is quasiprojective if X and Y are quasiprojective.

Lemma 9.7 *Let X be a quasi-projective algebraic set and let*

$$\Delta = \{(x, x) : x \in X\} \subset X \times X$$

be the diagonal. Then Δ is closed.

Ascending Chain Property

Theorem 9.8 (1) *Let M, N be irreducible affine algebraic sets, such that $M \subseteq N$. Then* $\dim M \leq \dim N$. *If* $\dim M = \dim N$ *then $M = N$.*
(2) *Let $X_1 \subset X_2 \subset \cdots$ be an increasing chain of irreducible affine algebraic subsets of an algebraic set X. Then there exists an index p so that $X_j = X_p$ for $j \geq p$.*

Exercises for Lecture 9.

1. Let $G = \mathrm{SL}(2, \mathbb{C})$ act on $\mathbb{C}^2$ by left multiplication as usual. This gives an action on $\mathbb{P}^1(\mathbb{C})$. Let $H = \{\mathrm{diag}[z, z^{-1}] : z \in \mathbb{C}^\times\}$ be the diagonal subgroup, let N be the subgroup of upper-triangular unipotent matrices $\begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix}$, $z \in \mathbb{C}$, and let $B = HN$ be the upper triangular subgroup.

   (a) Show that G acts transitively on $\mathbb{P}(\mathbb{C})$. Find a point whose stabilizer is B.

   (b) Show that H has one open dense orbit and two closed orbits on $\mathbb{P}(\mathbb{C})$. Show that N has one open dense orbit and one closed orbit on $\mathbb{P}(\mathbb{C})$.

   (c) Identify $\mathbb{P}(\mathbb{C})$ with the two-sphere $\mathbf{S}^2$ by stereographic projection and give geometric descriptions of the orbits in (b).

2. (*Same notation as previous exercise*) Let G act on $\mathfrak{g} = \{x \in M_2(\mathbb{C}) : \mathrm{tr}(x) = 0\}$ by the adjoint representation $\mathrm{Ad}(g)x = gxg^{-1}$. For $\mu \in \mathbb{C}$ define $X_\mu = \{x \in \mathfrak{g} : \mathrm{tr}(x^2) = 2\mu\}$. Use the Jordan canonical form to prove the following.

   (a) If $\mu \neq 0$ then X_μ is a G orbit and $X_\mu \cong G/H$ as a G-space.

(b) If $\mu = 0$ then $X_0 = \{0\} \cup Y$ is the union of two G orbits, where Y is the orbit of $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Show that $Y \cong G/\{\pm 1\}N$ and that Y is not closed in $\mathfrak{g}$.

3. *(Same notation as previous exercise)* Let $Z = \mathbb{P}(\mathfrak{g}) \cong \mathbb{P}^2(\mathbb{C})$ be the projective space of $\mathfrak{g}$, and let $\pi : \mathfrak{g} \to Z$ be the canonical mapping.

   (a) Show that G has two orbits on Z, namely $Z_1 = \pi(X_1)$ and $Z_0 = \pi(Y)$.

   (b) Find subgroups L_1 and L_0 of G so that $Z_i \cong G/L_i$ as a G space. (*Hint:* Be careful; from the previous problem you know that $H \subset L_1$ and $N \subset L_0$, but these inclusions are proper.)

   (c) Prove (without calculation) that one orbit must be closed in Z and one orbit must be dense in Z. Then calculate $\dim Z_i$ and identify the closed orbit. Find equations defining the closed orbit.

4. Let $G = \mathrm{SL}(n, \mathbb{C})$ and let V be the space of all symmetric quadratic forms $A(x) = \sum_{i,j} a_{ij} x_i x_j$ in n variables $x_1, \ldots, x_n$, with $n \geq 2$. The group G acts on V via its linear action on $x = [x_1, \ldots, x_n] \in \mathbb{C}^n$. In terms of the symmetric matrix $A = [a_{ij}]$, the action is

$$g \cdot A = (g^t)^{-1} A g^{-1} \quad \text{(matrix multiplication)}.$$

   (a) Show that the function $D(A) = \det A$ (the *discriminant* of the form) is G-invariant.

   (b) Show that every G-orbit in V contains exactly one of the forms

$$Q_{n,c}(x) = c x_1^2 + x_2^2 + \cdots + x_n^2, \quad \text{with } c \neq 0$$

$$Q_r(x) = x_1^2 + \cdots + x_r^2, \quad \text{with } 0 \leq r < n.$$

   (*Hint:* There exists $g \in \mathrm{GL}(n, \mathbb{C})$ so that $(g^t)^{-1} A g^{-1}$ is diagonal with all nonzero elements 1.)

   (c) Show that $\mathcal{P}(V)^G = \mathbb{C}[D]$. (*Hint:* Define $s : \mathbb{C} \to V$ by $s(c) = Q_{n,c}$. Given $f \in \mathcal{P}(V)^G$, show that when A is non-singular, $f(A) = \phi(D(A))$, where ϕ is the polynomial $f \circ s$.)

   (d) Show that $V//G \cong \mathbb{C}$, with the quotient map $\pi(x) = D(x)$. Show that the closed G-orbits are those on which $D \neq 0$ (non-singular forms) and the point $\{0\}$, and the quotient map takes all the non-closed orbits (the forms of rank $r < n$) to 0.

   (e) Show that the G-invariant polynomials can separate the G orbits of the nonsingular forms, but cannot separate the orbits of the singular forms. (*Hint:* Consider the sets $D^{-1}(c)$ for $c \in \mathbb{C}$.)

Lecture 10. Flag Manifolds and Solvable Groups

Grassmannian Manifolds

Let V be a finite-dimensional vector space, and let $\bigwedge^k V$ be the kth exterior power of V. We call an element of this space a *k-vector*. Given a k-vector u, we define a linear map

$$T_u : V \to \bigwedge^{k+1} V$$

by $T_u v = u \wedge v$ for $v \in V$. Set

$$V(u) = \{v \in V \,:\, u \wedge v = 0\} = \mathrm{Ker}(T_u)$$

(the *annihilator* of u in V). The non-zero k-vectors of the form $v_1 \wedge \ldots \wedge v_k$, with $v_i \in V$, are called *decomposable.*

Lemma 10.1 *Let* $\dim V = n$.
(1) *Let* $0 \neq u \in \bigwedge^k V$. *Then* $\dim V(u) \leq k$ *and* $\mathrm{Rank}(T_u) \geq n-k$. *Furthermore,* $\mathrm{Rank}(T_u) = n-k$ *if and only if u is decomposable.*
(2) *Suppose $u = v_1 \wedge \ldots \wedge v_k$ is decomposable. Then*

$$V(u) = \mathrm{Span}\{v_1, \ldots, v_k\}.$$

Furthermore, if $V(u) = V(w)$ then $w = cu$ for some $c \in \mathbb{C}^\times$. Hence the subspace $V(u) \subset V$ determines the point $[u] \in \mathbb{P}(\bigwedge^k V)$.
(3) *Let $0 < k < l < n$. Suppose $0 \neq u \in \bigwedge^k V$ and $0 \neq w \in \bigwedge^l V$ are decomposable. Then $V(u) \subset V(w)$ if and only if* $\mathrm{Rank}(T_u \oplus T_w)$ *is a minimum, namely $n-k$.*

Denote the set of all k-dimensional subspaces of V by $\mathrm{Grass}_k(V)$ (the kth *Grassmannian* manifold). Using part (2) of Lemma 10.1, we identify $\mathrm{Grass}_k(V)$ with the subset of the projective space $\mathbb{P}(\bigwedge^k V)$ corresponding to the decomposable k-vectors.

Proposition 10.2 $\mathrm{Grass}_k(V)$ *is an irreducible projective algebraic set.*

Take $V = \mathbb{C}^n$ and let $X \subset M_{n\times k}(\mathbb{C})$ be the open subset of matrices of maximal rank k. The k-dimensional subspaces of V then correspond to the column spaces of matrices $x \in X$. Since $x, y \in X$ have the same column space if and only if $x = yg$ for some $g \in \mathrm{GL}(k,\mathbb{C})$, we may view $\mathrm{Grass}_k(V)$ as the space of orbits of $\mathrm{GL}(k,\mathbb{C})$ on X. That is, we introduce the equivalence relation $x \sim y$ if $x = yg$; then $\mathrm{Grass}_k(V)$ is the set of equivalence classes.
For $J = (i_1, \ldots, i_k)$ with $1 \leq i_1 < \cdots < i_k \leq n$, let

$$\xi_J(x) = \det \begin{bmatrix} x_{i_1 1} & \cdots & x_{i_1 k} \\ \vdots & \ddots & \vdots \\ x_{i_k 1} & \cdots & x_{i_k k} \end{bmatrix}$$

be the minor determinant formed from rows $i_1, \ldots, i_k$ of $x \in M_{n\times k}(\mathbb{C})$. Set

$$X_J = \{x \in M_{n\times k}(\mathbb{C}) \,:\, \xi_J(x) \neq 0\}.$$

As J ranges over all $\binom{n}{k}$ increasing k-tuples the sets X_J cover X. The homogeneous polynomials ξ_J are the so-called *Plücker coordinates* on X (they are the restriction to X of the homogeneous linear coordinates on $\bigwedge^k \mathbb{C}^n$ relative to the standard basis). Under right multiplication they transform by

$$\xi_J(xg) = \xi_J(x) \det g, \quad g \in \mathrm{GL}(k, \mathbb{C}),$$

so the ratios of the Plücker coordinates are rational functions on $\mathrm{Grass}_k(V)$.
Every matrix in X_J is equivalent (under the right $\mathrm{GL}(k, \mathbb{C})$ action) to a matrix in the affine-linear subspace

$$A_J = \{x \in M_{n \times k}(\mathbb{C}) : x_{i_p q} = \delta_{pq} \text{ for } p, q = 1, \ldots, k\}.$$

Clearly if $x, y \in A_J$ and $x \sim y$ then $x = y$. Furthermore, $\xi_J = 1$ on A_J and the $k(n-k)$ matrix coordinates $\{x_{pq} : p \notin J\}$ are the restrictions to A_J of certain Plücker coordinates. In particular, $\dim \mathrm{Grass}_k(\mathbb{C}^n) = (n-k)k$.
Suppose that ω is a bilinear form on V (either symmetric or skew-symmetric). A subspace $W \subset V$ is *isotropic* relative to ω if $\omega(x, y) = 0$ for all $x, y \in W$. The *quadric grassmannian* $\mathcal{I}_k(V)$ is the subset of $\mathrm{Grass}_k(V)$ consisting of all isotropic subspaces. Then $\mathcal{I}_k(V)$ is closed in $\mathrm{Grass}_k(V)$, and hence is a projective algebraic set.

Flag Manifolds for Classical Groups

Let $0 < k_1 < \cdots < k_p < \dim V$ be integers, and set $\mathbf{k} = (k_1, \ldots, k_p)$. Let $\mathrm{Flag}_{\mathbf{k}}(V)$ consist of all nested chains $V_1 \subset \cdots \subset V_p \subset V$ of subspaces with $\dim V_i = k_i$. We can view $\mathrm{Flag}_{\mathbf{k}}(V)$ as a subset of the projective algebraic set

$$\mathrm{Grass}_{\mathbf{k}}(V) = \mathrm{Grass}_{k_1}(V) \times \cdots \times \mathrm{Grass}_{k_p}(V).$$

By part (3) of Lemma 10.1, $\mathrm{Flag}_{\mathbf{k}}(V)$ is closed in $\mathrm{Grass}_{\mathbf{k}}(V)$, since each inclusion $V(u) \subset V(w)$ between subspaces of V is defined by the vanishing of suitable minors in the matrix for $T_u \oplus T_w$.
The group $\mathrm{GL}(V)$ acts on $\mathrm{Grass}_{\mathbf{k}}(V)$. Fix a basis $\{e_i : i = 1, \ldots, n\}$ for V and set $V_i = \mathrm{span}\{e_1, \ldots, e_{k_i}\}$. Then $\mathrm{Flag}_{\mathbf{k}}(V)$ is the orbit of the flag $x_{\mathbf{k}} = \{V_i\}$. The isotropy group $P_{\mathbf{k}}$ of $x_{\mathbf{k}}$ consists of the block upper-triangular matrices

$$\begin{bmatrix} A_1 & \cdots & * \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_{p+1} \end{bmatrix}$$

where $A_i \in \mathrm{GL}(m_i, \mathbb{C})$, with $m_1 = k_1, m_2 = k_2 - k_1, \ldots, m_{p+1} = n - k_p$.
Let $G \subset \mathrm{GL}(n, \mathbb{C})$ be a classical group, in the matrix realization used in Theorem 7.2. Let H be the diagonal subgroup of G. Set $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$, where $\mathfrak{h} = \mathrm{Lie}(H)$ and

$$\mathfrak{n} = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$$

(recall that $\mathfrak{n}$ consists of strictly upper triangular matrices). Denote by $N_n(\mathbb{C})$ the group of all $n \times n$ upper triangular unipotent matrices.

Theorem 10.3 *Let G be a connected classical group. There is a projective algebraic set X_G on which G acts algebraically and transitively with the following properties.*
(1) *There is a point $x_0 \in X_G$ so that the stabilizer $B = G_{x_0}$ has Lie algebra $\mathfrak{b}$.*
(2) *The group $B = H \cdot N$, with N connected, unipotent and normal in B.*
(3) $\mathrm{Lie}(N) = \mathfrak{n}$ *and $N = G \cap N_n(\mathbb{C})$.*

$G = \mathrm{GL}(n, \mathbb{C})$ or $\mathrm{SL}(n, \mathbb{C})$:
Let X_A be the set of all *full flags* $\{V_i\}_{i=1}^n$, $\dim V_i = i$. Let $x_0 = \{V_i^0\}$ with $V_i^0 = \mathrm{Span}\{e_1, \ldots, e_i\}$, where $e_1, \ldots, e_n$ is the standard basis for $\mathbb{C}^n$.

$G = \mathrm{Sp}(l, \mathbb{C})$ or $\mathrm{SO}(n, \mathbb{C})$, $n = 2l$:
Let X be the set of all *isotropic flags* $\{V_i\}_{i=1}^l$, with $\dim V_i = i$ and V_i an isotropic subspace relative to the bilinear form defining G.

$G = \mathrm{SO}(n, \mathbb{C})$, $n = 2l + 1$:
Let X_G be the set of all flags $\{V_i\}_{i=1}^{l+1}$ such that $\dim V_i = i$ and V_i is isotropic for $i = 1, \ldots, l$.

In all cases B is the group of all upper triangular matrices in G and N is the group of all unipotent upper triangular matrices in G.

Solvable Groups

Let G be an (abstract) group. We say that G is *solvable* if there exists a series of subgroups

$$G = G_0 \supset G_1 \supset \cdots \supset G_d \supset G_{d+1} = \{1\}$$

with G_{i+1} a normal subgroup of G_i and G_i/G_{i+1} commutative, for $i = 0, 1, \ldots, d$.
The *commutator subgroup* $\mathcal{D}(G)$ of a group G is the group generated by the set of commutators $\{xyx^{-1}y^{-1} : x, y \in G\}$. If G_1 is a normal subgroup of G, then G/G_1 is commutative if and only if $G_1 \supset \mathcal{D}(G)$. It follows that G *is solvable if and only if $G \neq \mathcal{D}(G)$ and $\mathcal{D}(G)$ is solvable.*
Define the *derived series* $\{\mathcal{D}^n(G)\}$ of G inductively by

$$\mathcal{D}^0(G) = G, \qquad \mathcal{D}^{n+1}(G) = \mathcal{D}(\mathcal{D}^n(G)).$$

Then G is solvable if and only if $\mathcal{D}^{n+1}(G) = \{1\}$ for some n. In this case, the smallest such n is called the *solvable length* of G.
The archetypical example of a solvable group is the subgroup, B_n, of upper triangular matrices in $\mathrm{GL}(n, \mathbb{C})$ the we have already encountered in connection with the flag manifold. To see this we observe that the upper triangular matrices, N_n, with ones on the main diagonal form a normal subgroup of B_n such that B_n/N_n is isomorphic with the group of diagonal matrices. We set $N_{n,r}$ equal to the subgroup of N_n consisting of elements such that the second through the r-th diagonal are zero. Then $N_{n,r}$ is normal in B_n for $r \geq 2$ and $N_{n,r}/N_{n,r+1}$ is abelian. The isotropy group of any full flag in $\mathbb{C}^n$ is conjugate in $\mathrm{GL}(n, \mathbb{C})$ to B_n and hence is solvable.
We also note that if S is solvable and if $H \subset S$ is a subgroup then H is solvable. For example, let $G \subset \mathrm{GL}(n, \mathbb{C})$ be a connected classical group. Then the subgroup B in Theorem 10.3 is contained in the isotropy group of a full flag and hence is solvable.

Proposition 10.4 *Assume G is a connected algebraic group. Then $\mathcal{D}(G)$ is closed and connected.*

Exercises for Lecture 10.

1. Let $G = \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$. Let ρ be the representation of G on $\mathbb{M}_2 = M_2(\mathbb{C})$ given by $\rho(g,h)z = gzh^t$ and let B be the symmetric bilinear form on $\mathbb{M}_2$ such that $B(z,z) = 2\det(z)$.

   (a) Find $\mathrm{Ker}(\rho)$ and prove that $\rho(G) = \mathrm{SO}(\mathbb{M}_2, B)$ (*Hint:* Compare $\dim(G)$ and $\dim(\mathrm{SO}(\mathbb{M}_2, B)$.)

   (b) Use (a) to prove that $\mathfrak{so}(4,\mathbb{C}) \cong \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$.

2. (*Notation as in previous exercise)* Let $\pi : \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{M}_2$ by $\pi(x,y) = xy^t$. Identify $\mathbb{P}^3$ with $\mathbb{P}(\mathbb{M}_2)$ and let $\tilde{\pi} : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ be the map induced by π (the standard imbedding of $\mathbb{P}^m \times \mathbb{P}^n$ in $\mathbb{P}^{mn+m+n}$).

   (a) Show that the image of $\tilde{\pi}$ is $\{[z] : z \in \mathbb{M}_2 \setminus \{0\} \text{ and } \det(z) = 0\}$.

   (b) Let G act on $\mathbb{P}^1 \times \mathbb{P}^1$ by the natural action on $\mathbb{C}^2 \times \mathbb{C}^2$ and let G act on $\mathbb{P}^3$ by the representation ρ on $\mathbb{M}_2$. Show that $\tilde{\pi}$ intertwines the G actions.

   (c) Show that G has two orbits on $\mathbb{P}^3$ and describe the closed orbit.

3. (*Notation as in previous exercise*) Consider the subspaces $V_1 = \mathbb{C}E_{11} + \mathbb{C}E_{12}$ and $V_2 = \mathbb{C}E_{11} + \mathbb{C}E_{21}$ of $\mathbb{M}_2$, where E_{ij} are the usual elementary matrices.

   (a) Show that V_i are totally isotropic for the bilinear form B.

   (b) Let $B_i = \{g \in G : \rho(g)V_i = V_i\}$ for $i = 1, 2$. Describe B_1, B_2 and $B = B_1 \cap B_2$ in matrix form.

   (c) Show that $B = H \cdot N$ where H is a maximal torus in G and N is a connected unipotent normal subgroup of B.

4. Let $X = \{x \in M_{4\times 2}(\mathbb{C}) : \mathrm{rank}(x) = 2\}$. For $J = (i_1, i_2)$ with $1 \le i_1 < i_2 \le 4$ let $X_J = \{x \in X : \xi_J(x) \neq 0\}$, where

$$\xi_J(x) = \det \begin{bmatrix} x_{i_1 1} & x_{i_1 2} \\ x_{i_2 1} & x_{i_2 2} \end{bmatrix}$$

   is the Plücker coordinate corresponding to J.

   (a) Let $A_{\{1,2\}} = \{x \in X : x_{ij} = \delta_{ij} \text{ for } 1 \le i,j \le 2\}$. Calculate the restrictions of the Plücker coordinates to $A_{\{1,2\}}$.

   (b) Let $\mathrm{GL}(2,\mathbb{C})$ act by right multiplication on X. Show that $X_{\{1,2\}}$ is invariant under $\mathrm{GL}(2,\mathbb{C})$ and $A_{\{1,2\}}$ is a cross-section for the $\mathrm{GL}(2,\mathbb{C})$ orbits.

   (c) Let $\pi : X \to \mathrm{Grass}_2(\mathbb{C}^4)$ map x to its orbit under $\mathrm{GL}(2,\mathbb{C})$. Let $\mathrm{GL}(4,\mathbb{C})$ act by left multiplication on X and hence also on $\mathrm{Grass}_2(\mathbb{C}^4)$. Show that this action is transitive and calculate the stabilizer of $\pi([\, e_1 \quad e_2 \,])$, where e_i are the standard basis vectors for $\mathbb{C}^4$.

Lecture 11. Borel Subgroups

Lie-Kolchin Theorem

A single linear transformation on $\mathbb{C}^n$ can always be put into upper-triangular form by a suitable choice of basis. The same is true for a connected solvable algebraic group.

Theorem 11.1 *Let G be a connected solvable linear algebraic group, and let (π, V) be a regular representation of G. Then there exists a flag*

$$V = V_1 \supset V_2 \supset \cdots \supset V_n \supset V_{n+1} = \{0\}$$

and regular homomorphisms $\chi_i : G \to \mathbb{C}^\times$ for $i = 1, \ldots \chi_n$ so that for $v \in V_i$,

$$\pi(g)v \equiv \chi_i(g)v \mod V_{i+1}.$$

Corollary 11.2 *Assume $G \subset \mathrm{GL}(V)$ is connected and solvable. There exists a basis for V so that the elements of G are upper triangular matrices and the elements of $\mathcal{D}(G)$ have ones along the main diagonal relative to this basis. In particular, $\mathcal{D}(G)$ is unipotent.*

We have the following geometric generalization of the Lie-Kolchin theorem.

Theorem 11.3 (Borel Fixed-Point Theorem) *Let S be a connected solvable group that acts algebraically on a projective variety X. Then there exists a point $x_0 \in X$ such that $s \cdot x_0 = x_0$ for all $s \in S$.*

Existence and Conjugacy of Borel Subgroups

A *Borel subgroup* of an algebraic group G is a maximal connected solvable subgroup.

Theorem 11.4 *Let G be a connected linear algebraic group. Then G contains a Borel subgroup B, and all other Borel subgroups of G are conjugate to B. The homogeneous space G/B is a projective variety. Furthermore, if S is any connected solvable subgroup of G such that G/S is a projective variety, then S is a Borel subgroup.*

Example. Let G be a connected classical group and let B be the connected solvable subgroup in Theorem 10.3. The quotient space $X = G/B$ is a projective variety, and hence B is a Borel subgroup.

Theorem 11.5 *Let G be a connected linear algebraic group and B a fixed Borel subgroup of G. Then*

$$G = \bigcup_{x \in G} xBx^{-1}.$$

Thus every element of G is contained in a Borel subgroup.

Remark. When G is $\mathrm{GL}(n, \mathbb{C})$ this theorem is just the assertion that any (nonsingular) matrix can be conjugated into upper-triangular form.

Appendix: Algebraic Geometry for Lecture 11.

Let M be an irreducible affine set. Suppose $U \subset M$ is an open subset. Define the *regular functions on* U to be the restrictions to U of rational functions $f \in \mathrm{Rat}(M)$ such that $\mathcal{D}_f \supset U$. Replacing U by a point $x \in M$, we define the *local ring* $\mathcal{O}_x$ at x to consist of all rational functions on M that are defined at x. Clearly $\mathcal{O}_x$ is a subalgebra of $\mathrm{Rat}(M)$, and $\mathcal{O}_x = \bigcup_{x \in V} \mathcal{O}_M(V)$, where V runs over all open sets containing x.

This notion of regular function has two key properties:

(restriction) If $U \subset V$ are open subsets of M and $f \in \mathcal{O}_M(V)$, then $f|_U \in \mathcal{O}_M(U)$.

(locality) Suppose $f : U \to \mathbb{C}$ and for every $x \in U$ there exists $\phi \in \mathcal{O}_x$ with $\phi = f$ on some open neighborhood of x. Then $f \in \mathcal{O}_M(U)$.

Lemma 11.6 *Suppose X is a quasiprojective algebraic set. There is a finite open covering*

$$X = \bigcup_{\alpha \in A} U_\alpha$$

with the following properties:
(1) *There are irreducible affine algebraic sets M_α and homeomorphisms $\phi_\alpha : U_\alpha \to M_\alpha$ for $\alpha \in A$.*
(2) *The maps $\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ are regular, for all $\alpha, \beta \in A$.*

Let X be a quasiprojective algebraic set. We define the *local ring* $\mathcal{O}_x$ at $x \in X$ by carrying over the local rings of the affine open sets U_α via the maps ϕ_α:

$$\mathcal{O}_x = \phi_\alpha^*(\mathcal{O}_{\phi_\alpha(x)}), \quad \text{for } x \in U_\alpha.$$

If $x \in U_\alpha \cap U_\beta$ then $\mathcal{O}_x$ is the same, whether we use ϕ_α or ϕ_β, by the last statement in Lemma 11.6. For any open set $U \subset X$ we can now define the ring $\mathcal{O}_X(U)$ of *regular functions* on U using the local rings, just as in the affine case: a continuous function $f : U \to \mathbb{C}$ is *regular* if for each $x \in U$ there exists $g \in \mathcal{O}_x$ so that $f = g$ on an open neighborhood of x. One then verifies that the restriction and locality properties hold for the rings $\mathcal{O}_X(U)$.

Let X, Y be quasiprojective. A map $\phi : X \to Y$ will be called *regular* if ϕ is continuous and for all open sets $U \subset Y$, $\phi^*(\mathcal{O}(U)) \subset \mathcal{O}(\phi^{-1}(U))$. When X, Y are affine, this agrees with our earlier definition.

Lemma 11.7 *Let X, Y, Z be quasiprojective. A map $z \mapsto (f(z), g(z))$ from Z to $X \times Y$ is regular if and only if $f : Z \to X$ and $g : Z \to Y$ are regular.*

Proposition 11.8 *Suppose X, Y are quasiprojective algebraic sets. Let $\phi : X \to Y$ be regular. Then*

$$\Gamma_\phi = \{(x, \phi(x)) : x \in X\}$$

(the graph of ϕ) is closed in $X \times Y$.

Corollary 11.9 *Let X be a quasi-projective algebraic set and $\phi : X \to X$ a regular map. Then the fixed-point set*

$$\{x \in X \, : \, \phi(x) = x\}$$

is closed in X.

We denote by $\mathbb{C}[X] = \mathcal{O}_X(X)$ the ring of functions that are everywhere regular on X.

Theorem 11.10 *Let X be an irreducible projective algebraic set. Then $\mathbb{C}[X] = \mathbb{C}$.*

Corollary 11.11 *If X is an irreducible projective algebraic set which is also isomorphic to an affine algebraic set, then X is a single point.*

A map $\phi : X \to Y$ between quasiprojective algebraic sets is defined to be *regular* if $\phi^* \mathcal{O}_{\phi(x)} \subset \mathcal{O}_x$ for all $x \in X$. When X and Y are affine, this is consistent with the earlier terminology, by Lemma 3.10.

Theorems 3.6 and 3.9 are also valid when X, Y are quasiprojective algebraic sets. Furthermore, if f is a rational map between affine algebraic sets, then the open set $\mathcal{D}_f$ is a quasiprojective algebraic set, and $f : \mathcal{D}_f \to Y$ is a regular map in this new sense. Thus Theorem 3.12 is also valid for quasiprojective algebraic sets.

If X is quasiprojective and $x \in X$, we define $\dim_x(X) = \dim T(U_\alpha)_x$, where $x \in U_\alpha$ as in Lemma 11.6. It is easy to see that $\dim_x(X)$ only depends on the local ring $\mathcal{O}_x$ (cf. Theorem 4.11). We set

$$\dim X = \min_{x \in X} \dim_x(X).$$

It is clear from this definition of dimension that Theorem 9.8 holds for quasi-projective algebraic sets. Just as in the affine case, a point $x \in X$ is called *simple* if $\dim_x(X) = \dim X$. When X is irreducible, the simple points form a dense open set. If every point of X is simple then X is said to be *smooth* or *nonsingular*.

Theorem 11.12 *Let X, Y be quasiprojective sets with X projective. Let $p(x, y) = y$ for $(x, y) \in X \times Y$. If $C \subset X \times Y$ is closed then $p(C)$ is closed in Y.*

Corollary 11.13 *Let X be projective and $f : X \to Y$ be a regular map with Y quasiprojective. Then $f(X)$ is closed in Y.*

Exercises for Lecture 11.

1. Let $X = \mathbb{C}^2 \setminus \{0\}$ with its structure as a quasiprojective algebraic set. Then $X = X_1 \cup X_2$, where $X_1 = \mathbb{C}^\times \times \mathbb{C}$ and $X_2 = \mathbb{C} \times \mathbb{C}^\times$ are affine open subsets. Also $f \in \mathcal{O}(X)$ (the ring of regular functions on X) if and only if $f|_{X_i} \in \mathrm{Aff}(X_i)$ for $i = 1, 2$.

   (a) Prove that $\mathcal{O}(X) = \mathbb{C}[x_1, x_2]$, where x_i are the coordinate functions on $\mathbb{C}^2$. (*Hint:* Let $f \in \mathcal{O}(X)$. Write $f|_{X_1}$ as a polynomial in x_1, x_1^{-1}, x_2 and write $f|_{X_2}$ as a polynomial in x_1, x_2, x_2^{-1}. Then compare these expressions on $X_1 \cap X_2$.)

   (b) Prove that X is not a projective algebraic set. (*Hint:* Consider $\mathcal{O}(X)$.)

   (c) Prove that X is not an affine algebraic set. (*Hint:* By (a) there is a homomorphism $f \mapsto f(0)$ of $\mathcal{O}(X)$.)

   (d) Let $G = \mathrm{SL}(2, \mathbb{C})$ and N the upper-triangular unipotent matrices in G. Prove that $G/N \cong \mathbb{C}^2 \setminus \{0\}$, with G acting as usual on $\mathbb{C}^2$. (*Hint:* Find a vector in $\mathbb{C}^2$ whose stabilizer is N.)

2. Let $G = \mathrm{GL}(n, \mathbb{C})$, $H = D_n$ the diagonal matrices in G, N the upper-triangular unipotent matrices, and $B = HN$. Let X be the space of all flags in $\mathbb{C}^n$.

   (a) Suppose $x = \{V_1 \subset V_2 \subset \cdots \subset V_n\}$ is a flag that is invariant under H. Prove that there is a permutation $\sigma \in \mathfrak{S}_n$ so that

$$V_i = \mathrm{Span}\{e_{\sigma(1)}, \ldots, e_{\sigma(i)}\} \quad \text{for } i = 1, \ldots, n.$$

   (*Hint:* H is reductive and its action on $\mathbb{C}^n$ is multiplicity-free.)

   (b) Suppose the flag x in (a) is also invariant under N. Prove that $\sigma(i) = i$ for all i. (*Hint:* Use induction on i.)

   (c) Prove that if $g \in G$ and $gBg^{-1} = B$, then $g \in B$. (*Hint:* By (a) and (b), B has exactly one fixed point on $X = G/B$.)

3. Let G be a connected algebraic group and $B \subset G$ a Borel subgroup. Let $P \subset G$ be a closed subgroup.

   (a) Suppose that G/P is a projective algebraic set. Prove that there exists $g \in G$ such that $gBg^{-1} \subset P$. (*Hint:* B has a fixed point on G/P.)

   (b) Suppose that $B \subset P$. Prove that G/P is a projective algebraic set. (*Hint:* Consider the natural map $G/B \to G/P$.)

4. Let G be a classical group. Let B be the upper-triangular (Borel) subgroup of G, and H the diagonal subgroup of G. Suppose $P \subset G$ is a closed subgroup such that $B \subset P$.

   (a) Prove that $\mathrm{Lie}(P)$ is of the form

$$\mathfrak{b} + \sum_{\alpha \in S} \mathfrak{g}_{-\alpha} \qquad\qquad (*)$$

   where $\mathfrak{b} = \mathrm{Lie}(B)$ and $S \subset \Phi^+$ (the positive roots of $\mathfrak{g}$). (*Hint*: $\mathrm{Lie}(P)$ is invariant under $\mathrm{Ad}(H)$.)

   (b) Let $S \subset \Phi^+$ be any subset and let $\{\alpha_1, \ldots, \alpha_l\}$ be the simple roots in Φ^+. Prove that the subspace defined by $(*)$ is a Lie algebra if and only if S satisfies the properties

   (P1) If $\alpha, \beta \in S$ and $\alpha + \beta \in \Phi^+$, then $\alpha + \beta \in S$.
   (P2) If $\beta \in S$ and $\beta - \alpha_i \in \Phi^+$ then $\beta - \alpha_i \in S$.

   (*Hint*: $\mathfrak{b}$ is generated by $\mathfrak{h}$ and $\{\mathfrak{g}_{\alpha_i} : i = 1, \ldots, l\}$.)

   (c) Let R be any subset of the simple roots, and define S_R to be all the positive roots β so that no elements of R occur in β. Show that S_R satisfies **(P1)** and **(P2)**. Conversely, if S satisfies **(P1)** and **(P2)**, let R be the set of simple roots that do not occur in any $\beta \in S$. Prove that $S = S_R$.

   (d) Let $G = \mathrm{GL}(n, \mathbb{C})$. Use (c) to determine all subsets S of Φ^+ that satisfy **(P1)** and **(P2)**. (*Hint:* Use Exercise #4 from Lecture 8.)

   (e) For each subset S found in (d), show that there is a closed subgroup $P \supset B$ with $\mathrm{Lie}(P)$ given by $(*)$. (*Hint:* Show that S corresponds to a partition of n and consider the corresponding block decomposition of G.)

Part 4: Irreducible Representations

Lecture 12. Weyl Group and Weight Lattice

Weyl Group of a Classical Group

Let G be a connected classical group and let H be a maximal torus in G. Define the *normalizer* of H in G to be

$$\mathrm{Norm}_G(H) = \{g \in G \,:\, ghg^{-1} \in H \text{ for all } h \in H\},$$

and define the *Weyl group* $W_G = \mathrm{Norm}_G(H)/H$. Since all maximal torii of G are conjugate, the group W_G is uniquely defined (as an abstract group) by G, and it acts (by conjugation) as automorphisms of H.

Since H is abelian, there is a natural homomorphism $\phi : W_G \to \mathrm{Aut}(H)$ given by $\phi(sH)h = shs^{-1}$ for $s \in \mathrm{Norm}_G(H)$. This homomorphism gives an action of W_G on the character group $\mathcal{X}(H)$, where for $\theta \in \mathcal{X}(H)$ the character $s \cdot \theta$ is defined by

$$s \cdot \theta(h) = \theta(s^{-1}hs), \quad \text{for } h \in H.$$

Writing $\theta = e^{\lambda}$ for $\lambda \in P(G)$, we can describe this action as

$$s \cdot e^{\lambda} = e^{s \cdot \lambda},$$

where $\langle s \cdot \lambda, x\rangle = \langle \lambda, \mathrm{Ad}(s)^{-1}x\rangle$ for $x \in \mathfrak{h}$. This defines a linear action of W_G on $\mathfrak{h}^*$.

Theorem 12.1 *W_G is a finite group and the representation of W_G on $\mathfrak{h}^*$ is faithful.*

For $\sigma \in \mathfrak{S}_n$ let $s_\sigma \in \mathrm{GL}(n, \mathbb{C})$ be the matrix such that

$$s_\sigma e_i = e_{\sigma(i)} \quad \text{for } i = 1, \ldots, n.$$

This is the usual representation of $\mathfrak{S}_n$ on $\mathbb{C}^n$ as *permutation matrices*.

Suppose $G = \mathrm{GL}(n, \mathbb{C})$. Then H is the group of all $n \times n$ diagonal matrices. Every coset in W_G is of the form $s_\sigma H$ for some $\sigma \in \mathfrak{S}_n$. Hence $W_G \cong \mathfrak{S}_n$. The action of $\sigma \in \mathfrak{S}_n$ on the diagonal coordinate functions $x_1, \ldots, x_n$ for H is $\sigma \cdot x_i = x_{\sigma^{-1}(i)}$.

Let $G = \mathrm{SL}(n, \mathbb{C})$. Then H consists of all diagonal matrices of determinant 1 and $W_G \cong \mathfrak{S}_n$.

Next, consider the case $G = \mathrm{Sp}(\mathbb{C}^{2l}, \Omega)$, with Ω as in (4.3). Let $s_0 \in \mathrm{GL}(2l, \mathbb{C})$ be the matrix for the permutation $(1, l)(2, l-1)(3, l-2)\cdots$. For $\sigma \in \mathfrak{S}_l$ let $s_\sigma \in \mathrm{GL}(l, \mathbb{C})$ be the corresponding permutation matrix.Clearly $s_\sigma^t = s_\sigma^{-1}$, so if we define

$$\pi(\sigma) = \begin{bmatrix} s_\sigma & 0 \\ 0 & s_0 s_\sigma s_0 \end{bmatrix},$$

then $\pi(\sigma) \in G$ and hence $\pi(\sigma) \in \mathrm{Norm}_G(H)$. Consider the transpositions $(i, 2l+1-i)$ in $\mathfrak{S}_{2l}$, where $1 \le i \le l$. Set $e_{-i} = e_{2l+1-i}$, where $\{e_i\}$ is the standard basis for $\mathbb{C}^{2l}$. Define $\tau_i \in \mathrm{GL}(2l, \mathbb{C})$ by

$$\tau_i e_i = e_{-i}\,, \quad \tau_i e_{-i} = -e_i\,, \quad \tau_i e_k = e_k \text{ for } k \neq i, -i.$$

Given $F \subset \{1, \ldots, l\}$, define

$$\tau_F = \prod_{i \in F} \tau_i \in \mathrm{Norm}_G(H).$$

Then the H-cosets of the elements $\{\tau_F\}$ form an abelian subgroup $T_l \cong (\mathbb{Z}/2\mathbb{Z})^l$ of W_G. The action of τ_F on the coordinate functions $x_1, \ldots, x_l$ for H is $x_i \mapsto x_i^{-1}$ for $i \in F$ and $x_j \mapsto x_j$ for $j \notin F$.

Lemma 12.2 *For $G = \mathrm{Sp}(\mathbb{C}^{2l}, \Omega)$, the subgroup $T_l \subset W_G$ is normal, and W_G is the semidirect product of T_l and $\bar{\pi}(\mathfrak{S}_l)$. The action of W_G on the coordinate functions for $\mathrm{Aff}(H)$ is by $x_i \mapsto x_{\sigma(i)}^{\pm 1}$ $(i = 1, \ldots, l)$, for every permutation σ and choice ± 1 of exponents.*

Now consider the case $G = \mathrm{SO}(\mathbb{C}^{2l+1}, B)$, with the symmetric form B having 1s on the skew-diagonal and 0s elsewhere. For $\sigma \in \mathfrak{S}_l$ define

$$\phi(\sigma) = \begin{bmatrix} s_\sigma & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & s_0 s_\sigma s_0 \end{bmatrix}.$$

Then $\phi(\sigma) \in G$ and hence $\phi(\sigma) \in \mathrm{Norm}_G(H)$. Obviously $\phi(\sigma) \in H$ if and only if $\sigma = 1$, so again we have an injective homomorphism $\bar{\phi} : \mathfrak{S}_l \to W_G$.
We can construct other elements of W_G by the same method as for the symplectic group. Set

$$e_{-i} = e_{2l+2-i} \quad \text{for } i = 1, \ldots, l+1.$$

For each transposition $(i, 2l+2-i)$ in $\mathfrak{S}_{2l+1}$, where $1 \le i \le l$, we define $\gamma_i \in \mathrm{GL}(2l+1, \mathbb{C})$ by

$$\gamma_i e_i = e_{-i}\,, \quad \gamma_i e_{-i} = e_i\,, \quad \gamma_i e_0 = -e_0,$$

$$\gamma_i e_k = e_k \quad \text{for } k \neq i, 0, -i.$$

Then $\gamma_i \in \mathrm{Norm}_G(H)$. Furthermore, $\gamma_i^2 \in H$ and $\gamma_i \gamma_j = \gamma_j \gamma_i$ if $1 \le i, j \le l$. Given $F \subset \{1, \ldots, l\}$, we define

$$\gamma_F = \prod_{i \in F} \gamma_i \in \mathrm{Norm}_G(H).$$

Then the H-cosets of the elements $\{\gamma_F\}$ form an abelian subgroup $T_l \cong (\mathbb{Z}/2\mathbb{Z})^l$ of W_G. The action of γ_F on the coordinate functions $x_1, \ldots, x_l$ for $\mathrm{Aff}(H)$ is the same as that of τ_F for the symplectic group.

Lemma 12.3 *Let $G = \mathrm{SO}(\mathbb{C}^{2l+1}, B)$. The subgroup $T_l \subset W_G$ is normal, and W_G is the semidirect product of T_l and $\bar{\phi}(\mathfrak{S}_l)$. The action of W_G on the coordinate functions for $\mathrm{Aff}(H)$ is by $x_i \mapsto x_{\sigma(i)}^{\pm 1}$ $(i = 1, \ldots, l)$, for every permutation σ and choice ± 1 of exponents.*

Finally, we consider the case $G = \mathrm{SO}(\mathbb{C}^{2l}, B)$, with B as in (4.3). For $\sigma \in \mathfrak{S}_l$ define $\pi(\sigma)$ as in the symplectic case. Then $\pi(\sigma) \in \mathrm{Norm}_G(H)$. Obviously $\pi(\sigma) \in H$ if and only if $\sigma = 1$, so we have an injective homomorphism $\bar{\pi} : \mathfrak{S}_l \to W_G$. The automorphism of H induced by $\sigma \in \mathfrak{S}_l$ is the same as for the symplectic group.
Set

$$e_{-i} = e_{2l+1-i} \quad \text{for } i = 1, \ldots, l.$$

For each transposition $(i, 2l+1-i)$ in $\mathfrak{S}_{2l}$, where $1 \le i \le l$, we define $\beta_i \in \mathrm{GL}(2l, \mathbb{C})$ by

$$\beta_i e_i = e_{-i}\,, \quad \beta_i e_{-i} = e_i\,, \quad \beta_i e_k = e_k \quad \text{for } k \neq i, -i.$$

Then $\beta_i \in \mathrm{O}(\mathbb{C}^{2l}, B)$. Given $F \subset \{1, \ldots, l\}$, define

$$\beta_F = \prod_{i \in F} \beta_i.$$

If $\mathrm{card}(F)$ is *even,* then $\det \beta_F = 1$ and hence $\beta_F \in \mathrm{Norm}_G(H)$. Thus the H cosets of the elements $\{\beta_F : \mathrm{card}(F) \text{ even }\}$ form an abelian subgroup R_l of W_G.

Lemma 12.4 *Let $G = \mathrm{SO}(\mathbb{C}^{2l}, B)$. The subgroup $R_l \subset W_G$ is normal, and W_G is the semidirect product of R_l and $\bar{\pi}(\mathfrak{S}_l)$. The action of W_G on the coordinate functions for* $\mathrm{Aff}(H)$ *is by $x_i \mapsto x_{\sigma(i)}^{\pm 1}$ $(i = 1, \ldots, l)$, for every permutation σ and choice ± 1 of exponents with an* even *number of sign changes.*

Root Reflections

Let G be a connected classical group and let $\mathfrak{h}$ be the Lie algebra of the maximal torus H of G. Let $\Phi \subset \mathfrak{h}^*$ be the roots and Δ the simple roots of $\mathfrak{g}$ relative to the choice Φ^+ of positive roots. For each $\alpha \in \Phi$ let $h_\alpha \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ be the coroot to α.
Define the *root reflection* $s_\alpha : \mathfrak{h}^* \to \mathfrak{h}^*$ by

$$s_\alpha(\beta) = \beta - \langle \beta, h_\alpha \rangle \alpha, \quad \text{for } \beta \in \mathfrak{h}^*.$$

We can also write the formula for s_α as

$$s_\alpha(\beta) = \beta - \frac{2(\beta, \alpha)}{(\alpha, \alpha)} \alpha.$$

The linear transformation s_α satisfies

$$s_\alpha(\alpha) = -\alpha, \qquad s_\alpha(\beta) = \beta \qquad \text{if } \langle \beta, h_\alpha \rangle = 0.$$

Thus $s_\alpha^2 = I$. It can be described geometrically as the *reflection through the hyperplane* $(h_\alpha)^\perp$. Note that the roots α and $-\alpha$ define the same reflection.

Lemma 12.5 *Let $W = \mathrm{Norm}_G(H)/H$ be the Weyl group of G. Identify W with a subgroup of* $\mathrm{GL}(\mathfrak{h}^*)$ *by the natural action of W on $\mathcal{X}(H)$.*
(1) *For every $\alpha \in \Phi$ there exists $w \in W$ such that w acts on $\mathfrak{h}^*$ by the reflection s_α.*
(2) $W \cdot \Delta = \Phi$
(3) *W is generated by the reflections $\{s_\alpha : \alpha \in \Delta\}$.*
(4) *If $w \in W$ and $w\Phi^+ = \Phi^+$ then $w = 1$.*
(5) *There exists a unique element $w_0 \in W$ such that $w_0\Phi^+ = -\Phi^+$.*

Weight Lattice of a Classical Group

Proposition 12.6 *Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ and let $\{e, f, h\}$ be a TDS triple which is a basis for $\mathfrak{g}$. Let (ρ, V) be a finite-dimensional $\mathfrak{g}$-module and set $V^e = \mathrm{Ker}(\rho(e))$.*
(1) *$\rho(h)$ is diagonalizable with integral eigenvalues, while $\rho(e)$ and $\rho(f)$ are nilpotent.*
(2) *The eigenvalues of $\rho(h)$ on V^e are all non-negative.*
(3) *If $v \in V^e$ and $\rho(h)v = 0$ then $\rho(f)v = 0$.*
(4) *$V = V^e \oplus \rho(f)V$.*

Let G be a connected classical group. Fix a maximal torus H in G and let $\mathfrak{g} = \mathrm{Lie}(G)$, $\mathfrak{h} = \mathrm{Lie}(H)$. Let

$$\mathfrak{z}(\mathfrak{g}) = \{Z \in \mathfrak{g} \,:\, [X, Z] = 0 \text{ for all } X \in \mathfrak{g}\}$$

be the center of $\mathfrak{g}$. Then $\mathfrak{z}(\mathfrak{g}) \subset \mathfrak{h}$. Let $\Phi \subset \mathfrak{h}^*$ be the roots of H on $\mathfrak{g}$.

Theorem 12.7 *Let (π, V) be a finite-dimensional representation of $\mathfrak{g}$. For $\mu \in \mathfrak{h}^*$ set*

$$V(\mu) = \{v \in V \,:\pi(Y)v = \langle \mu, Y\rangle v \quad \textit{for all } Y \in \mathfrak{h}\}.$$

(1) *Suppose $V(\mu) \neq 0$. Then $\langle \mu, h_\alpha\rangle \in \mathbb{Z}$ for all $\alpha \in \Phi$, where h_α is the coroot to α.*
(2) *Suppose $\pi(Z)$ is diagonalizable for all $Z \in \mathfrak{z}(\mathfrak{g})$. Then*

$$V = \bigoplus_{\mu \in \mathfrak{h}^*} V(\mu).$$

Hence $\pi(Y)$ is diagonalizable for every $Y \in \mathfrak{h}$.

We define the *weight lattice* for $\mathfrak{g}$ as

$$P(\mathfrak{g}) = \{\mu \in \mathfrak{h}^* \,:\, \langle \mu, h_\alpha\rangle \in \mathbb{Z} \quad \text{for all } \alpha \in \Phi\}.$$

If V is a $\mathfrak{g}$-module and $V(\mu) \neq 0$, then we say that μ is a *weight* of V. In this case $\mu \in P(\mathfrak{g})$ by Theorem 12.7. For example, the weights of the adjoint representation are $\Phi \cup \{0\}$.
Clearly $P(\mathfrak{g})$ is an additive subgroup of $\mathfrak{h}^*$. We define the *root lattice* $Q(\mathfrak{g})$ to be the additive subgroup of $\mathfrak{h}^*$ generated by Φ. Thus $Q(\mathfrak{g}) \subset P(\mathfrak{g})$.

Lemma 12.8 *The lattices $P(\mathfrak{g})$ and $Q(\mathfrak{g})$ are invariant under the Weyl group W.*

We also denote by $s_\alpha \in \mathrm{GL}(\mathfrak{h})$ the transpose of the root reflection for α; it acts by

$$s_\alpha Y = Y - \langle \alpha, Y\rangle h_\alpha$$

for $Y \in \mathfrak{h}$.

Proposition 12.9 *Let (π, V) be a finite-dimensional representation of $\mathfrak{g}$. For $\alpha \in \Phi$ let $\{e_\alpha, f_\alpha, h_\alpha\}$ be a TDS triple associated with α, and define*

$$\tau_\alpha = \exp(\pi(e_\alpha))\exp(-\pi(f_\alpha))\exp(\pi(e_\alpha)) \in \mathrm{GL}(V).$$

Then
(1) *$\tau_\alpha \pi(Y)\tau_\alpha^{-1} = \pi(s_\alpha Y)$ for $Y \in \mathfrak{h}$,*
(2) *$\tau_\alpha V(\mu) = V(s_\alpha \mu)$ for all $\mu \in \mathfrak{h}^*$,*
(3) *$\dim V(\mu) = \dim V(s \cdot \mu)$ for all $s \in W$.*

Fundamental Weights and Dominant Weights

Let $\Delta = \{\alpha_1, \dots, \alpha_l\} \subset \Phi^+$ be the simple roots in Φ^+ and denote by H_i the coroot to α_i, as in Lemma 8.8. Let $\mathfrak{z}(\mathfrak{g})$ be the center of $\mathfrak{g}$. Then

$$\mathfrak{h} = \mathfrak{z}(\mathfrak{g}) \oplus (\mathfrak{h} \cap [\mathfrak{g}, \mathfrak{g}])$$

Thus we may identify $\mathfrak{z}(\mathfrak{g})^*$ with the subspace of $\mathfrak{h}^*$ that annihilates $\mathfrak{h} \cap [\mathfrak{g}, \mathfrak{g}]$. Since $\{H_1, \dots, H_l\}$ is a basis for $\mathfrak{h} \cap [\mathfrak{g}, \mathfrak{g}]$, there is a unique set $\{\varpi_1, \dots, \varpi_l\} \subset \mathfrak{h}^*$ such that

$$\langle \varpi_i, H_j \rangle = \delta_{ij} \quad \text{for } i, j = 1, \dots, l \quad \text{and } \varpi_i \perp \mathfrak{z}(\mathfrak{g}).$$

Then

$$P(\mathfrak{g}) = \mathfrak{z}(\mathfrak{g})^* \oplus \{n_1\varpi_1 + \cdots + n_l\varpi_l \,:\, n_i \in \mathbb{Z}\}. \tag{12.1}$$

The weights $\varpi_1, \dots, \varpi_l$ will be called the *fundamental weights* for $\mathfrak{g}$.
We now give the fundamental weights for each type of classical group in terms of the weights $\{\varepsilon_i\}$.

Type A: ($G = \mathrm{SL}(l+1, \mathbb{C})$)

$$\varpi_i = \varepsilon_1 + \cdots + \varepsilon_i - \frac{i}{l+1}(\varepsilon_1 + \cdots + \varepsilon_{l+1}) \quad \text{for } 1 \le i \le l.$$

Type B: ($G = \mathrm{SO}(2l+1, \mathbb{C})$)

$$\varpi_i = \varepsilon_1 + \cdots + \varepsilon_i \quad \text{for } 1 \le i \le l-1, \qquad \varpi_l = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_l)$$

Type C: ($G = \mathrm{Sp}(l, \mathbb{C})$)

$$\varpi_i = \varepsilon_1 + \cdots + \varepsilon_i \quad \text{for } 1 \le i \le l.$$

Type D: ($G = \mathrm{SO}(2l, \mathbb{C})$, with $l \ge 2$)

$$\varpi_i = \varepsilon_1 + \cdots + \varepsilon_i \quad \text{for } 1 \le i \le l-2,$$

$$\varpi_{l-1} = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{l-1} - \varepsilon_l), \quad \varpi_l = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{l-1} + \varepsilon_l).$$

Since the functionals ε_i are weights of the defining representation of G, we have $\varepsilon_i \in P(\mathfrak{g})$ for $i = 1, \dots, l$. Thus $P(G) \subset P(\mathfrak{g})$. For $\mathfrak{g}$ of type A or C all the fundamental weights are in $P(G)$, so

$$P(G) = P(\mathfrak{g}) \qquad (G = \mathrm{SL}(n, \mathbb{C}) \text{ or } \mathrm{Sp}(n, \mathbb{C})).$$

However, for $G = \mathrm{SO}(2l+1, \mathbb{C})$ we have

$$\varpi_i \in P(G) \quad \text{for } 1 \le i \le l-1, \quad 2\varpi_l \in P(G),$$

but $\varpi_l \notin P(G)$. For $G = \mathrm{SO}(2l, \mathbb{C})$ we have

$$\varpi_i \in P(G) \quad \text{for } 1 \le i \le l-2, \quad m\varpi_{l-1} + n\varpi_l \in P(G) \quad \text{if } m + n \in 2\mathbb{Z},$$

but ϖ_{l-1} and ϖ_l are not in $P(G)$. Thus

$$P(\mathfrak{g})/P(G) \cong \mathbb{Z}/2\mathbb{Z} \quad \text{when } G = \mathrm{SO}(n, \mathbb{C}).$$

This means that for the orthogonal groups in odd (*resp.* even) dimensions there is no single-valued character χ on the maximal torus whose differential is ϖ_l (*resp.* ϖ_{l-1} or ϖ_l). We will resolve this difficulty in Lecture 17 with the construction of the groups $\mathrm{Spin}(n, \mathbb{C})$ and the *spin* representations.

Define the *dominant weights* for $\mathfrak{g}$ (relative to the given choice of positive roots) to be

$$P_{++}(\mathfrak{g}) = \{\mu \in P(\mathfrak{g}) \, : \, \langle \mu, H_i \rangle \geq 0 \text{ for } i = 1, \ldots, l\}.$$

From (12.1) we see that

$$P_{++}(\mathfrak{g}) = \mathfrak{z}(\mathfrak{g}) + \mathbb{N}\varpi_1 + \cdots + \mathbb{N}\varpi_l \, .$$

where $\mathbb{N} = \{0, 1, 2, \ldots\}$. We say that $\mu \in P_{++}(\mathfrak{g})$ is *regular* if $\langle \mu, H_i \rangle > 0$ for $i = 1, \ldots, l$. This is equivalent to

$$\mu = \zeta + n_1\varpi_1 + \cdots + n_l\varpi_l, \qquad \text{with } \zeta \in \mathfrak{z}(\mathfrak{g})^* \text{ and } n_i \geq 1 \text{ for all } i.$$

We define the *dominant weights* for G to be

$$P_{++}(G) = P(G) \cap P_{++}(\mathfrak{g}).$$

Then $P_{++}(G) = P_{++}(\mathfrak{g})$ when G is $\mathrm{SL}(n, \mathbb{C})$ or $\mathrm{Sp}(n, \mathbb{C})$.

The definition of dominant weight depends on a choice of the system Φ^+ of positive roots. We now prove that any weight can be transformed into a unique dominant weight by the action of the Weyl group. This means that the dominant weights give a cross-section for the orbits of the Weyl group on the weight lattice.

Proposition 12.10 *For every $\lambda \in P(\mathfrak{g})$ there is $\mu \in P_{++}(\mathfrak{g})$ and $s \in W$ such that $\lambda = s \cdot \mu$. The weight μ is uniquely determined by λ. If μ is regular, then s is uniquely determined by λ and hence the orbit $W \cdot \mu$ has $|W|$ elements.*
For each type of classical group the dominant weights are given in terms of the weights $\{\varepsilon_i\}$ as follows:
(1) *Let $G = \mathrm{GL}(n, \mathbb{C})$ or $\mathrm{SL}(n, \mathbb{C})$. Then $P_{++}(\mathfrak{g})$ consists of all weights*

$$\mu = k_1\varepsilon_1 + \cdots + k_n\varepsilon_n \text{ with } k_1 \geq k_2 \geq \cdots \geq k_n \text{ and } k_i - k_{i+1} \in \mathbb{Z}. \tag{12.2}$$

(2) *Let $G = \mathrm{SO}(2l + 1, \mathbb{C})$. Then $P_{++}(\mathfrak{g})$ consists of all*

$$\mu = k_1\varepsilon_1 + \cdots + k_l\varepsilon_l \text{ with } k_1 \geq k_2 \geq \cdots \geq k_l \geq 0. \tag{12.3}$$

Here $2k_i$ and $k_i - k_j$ are integers for all i, j.
(3) *Let $G = \mathrm{Sp}(l, \mathbb{C})$. Then $P_{++}(\mathfrak{g})$ consists of all μ satisfying* (12.3) *with k_i integers for all i.*
(4) *Let $G = \mathrm{SO}(2l, \mathbb{C})$, $l \geq 2$. Then $P_{++}(\mathfrak{g})$ consists of all*

$$\mu = k_1\varepsilon_1 + \cdots + k_l\varepsilon_l \text{ with } k_1 \geq \cdots \geq k_{l-1} \geq |k_l|. \tag{12.4}$$

Here $2k_i$ and $k_i - k_j$ are integers for all i, j.
The weight μ is regular when all inequalities in (12.2), (12.3) *or* (12.4) *are strict.*

Exercises for Lecture 12.

1. Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ with standard basis $\{e, f, h\}$. Let (ρ, W) be a finite dimensional representation of $\mathfrak{g}$. For $k \in \mathbb{Z}$ set $f(k) = \dim\{w \in W : \rho(h)w = kw\}$.

   (a) Show that $f(k) = f(-k)$.

   (b) Let $g_{\text{even}}(k) = f(2k)$ and $g_{\text{odd}}(k) = f(2k+1)$. Show that g_{even} and g_{odd} are unimodal functions from $\mathbb{Z}$ to $\mathbb{N}$. Here a function ϕ is called *unimodal* if there exists k_0 such that $\phi(a) \le \phi(b)$ for all $a < b \le k_0$ and $\phi(a) \ge \phi(b)$ for all $k_0 \le a < b$.

   (c) Suppose $f(0) = f(2) = 4$, $f(1) = f(3) = 2$, $f(4) = 2$, $f(6) = 1$ and $f(k) = 0$ for other positive integers k. What is $\dim \mathrm{Ker}\rho(e)$? What are the irreducible $\mathfrak{g}$ submodules in W?

2. Let $G \subset \mathrm{GL}(n, \mathbb{C})$ be a classical group and let Φ be the root system of G. Set $V = \sum_{i=1}^{n} \mathbb{R}\varepsilon_i$. Give V the inner product $(\cdot\,|\,\cdot)$ so that $(\varepsilon_i|\varepsilon_j) = \delta_{ij}$.

   (a) Show that $(\alpha|\alpha)$, for $\alpha \in \Phi$, is 1, 2 or 4, and that at most two distinct lengths occur. (The system Φ is called *simply-laced* when all roots have the same length, because the Dynkin diagram has no double lines in this case.)

   (b) Let $\alpha, \beta \in \Phi$ with $(\alpha|\alpha) = (\beta|\beta)$. Show that there exists $w \in W_G$ so that $w \cdot \alpha = \beta$. (If $G = \mathrm{SO}(2l, \mathbb{C})$ assume that $l \ge 3$.)

3. Let $G = \mathrm{SL}(3, \mathbb{C})$, H the diagonal matrices in G, and let $V = \mathbb{C}^3 \otimes \mathbb{C}^3$.

   (a) Find the weights of H on V. Express the weights in terms of $\varepsilon_1, \varepsilon_2, \varepsilon_3$ and for each weight determine its multiplicity. Verify that the weight multiplicities are invariant under the Weyl group W of G.

   (b) Verify that each Weyl group orbit in the set of weights of V contains exactly one dominant weight. Find the *extreme* dominant weights β (those such that $\beta + \alpha$ is not a weight, for any positive root α).

   (c) Write the weights of V in terms of the fundamental weights $\{\varpi_1, \varpi_2\}$ and plot the set of weights in the $\mathfrak{h}^*$ plane. Indicate multiplicities and W-orbits in the plot. (Show that $||\varpi_1|| = ||\varpi_2||$ and that the angle between ϖ_1 and ϖ_2 is 60°. Note that $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0$ on $\mathfrak{h}^*$.)

   (d) V decomposes into G-invariant subspaces $V = V_+ \oplus V_-$, where V_+ consists of the symmetric 2-tensors, and V_- is the skew-symmetric 2-tensors. Determine the weights and multiplicities of $V_\pm$ and verify that the weight multiplicities are invariant under W.

4. Let $G = \mathrm{Sp}(\mathbb{C}^4, \Omega)$, where $\Omega = \begin{bmatrix} 0 & s_0 \\ -s_0 & 0 \end{bmatrix}$ and s_0 has antidiagonal 1 as usual. Let H be the diagonal matrices in G, and let $V = \bigwedge^2 \mathbb{C}^4$.

   (a) Find all the weights of H on V. Express the weights in terms of $\varepsilon_1, \varepsilon_2$ and for each weight determine its multiplicity (note that $\varepsilon_3 = -\varepsilon_2$ and $\varepsilon_4 = -\varepsilon_1$ as elements of $\mathfrak{h}^*$). Verify that the weight multiplicities are invariant under the Weyl group W of G.

   (b) Verify that each Weyl group orbit in the set of weights of V contains exactly one dominant weight. Find the *extreme* dominant weights β (those such that $\beta + \alpha$ is not a weight, for any positive root α).

(c) Write the weights of V in terms of the fundamental weights $\{\varpi_1, \varpi_2\}$ and plot the set of weights in the $\mathfrak{h}^*$ plane. Indicate multiplicities and W orbits in the plot. (Show that $||\varpi_2||^2 = 2||\varpi_1||$ and that the angle between ϖ_1 and ϖ_2 is $45°$ relative to a W-invariant inner product on $\mathfrak{h}^*$.

Lecture 13. Highest Weight Theory

Extreme Vectors and Highest Weights

Let G be a classical group whose Lie algebra is semisimple. We fix a set Φ^+ of positive roots and the associated triangular decomposition

$$\mathfrak{g} = \bar{\mathfrak{n}} + \mathfrak{h} + \mathfrak{n}$$

as in Theorem 8.9. We set $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$ and call $\mathfrak{b}$ a *Borel subalgebra* of $\mathfrak{g}$. We have

$$[\mathfrak{b}, \mathfrak{b}] = \mathfrak{n}, \qquad [\mathfrak{h}, \mathfrak{n}] = \mathfrak{n}.$$

Let $P(\mathfrak{g})$ be the weight lattice and $P_{++}(\mathfrak{g})$ the dominant weights (relative to the choice of Φ^+). If (π, V) is a finite-dimensional representation of $\mathfrak{g}$, then V has a weight-space decomposition

$$V = \bigoplus_{\mu \in P(\mathfrak{g})} V(\mu), \tag{13.1}$$

where $V(\mu) = \{v \in V \,:\, \pi(Y)v = \mu(Y)v \quad \text{for all } Y \in \mathfrak{h}\}$. We denote by

$$\mathcal{X}(V) = \{\mu \in P(\mathfrak{g}) \,:\, V(\mu) \neq 0\}$$

the set of weights of the $\mathfrak{g}$-module V.

Let $\{\alpha_1, \ldots, \alpha_l\}$ be the simple roots in Φ^+ and let $Q_+(\mathfrak{g}) = \mathbb{N}\alpha_1 + \cdots + \mathbb{N}\alpha_l$ be the semigroup generated by the positive roots. We define a *partial order* on $P(\mathfrak{g})$ by

$$\lambda \prec \mu \quad \text{if } \lambda = \mu - \beta \quad \text{for some} \;\; \beta \in Q_+(\mathfrak{g}) \setminus \{0\}.$$

Let (π, V) be a representation of $\mathfrak{g}$ (not necessarily finite-dimensional). A non-zero vector $v_0 \in V$ is called $\mathfrak{b}$-*extreme* if $\pi(\mathfrak{b})v_0 \subset \mathbb{C}v_0$. A vector $v_0 \in V$ is $\mathfrak{g}$-cyclic if V is spanned by v_0 together with the vectors $\pi(x_1)\cdots\pi(x_p)v_0$, where $x_i \in \mathfrak{g}$ and $p = 1, 2, \ldots$.

Proposition 13.1 *Let (π, V) be a finite-dimensional representation of $\mathfrak{g}$.*
(1) *A vector v_0 is $\mathfrak{b}$-extreme if and only if $\pi(\mathfrak{n})v_0 = 0$ and there exists $\mu \in P_{++}(\mathfrak{g})$ such that $\pi(H)v_0 = \langle \mu, H\rangle v_0$ for all $H \in \mathfrak{h}$.*
(2) *The $\mathfrak{b}$-extreme vectors in V span the subspace*

$$V^{\mathfrak{n}} = \{v \in V \,:\, \pi(\mathfrak{n})v = 0\}.$$

(3) *Suppose μ is a maximal element of $\mathcal{X}(V)$ relative to the partial order $\prec$. Then μ is dominant and $V(\mu) \subset V^{\mathfrak{n}}$. In particular, $V^{\mathfrak{n}} \neq 0$.*
(4) *Suppose $v_0 \in V$ is $\mathfrak{b}$-extreme of weight μ and is cyclic under $\mathfrak{g}$. Then π is irreducible, $V(\mu) = \mathbb{C}v_0$, and $\mathcal{X}(V) \subset \mu - Q_+(\mathfrak{g})$.*

Theorem 13.2 (Highest Weight) *Suppose (π, V) is an irreducible finite-dimensional representation of $\mathfrak{g}$. Then V has a unique* highest weight *μ such that $\lambda \prec \mu$ for all other weights λ of V. One has $\mu \in P_{++}(\mathfrak{g})$ and* $\dim V(\mu) = 1$. *A nonzero vector $v_0 \in V(\mu)$ is called a* highest weight vector *of V. If U is another irreducible finite-dimensional $\mathfrak{g}$-module with highest weight μ, then $U \cong V$.*

The definition of *highest weight* depends on the choice of a set of positive roots. However, the elements of $P_{++}(\mathfrak{g})$ are in one-to-one correspondence with the Weyl group orbits in $P(\mathfrak{g})$. Thus every irreducible finite-dimensional representation of $\mathfrak{g}$ corresponds to a unique W_G-orbit in $P^{\mathfrak{g}}$, namely the orbit of the highest weight.

Commuting Algebra and $\mathfrak{n}$-invariants

If V is a $\mathfrak{g}$-module we set

$$V^{\mathfrak{n}} = \{v \in V \;:\; X \cdot v = 0 \quad \text{for all } X \in \mathfrak{n}\}.$$

Lemma 13.3 *Let V be a finite-dimensional $\mathfrak{g}$-module. Then V is irreducible if and only if $\dim V^{\mathfrak{n}} = 1$.*

Let V be a finite-dimensional $\mathfrak{g}$-module. We shall apply the theorem of the highest weight to obtain the following decomposition of the commuting algebra $\mathrm{End}_{\mathfrak{g}}(V)$ as a direct sum of full matrix algebras. Note that if $T \in \mathrm{End}_{\mathfrak{g}}(V)$ then it preserves $V^{\mathfrak{n}}$ and it preserves the weight space decomposition

$$V^{\mathfrak{n}} = \bigoplus_{\mu \in \mathcal{S}} V^{\mathfrak{n}}(\mu).$$

Here $\mathcal{S} = \{\mu \in P_{++}(\mathfrak{g}) \;:\; V^{\mathfrak{n}}(\mu) \neq 0\}$. By Theorem 13.2 we can label the equivalence classes of irreducible $\mathfrak{g}$-modules by their highest weights. For each $\mu \in \mathcal{S}$ choose an irreducible representation (π^{μ}, V^{μ}) with highest weight μ.

Theorem 13.4 *The map $\phi(T) = T|_{V^{\mathfrak{n}}}$ gives an algebra isomorphism*

$$\mathrm{End}_{\mathfrak{g}}(V) \cong \bigoplus_{\mu \in \mathcal{S}} \mathrm{End}(V^{\mathfrak{n}}(\mu)). \tag{13.2}$$

For every $\mu \in \mathcal{S}$ the space $V^{\mathfrak{n}}(\mu)$ is an irreducible module for $\mathrm{End}_{\mathfrak{g}}(V)$ and distinct values of μ give inequivalent modules for $\mathrm{End}_{\mathfrak{g}}(V)$. Under the joint action of $\mathfrak{g}$ and $\mathrm{End}_{\mathfrak{g}}(V)$ the space V decomposes as

$$V \cong \bigoplus_{\mu \in \mathcal{S}} V^{\mu} \otimes V^{\mathfrak{n}}(\mu), \tag{13.3}$$

where V^{μ} is the irreducible $\mathfrak{g}$-module with highest weight μ.

Appendix: Linear and Associative Algebra for Lecture 13.

Representations of Associative Algebras

Let cA be an associative algebra over $\mathbb{C}$ with identity 1.

Lemma 13.5 (Schur) *If (ρ, V) and (τ, W) are finite-dimensional irreducible representations of cA, then*

$$\dim \mathrm{Hom}_{\mathcal{A}}(V, W) = \begin{cases} 1 & \text{if } (\rho, V) \cong (\tau, W) \\ 0 & \text{otherwise.} \end{cases}$$

Let (ρ, V) be a finite-dimensional representation of $\mathcal{A}$. We say that V is *completely reducible* as an $\mathcal{A}$ module if for every $\mathcal{A}$-invariant subspace $W \subset V$ there exists a complementary invariant subspace $U \subset V$ such that $V = W \oplus U$. If U is a finite-dimensional irreducible $\mathcal{A}$-module, we denote by $[U]$

the equivalence class of all $\mathcal{A}$-modules equivalent to U. Let $\widehat{\mathcal{A}}$ be the set of all equivalence classes of finite-dimensional irreducible $\mathcal{A}$-modules.

Suppose that V is a completely reducible $\mathcal{A}$-module. For each $\xi \in \widehat{\mathcal{A}}$ we define

$$V_{(\xi)} = \sum_{U \subset V,\, [U]=\xi} U,$$

where the subspaces U are invariant and irreducible under $\mathcal{A}$ and furnish representations of $\mathcal{A}$ in the equivalence class ξ. We call $V_{(\xi)}$ the ξ-*isotypic subspace* of V.

For each $\xi \in \widehat{\mathcal{A}}$ fix a module E_ξ in the class ξ. There is a linear map

$$S_\xi : \mathrm{Hom}_{\mathcal{A}}(E_\xi, V) \otimes E_\xi \to V, \qquad S_\xi(u \otimes w) = u(w)$$

for $u \in \mathrm{Hom}_{\mathcal{A}}(E_\xi, V)$ and $w \in E_\xi$. If we make $\mathrm{Hom}_{\mathcal{A}}(E_\xi, V) \otimes E_\xi$ into an $\mathcal{A}$-module with action $x \cdot (u \otimes w) = u \otimes (x \cdot w)$ for $x \in \mathcal{A}$, then S_ξ is an $\mathcal{A}$-intertwining map. If $0 \neq u \in \mathrm{Hom}_{\mathcal{A}}(E_\xi, V)$ then Schur's Lemma implies that $u(E_\xi)$ is an irreducible $\mathcal{A}$-submodule of V isomorphic to E_ξ. Hence

$$S_\xi(\mathrm{Hom}_{\mathcal{A}}(E_\xi, V) \otimes E_\xi) \subset V_{(\xi)}$$

for every $\xi \in \widehat{\mathcal{A}}$.

Proposition 13.6 *Let V be a completely reducible $\mathcal{A}$-module. Let*

$$V = V_1 \oplus \cdots \oplus V_d \tag{13.4}$$

be any decomposition with each V_i invariant and irreducible. Then

$$V_{(\xi)} = \bigoplus_{[V_j]=\xi} V_j \tag{13.5}$$

for all $\xi \in \widehat{\mathcal{A}}$, and hence

$$V = \bigoplus_{\xi \in \widehat{\mathcal{A}}} V_{(\xi)}. \tag{13.6}$$

The map S_ξ gives an $\mathcal{A}$-module isomorphism

$$\mathrm{Hom}_{\mathcal{A}}(E_\xi, V) \otimes E_\xi \cong V_{(\xi)}$$

for each $\xi \in \widehat{\mathcal{A}}$.

We call (13.6) the *primary decomposition* of V. The cardinality $m_V(\xi)$ of the set $\{j : [V_j] = \xi\}$ is called the multiplicity of ξ in V. We have

$$m_V(\xi) = \dim \mathrm{Hom}_{\mathcal{A}}(E_\xi, V) = \dim \mathrm{Hom}_{\mathcal{A}}(V, E_\xi).$$

Simple Associative Algebras

An associative algebra $\mathcal{A}$ is called *simple* if the only two-sided ideals in $\mathcal{A}$ are 0 and $\mathcal{A}$.

Theorem 13.7 (Wedderburn) *The algebra* $\mathrm{End}(V)$ *is simple for every finite dimensional complex vector space* V*. Conversely, if* $\mathcal{A}$ *is any finite dimensional simple algebra over* $\mathbb{C}$ *with unit, then there is a finite dimensional complex vector space* V *such that* $\mathcal{A} \cong \mathrm{End}(V)$*.*

Theorem 13.8 (Burnside) *Let* (ρ, V) *be an irreducible representation of an associative algebra* $\mathcal{A}$*. If* $\dim V$ *is finite and* $\rho(\mathcal{A}) \neq 0$ *then* $\rho(\mathcal{A}) = \mathrm{End}(V)$*.*

Proposition 13.9 *Up to equivalence, the only irreducible representation of* $\mathrm{End}(V)$ *is the representation* τ *on* V *given by* $\tau(x)v = xv$*.*

Theorem 13.10 *Let* $\mathcal{A} = \mathrm{End}(V)$ *and suppose* (ρ, W) *is a finite-dimensional representation of* $\mathcal{A}$*. Then* $\dim W = m \dim V$*, where* $m = \dim \mathrm{Hom}_{\mathcal{A}}(V, W)$*, and there exists a linear bijection*

$$T : W \to V^m, \qquad \text{with } Tw = (v_1, \ldots, v_m),$$

such that $T\rho(x)w = (xv_1, \ldots, xv_m)$ *for* $x \in \mathcal{A}$ *and* $w \in W$*. Hence* W *is equivalent to the* $\mathcal{A}$*-module* $\mathrm{Hom}_{\mathcal{A}}(V, W) \otimes V$*, where* $x \in \mathcal{A}$ *acts by* $x \cdot (u \otimes v) = u \otimes (xv)$ *for* $u \in \mathrm{Hom}_{\mathcal{A}}(V, W)$ *and* $v \in V$*.*

Semisimple Associative Algebras

A finite-dimensional associative algebra $\mathcal{A}$ with unit is said to be *semisimple* if it is the direct sum of simple algebras. Throughout this section we assume that $\mathcal{A}$ is semisimple with unit $1_{\mathcal{A}}$. By Wedderburn's theorem, there exist finite-dimensional vector spaces V^λ, with λ running over some finite set L, and an algebra isomorphism

$$\Phi : \mathcal{A} \xrightarrow{\cong} \bigoplus_{\lambda \in L} \mathrm{End}(V^\lambda). \tag{13.7}$$

Conversely, every direct sum of matrix algebras is semisimple.
The isomorphism Φ in (13.7) gives representations (π^λ, V^λ) of $\mathcal{A}$, where $\pi^\lambda(x)$ is the restriction of $\Phi(x)$ to V^λ for $x \in \mathcal{A}$.

Proposition 13.11 *The representations* (π^λ, V^λ) *are irreducible and mutually inequivalent. Every irreducible representation of* $\mathcal{A}$ *is equivalent to some* π^λ*.*

An arbitrary representation of $\mathcal{A}$ can be described as follows.

Proposition 13.12 *Let* $\mathcal{A}$ *be given by* (13.7) *and suppose* (ρ, W) *is a finite-dimensional representation of* $\mathcal{A}$*. Set* $U^\lambda = \mathrm{Hom}_{\mathcal{A}}(V^\lambda, W)$ *for* $\lambda \in \widehat{\mathcal{A}}$ *and define a linear map*

$$S : \bigoplus_{\lambda \in \widehat{\mathcal{A}}} U^\lambda \otimes V^\lambda \to W, \qquad S\Big(\sum_{\lambda \in \widehat{\mathcal{A}}} u_\lambda \otimes v_\lambda\Big) = \sum_{\lambda \in \widehat{\mathcal{A}}} u_\lambda(v_\lambda).$$

Then S *is an* $\mathcal{A}$*-module isomorphism and*

$$S^{-1}\rho(x)S = \bigoplus_{\lambda \in \widehat{\mathcal{A}}} I_{U^\lambda} \otimes \pi^\lambda(x). \tag{13.8}$$

Double Commutant Theorem

Let V be a finite dimensional vector space. For any subset $\mathcal{S} \subset \mathrm{End}(V)$ we define

$$\mathrm{Comm}(\mathcal{S}) = \{x \in \mathrm{End}(V) \,:\, xs = sx \quad \text{for all } s \in \mathcal{S}\}$$

and call it the *commutant* of $\mathcal{S}$. We observe that $\mathrm{Comm}(\mathcal{S})$ is an associative algebra with unit I_V. Suppose now that $\mathcal{A} \subset \mathrm{End}(V)$ is a semisimple algebra with $I_V \in \mathcal{A}$. Set $\mathcal{B} = \mathrm{Comm}(\mathcal{A})$. The vector space $\mathcal{A} \otimes \mathcal{B}$ is an associative algebra under the multiplication

$$(a \otimes b)(a' \otimes b') = aa' \otimes bb',$$

and $\mathcal{A}$ (*resp.* $\mathcal{B}$) is isomorphic to the subalgebra $\mathcal{A} \otimes 1$ (*resp.* $1 \otimes \mathcal{B}$) of $\mathcal{A} \otimes \mathcal{B}$.
By Proposition 13.12 there is an $\mathcal{A}$-module isomorphism

$$V \cong \bigoplus_{i=1}^{r} V_i \otimes U_i \tag{13.9}$$

where V_i is an irreducible $\mathcal{A}$-module, $V_i \ncong V_j$ for $i \neq j$ and $U_i = \mathrm{Hom}_{\mathcal{A}}(V_i, V)$. Under this isomorphism

$$\mathcal{A} \cong \bigoplus_{i=1}^{r} \mathrm{End}(V_i) \otimes I_{U_i}. \tag{13.10}$$

We now use this isomorphism to obtain the basic dual relationship between the algebras $\mathcal{A}$ and $\mathrm{Comm}(\mathcal{A})$.

Theorem 13.13 (Double Commutant) *Let V be a finite-dimensional vector space and $\mathcal{A} \subset \mathrm{End}(V)$ a semisimple algebra. Then the algebra $\mathcal{B} = \mathrm{Comm}(\mathcal{A})$ is semisimple and $\mathrm{Comm}(\mathcal{B}) = \mathcal{A}$. Furthermore, relative to the isomorphisms* (13.9), (13.10)*, one has*

$$\mathcal{B} \cong \bigoplus_{i=1}^{r} I_{V_i} \otimes \mathrm{End}(U_i). \tag{13.11}$$

Hence the subspaces $V_i \otimes U_i$ are irreducible and mutually inequivalent representations of the algebra $\mathcal{A} \otimes \mathcal{B}$.

We can view (13.9) in two ways: as a decomposition of V into isotypic subspaces for $\mathcal{A}$ (where the representation V_i occurs with multiplicity $\dim U_i$), or as a decomposition of V into isotypic subspaces for $\mathcal{B}$ (where the representation U_i occurs with multiplicity $\dim V_i$). This *dual* point of view sets up a correspondence between irreducible representations of $\mathcal{A}$ and irreducible representations of $\mathcal{B}$, where V_i is paired with U_i.

Exercises for Lecture 13.

1. Let $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{C})$. Fix the positive roots $\Phi^+ = \{\varepsilon_1 - \varepsilon_2,\, \varepsilon_2 - \varepsilon_3,\, \varepsilon_1 - \varepsilon_3\}$ as usual. Let $\pi = \mathrm{ad}$ be the adjoint representation on $\mathfrak{g}$.

(a) Express the highest weight λ of π in terms of the fundamental weights ϖ_1 and ϖ_2. What is the highest weight vector?

(b) Find all $\beta \in P_{++}(\mathfrak{g})$ of the form $\beta = \lambda - \gamma$, where $\gamma \in Q_+(\mathfrak{g})$. (Here $P_{++}(\mathfrak{g})$ are the dominant weights, and $Q_+(\mathfrak{g})$ are the sums of positive roots.) Verify that for every such β, the corresponding weight space $\mathfrak{g}_\beta \neq 0$.

(c) Find the orbit $W \cdot \beta$ of each weight β in (b), where W is the Weyl group of $\mathfrak{g}$. Verify that the union of these orbits is the set of weights of π.

(d) Plot the set of weights of π as points in the $\mathfrak{h}^*$ plane. Observe that this set is in the convex hull of the orbit $W \cdot \lambda$ of the highest weight.

2. Let $\mathfrak{g} = \mathfrak{sp}(2, \mathbb{C})$. Fix the positive roots $\Phi^+ = \{\varepsilon_1 - \varepsilon_2, \varepsilon_1 + \varepsilon_2, 2\varepsilon_1, 2\varepsilon_2\}$ as usual. Let $\pi = \mathrm{ad}$ be the adjoint representation on $\mathfrak{g}$. Carry out parts (a), (b), (c), (d) of the previous exercise in this case.

3. Let $\mathfrak{g} = \mathfrak{sp}(2, \mathbb{C})$. Suppose (π, V) is the irreducible representation of $\mathfrak{g}$ with highest weight $\rho = \varpi_1 + \varpi_2$ (the smallest regular dominant weight).

   (a) Show that there is exactly one $\beta \in P_{++}(\mathfrak{g})$ of the form $\beta = \rho - \gamma$, where $0 \neq \gamma \in Q_+(\mathfrak{g})$. Show that $V_\beta \neq 0$ and find a *spanning set* for it. (*Hint*: Use the representation theory of $\mathfrak{sl}(2, \mathbb{C})$ and the action of $U(\mathfrak{g})$ on the highest weight vector.)

   (b) Find the orbits $W \cdot \rho$ and $W \cdot \beta$, where W is the Weyl group of $\mathfrak{g}$.

   (c) Plot the weights of π in the $\mathfrak{h}^*$ plane. Observe that all the weights are contained in the convex hull of the orbit $W \cdot \rho$ of the highest weight.

   (d) The Weyl dimension formula implies that $\dim V = 2^{|\Phi^+|} = 16$. Use this result to determine the dimension of the weight space V_β in (a).

4. Let $\mathfrak{g}$ be the Lie algebra of a classical group of rank l and let $\varpi_1, \ldots, \varpi_l$ be the fundamental weights. Suppose $\lambda = m_1\varpi_1 + \cdots + m_l\varpi_l$ is the highest weight of an irreducible $\mathfrak{g}$-module V. Let λ^* be the highest weight of the dual module V^*. Use the formula $\lambda^* = -w_0 \cdot \lambda$ ($w_0(\Phi^+ = -\Phi^+$) and the results of Lecture 12 to show that λ^* is given as follows:

   Type A_l: $\lambda^* = m_l\varpi_1 + m_{l-1}\varpi_2 + \cdots + m_2\varpi_{l-1} + m_1\varpi_l$

   Type B_l or C_l: $\lambda^* = \lambda$

   Type D_l: $\lambda^* = \begin{cases} \lambda & \text{if } l \text{ is even} \\ m_1\varpi_1 + \cdots + m_{l-2}\varpi_{l-2} + m_l\varpi_{l-1} + m_{l-1}\varpi_l & \text{if } l \text{ is odd.} \end{cases}$

Part 5: Invariant Theory and Irreducible Representations

Lecture 14. Invariants for Classical Groups

First Fundamental Theorem of Invariants

Let G be a reductive linear algebraic group and (ρ, V) a regular representation of G. For each positive integer k, let

$$V^k = \underbrace{V \oplus \cdots \oplus V}_{k \text{ copies}}.$$

(This should not be confused with the k-fold tensor power $V^{\otimes k} = \bigotimes^k V$.) Likewise, let $(V^*)^k$ be the sum of k copies of V^*. Given positive integers k and m, consider the algebra $\mathcal{P}((V^*)^k \times V^m)$ of polynomials with k *covector* arguments (elements of V^*) and m *vector* arguments (elements of V). The induced action of G on $\mathcal{P}((V^*)^k \times V^m)$ is

$$\begin{aligned} & g \cdot f(v_1^*, \ldots, v_k^*, v_1, \ldots, v_m) \\ = \; & f(v_1^* \circ \rho(g), \ldots, v_k^* \circ \rho(g), \rho(g^{-1})v_1, \ldots, \rho(g^{-1})v_n). \end{aligned}$$

We shall refer to a description of (finite) generating sets for $\mathcal{P}((V^*)^k \times V^m)^G$, for all k, m, as a *First Fundamental Theorem* (FFT) for the pair (G, ρ). Here the emphasis is on an *explicit* listing of generating sets; the existence of a finite generating set of invariants (for each k, m) is a consequence of Theorem 9.4. In this lecture we will state the FFT when G is a classical group and V is its defining representation.
Since $\mathcal{P}((V^*)^k \times V^m)^G \supset \mathcal{P}((V^*)^k \times V^m)^{\mathrm{GL}(V)}$, a FFT for $\mathrm{GL}(V)$ gives some information about invariants for the group $\rho(G)$, so we first consider this case. The key observation is that $\mathrm{GL}(V)$-invariant polynomials on $(V^*)^k \times V^m$ come from the following geometric construction.
There are natural isomorphisms

$$(V^*)^k \cong \mathrm{Hom}(V, \mathbb{C}^k), \qquad V^m \cong \mathrm{Hom}(\mathbb{C}^m, V).$$

Here the direct sum $v_1^* \oplus \cdots \oplus v_k^*$ of k covectors corresponds to the linear map

$$v \mapsto [\langle v_1^*, v\rangle, \ldots, \langle v_k^*, v\rangle]$$

from V to $\mathbb{C}^k$, while the direct sum $v_1 \oplus \cdots \oplus v_m$ of m vectors corresponds to the linear map

$$[c_1, \ldots, c_m] \mapsto c_1 v_1 + \cdots + c_m v_m$$

from $\mathbb{C}^m$ to V. This gives an algebra isomorphism

$$\mathcal{P}((V^*)^k \times V^m) \cong \mathcal{P}(\mathrm{Hom}(V, \mathbb{C}^k) \times \mathrm{Hom}(\mathbb{C}^m, V))$$

with the action of $g \in \mathrm{GL}(V)$ on $f \in \mathcal{P}(\mathrm{Hom}(V, \mathbb{C}^k) \times \mathrm{Hom}(\mathbb{C}^m, V))$ becoming

$$g \cdot f(x, y) = f(x\rho(g^{-1}), \rho(g)y), \qquad x \in X,\, y \in Y. \tag{14.1}$$

We denote the vector space of $k \times m$ complex matrices as $M_{k,m}$. Define a map

$$\mu : \mathrm{Hom}(V, \mathbb{C}^k) \times \mathrm{Hom}(\mathbb{C}^m, V) \to M_{k,m}$$

by $\mu(x, y) = xy$ (composition of linear transformations). Then

$$\mu(x\rho(g^{-1}), \rho(g)y) = x\rho(g)^{-1}\rho(g)y = \mu(x, y)$$

for $g \in G$ and $x \in X$, $y \in Y$. The induced homomorphism μ^* on $\mathcal{P}(M_{k,m})$ has range in the $\mathrm{GL}(V)$-invariant polynomials:

$$\mu^* : \mathcal{P}(M_{k,m}) \to \mathcal{P}(\mathrm{Hom}(V, \mathbb{C}^k) \times \mathrm{Hom}(\mathbb{C}^m, V))^{\mathrm{GL}(V)},$$

where, as usual, $\mu^*(f) = f \circ \mu$ for $f \in \mathcal{P}(M_{k,m})$. Thus if we let $z_{ij} = \mu^*(x_{ij})$ be the image of the matrix entry function x_{ij} on $M_{k,m}$, then z_{ij} is the *contraction* of the ith covector position with the jth vector position:

$$z_{ij}(v_1^*, \ldots, v_k^*, v_1, \ldots, v_m) = \langle v_i^*, v_j \rangle.$$

Theorem 14.1 (polynomial FFT for $\mathrm{GL}(V)$) *The map*

$$\mu^* : \mathcal{P}(M_{k,m}) \to \mathcal{P}((V^*)^k \times V^m)^{\mathrm{GL}(V)}$$

is surjective. Hence $\mathcal{P}((V^*)^k \times V^m)^{\mathrm{GL}(V)}$ is generated (as an algebra) by the contractions $\{\langle v_i^*, v_j \rangle : i = 1, \ldots, m, j = 1, \ldots, k\}$.

Consider now the orthogonal or symplectic groups acting in their defining representations. Here we obtain the invariant polynomials by the following modification of the geometric construction used for $\mathrm{GL}(V)$.

Let $V = \mathbb{C}^n$ and define the symmetric form

$$(x, y) = \sum_i x_i y_i \quad \text{for } x, y \in \mathbb{C}^n. \tag{14.2}$$

Write O_n for the orthogonal group for this form. Thus $g \in \mathrm{O}_n$ if and only if $g^t g = I_n$. Let SM_k be the vector space of $k \times k$ complex *symmetric* matrices B (so $B = B^t$). Define a map $\tau : M_{n,k} \to SM_k$ by $\tau(X) = X^t X$. Then

$$\tau(gX) = X^t g^t g X = \tau(X) \quad \text{for } g \in \mathrm{O}_n \text{ and } X \in M_{n,k}.$$

Hence $\tau^*(f)(gX) = \tau^*(f)(X)$ for $f \in \mathcal{P}(SM_k)$, so we obtain an algebra homomorphism

$$\tau^* : \mathcal{P}(SM_k) \to \mathcal{P}(V^k)^{\mathrm{O}_n}$$

For example, given $v_1, \ldots, v_k \in \mathbb{C}^n$, we can form the $n \times k$ matrix

$$X = [v_1, \ldots, v_k] \in M_{n,k}$$

(we always take $\mathbb{C}^n$ to consist of *column vectors* with n components). Then $X^t X$ is the $k \times k$ symmetric matrix with entries (v_i, v_j). Hence under the map τ^* the matrix entry function x_{ij} on SM_k pulls back to the O_n-invariant quadratic polynomial

$$\tau^*(x_{ij})(v_1, \ldots, v_k) = (v_i, v_j)$$

on $(\mathbb{C}^n)^k$ (the contraction of the ith and jth vector position using the symmetric form).

When n is even, let J_n be the $n \times n$ block-diagonal matrix

$$J_n = \begin{bmatrix} \kappa & 0 & \cdots & 0 \\ 0 & \kappa & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \kappa \end{bmatrix}, \qquad \kappa = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Define the skew-symmetric form

$$\omega(x, y) = (x, J_n y) \tag{14.3}$$

for $x, y \in \mathbb{C}^n$ and let Sp_n be the invariance group of this form. Thus $g \in \mathrm{Sp}_n$ if and only if $g^t J_n g = J_n$. Let AM_k be the vector space of $k \times k$ complex *skew-symmetric* matrices A (so $A^t = -A$). Define a map

$$\gamma : M_{n,k} \to AM_k$$

by $\gamma(X) = X^t J_n X$. Then

$$\gamma(gX) = X^t g^t J_n g X = \gamma(X) \quad \text{for } g \in \mathrm{Sp}_n \text{ and } X \in M_{n,k}.$$

Hence $\gamma^*(f)(gX) = \gamma^*(f)(X)$ for $f \in \mathcal{P}(AM_k)$, so we obtain an algebra homomorphism

$$\gamma^* : \mathcal{P}(AM_k) \to \mathcal{P}(V^k)^{\mathrm{Sp}_n}.$$

As in the orthogonal case, given $v_1, \ldots, v_k \in \mathbb{C}^n$, we form the matrix $X = [v_1, \ldots, v_k] \in M_{n,k}$. Then the skew-symmetric $k \times k$ matrix $X^t J_n X$ has entries $(v_i, J_n v_j)$. Hence the matrix entry function x_{ij} on AM_k pulls back to the Sp_n-invariant quadratic polynomial

$$\gamma^*(x_{ij})(v_1, \ldots, v_k) = \omega(v_i,\, v_j)$$

(the contraction of the ith and jth positions, $i \neq j$, using the skew form).

Theorem 14.2 (polynomial FFT for O_n and Sp_n)
(1) *The homomorphism*

$$\tau^* : \mathcal{P}(SM_k) \to \mathcal{P}((\mathbb{C}^n)^k)^{\mathrm{O}_n}$$

is surjective. Hence $\mathcal{P}((\mathbb{C}^n)^k)^{\mathrm{O}_n}$ *is generated (as an algebra) by the orthogonal contractions* $\{(v_i, v_j) \,:\, 1 \le i \le j \le k\}$.
(2) *Suppose n is even. The homomorphism*

$$\gamma^* : \mathcal{P}(AM_k) \to \mathcal{P}((\mathbb{C}^n)^k)^{\mathrm{Sp}_n}$$

is surjective. Hence $\mathcal{P}((\mathbb{C}^n)^k)^{\mathrm{Sp}_n}$ *is generated (as an algebra) by the symplectic contractions* $\{\omega(v_i, v_j) \,:\, 1 \le i < j \le k\}$.

Corollary 14.3 (1) *Let $G = \mathrm{O}_n$ and $V = \mathbb{C}^n$. Then $\mathcal{P}((V^*)^k \times V^m)^G$ is generated (as an algebra) by the quadratic polynomials*

$$(v_i, v_j), \quad (v_p^*, v_q^*), \quad \langle v_p^*, v_i \rangle, \qquad \text{for } 1 \le i, j \le m \text{ and } 1 \le p, q \le k.$$

(2) *Let $G = \mathrm{Sp}_n$ and $V = \mathbb{C}^n$ (with n even). Then $\mathcal{P}((V^*)^k \times V^m)^G$ is generated (as an algebra) by the quadratic polynomials*

$$\omega(v_i, v_j), \quad \omega(v_p^*, v_q^*), \quad \langle v_p^*, v_i \rangle, \qquad \text{for } 1 \le i, j \le m \text{ and } 1 \le p, q \le k.$$

Tensor Invariants and Schur Duality

Let $\mathrm{GL}(V)$ act on V by the defining representation ρ, and let ρ^* be the dual representation on V^*. For all integers $k, m \geq 0$ we have the representations $\rho_k = \rho^{\otimes k}$ on $V^{\otimes k}$ and $\rho_m^* = \rho^{*\otimes k}$ on $V^{*\otimes m}$. Since there is a natural isomorphism

$$(V^*)^{\otimes m} \cong (V^{\otimes m})^*$$

as $\mathrm{GL}(V)$ modules, we may view ρ_m^* as acting on $(V^{\otimes m})^*$. We set $\rho_{k,m} = \rho^{\otimes k} \otimes \rho^{*\otimes m}$, acting on $V^{\otimes k} \otimes (V^{\otimes m})^*$.

To obtain the tensor form of the FFT for $\mathrm{GL}(V)$, we must find an explicit spanning set for the space of $\mathrm{GL}(V)$ invariants in $V^{\otimes k} \otimes (V^{\otimes m})^*$. For $x \in V^{\otimes k} \otimes (V^{\otimes m})^*$ and $\lambda \in \mathbb{C}^\times$ we have

$$\rho_{k,m}(\lambda I)x = \lambda^{k-m}x.$$

Hence there are no invariants if $k \neq m$, so we only need to consider the representation $\rho_{k,k}$ on $V^{\otimes k} \otimes (V^{\otimes k})^*$.

Recall that when W is a finite-dimensional vector space, then $W \otimes W^* \cong \mathrm{End}(W)$ as a $\mathrm{GL}(W)$ module, where $w \otimes w^*$ gives the linear transformation

$$u \mapsto \langle w^*, u\rangle w.$$

We apply this to the case $W = V^{\otimes k}$. The action of $g \in \mathrm{GL}(V)$ on $\mathrm{End}(V^{\otimes k})$ is given by

$$T \mapsto \rho_k(g) T \rho_k(g)^{-1}.$$

Thus the space of $\mathrm{GL}(V)$ invariants in $\mathrm{End}(V^{\otimes k})$ is the *commutant* of the set of operators $\rho_k(\mathrm{GL}(V))$
.

Let $\mathfrak{S}_k$ be the group of permutations of $\{1, 2, \ldots, k\}$. Define a representation σ_k of $\mathfrak{S}_k$ on $V^{\otimes k}$ by

$$\sigma_k(s)(v_1 \otimes \cdots \otimes v_k) = v_{s^{-1}(1)} \otimes \cdots \otimes v_{s^{-1}(k)}.$$

Theorem 14.4 (Schur Duality) *Set* $\mathcal{A} = \rho_k(\mathbb{C}[\mathrm{GL}(V)])$ *and* $\mathcal{B} = \sigma_k(\mathbb{C}[\mathfrak{S}_k])$. *Then* $\mathrm{Comm}(\mathcal{B}) = \mathcal{A}$ *and* $\mathrm{Comm}(\mathcal{A}) = \mathcal{B}$.

We now apply this result to obtain the tensor version of the FFT for $\mathrm{GL}(V)$. Let $e_1, \ldots, e_n$ be a basis for V and let $e_1^*, \ldots, e_n^*$ be the dual basis for V^*. For a multi-index $I = (i_1, \ldots, i_k)$ with $1 \leq i_j \leq n$, set $|I| = k$ and

$$e_I = e_{i_1} \otimes \cdots \otimes e_{i_k}.$$

The elements e_I form a basis for $\bigotimes^k V$ as I ranges over the finite set of all such multi-indices. For $s \in \mathfrak{S}_k$ and $I = (i_1, \ldots, i_k)$ we set

$$s \cdot (i_1, \ldots, i_k) = (i_{s^{-1}(1)}, \ldots, i_{s^{-1}(k)}).$$

Then we have $\sigma_k(s)e_I = e_{s\cdot I}$. Let Ξ be the set of all ordered pairs (I, J) of multi-indices with $|I| = |J| = k$. The set

$$\{e_I \otimes e_J^* \,:\, (I, J) \in \Xi\}$$

is a basis for $V^{\otimes k} \otimes (V^{\otimes k})^*$.

For $s \in \mathfrak{S}_k$ define a tensor C_s of type (k, k) by

$$C_s = \sum_{|I|=k} e_{s\cdot I} \otimes e_I^* \,. \tag{14.4}$$

Theorem 14.5 *Let $G = \mathrm{GL}(V)$. The space of G-invariants in $V^{\otimes k} \otimes V^{*\otimes k}$ is spanned by the tensors $\{C_s : s \in \mathfrak{S}_k\}$.*

Tensor Invariants for Orthogonal and Symplectic Groups

Let $G \subset \mathrm{GL}(V)$ be the group leaving invariant a nondegenerate bilinear form ω (which we assume is either symmetric or skew-symmetric). Since $V \cong V^*$ as a G-module via the form ω, we only need to consider tensor invariants in $(V^{\otimes m})^G$ when $m = 1, 2, \ldots$. Clearly there are no invariants if m is odd, since $-I \in G$, so we may assume that $m = 2k$ is even.
The $\mathrm{GL}(V)$ isomorphism $V^* \otimes V \cong \mathrm{End}(V)$ and the G-module isomorphism $V \cong V^*$ combine to give a G-module isomorphism

$$T : V^{\otimes 2k} \cong \mathrm{End}(V^{\otimes k}) \tag{14.5}$$

which we take in the following explicit form: If $u = v_1 \otimes \cdots \otimes v_{2k}$ with $v_i \in V$, then $T(u)$ is the linear transformation

$$T(u)(x_1 \otimes \cdots \otimes x_k) = \omega(x_1, v_2)\omega(x_2, v_4) \cdots \omega(x_k, v_{2k}) v_1 \otimes v_3 \cdots \otimes v_{2k-1}$$

for $x_i \in V$. That is, we use the invariant form to change each v_{2i} into a covector, pair it with v_{2i-1} to get a rank-one linear transformation on V, and then take the tensor product of these transformations to get $T(u)$. We extend ω to a nondegenerate bilinear form on $V^{\otimes k}$ for every k by

$$\omega(x_1 \otimes \cdots \otimes x_k, y_1 \otimes \cdots \otimes y_k) = \prod_{i=1}^{k} \omega(x_i, y_i).$$

Then we can write the formula for T as

$$T(v_1 \otimes \cdots \otimes v_{2k})x = \omega(x, v_2 \otimes v_4 \cdots \otimes v_{2k})\, v_1 \otimes v_3 \otimes \cdots \otimes v_{2k-1}$$

for $x \in V^{\otimes k}$.
The identity operator $I_V^{\otimes k}$ on $V^{\otimes k}$ is G-invariant, of course. We can express this operator in tensor form as follows. Fix a basis $\{f_p\}$ for V and let $\{f^p\}$ be the dual basis for V relative to the invariant form ω:

$$\omega(f_p,\, f^q) = \delta_{pq}.$$

Set $\theta = \sum_{p=1}^{n} f_p \otimes f^p$ (where $n = \dim V$). Then $T(\theta) = I_V$. Hence the $2k$-tensor

$$\theta_k = \underbrace{\theta \otimes \cdots \otimes \theta}_{k} = \sum_{p_1, \ldots, p_k} f_{p_1} \otimes f^{p_1} \otimes \cdots \otimes f_{p_k} \otimes f^{p_k}$$

satisfies $T(\theta_k) = I_V^{\otimes k}$. It follows that θ_k is G-invariant. Since the action of G on $V^{\otimes 2k}$ commutes with the action of $\mathfrak{S}_{2k}$, the tensors $\sigma_{2k}(s)\theta_k$ are also G-invariant, for any $s \in \mathfrak{S}_{2k}$. The *first fundamental theorem* asserts that all G-invariant tensors are linear combinations of these tensors.

Theorem 14.6 *Let G be $\mathrm{O}(V)$ or $\mathrm{Sp}(V)$. Then $[V^{\otimes m}]^G = 0$ if m is odd, and*

$$[V^{\otimes 2k}]^G = \mathrm{Span}\{\sigma_{2k}(s)\theta_k \,:\, s \in \mathfrak{S}_{2k}\}.$$

Exercises for Lecture 14.

In all these problems $G = \mathrm{GL}(n,\mathbb{C})$, $V = \mathbb{C}^n = M_{n\times 1}$ with left G action, and $V^* = M_{1\times n}$ with right G action.

1. Let $X = M_{k\times n} \times M_{n\times m}$ and $Y = M_{k\times m}$. Let G act on X by $g \cdot (x, y) = (xg^{-1}, gy)$. Map $\mu : X \to Y$ by matrix multiplication: $\mu(x, y) = xy$.

   (a) Assume that $n \geq \min(k, n)$, so μ is surjective. Prove that (μ, Y) is the algebraic quotient $X//G$. (*Hint:* Use the first fundamental theorem of invariant theory for G to prove that $\mu^* : \mathcal{P}(Y) \to \mathcal{P}(X)^G$ is bijective. Note that $X \cong (V^*)^k \times V^m$ as a G module.)

   (b) Let n be arbitrary. Let $K = \mathrm{GL}(k,\mathbb{C}) \times \mathrm{GL}(m,\mathbb{C})$ act on X and on Y by matrix multiplication: $(g, h) \cdot (u, v) = (gu, vh^{-1})$ and $(g, h) \cdot y) = gyh^{-1}$ for $(g, h) \in K$, $(u, v) \in X$, and $y \in Y$. Let $\mathcal{I} = \mathrm{Ker}(\mu^*)$, where $\mu^* : \mathcal{P}(Y) \to \mathcal{P}(X)$. Prove that $\mathcal{I}$ is invariant under K and that $\mathcal{P}(X)^G \cong \mathcal{P}(Y)/\mathcal{I}$. (*Hint:* Show that K commutes with the action of G on X and that the map μ is K equivariant.)

2. For $v \in V$ and $v^* \in V^*$, let $T(v \otimes v^*) = vv^* \in M_n$. This defines the canonical isomorphism $u \mapsto T(u)$ between $V \otimes V^*$ and M_n. Let $T_k = T^{\otimes k}$ be the canonical isomorphism $(V \otimes V^*)^{\otimes k} \to (M_n)^{\otimes k}$. Let $g \in G$ act on $x \in M_n$ by $g \cdot x = gxg^{-1}$.

   (a) Show that T_k intertwines the action of G on $(V \otimes V^*)^{\otimes k}$ and $(M_n)^{\otimes k}$.

   (b) Let $\sigma \in \mathfrak{S}_k$ be a cyclic permutation $m_1 \to m_2 \to \cdots \to m_k \to m_{k+1} = m_1$. Let $C_\sigma : (V \otimes V^*)^{\otimes k} \to \mathbb{C}$ be the G-invariant contraction

   $$C_\sigma(v_1 \otimes v_1^* \otimes \cdots \otimes v_k \otimes v_k^*) = \prod_{j=1}^{k} \langle v_{m_j}^*, v_{m_{j+1}} \rangle$$

   Set $X_j = T(v_j \otimes v_j^*)$. Prove that

   $$C_\sigma(v_1 \otimes v_1^* \otimes \cdots \otimes v_k \otimes v_k^*) = \mathrm{tr}(X_{m_1} X_{m_2} \cdots X_{m_k}).$$

   (*Hint:* Note that for $X \in M_n$, one has $T(v^* \otimes Xv) = XT(v^* \otimes v)$ and $\mathrm{tr}(T(v^* \otimes v)) = v^*v$.)

   (c) Let $\sigma \in \mathfrak{S}_k$ be a product of disjoint cyclic permutations $c_1, \ldots, c_r$, where c_i is the cycle $m_{1,i} \to m_{2,i} \to \cdots \to m_{p_i,i} \to m_{1,i}$. Let $C_\sigma : (V \otimes V^*)^{\otimes k} \to \mathbb{C}$ be the G-invariant contraction

   $$C_\sigma(v_1 \otimes v_1^* \otimes \cdots \otimes v_k \otimes v_k^*) = \prod_{i=1}^{r} \prod_{j=1}^{p_i} \langle v_{m_{j,i}}^*, v_{m_{j+1,i}} \rangle$$

   Set $X_j = T(v_j \otimes v_j^*)$. Prove that

   $$C_\sigma(v_1 \otimes v_1^* \otimes \cdots \otimes v_k \otimes v_k^*) = \prod_{i=1}^{r} \mathrm{tr}(X_{m_{1,i}} X_{m_{2,i}} \cdots X_{m_{p_i,i}}).$$

3. (a) Use the previous exercise to find a basis for the G-invariant linear functionals on $M_n^{\otimes 2}$ (assume $n \geq 2$).

   (b) Prove that there are no nonzero skew-symmetric G invariant bilinear forms on M_n. (*Hint:* Use the result in (a) and the projection from $(M_n)^{\otimes 2}$ onto $(M_n)^{\wedge 2}$.)

4. (a) Find a spanning set for the G-invariant linear functionals on $M_n^{\otimes 3}$.

   (b) Define $\omega(X_1, X_2, X_3) = \mathrm{tr}([X_1, X_2]X_3)$ for $X_i \in M_n$. Prove that ω is skew-symmetric and G invariant.

   (c) Prove that ω is the unique G invariant skew-symmetric linear functional on $M_n^{\otimes 3}$, up to a scalar multiple. (*Hint:* Use the result in (a) and the projection from $(M_n)^{\otimes 3}$ onto $(M_n)^{\wedge 3}$.)

Lecture 15. Skew-Duality for Classical Groups

Representations on Exterior Algebras

We now use the FFT for a classical group G to find the commuting algebra of G on the exterior algebra of its defining representation.
We denote by ρ the representation of $\mathrm{GL}(V)$ on $\bigwedge V$:

$$\rho(g)(v_1 \wedge \cdots \wedge v_p) = gv_1 \wedge \cdots \wedge gv_p$$

for $g \in \mathrm{GL}(V)$ and $v_i \in V$. It is easy to check from the definition of interior and exterior products that

$$\rho(g)\epsilon(v)\rho(g^{-1}) = \epsilon(gv), \quad \rho(g)\iota(v^*)\rho(g^{-1}) = \iota((g^t)^{-1}v^*). \tag{15.1}$$

We define the *skew Euler operator* E on $\bigwedge V$ by

$$E = \sum_{j=1}^{d} \epsilon(f_j)\iota(f_j^*),$$

where $d = \dim V$ and $\{f_1, \ldots, f_d\}$ is a basis for V with dual basis $\{f_1^*, \ldots, f_d^*\}$.

Lemma 15.1 *The operator E commutes with* GL*(V) and acts by the scalar k on $\bigwedge^k V$. Hence E does not depend on the choice of basis for V. If $T \in \mathrm{End}(\bigwedge V)$ and $T : \bigwedge^k V \to \bigwedge^{k+p} V$ for all k, then $[E, T] = pT$.*

As a particular case of the commutation relations in Lemma 15.1, we have

$$[E, \epsilon(v)] = \epsilon(v), \quad [E, \iota(v^*)] = -\iota(v^*) \quad \text{for } v \in V \text{ and } v^* \in V^*. \tag{15.2}$$

Now suppose $G \subset \mathrm{GL}(V)$ is an algebraic group. The action of G on V extends to regular representations on $V^{\otimes m}$ and on $\bigwedge V$. Denote by Q_k the projection from $\bigwedge V$ onto $\bigwedge^k V$. Then Q_k commutes with G and we may identify $\mathrm{Hom}(\bigwedge^l V, \bigwedge^k V)$ with the subspace of $\mathrm{End}_G(\bigwedge V)$ consisting of the operators $Q_k A Q_l$, where $A \in \mathrm{End}_G(\bigwedge V)$ (these are the G-intertwining operators that map $\bigwedge^l V$ to $\bigwedge^k V$ and are zero on $\bigwedge^r V$ for $r \neq l$). Thus

$$\mathrm{End}_G(\textstyle\bigwedge V) = \bigoplus_{0 \le l,k \le d} \mathrm{Hom}_G(\bigwedge^l V, \bigwedge^k V).$$

Let $\mathcal{T}(V)$ be the tensor algebra over V and let $P : \mathcal{T}(V) \to \bigwedge V$ be the projection operator:

$$Pu = \frac{1}{m!} \sum_{s \in \mathfrak{S}_m} \mathrm{sgn}(s)\sigma_m(s)u \qquad \text{for } u \in V^{\otimes m}.$$

Then we have

$$\mathrm{Hom}_G(\textstyle\bigwedge^l V, \bigwedge^k V) = \{PRP \,:\, R \in \mathrm{Hom}_G(V^{\otimes l}, V^{\otimes k})\}. \tag{15.3}$$

We now use these results and the FFT to find generators for $\mathrm{End}_G(\bigwedge V)$ when $G \subset \mathrm{GL}(V)$ is a classical group.

General Linear Group

Theorem 15.2 *Let $G = \mathrm{GL}(V)$. Then $\mathrm{End}_G(\bigwedge V)$ is generated by the skew Euler operator E.*

Corollary 15.3 *In the decomposition $\bigwedge V = \bigoplus_{p=1}^n \bigwedge^p V$, where $n = \dim V$, the summands are irreducible and mutually inequivalent $\mathrm{GL}(V)$-modules.*

Orthogonal and Symplectic Groups

Now let Ω be a non-degenerate bilinear form on V that is either symmetric or skew-symmetric. Let G be the subgroup of $\mathrm{GL}(V)$ that preserves Ω. In order to pass from the FFT for G to a description of the commutant of G in $\mathrm{End}(\bigwedge V)$, we need to introduce some operators on the tensor algebra over V.
Define $C : V^{\otimes m} \to V^{\otimes(m+2)}$ by

$$Cu = \theta \otimes u \quad \text{for } u \in V^{\otimes m},$$

where $\theta \in \left(\bigotimes^2 V\right)^G$ is the invariant 2-tensor corresponding to the bilinear form Ω. Define $C^* : V^{\otimes m} \to V^{\otimes(m-2)}$ by

$$C^*(v_1 \otimes \cdots \otimes v_m) = \Omega(v_{m-1}, v_m) v_1 \otimes \cdots \otimes v_{m-2}.$$

Clearly C and C^* commute with the action of G.
For $v^* \in V^*$ define $\kappa(v^*) : V^{\otimes m} \to V^{\otimes(m-1)}$ by evaluation on the first tensor place:

$$\kappa(v^*)(v_1 \otimes \cdots \otimes v_m) = \langle v^*, v_1 \rangle v_2 \otimes \cdots \otimes v_m.$$

For $v \in V$ define $\mu(v) : V^{\otimes m} \to V^{\otimes(m+1)}$ by left tensor multiplication:

$$\mu(v)(v_1 \otimes \cdots \otimes v_m) = v \otimes v_1 \otimes \cdots \otimes v_m.$$

For $v \in V$ let $v^\sharp \in V^*$ be defined by

$$\langle v^\sharp, w \rangle = \Omega(v, w) \quad \text{for all } w \in V.$$

Then $v \mapsto v^\sharp$ is G-module isomorphism. We extend Ω to a bilinear form on $V^{\otimes k}$ for all k. Then

$$\Omega(Cu, w) = \Omega(u, C^* w), \quad \Omega(\mu(v)u, w) = \Omega(u, \kappa(v^\sharp)w).$$

The intertwining operators for G on tensor spaces have the following form.

Lemma 15.4 *Let G be $\mathrm{O}(V, \Omega)$ (if Ω is symmetric) or $\mathrm{Sp}(V, \Omega)$ (if Ω is skew-symmetric). Then the space $\mathrm{Hom}_G(V^{\otimes l}, V^{\otimes k})$ is zero if $k + l$ is odd. If $k + l$ is even, this space is spanned by the operators $\sigma_k(s) A \sigma_l(t)$, where $s \in \mathfrak{S}_k$, $t \in \mathfrak{S}_l$ and A is one of the following operators:*
(1) *CB with $B \in \mathrm{Hom}_G(V^{\otimes l}, V^{\otimes(k-2)})$.*
(2) *BC^* with $B \in \mathrm{Hom}_G(V^{\otimes(l-2)}, V^{\otimes k})$.*
(3) *$\sum_{p=1}^d \mu(f_p) B \kappa(f_p^*)$ with $B \in \mathrm{Hom}_G(V^{\otimes(l-1)}, V^{\otimes(k-1)})$ (here $d = \dim V$).*
In (3) $\{f_p\}$ is any basis for V and $\{f_p^*\}$ is the dual basis for V^*.

From this lemma, we obtain the commuting algebra of G.

Theorem 15.5 *Assume the form Ω is symmetric and $G = \mathrm{O}(V, \Omega)$. Then $\mathrm{End}_G(\bigwedge V)$ is generated by the skew Euler operator E.*

Corollary 15.6 (Ω symmetric) *In the decomposition $\bigwedge V = \bigoplus_{p=1}^{d} \bigwedge^p V$, the summands are irreducible and mutually inequivalent $\mathrm{O}(V, \Omega)$-modules.*

Now assume that $\dim V = 2n$ and Ω is skew-symmetric. Let $G = \mathrm{Sp}(V, \Omega)$ and define

$$X = -\frac{1}{2} PC^* P, \quad Y = \frac{1}{2} PCP.$$

These operators on $\bigwedge V$ commute with the action of G, since C, C^* and P commute with G on tensor space.

Lemma 15.7 *One has the commutation relations*

$$[Y, \epsilon(v)] = [X, \iota(v^*)] = 0, \quad [Y, \iota(v^\sharp)] = \epsilon(v), \quad [X, \epsilon(v)] = \iota(v^\sharp)$$

for $v \in V$ and $v^* \in V^*$. Furthermore,

$$[E, Y] = 2Y, \quad [E, X] = -2X, \quad \text{and} \quad [Y, X] = E - nI.$$

Define

$$\mathfrak{g}' = \mathrm{Span}\{X, Y, E - nI\}.$$

From Lemma 15.7 we see that $\mathfrak{g}'$ is a Lie algebra isomorphic to $\mathfrak{sl}(2, \mathbb{C})$.

Theorem 15.8 (Ω skew-symmetric) *The commutant of $G = \mathrm{Sp}(V, \Omega)$ in $\mathrm{End}(\bigwedge V)$ is generated by $\mathfrak{g}'$.*

Corollary 15.9 ($G = \mathrm{Sp}(V, \Omega)$) *There is a canonical decomposition*

$$\bigwedge V \cong \bigoplus_{k=0}^{n} \mathcal{V}^{n-k} \otimes \mathcal{H}^k, \tag{15.4}$$

as a $(G, \mathfrak{g}')$-module, where $\dim V = 2n$ and $\mathcal{V}^k$ is the irreducible $\mathfrak{g}'$-module of dimension $k+1$. Here $\mathcal{H}^k$ is an irreducible G module and $\mathcal{H}^k \ncong \mathcal{H}^l$ for $k \neq l$.

Lemma 15.10 *The space $\mathrm{Hom}_G(\bigwedge^l V, \bigwedge^k V)$, for $k + l$ an even integer, is spanned by operators of the following forms:*
(1) *YQ with $Q \in \mathrm{Hom}_G(\bigwedge^l V, \bigwedge^{k-2} V)$.*
(2) *QX with $Q \in \mathrm{Hom}_G(\bigwedge^{l-2} V, \bigwedge^k V)$.*
(3) *$\sum_{p=1}^{2n} \epsilon(f_p) Q \iota(f_p^*)$ with $Q \in \mathrm{Hom}_G(\bigwedge^{l-1} V, \bigwedge^{k-1} V)$. Here $\{f_p\}$ is any basis for V and $\{f_p^*\}$ is the dual basis for V^*.*

Appendix: Linear and Associative Algebra for Lecture 15.

Interior and Exterior Product Operators

Let V be a finite-dimensional vector space and $\bigwedge^{\bullet} V$ the exterior algebra over V. For $v \in V$ and $v^* \in V^*$ we have the *exterior product operator* $\epsilon(v)$ and the *interior product operator* $\iota(v^*)$ on $\bigwedge^{\bullet} V$ that act by

$$\begin{aligned} \epsilon(v)u &= v \wedge u, \\ \iota(v^*)(v_1 \wedge \cdots \wedge v_k) &= \sum_{j=1}^{k} (-1)^{j-1} \langle v^*, v_j \rangle \, v_1 \wedge \cdots \wedge \widehat{v_j} \wedge \cdots \wedge v_k \end{aligned}$$

for $u \in \bigwedge V$ and $v_i \in V$ (here $\widehat{v_j}$ means to omit v_j). Note that $\epsilon(v) : \bigwedge^p V \to \bigwedge^{p+1} V$ and $\iota(v^*) : \bigwedge^p V \to \bigwedge^{p-1} V$. Also

$$\iota(v^*)(w \wedge u) = (\iota(v^*)w) \wedge u + (-1)^k w \wedge (\iota(v^*)u) \quad \text{for } w \in \textstyle\bigwedge^k V,\ u \in \bigwedge V.$$

Define the *anti-commutator*

$$\{a, b\} = ab + ba$$

for elements a, b of an associative algebra. Then the exterior product and interior product operators satisfy the *canonical anti-commutation relations*

$$\{\epsilon(x), \epsilon(y)\} = 0, \quad \{\iota(x^*), \iota(y^*)\} = 0, \quad \{\epsilon(x), \iota(x^*)\} = \langle x^*, x \rangle I \tag{15.5}$$

for $x, y \in V$ and $x^*, y^* \in V^*$. Interchanging V and V^*, we also have the exterior and interior multiplication operators $\epsilon(v^*)$ and $\iota(v)$ on $\bigwedge^{\bullet} V^*$ for $v \in V$ and $v^* \in V^*$. They satisfy

$$\epsilon(v^*) = \iota(v^*)^t, \qquad \iota(v) = \epsilon(v)^t \tag{15.6}$$

Exercises for Lecture 15.

1. Let $G = \mathrm{O}(V, B)$, where B is a symmetric bilinear form on V (assume $\dim V \geq 3$). Let $\{e_i\}$ be a basis for V such that $B(e_i, e_j) = \delta_{ij}$.

   (a) Let $R \in \left(V^{\otimes 4}\right)^G$. Show that there are constants $a, b, c \in \mathbb{C}$ so that

   $$R = \sum_{i,j,k,l} \{a\delta_{ij}\delta_{kl} + b\delta_{ik}\delta_{jl} + c\delta_{il}\delta_{jk}\} \, e_i \otimes e_j \otimes e_k \otimes e_l$$

   (*Hint:* Determine all the two-partitions of $\{1, 2, 3, 4\}$).

   (b) Use (a) to find a basis for the space $\left[S^2(V) \otimes S^2(V)\right]^G$. (*Hint:* Symmetrize relative to tensor positions 1, 2 and positions 3, 4.)

   (c) Use (b) to show that $\dim \mathrm{End}_G(S^2(V)) = 2$ and that $S^2(V)$ decomposes into the sum of two inequivalent irreducible G modules. (*Hint:* $S^2(V) \cong S^2(V)^*$ as G modules.)

   (d) Find the dimensions of the irreducible modules in (c). (*Hint*: There is an obvious irreducible submodule in $S^2(V)$.)

2. Let $G = \mathrm{O}(V, B)$ as in the previous exercise.

   (a) Use part (a) of the previous exercise to find a basis for the space $\left[\bigwedge^2 V \otimes \bigwedge^2 V\right]^G$. (*Hint:* Skew-symmetrize relative to tensor positions 1, 2 and positions 3, 4.)

   (b) Use (a) to show that $\dim \mathrm{End}_G(\bigwedge^2 V) = 1$ and hence $\bigwedge^2 V$ is irreducible under G. (*Hint:* $\bigwedge^2 V \cong \bigwedge^2 V^*$ as G modules.)

3. Let $G = \mathrm{Sp}(V, \Omega)$, where Ω is a nonsingular skew form on V (assume $\dim V \geq 4$ is even). Let $\{f_i\}$ and $\{f^j\}$ be bases for V such that $\Omega(f_i, f^j) = \delta_{ij}$.

   (a) Show that $(V^{\otimes 4})^G$ is spanned by the tensors

$$\sum_{i,j} f_i \otimes f^i \otimes f_j \otimes f^j, \quad \sum_{i,j} f_i \otimes f_j \otimes f^i \otimes f^j, \quad \sum_{i,j} f_i \otimes f_j \otimes f^j \otimes f^i.$$

   (b) Use (a) to find a basis for the space $\left[\bigwedge^2 V \otimes \bigwedge^2 V\right]^G$. (*Hint:* Skew-symmetrize relative to tensor positions 1, 2 and positions 3, 4.)

   (c) Use (b) to show that $\dim \mathrm{End}_G(\bigwedge^2 V) = 2$ and that $\bigwedge^2 V$ decomposes into the sum of two inequivalent irreducible G modules. (*Hint:* $\bigwedge^2 V \cong \bigwedge^2 V^*$ as a G-module.)

   (d) Find the dimensions of the irreducible modules in (c). (*Hint*: There is an obvious irreducible submodule in $\bigwedge^2 V$.)

4. Let $G = \mathrm{Sp}(V, \Omega)$ as in the previous exercise.

   (a) Use part (a) of the previous exercise to find a basis for the space $\left[S^2(V) \otimes S^2(V)\right]^G$. *(Hint:* Symmetrize relative to tensor positions 1, 2 and positions 3, 4.)

   (b) Use (a) to show that $\dim \mathrm{End}_G(S^2(V)) = 1$ and hence $S^2(V)$ is irreducible under G. (*Hint:* $S^2(V) \cong S^2(V)^*$ as a G-module.)

Lecture 16. Tensor Models for Irreducible Representations

Fundamental Representations

Let G be a classical group whose Lie algebra $\mathfrak{g}$ is semisimple. The irreducible finite-dimensional representations of $\mathfrak{g}$ are parameterized by their highest weights. We shall prove that for every $\mu \in P_{++}(\mathfrak{g})$, there exists an irreducible finite-dimensional $\mathfrak{g}$-module V with highest weight μ. We begin with the so-called *fundamental representations.* The elements of $P_{++}(\mathfrak{g})$ are of the form

$$n_1\varpi_1 + \cdots + n_l\varpi_l, \qquad \text{with } n_i \in \mathbb{N},$$

where $\varpi_1, \ldots, \varpi_l$ are the fundamental weights. An irreducible finite-dimensional representation of $\mathfrak{g}$ whose highest weight is ϖ_k for some k is called a *fundamental representation* . We now prove the existence of the fundamental representations by giving explicit models for them (for the orthogonal groups this construction will be completed in Lecture 18 with the construction of the spin representations).

Special Linear Group

We construct the fundamental representations when G is $\mathrm{SL}(n, \mathbb{C})$. Let $(\sigma_r, \bigwedge^r \mathbb{C}^n)$ be the rth exterior power of the defining representation of G on $\mathbb{C}^n$, for $r = 1, 2, \ldots, n$.

Theorem 16.1 *Let $G = \mathrm{SL}(n, \mathbb{C})$. The representation σ_r on the* rth *exterior power $\bigwedge^r \mathbb{C}^n$ is regular, irreducible and has highest weight ϖ_r for $1 \le r < n$.*

Remark. For $r = n$ the space $\bigwedge^n \mathbb{C}^n$ is one-dimensional and σ_n is the trivial representation of $\mathrm{SL}(n, \mathbb{C})$.

Special Orthogonal Group

Let $G = \mathrm{SO}(n, \mathbb{C})$. Let σ_1 be the defining representation of G on $\mathbb{C}^n$ and denote by σ_r the representation of G on the rth exterior power $\bigwedge^r \mathbb{C}^n$.

Theorem 16.2 (1) *Let $n = 2l + 1 \ge 3$ be odd. For $1 \le r \le l$, $(\sigma_r, \bigwedge^r \mathbb{C}^n)$ is an irreducible representation of* $\mathrm{SO}(n, \mathbb{C})$ *with highest weight ϖ_r for $r \le l - 1$ and highest weight $2\varpi_l$ for $r = l$.*

(2) *Let $n = 2l \ge 4$ be even.*
(a) *For $1 \le r \le l - 1$, $(\sigma_r, \bigwedge^r \mathbb{C}^n)$ is an irreducible representation of* $\mathrm{SO}(n, \mathbb{C})$ *with highest weight ϖ_r for $r \le l - 2$ and highest weight $\varpi_{l-1} + \varpi_l$ for $r = l - 1$.*
(b) *For $r = l$, the space $\bigwedge^l \mathbb{C}^n$ is irreducible under the action of* $\mathrm{O}(n, \mathbb{C})$. *As a module for* $\mathrm{SO}(n, \mathbb{C})$ *it decomposes into the sum of two irreducible representations with highest weights $2\varpi_{l-1}$ and $2\varpi_l$.*

Symplectic Group

Let $G = \mathrm{Sp}(\mathbb{C}^{2l}, \Omega)$, where Ω is a non-degenerate symplectic form. We recall the decomposition of $\bigwedge \mathbb{C}^{2l}$ under G (Corollary 15.9). Let $\theta \in (\bigwedge^2 V)^G$ be the G-invariant skew 2-tensor corresponding to Ω. Let Y be the operator of exterior multiplication by $\frac{1}{2}\theta$, and let $X = -Y^*$ (adjoint operator relative to the skew-bilinear form on $\bigwedge V$ obtained from Ω). Set $H = lI - E$, where E is the skew-Euler operator. Then

$$[H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H$$

by Lemma 15.7. Set $\mathfrak{g}' = \mathrm{Span}\{X, Y, H\}$. Then $\mathfrak{g}' \cong \mathfrak{sl}(2, \mathbb{C})$ and $\mathfrak{g}'$ generates the commuting algebra $\mathrm{End}_G(\bigwedge V)$, by Theorem 15.8.

We say that an element $u \in \bigwedge \mathbb{C}^{2l}$ is *Ω-harmonic* if $Xu = 0$. Let $\mathcal{H}(\bigwedge \mathbb{C}^{2l}, \Omega)$ be the space of Ω-harmonic elements in $\bigwedge \mathbb{C}^{2l}$. Since $X : \bigwedge^p \mathbb{C}^{2l} \to \bigwedge^{2p-2} \mathbb{C}^{2l}$, an element u is Ω-harmonic if and only if each homogeneous component of u is Ω-harmonic. Thus

$$\mathcal{H}(\textstyle\bigwedge \mathbb{C}^{2l}, \Omega) = \bigoplus_{p \geq 0} \mathcal{H}(\bigwedge^p \mathbb{C}^{2l}, \Omega),$$

where $\mathcal{H}(\bigwedge^p \mathbb{C}^{2l}, \Omega) = \{u \in \bigwedge^p \mathbb{C}^{2l} : Xu = 0\}$. Because X commutes with G, the spaces $\mathcal{H}(\bigwedge^p \mathbb{C}^{2l}, \Omega)$ are G-invariant.

Theorem 16.3 (1) *If $p > l$ then $\mathcal{H}(\bigwedge^p \mathbb{C}^{2l}, \Omega) = 0$.*
(2) *Let $\mathcal{V}^k$ be the irreducible $\mathfrak{g}'$-module of dimension $k+1$. Then*

$$\textstyle\bigwedge \mathbb{C}^{2l} \cong \bigoplus_{p=0}^{l} \left\{ \mathcal{V}^{l-p} \otimes \mathcal{H}(\bigwedge^p \mathbb{C}^{2l}, \Omega) \right\} \tag{16.1}$$

as a $(\mathfrak{g}', G)$-module.
(3) *If $1 \leq p \leq l$, then $\mathcal{H}(\bigwedge^p \mathbb{C}^{2l}, \Omega)$ is an irreducible G-module with highest weight ϖ_p.*

Corollary 16.4 *The map $\mathbb{C}[\theta] \otimes \mathcal{H}(\bigwedge \mathbb{C}^{2l}, \Omega) \to \bigwedge \mathbb{C}^{2l}$ given by $f(\theta) \otimes u \mapsto f(\theta) \wedge u$ (exterior multiplication) is a G-module isomorphism. Thus*

$$\textstyle\bigwedge^k \mathbb{C}^{2l} = \bigoplus_{p=0}^{[k/2]} \theta^p \wedge \mathcal{H}(\bigwedge^{k-2p} \mathbb{C}^{2l}, \Omega). \tag{16.2}$$

Hence $\bigwedge^k \mathbb{C}^{2l}$ is multiplicity-free as a G-module and has highest weights ϖ_{k-2p} for $p = 0, 1, \ldots, [k/2]$.

Corollary 16.5 *For $k = 1, \ldots, l$ one has* $\dim \mathcal{H}(\bigwedge^k \mathbb{C}^{2l}, \Omega) = \binom{2l}{k} - \binom{2l}{k-2}$.

We can describe the space $\mathcal{H}(\bigwedge^p \mathbb{C}^{2l}, \Omega)$ in another way. Let $v_i \in \mathbb{C}^{2l}$. Call $v_1 \wedge \cdots \wedge v_r$ an *isotropic r-vector* if $\Omega(v_i, v_j) = 0$ for $i, j = 1, \ldots, r$.

Proposition 16.6 *For $p = 1, \ldots, l$ the space $\mathcal{H}(\bigwedge^p \mathbb{C}^{2l}, \Omega)$ is spanned by the isotropic p-vectors.*

Cartan Product

Now that we have constructed the fundamental representations of $\mathfrak{g}$ (with three exceptions in the case of the orthogonal groups), we show how to obtain more irreducible representations by decomposing tensor products of representations already constructed.

Given finite-dimensional representations (ρ, U) and (σ, V) of $\mathfrak{g}$, we can form the tensor product $(\rho \otimes \sigma, U \otimes V)$ of these representations. The weight spaces of $\rho \otimes \sigma$ are

$$(U \otimes V)(\nu) = \sum_{\lambda + \mu = \nu} U(\lambda) \otimes V(\mu). \tag{16.3}$$

In particular, for $\nu \in P^{\mathfrak{g}}$ we have

$$\dim(U \otimes V)(\nu) = \sum_{\lambda + \mu = \nu} \dim U(\lambda) \dim V(\mu) \tag{16.4}$$

Proposition 16.7 *Let* (π^λ, V^λ) *and* (π^μ, V^μ) *be finite dimensional irreducible representations of* $\mathfrak{g}$ *with highest weights* $\lambda, \mu \in P_{++}(\mathfrak{g})$*.*
(1) *Fix highest weight vectors* $v_\lambda \in V^\lambda$ *and* $v_\mu \in V^\mu$*. Then the* $\mathfrak{g}$*-cyclic subspace* $U \subset V^\lambda \otimes V^\mu$ *generated by* $v_\lambda \otimes v_\mu$ *is an irreducible* $\mathfrak{g}$*-module with highest weight* $\lambda + \mu$*.*
(2) *If* ν *occurs as the highest weight of a* $\mathfrak{g}$*-submodule of* $V^\lambda \otimes V^\mu$ *then* $\nu \preceq \lambda + \mu$*.*
(3) *The irreducible representation* $(\pi^{\lambda+\mu}, V^{\lambda+\mu})$ *occurs with multiplicity one in* $V^\lambda \otimes V^\mu$*.*

We call the submodule U in (1) of Proposition 16.7 the *Cartan product* of the representations (π^λ, V^λ) and (π^μ, V^μ).

Corollary 16.8 (1) *The set of highest weights of irreducible finite-dimensional* $\mathfrak{g}$*-modules is closed under addition.*
(2) *Suppose* G *is connected and has Lie algebra* $\mathfrak{g}$*. If* π^λ *and* π^μ *are differentials of irreducible regular representations of* G*, then the Cartan product of* π^λ *and* π^μ *is the differential of an irreducible regular representation of* G *with highest weight* $\lambda + \mu$*.*
(3) *The set of highest weights of irreducible regular* G*-modules is closed under addition.*

Theorem 16.9 *Let* G *be the group* $\mathrm{SL}(V)$*,* $\mathrm{Sp}(V)$ *or* $\mathrm{SO}(V)$ *(in the last case assume* $\dim V > 2$*). For every dominant weight* $\mu \in P_{++}(G)$ *there exists an integer* k *so that* $V^{\otimes k}$ *contains an irreducible* G*-module with highest weight* μ*. Hence every irreducible regular representation of* G *occurs in the tensor algebra of* V*.*

Irreducible Representations of $\mathrm{GL}(n, \mathbb{C})$

We shall extend the theorem of the highest weight to the group $G = \mathrm{GL}(n, \mathbb{C})$. Recall from Lecture #7 that $P_{++}(G)$ consists of all weights

$$\mu = m_1\varepsilon_1 + \cdots + m_n\varepsilon_n, \qquad m_1 \geq \cdots \geq m_n, \quad m_i \in \mathbb{Z}. \tag{16.5}$$

Define the dominant weights

$$\lambda_i = \varepsilon_1 + \cdots + \varepsilon_i \tag{16.6}$$

for $i = 1, \ldots, n$. Note that the restriction of λ_i to the diagonal matrices of trace zero is the fundamental weight ϖ_i of $\mathfrak{sl}(n, \mathbb{C})$ for $i = 1, \ldots, n-1$. If μ is given by (16.5) then

$$\mu = (m_1 - m_2)\lambda_1 + (m_2 - m_3)\lambda_2 + \cdots + (m_{n-1} - m_n)\lambda_{n-1} + m_n\lambda_n.$$

Hence $P_{++}(G)$ consists of all weights

$$\mu = k_1\lambda_1 + \cdots + k_n\lambda_n, \qquad k_i \in \mathbb{Z}, \quad k_1 \geq 0, \ldots, k_{n-1} \geq 0.$$

The restriction of μ to the diagonal matrices of trace zero is the weight

$$\mu_0 = (m_1 - m_2)\varpi_1 + (m_2 - m_3)\varpi_2 + \cdots + (m_{n-1} - m_n)\varpi_{n-1}. \tag{16.7}$$

Theorem 16.10 *Let* $G = \mathrm{GL}(n, \mathbb{C})$ *and let* μ *be given by* (16.5)*. Then there exists a unique irreducible regular representation* (π, V) *of* G *such that*
(1) *the restriction of* π *to* $\mathrm{SL}(n, \mathbb{C})$ *has highest weight* μ_0 *given by* (16.7)*;*
(2) $\pi(zI_n) = z^{m_1 + \cdots + m_n} I_V$ *for* $z \in \mathbb{C}^\times$*.*
Furthermore, the representation $(\check{\pi}, V)$*, where* $\check{\pi}(g) = \pi(g^t)^{-1}$*, is equivalent to the dual representation* (π^*, V^*)*.*

Lecture 17. Spinors

Clifford Algebras

Let V be a finite-dimensional complex vector space with a symmetric bilinear form β (for the moment we allow β to be degenerate). A *Clifford algebra* for (V, β) is an associative algebra $\mathrm{Cliff}(V, \beta)$ with unit 1 over $\mathbb{C}$ and a linear map

$$\gamma : V \to \mathrm{Cliff}(V, \beta)$$

satisfying the following properties:

(C1) $\{\gamma(x), \gamma(y)\} = \beta(x, y)1$ for $x, y \in V$, where $\{a, b\} = ab + ba$ is the *anticommutator* of a, b.

(C2) $\gamma(V)$ generates $\mathrm{Cliff}(V, \beta)$ as an algebra.

(C3) Given any complex associative algebra $\mathcal{A}$ with unit and a linear map $\phi : V \to \mathcal{A}$ such that $\{\phi(x), \phi(y)\} = \beta(x, y)1$, there exists an associative algebra homomorphism

$$\widetilde{\phi} : \mathrm{Cliff}(V, \beta) \to \mathcal{A}$$

such that $\phi = \widetilde{\phi} \circ \gamma$:

$$\begin{tikzcd} V \arrow[r, "\phi"] \arrow[d, "\gamma"'] & \mathcal{A} \\ \mathrm{Cliff}(V, \beta) \arrow[ur, "\widetilde{\phi}"'] & \end{tikzcd}$$

Using the tensor algebra over V, one proves that an algebra satisfying properties (C1), (C2), and (C3) exists and is unique (up to isomorphism).
Let $\mathrm{Cliff}_k(V, \beta)$ be the span of 1 and the operators

$$\gamma(a_1) \cdots \gamma(a_p) \quad \text{for } a_i \in V \text{ and } p \leq k.$$

The subspaces $\mathrm{Cliff}_k(V, \beta)$, for $k = 0, 1, \ldots$, give a *filtration* of the Clifford algebra:

$$\mathrm{Cliff}_k(V, \beta) \cdot \mathrm{Cliff}_m(V, \beta) \subset \mathrm{Cliff}_{k+m}(V, \beta)$$

Let $\{v_i : i = 1, \ldots, n\}$ be a basis for V. Since $\{\gamma(v_i), \gamma(v_j)\} = \beta(v_i, v_j)$, we see from (C1) that $\mathrm{Cliff}_k(V, \beta)$ is spanned by 1 and the products

$$\gamma(v_{i_1}) \cdots \gamma(v_{i_p}), \quad i_1 < i_2 < \cdots < i_p$$

for $p \leq k$. In particular, we have

$$\mathrm{Cliff}(V, \beta) = \mathrm{Cliff}_n(V, \beta), \quad n = \dim V.$$

and $\dim \mathrm{Cliff}(V, \beta) \leq 2^{\dim V}$.

The linear map $v \mapsto -\gamma(v)$ satisfies (C3), so it extends to an algebra homomorphism

$$\alpha : \mathrm{Cliff}(V, \beta) \to \mathrm{Cliff}(V, \beta)$$

such that

$$\alpha(\gamma(v_1) \cdots \gamma(v_k)) = (-1)^k \gamma(v_1) \cdots \gamma(v_k).$$

Obviously $\alpha^2(u) = u$ for all $u \in \mathrm{Cliff}(V, \beta)$. Hence α is an automorphism, which we call the *main involution* of $\mathrm{Cliff}(V, \beta)$. There is a decomposition

$$\mathrm{Cliff}(V, \beta) = \mathrm{Cliff}^+(V, \beta) \oplus \mathrm{Cliff}^-(V, \beta),$$

where $\mathrm{Cliff}^+(V, \beta)$ is spanned by products of an even number of elements of V, $\mathrm{Cliff}^-(V, \beta)$ is spanned by products of an odd number of elements of V, and α acts by ± 1 on $\mathrm{Cliff}^\pm(V, \beta)$.

Spaces of Spinors

Let V be a finite-dimensional complex vector space with nondegenerate symmetric bilinear form β. Let S be a complex vector space and let $\gamma : V \to \mathrm{End}(S)$ be a linear map. We say that (S, γ) is a *space of spinors* for (V, β) if

(S1) $\{\gamma(x), \gamma(y)\} = \beta(x, y)I$ for all $x, y \in V$.

(S2) The only subspaces of S that are invariant under $\gamma(V)$ are 0 and S.

If (S, γ) is a space of spinors, then the map γ extends to an irreducible representation

$$\widetilde{\gamma} : \mathrm{Cliff}(V, \beta) \to \mathrm{End}(S),$$

and every irreducible representation of $\mathrm{Cliff}(V, \beta)$ arises this way. Since $\mathrm{Cliff}(V, \beta)$ is a finite-dimensional algebra, a space of spinors for (V, β) must also be finite-dimensional.
If (γ, S) and (γ', S') are spaces of spinors for (V, β) then (S, γ) is said to be isomorphic to (S', γ') if there exists a linear bijection $T : S \to S'$ such that $T\gamma(v) = \gamma'(v)T$ for all $v \in V$.

Theorem 17.1 *Let* $n = \dim V$.
(1) *If* n *is even then up to isomorphism there is exactly one space of spinors* (γ, S) *for* (V, β) *and* $\dim S = 2^{n/2}$.
(2) *If* n *is odd, then up to isomorphism there are two spaces of spinors for* (V, β) *and they are each of dimension* $2^{[n/2]}$.

Structure of Clifford Algebras

Proposition 17.2 *Suppose* $\dim V = n$ *is even. Let* (S, γ) *be a space of spinors for* (V, β). *Then* $(\mathrm{End}(S), \gamma)$ *is a Clifford algebra for* (V, β). *Thus* $\mathrm{Cliff}(V, \beta)$ *is a simple algebra of dimension* 2^n. *The map* $\gamma : V \to \mathrm{Cliff}(V, \beta)$ *is injective. For any basis* $\{v_1, \ldots, v_n\}$ *of* V *the set of all ordered products*

$$\gamma(v_{i_1}) \cdots \gamma(v_{i_p}) \qquad 1 \le i_1 < \ldots < i_p \le n \tag{17.1}$$

(empty product $= 1$*) is a basis for* $\mathrm{Cliff}(V, \beta)$.

Before considering the Clifford algebra for an odd-dimensional space, we introduce another model for the spin spaces which is useful for calculations. Assume that $\dim V = 2l$ is even. Take the β-isotropic spaces W, W^* and the basis $e_{\pm i}$ for V as above. Set

$$U_i = \bigwedge \mathbb{C}e_{-i} = \mathbb{C}1 \oplus \mathbb{C}e_{-i}$$

for $i = 1, \ldots, l$. Then U_i is a graded algebra with ordered basis $\{1, e_{-i}\}$ and relation $e_{-i}^2 = 0$. Since $W^* = \mathbb{C}e_{-1} \oplus \cdots \oplus \mathbb{C}e_{-l}$, there is an isomorphism of graded algebras

$$\bigwedge\nolimits^{\bullet}(W^*) \cong U_1 \hat{\otimes} \cdots \hat{\otimes} U_l \quad \text{(skew-commutative tensor product).} \tag{17.2}$$

If we ignore the algebra structure and consider $\bigwedge W^*$ as a vector space, we have an isomorphism $\bigwedge W^* \cong U_1 \otimes \cdots \otimes U_l$. Hence

$$\mathrm{End}(\bigwedge W^*) \cong \mathrm{End}(U_1) \otimes \cdots \otimes \mathrm{End}(U_l) \tag{17.3}$$

(algebra isomorphism). Notice that in this isomorphism the factors on the right mutually commute. To describe the operators $\gamma(x)$ in this tensor product model, let $J = \{j_1, \ldots, j_p\}$ with $1 \leq j_1 < \cdots < j_p \leq l$. Under the isomorphism (17.2) the element $e_{-j_1} \wedge \ldots \wedge e_{-j_p}$ corresponds to $u_J = u_1 \otimes \cdots \otimes u_l$, where

$$u_i = \begin{cases} e_{-i} & \text{if } i \in J \\ 1 & \text{if } i \notin J. \end{cases}$$

We have

$$e_{-i} \wedge e_{-j_1} \wedge \ldots \wedge e_{-j_p} = \begin{cases} 0 & \text{if } i \in J \\ (-1)^r e_{-j_1} \wedge \ldots \wedge e_{-i} \wedge \ldots \wedge e_{-j_p} & \text{if } i \notin J, \end{cases}$$

where r is the number of indices in J that are less than i. Thus the exterior multiplication operator $\epsilon(e_{-i})$ acts on the basis $\{u_J\}$ by

$$A_{-i} = H \otimes \cdots \otimes H \otimes \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{i\text{th place}} \otimes I \otimes \cdots \otimes I,$$

where $H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, I is the 2×2 identity matrix and we enumerate the basis for U_i in the order 1, e_{-i}. On the other hand,

$$\iota(e_i)(e_{-j_1} \wedge \ldots \wedge e_{-j_p}) = \begin{cases} (1)^r e_{-j_1} \wedge \ldots \wedge \widehat{e_{-i}} \wedge \ldots \wedge e_{-j_p} & \text{if } i \in J \\ 0 & \text{if } i \notin J. \end{cases}$$

Thus the interior product operator $\iota(e_i)$ acts on the basis $\{u_J\}$ by

$$A_i = H \otimes \cdots \otimes H \otimes \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{i\text{th place}} \otimes I \otimes \cdots \otimes I.$$

It is easy to check that the operators $\{A_{\pm i}\}$ satisfy the canonical anticommutation relations (the factors of H in the tensor product ensure that $A_i A_j = -A_j A_i$). This gives a direct proof that $S = U_1 \otimes \cdots \otimes U_l$ together with the map $e_{\pm i} \mapsto A_{\pm i}$ furnishes a space of spinors for (V, β).

When $\dim V = 2l+1$ is odd, set

$$A_0 = H \otimes \cdots \otimes H \qquad (l \text{ factors}).$$

Then $A_0^2 = 1$ and $A_0 A_{\pm i} = -A_{\pm i} A_0$ for $i = 1, \ldots, l$. Hence we can obtain models for the spinor spaces $(S, \gamma_\pm)$ by seting $S = U_1 \otimes \cdots \otimes U_l$, with $e_{\pm i}$ acting by $A_{\pm i}$ and e_0 acting by $\pm A_0$.

Proposition 17.3 *Suppose* $\dim V = 2l+1$ *is odd. Let* (S, γ_+) *and* (S, γ_-) *be the two inequivalent spaces of spinors for* (V, β)*, and let*

$$\gamma : V \to \mathrm{End}(S) \oplus \mathrm{End}(S), \qquad \gamma(v) = \gamma_+(v) \oplus \gamma_-(v).$$

Then $(\mathrm{End}(S) \oplus \mathrm{End}(S), \gamma)$ *is a Clifford algebra for* (V, β)*. Thus* $\mathrm{Cliff}(V, \beta)$ *is a semisimple algebra and is the sum of two simple ideals of dimension* 2^{n-1}*. The map* $\gamma : V \to \mathrm{Cliff}(V, \beta)$ *is injective. For any basis* $\{v_1, \ldots, v_n\}$ *of* V *the set of all ordered products*

$$\gamma(v_{i_1}) \cdots \gamma(v_{i_p}) \qquad 1 \le i_1 < \ldots < i_p \le n$$

(empty product $= 1$*) is a basis for* $\mathrm{Cliff}(V, \beta)$*.*

Let V be odd-dimensional. Decompose $V = W \oplus \mathbb{C}e_0 \oplus W^*$ as above. Set $V_0 = W \oplus W^*$ and let β_0 be the restriction of β to V_0. Recall that $\mathrm{Cliff}^+(V, \beta)$ is the subalgebra of $\mathrm{Cliff}(V, \beta)$ spanned by the products of an even number of elements of V.

Lemma 17.4 *There is an algebra isomorphism*

$$\mathrm{Cliff}(V_0, \beta_0) \cong \mathrm{Cliff}^+(V, \beta)$$

Hence $\mathrm{Cliff}^+(V, \beta)$ *is a simple algebra.*

Exercises for Lecture 17.

1. Let $V = W \oplus W^*$ be an even-dimensional space, and β a bilinear form on V for which W and W^* are β-isotropic and in duality.

   (a) Let (S, γ) be a space of spinors for (V, β). Show that $\bigcap_{w^* \in W^*} \mathrm{Ker}(\gamma(w^*))$ is one-dimensional.

   (b) Let $S' = \bigwedge W$ and for $w \in W$, $w^* \in W^*$ define $\gamma'(w + w^*) = \epsilon(w) + \iota(w^*)$ on S', where $\epsilon(w)$ is the exterior product operator and $\iota(w^*)$ is the interior product operator. Show that (S', γ') is a space of spinors for (V, β).

   (c) Fix $0 \ne u \in \bigwedge^l W$, where $l = \dim W$. Show that there is a unique spinor-space isomorphism T from $(\bigwedge W^*, \gamma)$ to $(\bigwedge W, \gamma')$ such that $T(1) = u$. Here $\gamma(w + w^*) = \iota(w) + \epsilon(w^*)$ and γ' is the map in (b).

   (d) Let $\{e_1, \ldots, e_l\}$ be a basis for W and $\{e_{-1}, \ldots, e_{-l}\}$ a basis for W^* such that $\beta(e_i, e_{-j}) = \delta_{ij}$. For $J = \{1 \le j_1 < \cdots < j_p \le l\}$ set $e_J = e_{j_1} \wedge \cdots \wedge e_{j_p}$ and $e_{-J} = e_{-j_1} \wedge \cdots \wedge e_{-j_p}$. Let T be the map in (c) defined using $u = e_1 \wedge \cdots \wedge e_l$. Prove that $T(e_{-J}) = (-1)^{q-p} e_{J^c}$, where $q = j_1 + \cdots + j_p$ and J^c is the complement to J in $\{1, \ldots, l\}$, arranged in increasing order.

2. Let V be a complex vector space with a symmetric bilinear form β. Let $\{e_1, \ldots, e_n\}$ be a basis for V such that $\beta(e_i, e_j) = \delta_{ij}$.

   (a) Show that if i, j, k are distinct, then
$$e_i e_j e_k = e_j e_k e_i = e_k e_i e_j,$$
   where the product is in the Clifford algebra for (V, β).

   (b) Show that if $A = [a_{ij}]$ is a symmetric $n \times n$ matrix, then
$$\sum_{i,j=1}^{n} a_{ij}\, e_i\, e_j = \frac{1}{2}\mathrm{tr}(A)$$
   (product in the Clifford algebra for (V, β)).

   (c) Show that if $A = [a_{ij}]$ is a skew-symmetric $n \times n$ matrix, then
$$\sum_{i,j=1}^{n} a_{ij}\, e_i\, e_j = 2 \sum_{1 \le i < j \le n} a_{ij}\, e_i\, e_j$$
   (product in the Clifford algebra for (V, β)).

3. Let (V, β) and $e_1, \ldots, e_n$ be as in the previous exercise. Let $R_{ijkl} \in \mathbb{C}$ for $1 \le i, j, k, l \le n$ be such that

   (i) $R_{ijkl} = R_{klij}$,

   (ii) $R_{jikl} = -R_{ijkl}$,

   (iii) $R_{ijkl} + R_{kijl} + R_{jkil} = 0$.

   (a) Show that $\sum R_{ijkl} e_i e_j e_k e_l = (1/2) \sum R_{ijji}$, where the multiplication of the e_i is in the Clifford algebra for (V, β). (*Hint*: Use part (a) of the previous exercise to show that for each l, the sum over distinct triples i, j, k is zero. Then use the anticommutation relations to show that the sum with $i = j$ is also zero. Finally, use part (b) of the previous exercise to simplify the remaining sum.)

   (b) Let $\mathfrak{g}$ be a Lie algebra and B a symmetric non-degenerate bilinear form on $\mathfrak{g}$ such that $B([x, y], z) = -B(y, [x, z])$. Let $e_1, ..., e_n$ be an orthonormal basis of $\mathfrak{g}$ relative to B. Show that $R_{ijkl} = B([e_i, e_j], [e_k, e_l])$ satsifies (i), (ii), and (iii).

4. Let $V = \mathbb{C}^n$ and let $\beta(x, y) = x^t y$ for $x, y \in V$.

   (a) Show that when $n \ge 3$, the polynomial $x_1^2 + \cdots + x_n^2$ in the commuting variables $x_1, \ldots, x_n$ cannot be factored into a product of linear factors with coefficients in $\mathbb{C}$.

   (b) Show that $x_1^2 + \cdots + x_n^2 = 2(x_1 e_1 + \cdots + x_n e_n)^2$ when the multiplication on the right is done in the Clifford algebra $\mathrm{Cliff}(\mathbb{C}^n, \beta)$ and $e_1, \ldots, e_n$ is a β-orthonormal basis for $\mathbb{C}^n$.

   (c) Let (S, γ) be a space of spinors for $(\mathbb{C}^n, \beta)$. Consider the Laplace operator $\Delta = \frac{1}{2}\sum_{i=1}^{n}(\partial/\partial x_i)^2$ acting on $\mathcal{P}(\mathbb{C}^n, S)$ (polynomial functions with values in S). Show that Δ can be factored as D^2, where
$$D = \gamma(e_1)\frac{\partial}{\partial x_1} + \cdots + \gamma(e_n)\frac{\partial}{\partial x_n}$$
   (D is called the *Dirac operator*).

Lecture 18. Spin Representations

Embedding $\mathfrak{so}(V)$ in $\mathrm{Cliff}(V)$

For $a, b \in V$ define $R_{a,b} \in \mathrm{End}(V)$ by

$$R_{a,b}v = \beta(b,v)a - \beta(a,v)b.$$

Since

$$\beta(R_{a,b}x, y) = \beta(b,x)\beta(a,y) - \beta(a,x)\beta(b,y) = -\beta(x, R_{a,b}y),$$

we have $R_{a,b} \in \mathfrak{so}(V,\beta)$.

Lemma 18.1 $\mathfrak{so}(V,\beta) = \mathrm{Span}\{R_{a,b} \, : \, a, b \in V\}$.

Since $R_{a,b}$ is a skew-symmetric bilinear function of the vectors a, b, it defines a linear map

$$R : \bigwedge^2 V \to \mathfrak{so}(V,\beta), \quad a \wedge b \mapsto R_{a,b}.$$

This map is easily seen to be injective, and by Lemma 18.1 it is surjective. We calculate that

$$[R_{a,b}, R_{x,y}] = R_{R_{a,b}x,y} + R_{x,R_{a,b}y} \tag{18.1}$$

for $a, b, x, y \in V$, which shows that R intertwines the representation of $\mathfrak{so}(V,\beta)$ on $\bigwedge^2 V$ with the adjoint representation of $\mathfrak{so}(V,\beta)$.

Lemma 18.2 *Define a linear map* $\phi : \mathfrak{so}(V,\beta) \to \mathrm{Cliff}_2(V,\beta)$ *by*

$$\phi(R_{a,b}) = \frac{1}{2}[\gamma(a), \gamma(b)], \qquad \text{for } a, b \in V.$$

Then ϕ is an injective Lie algebra homomorphism, and

$$[\phi(X), \gamma(v)] = \gamma(Xv). \tag{18.2}$$

for $X \in \mathfrak{so}(V,\beta)$ and $v \in V$.

Spin Representations of $\mathfrak{so}(V)$

Assume V is even dimensional and fix a decomposition

$$V = W \oplus W^*$$

with W and W^* maximal β-isotropic subspaces. Let $(C^\bullet(W), \gamma)$ be the space of spinors defined in the proof of Theorem 17.1. Define the *even* and *odd* spin spaces

$$C^+(W) = \bigoplus_{p \text{ even}} C^p(W), \qquad C^-(W) = \bigoplus_{p \text{ odd}} C^p(W).$$

Then

$$\gamma(v) : C^\pm(W) \to C^\mp(W), \quad \text{for } v \in V, \tag{18.3}$$

so the action of $\gamma(V)$ interchanges the even and odd spin spaces. Denote by $\widetilde{\gamma}$ the extension of γ to a representation of $\mathrm{Cliff}(V, \beta)$ on $C^\bullet(W)$.
Let $\phi : \mathfrak{so}(V, \beta) \to \mathrm{Cliff}(V, \beta)$ be the Lie algebra homomorphism in Lemma 18.2. Set

$$\pi(X) = \widetilde{\gamma}(\phi(X)), \qquad \text{for } X \in \mathfrak{so}(V, \beta).$$

Since $\phi(X)$ is an even element in the Clifford algebra, (18.3) implies that $\pi(X)$ preserves the even and odd subspaces $C^\pm(W)$. We define

$$\pi^\pm(X) = \pi(X)|_{C^\pm(W)}$$

and call $\pi^\pm$ the *half-spin representations* of $\mathfrak{so}(V, \beta)$. Notice that the labeling of these representations by $\pm$ depends on a particular choice of the space of spinors. In both cases the representation space has dimension 2^{l-1}, when $\dim V = 2l$.

Proposition 18.3 ($\dim V = 2l$) *The representations $\pi^\pm$ of $\mathfrak{so}(V, \beta)$ are irreducible and have highest weights $\varpi_\pm = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{l-1} \pm \varepsilon_l)$. The weights of $\pi^\pm$ are*

$$\frac{1}{2}(\pm\varepsilon_1 \pm \cdots \pm \varepsilon_l) \tag{18.4}$$

(with an even number of minus signs for π^+ and an odd number of minus signs for π^-), and each weight has multiplicity one.

Now assume $\dim V = 2l + 1$. Fix a decomposition

$$V = W \oplus \mathbb{C}e_0 \oplus W^*$$

with W and W^* maximal β-isotropic subspaces, as above. Let $(C^\bullet(W), \gamma_+)$ be the space of spinors defined in the proof of Theorem 17.1. Define a representation of $\mathfrak{so}(V, \beta)$ on $C^\bullet(W)$ by

$$\pi = \widetilde{\gamma}_+ \circ \phi,$$

where $\phi : \mathfrak{so}(V, \beta) \to \mathrm{Cliff}(V, \beta)$ is the homomorphism in Lemma 18.2 and $\widetilde{\gamma}_+$ is the canonical extension of γ_+ to a representation of $\mathrm{Cliff}(V, \beta)$ on $C^\bullet(W)$. We call π the *spin representation* of $\mathfrak{so}(V, \beta)$. The representation space has dimension 2^l when $\dim V = 2l + 1$.

Proposition 18.4 ($\dim V = 2l + 1$) *The spin representation of $\mathfrak{so}(V, \beta)$ is irreducible and has highest weight $\varpi_l = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{l-1} + \varepsilon_l)$. The weights of the spin representation are*

$$\frac{1}{2}(\pm\varepsilon_1 \pm \cdots \pm \varepsilon_l) \tag{18.5}$$

and each weight has multiplicity one.

Spin Groups

On $\mathrm{Cliff}(V, \beta)$ there is the *main anti-automorphism* τ ('transpose') that acts by

$$\tau(\gamma(v_1)\cdots\gamma(v_p)) = \gamma(v_p)\cdots\gamma(v_1), \quad \text{for } v_i \in V.$$

We define the *conjugation* $u \mapsto u^*$ on $\mathrm{Cliff}(V, \beta)$ by

$$u^* = \tau(\alpha(u)),$$

where α is the main involution. For $v_1, \ldots, v_p \in V$ we have

$$(\gamma(v_1)\cdots\gamma(v_p))^* = (-1)^p \gamma(v_p)\cdots\gamma(v_1).$$

In particular,

$$\gamma(v)^* = -\gamma(v), \qquad \gamma(v)\gamma(v)^* = -\frac{1}{2}\beta(v, v) \quad \text{for } v \in V.$$

Suppose v is non-isotropic and normalized so that $\beta(v, v) = -2$. Then

$$\gamma(v)\gamma(v)^* = \gamma(v)^*\gamma(v) = 1,$$

so we see that $\gamma(v)$ is an invertible element of $\mathrm{Cliff}(V, \beta)$ with $\gamma(v)^{-1} = \gamma(v)^*$. Furthermore, for $y \in V$ we can use the Clifford relations to write

$$\begin{aligned} \alpha(\gamma(v))\gamma(y)\gamma(v)^* &= \gamma(v)\gamma(y)\gamma(v) = (\beta(v, y) - \gamma(y)\gamma(v))\gamma(v) \\ &= \gamma(y) + \beta(v, y)\gamma(v) = \gamma(s_v y), \end{aligned}$$

where $s_v y = y + \beta(v, y)v$ is the orthogonal reflection through the hyperplane $(v)^\perp$. Thus the (twisted) conjugation

$$\gamma(y) \mapsto \alpha(\gamma(v))\gamma(y)\gamma(v)^*$$

on the Clifford algebra corresponds to the reflection s_v on V.
In general, we define

$$\mathrm{Pin}(V, \beta) = \{x \in \mathrm{Cliff}(V, \beta) \,:\, x \cdot x^* = 1 \text{ and } \alpha(x)\gamma(V)x^* = \gamma(V)\}.$$

Since $\mathrm{Cliff}(V, \beta)$ is finite-dimensional, the condition $x \cdot x^* = 1$ implies that x is invertible, with $x^{-1} = x^*$. Thus $\mathrm{Pin}(V, \beta)$ is a subgroup of the group of invertible elements of $\mathrm{Cliff}(V, \beta)$. The defining conditions are given by polynomial equations in the components of x, so $\mathrm{Pin}(V, \beta)$ is an algebraic group. The calculation above shows that $\gamma(v) \in \mathrm{Pin}(V, \beta)$ when $v \in V$ and $\beta(v, v) = -2$.

Theorem 18.5 *There is a unique regular homomorphism*

$$\pi : \mathrm{Pin}(V, \beta) \to \mathrm{O}(V, \beta)$$

such that $\alpha(x)\gamma(v)x^* = \gamma(\pi(x)v)$ *for* $v \in V$ *and* $x \in \mathrm{Pin}(V, \beta)$*. Furthermore,* π *is surjective and* $\mathrm{Ker}(\pi) = \pm 1$.

Since $\mathrm{O}(V, \beta)$ is generated by reflections, the surjectivity of the map π furnishes an alternate description of the Pin group:

Corollary 18.6 *The elements* -1 *and* $\gamma(v)$, *with* $v \in V$ *and* $\beta(v,v) = -2$, *generate the group* $\mathrm{Pin}(V,\beta)$.

Finally, we introduce the spin group. Assume $\dim V \geq 3$. Define

$$\mathrm{Spin}(V,\beta) = \mathrm{Pin}(V,\beta) \cap \mathrm{Cliff}^{+}(V,\beta).$$

Let $l = [\dim V/2]$. When $\dim V$ is even, we fix a β-isotropic basis $\{e_1,\ldots,e_l,e_{-1},\ldots,e_{-l}\}$ for V with

$$\beta(e_i,e_j) = \delta_{i+j} \tag{18.6}$$

for $i,j = \pm 1,\ldots,\pm l$. If $\dim V$ is odd, we take a basis $\{e_0,e_1,\ldots,e_l,e_{-1},\ldots,e_{-l}\}$ for V so that (18.6) holds for $i,j = 0,\pm 1,\ldots,\pm l$. For $i = 1,\ldots,l$ and $z \in \mathbb{C}^{\times}$, define

$$c_i(z) = z\gamma(e_i)\gamma(e_{-i}) + z^{-1}\gamma(e_{-i})\gamma(e_i).$$

For $z = [z_1,\ldots,z_l] \in (\mathbb{C}^{\times})^l$ set $c(z) = c_1(z_1)\cdots c_l(z_l)$.

Lemma 18.7 *The map* $z \mapsto c(z)$ *is a regular injective homomorphism from* $(\mathbb{C}^{\times})^l$ *to* $\mathrm{Spin}(V,\beta)$.

Let $H \subset \mathrm{SO}(V,\beta)$ be the maximal torus that is diagonalized by the β-isotropic basis $\{e_i\}$ for V. Define

$$\widetilde{H} = \{c(z) \,:\, z \in (\mathbb{C}^{\times})^l\}.$$

Then $\widetilde{H}$ is a torus of rank l in $\mathrm{Spin}(V,\beta)$, by Lemma 18.7.

Theorem 18.8 *The group* $\mathrm{Spin}(V,\beta)$ *is the identity component of the group* $\mathrm{Pin}(V,\beta)$, *and*

$$\pi : \mathrm{Spin}(V,\beta) \to \mathrm{SO}(V,\beta)$$

is surjective with $\mathrm{Ker}(\pi) = \{\pm 1\}$. *One has* $\widetilde{H} = \pi^{-1}(H)$ *and*

$$\pi(c(z)) = \begin{cases} \mathrm{diag}[z_1^2,\ldots,z_l^2,z_l^{-2},\ldots,z_1^{-2}] & (\dim V = 2l), \\ \mathrm{diag}[z_1^2,\ldots,z_l^2,1,z_l^{-2},\ldots,z_1^{-2}] & (\dim V = 2l+1). \end{cases}$$

Hence $\widetilde{H}$ *is a maximal torus in* $\mathrm{Spin}(V,\beta)$ *and every semisimple element of* $\mathrm{Spin}(V,\beta)$ *is conjugate to an element of* $\widetilde{H}$.

Theorem 18.9 *The Lie algebra of* $\mathrm{Spin}(V,\beta)$ *is* $\phi(\mathfrak{so}(V,\beta))$, *where* ϕ *is the isomorphism of Lemma 18.2.*

Corollary 18.10 *Let* P *be the weight lattice of* $\mathfrak{so}(V,\beta)$. *For* $\lambda \in P_{++}$ *there is an irreducible regular representation of* $\mathrm{Spin}(V,\beta)$ *and* $\mathfrak{so}(V,\beta)$ *with highest weight* λ.

Exercises for Lecture 18.

1. (a) Show that $\mathrm{Spin}(3,\mathbb{C}) \cong \mathrm{SL}(2,\mathbb{C})$ and the spin representation is the representation on $\mathbb{C}^2$. (*Hint:* Consider the adjoint representation of $\mathrm{SL}(2,\mathbb{C})$.)

   (b) Show that $\mathrm{Spin}(5,\mathbb{C}) \cong \mathrm{Sp}(\mathbb{C}^4)$ and the spin representation is the defining representation of $\mathrm{Sp}(\mathbb{C}^4)$. (*Hint:* Use Exercise # 4 from Lecture 7.)

2. (a) Show that $\mathrm{Spin}(4,\mathbb{C}) \cong \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$ and the half-spin representations are the two representations $(x, y) \mapsto x$ and $(x, y) \mapsto y$ on $\mathbb{C}^2$.

   (b) Show that $\mathrm{Spin}(6,\mathbb{C}) \cong \mathrm{SL}(4,\mathbb{C})$ and the half-spin representations are the representation of $\mathrm{SL}(4,\mathbb{C})$ on $\mathbb{C}^4$ and its dual. (*Hint:* Use Exercise #3 of Lecture 7.)

3. Let $V = \mathbb{C}^n$ with nondegenerate bilinear form β. Let $\mathcal{C} = \mathrm{Cliff}(V, \beta)$ and identify V with $\gamma(V) \subset \mathcal{C}$ by the canonical map γ. Let α be the automorphism of $\mathcal{C}$ such that $\alpha(v) = -v$ for $v \in V$, let τ be the antiautomorphism of $\mathcal{C}$ such that $\tau(v) = v$ for $v \in V$, and let $x \mapsto x^*$ be the antiautomorphism $\alpha \circ \tau$ of $\mathcal{C}$. Define the *norm function* $\Delta : \mathcal{C} \to \mathcal{C}$ by $\Delta(x) = x^* x$. Let $\mathcal{L} = \{x \in \mathcal{C} \,:\, \Delta(x) \in \mathbb{C}\}$.

   (a) Show that $\lambda + v \in \mathcal{L}$ for all $\lambda \in \mathbb{C}$ and $v \in V$.

   (b) Show that if $x, y \in \mathcal{L}$ and $\lambda \in \mathbb{C}$ then $\lambda x \in \mathcal{L}$ and

$$\Delta(xy) = \Delta(x)\Delta(y), \quad \Delta(\tau(x)) = \Delta(\alpha(x)) = \Delta(x^*) = \Delta(x).$$

   Hence $xy \in \mathcal{L}$ and $\mathcal{L}$ is invariant under τ and α. Prove that $x \in \mathcal{L}$ is invertible if and only if $\Delta(x) \neq 0$. In this case $x^{-1} = \Delta(x)^{-1} x^*$ and $\Delta(x^{-1}) = 1/\Delta(x)$.

   (c) Let $\Gamma(V, \beta) \subset \mathcal{L}$ be the set of all products $w_1 \cdots w_k$, where $w_j \in \mathbb{C} + V$ and $\Delta(w_j) \neq 0$ for all $1 \leq j \leq k$ (k arbitrary). Prove that $\Gamma(V, \beta)$ is a group (under multiplication) that is stable under α and τ.

   (d) Prove that if $g \in \Gamma(V, \beta)$ then $\alpha(g)(\mathbb{C} + V)g^* = \mathbb{C} + V$. ($\Gamma(V, \beta)$ is called the *Clifford group*; note that it contains $\mathrm{Pin}(V, \beta)$.)

4. Let $\mathfrak{g}$ be the Lie algebra of a classical group. Assume that $\mathfrak{g} = \bar{\mathfrak{n}} + \mathfrak{h} + \mathfrak{n}$ is simple. Let $l = \dim \mathfrak{h}$ be the rank of $\mathfrak{g}$ and let $B(X, Y) = \mathrm{tr}(XY)$ for $X, Y \in \mathfrak{g}$. Then B is a nondegenerate symmetric form on $\mathfrak{g}$, and $\mathrm{ad} : \mathfrak{g} \longrightarrow \mathfrak{so}(\mathfrak{g}, B)$.

   (a) Set $W = \mathfrak{n} + \mathfrak{u}$, where $\mathfrak{u}$ is a maximal B-isotropic subspace in $\mathfrak{h}$. Show that W is a maximal B-isotropic subspace of $\mathfrak{g}$. Note that the weights of $\mathrm{ad}(\mathfrak{h})$ on W are the positive roots with multiplicity one and 0 with multiplicity $[l/2]$.

   (b) Let π be the spin representation of $\mathfrak{so}(\mathfrak{g}, B)$ if l is odd or either of the half-spin representations of $\mathfrak{so}(\mathfrak{g}, B)$ if l is even. Show that the representation $\pi \circ \mathrm{ad}$ of $\mathfrak{g}$ is $2^{[l/2]}$ copies of the irreducible representation of $\mathfrak{g}$ with highest weight $\rho = \varpi_1 + \cdots + \varpi_l$. (*Hint:* Use (a) and Propositions 18.3 and 18.4 to show that ρ is the only highest weight of $\pi \circ \mathrm{ad}$ and that it occurs with multiplicity $2^{[l/2]}$. Now apply Theorem 13.4.)

Part 6: Representations on Spaces of Regular Functions

Lecture 19. Multiplicity Free Spaces

Isotypic Decomposition of $\mathrm{Aff}(X)$

Let X be an affine algebraic set on which the reductive algebraic group G acts regularly. We denote by ρ_X the associated representation of G on $\mathrm{Aff}(X)$, given by

$$\rho_X(g)f(x) = f(g^{-1}x), \quad \text{for } f \in \mathrm{Aff}(X).$$

This representation is *locally regular:* for any finite-dimensional subspace $U \subset \mathrm{Aff}(X)$, the G-invariant space

$$\mathbb{C}[G]U = \sum_{g \in G} \rho_X(g)U$$

that it generates is finite-dimensional, and the representation of G on $\mathbb{C}[G]U$ is regular.
Let $\widehat{G}$ denote the set of equivalence classes of irreducible regular finite-dimensional representations of G. For $\omega \in \widehat{G}$ let (π_ω, V_ω) be a representation in the class ω. Let (ρ, E) be a locally-regular representation of G, for example the representation $(\rho_X, \mathrm{Aff}(X))$ as above. Denote by $E_{(\omega)}$ the sum of all the G-irreducible subspaces V of E such that $\rho|_V$ is in the class ω.

Proposition 19.1 *One has* $E = \bigoplus_{\omega \in \widehat{G}} E_{(\omega)}$.

Let $\omega \in \widehat{G}$. We can decompose the isotypic subspace $E_{(\omega)}$ as a direct sum of irreducible representations in the class ω (usually in a non-unique way). The number of summands (which can be finite or infinite) is uniquely determined and is called the *multiplicity* of ω in E, denoted as $\mathrm{mult}_\rho(\omega)$. A linear G-intertwining map $T : V_\omega \to E$ is called a *covariant of type* ω for the representation (ρ, E). We denote the space of all covariants of type ω by $\mathrm{Hom}_G(\omega, \rho)$. It is a G-module with trivial action.

Lemma 19.2 *Let $\omega \in \widehat{G}$. The map $T \otimes v \mapsto T(v)$ for $T \in \mathrm{Hom}_G(\omega, \rho)$ and $v \in V_\omega$ gives a G-module isomorphism*

$$\mathrm{Hom}_G(\omega, \rho) \otimes V_\omega \cong E_{(\omega)}. \tag{19.1}$$

In particular,

$$\mathrm{mult}_\rho(\omega) = \dim \mathrm{Hom}_G(\omega, \rho). \tag{19.2}$$

We say that (ρ, E) is *multiplicity-free* if $\mathrm{mult}_\rho(\omega) \le 1$ for all $\omega \in \widehat{G}$. When $(\rho_X, \mathrm{Aff}(X))$ is multiplicity-free, where X is an affine G-space, we also say that X is a *multiplicity-free G-space.*
Now suppose that G is a connected classical group. Fix a Borel subgroup $B = HN$ of G, with H a maximal torus in G and N the unipotent radical of B. Taking $G \subset \mathrm{GL}(n, \mathbb{C})$, we can always conjugate G so that H consists of the diagonal matrices in G and N consists of the upper-triangular unipotent matrices in G. Write $P(G) \subset \mathfrak{h}^*$ for the *weight lattice* of G and $P_{++}(G)$ for the *dominant weights*, relative to the system of positive roots determined by N. For $\lambda \in P(G)$ we denote by $h \mapsto h^\lambda$ the corresponding character of H. We extend this to a character of B by setting $(hn)^\lambda = h^\lambda$ for $h \in H$ and $n \in N$.

Recall from Theorem 13.2 that an irreducible representation (π, V) of G is completely determined (up to equivalence) by its highest weight, relative to the subgroup B. The subspace V^N of N-fixed vectors in V is one-dimensional, and H acts on it by a character $h \mapsto h^\lambda$ where $\lambda \in P_{++}(G)$. For each such λ we fix a model (π^λ, V^λ) for the irreducible representation with highest weight λ, and we fix a non-zero highest weight vector $v_\lambda \in (V^\lambda)^N$. Let $\mathrm{Aff}(X)^N$ be the space of N-fixed regular functions on X. For every regular character $b \mapsto b^\lambda$ of B, let $\mathrm{Aff}(X)^N(\lambda)$ be the N-fixed regular functions f of weight λ:

$$\rho_X(b)f = b^\lambda f \qquad \text{for } b \in B. \tag{19.3}$$

We can then describe the G-isotypic decomposition of $\mathrm{Aff}(X)$ as follows.

Theorem 19.3 *For $\lambda \in P_{++}(G)$, the isotypic subspace of type π^λ in $\mathrm{Aff}(X)$ is the span of $\rho_X(G)\mathrm{Aff}(X)^N(\lambda)$. This subspace is isomorphic to $V^\lambda \otimes \mathrm{Aff}(X)^N(\lambda)$ as a G-module, with action $\pi^\lambda(g) \otimes 1$. Thus*

$$\mathrm{Aff}(X) \cong \bigoplus_{\lambda \in P_{++}(G)} V^\lambda \otimes \mathrm{Aff}(X)^N(\lambda)$$

This theorem shows that the G-multiplicities in $\mathrm{Aff}(X)$ are the dimensions of the spaces $\mathrm{Aff}(X)^N(\lambda)$. We have $\mathrm{Aff}(X)^N(\lambda) \cdot \mathrm{Aff}(X)^N(\mu) \subset \mathrm{Aff}(X)^N(\lambda+\mu)$ under pointwise multiplication. Hence the set

$$\mathcal{S}(X) = \{\lambda \in P_{++}(G) \,:\, \mathrm{Aff}(X)^N(\lambda) \neq 0\} \quad \text{(the \textit{spectrum} of } X\text{)}$$

is an additive semigroup that completely determines the G-isotypic decomposition of $\mathrm{Aff}(X)$.

Multiplicities and B-Orbits

We now obtain a geometric condition for an affine G-space X to be multiplicity free. For a subgroup $M \subset G$ and $x \in X$ we write $M_x = \{m \in M \,:\, m \cdot x = x\}$ for the isotropy group at x. Note that if $\mathfrak{m} = \mathrm{Lie}(M)$, then the Lie algebra of M_x is

$$\mathfrak{m}_x = \{Y \in \mathfrak{m} \,:\, d\rho(Y)_x = 0\}.$$

(Here $d\rho$ denotes the differential of the representation ρ of G on $\mathrm{Aff}(X)$. For $Y \in \mathfrak{g}$ the operator $d\rho(Y)$ is a vector field on X, and $d\rho(Y)_x$ is the corresponding tangent vector at x. When X is a vector space and the G-action is linear, then $d\rho(Y)_x = d\rho(Y)x$.)

Theorem 19.4 *Let X be an irreducible affine G-space. Suppose there is a point $x_0 \in X$ such that $B \cdot x_0$ is open in X (this is equivalent to the condition $\dim \mathfrak{b} = \dim X + \dim \mathfrak{b}_{x_0}$). Then*
(1) X is multiplicity-free as a G-space.
(2) If $\lambda \in \mathcal{S}(X)$ then $h^\lambda = 1$ for all $h \in H_{x_0}$.

B-eigenfunctions for Linear Actions

Let (σ, X) be a regular representation of G. Let $\rho(g)f(x) = f(\sigma(g^{-1})x)$ be the corresponding representation of G on $\mathcal{P}(X)$.

Theorem 19.5 *Assume there is an $x_0 \in X$ with $\sigma(B)x_0$ open in X. Let*

$$H_0 = \{h \in H \,:\, h \cdot x_0 = x_0\}.$$

Let $\mathcal{E}(X)$ be the set of all irreducible polynomials $f \in \mathcal{P}(X)$ such that f is a B-eigenfunction and $f(x_0) = 1$. Then the following holds.
(1) *The set $\mathcal{E}(X) = \{f_1, \ldots, f_k\}$ is finite with $k \le \dim(H/H_0)$, where the polynomial f_i has B-weight λ_i and is homogeneous of degree d_i. Furthermore, the set of weights $\{\lambda_1, \ldots, \lambda_k\}$ is linearly independent over $\mathbb{Q}$ and $h^{\lambda_i} = 1$ for all $h \in H_0$.*
(2) *The B-eigenfunctions $f \in \mathcal{P}(X)$, normalized by $f(x_0) = 1$, are the functions*

$$f_{\mathbf{m}} = \prod_{i=1}^{k} f_i^{m_i} \tag{19.4}$$

with $\mathbf{m} = (m_1, \ldots, m_k) \in \mathbb{N}^k$ arbitrary.
(3) *For $r \ge 0$ the space $\mathcal{P}^r(X)$ of homogeneous polynomials of degree r decomposes under G as*

$$\mathcal{P}^r(X) = \bigoplus_{\lambda} V^{\lambda},$$

where the sum is over all $\lambda = \sum m_i\lambda_i$ with $r = \sum d_i m_i$, and V^{λ} is the irreducible G-module generated by $f_{\mathbf{m}}$.

Corollary 19.6 *The algebra $\mathcal{P}(X)^N \cong \mathbb{C}[f_1, \ldots, f_k]$ is a polynomial ring with generators $\mathcal{E}(X)$.*

Exercises for Lecture 19.

1. Suppose the reductive group G acts linearly on a vector space V. The group $\mathbb{C}^\times$ acts on $\mathcal{P}(V)$ via scalar multiplication on V, and commutes with G. Hence one has a representation of the group $G \times \mathbb{C}^\times$ on $\mathcal{P}(V)$. Prove that the isotypic decomposition of $\mathcal{P}(V)$ under $G \times \mathbb{C}^\times$ is

$$\mathcal{P}(V) = \bigoplus_{k \ge 0} \bigoplus_{\omega \in \widehat{G}} \mathcal{P}^k(V)_{(\omega)}$$

   where $\mathcal{P}^k(V)_{(\omega)}$ is the ω-isotypic component in the homogeneous polynomials of degree k.

2. Let $G = \mathrm{SL}(n, \mathbb{C})$ acting on $X = \mathbb{C}^n$ by the defining representation ($n \ge 2$). Let B be the Borel subgroup of upper-triangular matrices in G and H the subgroup of diagonal matrices in G.

   (a) Let $x_0 = e_n$. Show that Bx_0 is Zariski open in $\mathbb{C}^n$ and find the stabilizer H_{x_0}.

   (b) Let $\lambda \in P_{++}(G)$. Show that $h^\lambda = 1$ for all $h \in H_{x_0}$ if and only if $\lambda = k\varpi_{n-1}$ for some $k \in \mathbb{N}$, where ϖ_{n-1} is the highest weight of the representation of G on $(\mathbb{C}^n)^*$.

   (c) Show that the only irreducible normalized B eigenfunction on $\mathbb{C}^n$ is $f(x) = x_n$ and the G spectrum of X is $\{k\varpi_{n-1} \,:\, k \in \mathbb{N}\}$.

   (d) Show that the space $\mathcal{P}^k(\mathbb{C}^n)$ is an irreducible G module with highest weight $k\varpi_{n-1}$.

3. Let $G = \mathrm{Sp}(\mathbb{C}^{2n}, \Omega)$, where Ω is the bilinear form with matrix $\begin{bmatrix} 0 & s_0 \\ -s_0 & 0 \end{bmatrix}$ (where s_0 has 1 on the antidiagonal, 0 elsewhere). Take as Borel subgroup B the upper-triangular matrices in G with maximal torus H the diagonal matrices in G.

   (a) Show that the action of G on $\mathbb{C}^{2n}$ is multiplicity-free. (*Hint*: Consider the B-orbit of $e_1 + e_{2n}$.)

   (b) Show that there is one irreducible B-eigenfunction. namely x_{2n}. (*Hint*: Calculate the stabilizer of $e_1 + e_{2n}$ in H.)

   (c) Show that for $k \geq 1$ the space $\mathcal{P}^k(\mathbb{C}^{2n})$ is irreducible under G, with highest weight $k\varpi_1$ and highest weight eigenfunction $(x_{2n})^k$.

4. Let $G = \mathrm{SO}(\mathbb{C}^n, \omega)$ with $n \geq 3$, where the symmetric form ω has matrix with 1 on the antidiagonal and 0 elsewhere. Let $Q(x) = \omega(x, x)$ be the G-invariant quadratic form on $\mathbb{C}^n$. Take as Borel subgroup B the upper-triangular matrices in G with maximal torus H the diagonal matrices in G.

   (a) Show that the action of $\mathbb{C}^\times \times G$ on $\mathbb{C}^n$ is multiplicity-free, where $\mathbb{C}^\times$ acts by scalar multiplication. (*Hint*: Consider the $\mathbb{C}^\times \times B$-orbit of $x_0 = e_1 + e_n$ when n is even, or $x_0 = e_1 + e_{l+1} + e_n$ when $n = 2l + 1$ is odd.)

   (b) Show that the irreducible $\mathbb{C}^\times \times B$-eigenfunctions are x_n and Q. (*Hint*: Calculate the stabilizer in $\mathbb{C}^\times \times H$ of the vector x_0 in (a).)

   (c) Show that for $r \geq 1$

$$\mathcal{P}^r(\mathbb{C}^n) = \bigoplus_{k+2m=r} Q^m V^{k\varpi_1} \qquad (k \geq 0,\, m \geq 0),$$

   where $V^{k\varpi_1}$ is the G cyclic subspace generated by $(x_n)^k$ and is an irreducible representation of highest weight $k\varpi_1$.

Lecture 20. Maximal Parabolic Subgroups and Multiplicity Free Spaces

Maximal Parabolic Subalgebras

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$. Fix a Cartan subalgebra $\mathfrak{h}$ and a set Φ^+ of positive roots of $\mathfrak{h}$ on $\mathfrak{g}$. Let Δ be the simple roots in Φ^+. Fix an element $\alpha_0 \in \Delta$ and set $\Delta_0 = \Delta \setminus \{\alpha_0\}$. Then there exists a unique element $H_0 \in \mathfrak{h}$ such that

$$\langle \alpha_0, H_0 \rangle = 1, \qquad \langle \alpha, H_0 \rangle = 0 \quad \text{for all } \alpha \in \Delta_0.$$

Set $\Phi_0 = \{\gamma \in \Phi : \langle \gamma, H_0 \rangle = 0\}$ and $\Psi = \{\beta \in \Phi^+ : \langle H_0, \beta \rangle > 0\}$. Then Φ_0 consists of all roots that do not contain α_0, and Ψ consists of all positive roots that contain α_0. Define $\mathfrak{h}_0 = \mathrm{Span}\{h_\alpha : \alpha \in \Delta_0\}$, $\mathfrak{a} = \mathbb{C}H_0$, and

$$\mathfrak{m} = \mathfrak{h}_0 + \sum_{\gamma \in \Phi_0} \mathfrak{g}_\gamma\,, \quad \mathfrak{p}_+ = \sum_{\beta \in \Psi} \mathfrak{g}_\beta, \quad \mathfrak{p}_- = \sum_{\beta \in \Psi} \mathfrak{g}_{-\beta}.$$

Then $\mathfrak{g} = \mathfrak{p}_+ + \mathfrak{m} + \mathfrak{a} + \mathfrak{p}_-$ (direct sum of vector spaces) and $\mathfrak{m} + \mathfrak{a} + \mathfrak{p}_+$ is the maximal parabolic subalgebra associated with the subset $\{\alpha_0\}$ of Δ.

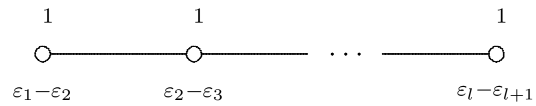
Proposition 20.1
(1) *The subalgebra $\mathfrak{m} + \mathfrak{a}$ is reductive with center $\mathfrak{a}$ and semisimple derived algebra $\mathfrak{m}$. The Dynkin diagram for $\mathfrak{m}$ is obtained by removing the vertex for α_0 from the diagram for $\mathfrak{g}$.*
(2) *The subalgebras $\mathfrak{p}_+$ and $\mathfrak{p}_-$ are nilpotent, and $\mathfrak{m}+\mathfrak{a}$ normalizes $\mathfrak{p}_\pm$. Also $\mathfrak{p}_- \cong (\mathfrak{p}_+)^*$ as a module for $\mathfrak{m} + \mathfrak{a}$.*
(3) *Let $\widetilde{\alpha}$ be the highest positive root. Suppose $\alpha_0 \in \Delta$ appears in $\widetilde{\alpha}$ with coefficient 1. Then $[\mathfrak{p}_+, \mathfrak{p}_+] = 0$ and $\mathfrak{p}_+$ is the irreducible $\mathfrak{m}$ module with highest weight $\widetilde{\alpha}|_{\mathfrak{h}_0}$. Furthermore, $\mathrm{ad}H_0$ has eigenvalues ± 1, with eigenspaces $\mathfrak{p}_\pm$.*

Proof.
(1): $\mathfrak{h}_0$ is a Cartan subalgebra of $\mathfrak{m}$. The root system of $\mathfrak{m}$ is the restrictions of Φ_0 to $\mathfrak{h}_0$.
(2): This is clear from the root space decomposition; the Killing form gives the duality between $\mathfrak{p}_+$ and $\mathfrak{p}_-$.
(3): If $\beta \in \Psi$, then $\beta = c_0\alpha_0 + \cdots$ with $c_0 \geq 1$. But $\beta \leq \widetilde{\alpha}$ (in the partial order defined by the positive roots). Hence $c_0 = 1$. This shows that $\mathrm{ad}(H_0) = 1$ on $\mathfrak{p}_+$. Also, if $\beta, \gamma \in \Psi$ then $\beta + \gamma = 2\alpha_0 + \cdots$, so $\beta + \gamma \notin \Phi$. Thus $[\mathfrak{p}_+, \mathfrak{p}_+] = 0$.
To prove irreducibility of $\mathfrak{p}_+$, suppose $0 \neq V \subset \mathfrak{p}_+$ is invariant under $\mathrm{ad}\,\mathfrak{m}$. Then $V^* \subset \mathfrak{p}_-$ is also $\mathfrak{m}$ invariant. Since $[\mathfrak{p}_+, \mathfrak{p}_-]$ is contained in the zero eigenspace of J, which is $\mathfrak{a} + \mathfrak{m}$, it follows that $V^* + \mathfrak{a} + \mathfrak{m} + V$ is an ideal in $\mathfrak{g}$. But $\mathfrak{g}$ is simple, so $V = \mathfrak{p}_+$. The $\widetilde{\alpha}$ root space is in $\mathfrak{p}_+$. Since it is annihilated by $\mathrm{ad}\,\mathfrak{g}_\beta$ for all $\beta \in \Phi^+$, it is the highest weight space for $\mathfrak{p}_+$ as an $\mathfrak{m}$ module. $\square$

Classical Examples

For each of the four types of classical simple Lie algebras we give the Dynkin diagram with the coefficients of $\widetilde{\alpha}$ written above each vertex. We determine $\mathfrak{m}$ and $\mathfrak{p}_+$ for all the maximal parabolic subalgebras defined by simple roots α_0 having coefficient 1 in $\widetilde{\alpha}$.

Type A_l ($\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$, with $n = l + 1 \geq 2$): The Dynkin diagram is



We may take $\alpha_0 = \varepsilon_p - \varepsilon_{p+1}$ for any $1 \leq p \leq l$. Then

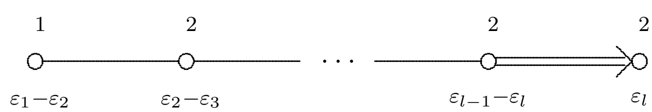
$$H_0 = \begin{bmatrix} \frac{q}{n} I_p & 0 \\ 0 & -\frac{p}{n} I_q \end{bmatrix}, \quad \text{where } p + q = n$$

(here I_p is the $p \times p$ identity matrix). Removing α_0 from the Dynkin diagram, we obtain the diagram for $\mathfrak{m} = \mathfrak{sl}_p \oplus \mathfrak{sl}_q$. In matrix form, $\mathfrak{m}$ is block diagonal, corresponding to H_0.
We have $\Psi = \{\varepsilon_i - \varepsilon_{p+j} \,:\, 1 \leq i \leq p \text{ and } 1 \leq j \leq q\}$. The Cartan subalgebra of $\mathfrak{m}$ is $\mathfrak{h}_0 \cong \mathfrak{h}_p \oplus \mathfrak{h}_q$, where $\mathfrak{h}_p$ consists of diagonal matrices in $\mathfrak{sl}_p$. The root $\varepsilon_i - \varepsilon_{p+j}$ restricts to ε_i on $\mathfrak{h}_p$ and to $-\varepsilon_j$ on $\mathfrak{h}_q$. In this case $\widetilde{\alpha} = \varepsilon_1 - \varepsilon_n$ and $\widetilde{\alpha}|_{\mathfrak{h}_0} = \varpi_1 \oplus \varpi_{q-1}$ (the first fundamental weight of $\mathfrak{sl}_p$ and the last fundamental weight of $\mathfrak{sl}_q$). Thus

$$\mathfrak{p}_+ \cong \mathbb{C}^p \otimes (\mathbb{C}^q)^* \cong M_{p \times q}$$

as an $\mathfrak{m}$ module (left multiplication by $\mathfrak{sl}_p$ and right multiplication by $\mathfrak{sl}_q$).

Type B_l ($\mathfrak{g} = \mathfrak{so}(\mathbb{C}^n, B)$, with $n = 2l + 1 \geq 7$): We take the bilinear form B to have antidiagonal 1, as usual, and $\mathfrak{h}$ the diagonal matrices in $\mathfrak{g}$. The Dynkin diagram is



The only choice for α_0 is $\varepsilon_1 - \varepsilon_2$. Then

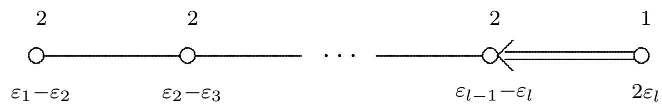
$$H_0 = \mathrm{diag}[1, 0, \ldots, 0, -1].$$

Removing α_0 from the Dynkin diagram, we obtain the diagram for $\mathfrak{m} = \mathfrak{so}_{n-2}$. We have $\Psi = \{\varepsilon_1\} \cup \{\varepsilon_1 - \varepsilon_j \,:\, 2 \leq j \leq l\}$. The Cartan subalgebra of $\mathfrak{m}$ is

$$\mathfrak{h}_0 = \{\mathrm{diag}[0, x_2, \ldots, x_l, 0, -x_l, \ldots, -x_2, 0]\},$$

so ε_1 restricts to 0 on $\mathfrak{h}_0$. Thus $\mathfrak{h}_0$ has weights $0, \pm\varepsilon_j$ (with $j = 2, \ldots, l$) on $\mathfrak{p}_+$, each with multiplicity one. In this case $\widetilde{\alpha} = \varepsilon_1 + \varepsilon_2$ and $\widetilde{\alpha}|_{\mathfrak{h}_0} = \varpi_1$, the first fundamental weight of $\mathfrak{m}$. Hence $\mathfrak{p}_+ \cong \mathbb{C}^{n-2}$ is the defining representation for $\mathfrak{so}_{n-2}$.

Type C_l ($\mathfrak{g} = \mathfrak{sp}(\mathbb{C}^n, \Omega)$, with $n = 2l \geq 4$): We take the bilinear form Ω to have matrix $\begin{bmatrix} 0 & s_0 \\ -s_0 & 0 \end{bmatrix}$, where s_0 has 1 on the antidiagonal. We take $\mathfrak{h}$ as the diagonal matrices in $\mathfrak{g}$. The Dynkin diagram is



The only choice for α_0 is $2\varepsilon_l$. Then

$$H_0 = \begin{bmatrix} \frac{1}{2}I & 0 \\ 0 & -\frac{1}{2}I \end{bmatrix}$$

Removing α_0 from the Dynkin diagram, we obtain the diagram for $\mathfrak{m} \cong \mathfrak{sl}(l, \mathbb{C})$. In matrix form, $\mathfrak{m}$ consists of the block diagonal matrices

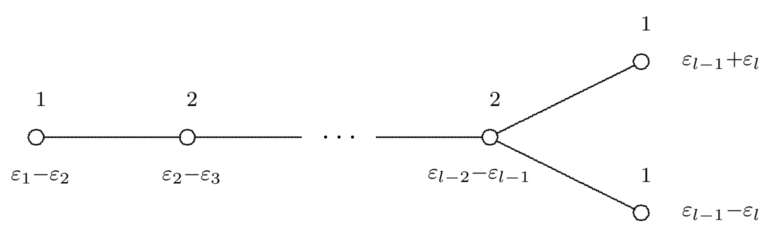
$$X = \begin{bmatrix} A & 0 \\ 0 & -s_0 A^t s_0 \end{bmatrix}, \quad A \in \mathfrak{sl}(l, \mathbb{C}).$$

We have $\Psi = \{\varepsilon_i + \varepsilon_j \,:\, 1 \le i \le j \le l\}$. The Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{m}$ consists of all X with A diagonal. In this case $\widetilde{\alpha} = 2\varepsilon_1$ and $\widetilde{\alpha}|_{\mathfrak{h}_0} = 2\varpi_1$, where ϖ_1 is the first fundamental weight of $\mathfrak{m}$. Hence $\mathfrak{p}_+ \cong SM_l(\mathbb{C})$ (the $l \times l$ symmetric matrices) as an $\mathfrak{m}$ module. In matrix form, $\mathfrak{p}_+$ consists of all matrices

$$\begin{bmatrix} 0 & s_0 Z s_0 \\ 0 & 0 \end{bmatrix}, \quad Z \in SM_l(\mathbb{C})$$

and the action of $\mathfrak{m}$ on $\mathfrak{p}_+$ is by $Z \mapsto AZ + ZA^t$, for $A \in \mathfrak{sl}(l, \mathbb{C})$.

Type D_l ($\mathfrak{g} = \mathfrak{so}(\mathbb{C}^n, B)$, with $n = 2l \ge 8$): We take the bilinear form B to have matrix $\begin{bmatrix} 0 & s_0 \\ s_0 & 0 \end{bmatrix}$. We take $\mathfrak{h}$ as the diagonal matrices in $\mathfrak{g}$. The Dynkin diagram is



There are three choices for α_0. Consider first the case $\alpha_0 = \varepsilon_{l-1} + \varepsilon_l$. Then, just as for type C_l,

$$H_0 = \begin{bmatrix} \frac{1}{2}I & 0 \\ 0 & -\frac{1}{2}I \end{bmatrix}$$

Removing α_0 from the Dynkin diagram, we obtain the diagram for $\mathfrak{m} \cong \mathfrak{sl}(l, \mathbb{C})$. As in the type C_l case, $\mathfrak{m}$ consists of the block diagonal matrices

$$X = \begin{bmatrix} A & 0 \\ 0 & -s_0 A^t s_0 \end{bmatrix}, \quad A \in \mathfrak{sl}(l, \mathbb{C}).$$

However, now we have $\Psi = \{\varepsilon_i + \varepsilon_j \,:\, 1 \le i < j \le l\}$, since $2\varepsilon_i$ is not a root. The Cartan subalgebra $\mathfrak{h}_0$ consists of all X as above with A diagonal. In this case $\widetilde{\alpha} = \varepsilon_1 + \varepsilon_2$ and so $\widetilde{\alpha}|_{\mathfrak{h}_0} = \varpi_2$, the second fundamental weight of $\mathfrak{m}$. Hence $\mathfrak{p}_+ \cong AM_l(\mathbb{C})$ (the $l \times l$ skew-symmetric matrices) as an $\mathfrak{m}$ module. In matrix form, $\mathfrak{p}_+$ consists of all

$$\begin{bmatrix} 0 & s_0 Z s_0 \\ 0 & 0 \end{bmatrix}, \quad Z \in AM_l(\mathbb{C}).$$

The action of $\mathfrak{m}$ on $\mathfrak{p}_+$ is by $Z \mapsto AZ + ZA^t$, for $A \in \mathfrak{sl}(l, \mathbb{C})$.
The choice $\alpha_0 = \varepsilon_{l-1} - \varepsilon_l$ gives a pair $(\mathfrak{m}, \mathfrak{p}_+)$ isomorphic to $(\mathfrak{sl}(l, \mathbb{C}), AM_l(\mathbb{C}))$, since there is an outer automorphism of $\mathfrak{g}$ that interchanges ε_l and $-\varepsilon_l$.
Finally, consider the choice $\alpha_0 = \varepsilon_1 - \varepsilon_2$. Then

$$H_0 = \mathrm{diag}[1, 0, \ldots, 0, -1],$$

just as for Type B. Removing α_0 from the Dynkin diagram, we obtain the diagram for $\mathfrak{m} = \mathfrak{so}_{n-2}$. We have $\Psi = \{\varepsilon_1 \pm \varepsilon_j \,:\, 2 \le j \le l\}$. The Cartan subalgebra

$$\mathfrak{h}_0 = \{\mathrm{diag}[0, x_2, \ldots, x_l, -x_l, \ldots, -x_2, 0]\},$$

so $\varepsilon_1 = 0$ on $\mathfrak{h}_0$. Thus $\mathfrak{h}_0$ has weights $\pm\varepsilon_j$ (with $j = 2, \ldots, l$) on $\mathfrak{p}_+$, each with multiplicity one. In this case $\widetilde{\alpha}|_{\mathfrak{h}_0} = \varpi_1$, the first fundamental weight of $\mathfrak{m}$. Hence $\mathfrak{p}_+ \cong \mathbb{C}^{n-2}$ is the defining representation for $\mathfrak{so}_{n-2}$, as for Type B.

Remarks. Among the five exceptional simple Lie algebras, only E_6 and E_7 have simple roots with coefficient 1 in $\widetilde{\alpha}$. For E_6 there are two such roots, which are interchanged by an outer automorphism (just as for D_l). Thus there is one pair $(\mathfrak{m}, \mathfrak{p}_+)$ associated with E_6, up to isomorphism. Here $\mathfrak{m} = \mathfrak{so}_{10}$. For E_7 there is a unique simple root with coefficient 1 in $\widetilde{\alpha}$. In this case $\mathfrak{m}$ is of type E_6.

Multiplicity Free Spaces from Hermitian Symmetric Spaces

Let $\mathfrak{g} = \mathfrak{p}_- + \mathfrak{a} + \mathfrak{m} + \mathfrak{p}_+$ as in Proposition 20.1. We assume that the simple root α_0 occurs with coefficient 1 in the highest root. Let G be the adjoint group of $\mathfrak{g}$, and let $K \subset G$ be a connected subgroup with Lie algebra $\mathfrak{m} + \mathfrak{a}$. Then $\mathfrak{p}_+$ is a K module.

Theorem 20.2 *The space $\mathfrak{p}_+$ is multiplicity free for K.*

This result has many important applications to geometry, function theory, and representation theory for the following reason. Set $\mathfrak{k} = \mathfrak{m} + \mathfrak{a}$ and $\mathfrak{p} = \mathfrak{p}_+ + \mathfrak{p}_-$. Then $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is the complexified Cartan decomposition associated with a Hermitian symmetric space $X = G_0/K_0$ of noncompact type. Here K_0 is the compact real form of K and G_0 is a noncompact real form of G. The space X can be holomorphically embedded in the complex vector space $\mathfrak{p}_+$ as a bounded, convex open set (the Harish-Chandra embedding), with the action of K_0 on X becoming the linear action of $\mathrm{Ad}(K_0)$ on $\mathfrak{p}_+$.
Theorem 20.2 was first obtained by L.K. Hua when X is a classical bounded domain (Cartan domain) by elaborate calculations involving integration on compact groups. It was proved in general by W. Schmid by a lengthy root system argument. A much simpler proof was later given by K. Johnson, using a mixture of general invariant theory results and case-by-case arguments. In our treatment we use the geometric criterion (Theorem 19.4) for multiplicity free actions together with Theorem 19.5 to obtain a basis of highest weight vectors. We give full details for three of the four types of classical domains. The remaining case ($\mathfrak{m} = \mathfrak{so}_{n-2}$) we leave as an exercise.

Decomposition of $\mathcal{P}(M_{p\times q})$ under $\mathrm{GL}_p \times \mathrm{GL}_q$

Let $G = \mathrm{GL}(p,\mathbb{C})\times\mathrm{GL}(q,\mathbb{C})$ and let $M_{p\times q}$ be the $p\times q$ complex matrices. Let ρ be the representation of G on $\mathcal{P}(M_{p\times q})$ given by

$$\rho(y,z)f(x) = f(y^{-1}xz) \quad \text{for } f \in \mathcal{P}(M_{p\times q}),\ (y,z)\in G$$

In $\mathrm{GL}(n,\mathbb{C})$ we have the subgroups D_n of invertible diagonal matrices, N_n of upper-triangular unipotent matrices, $\bar{N}_n$ of lower-triangular unipotent matrices. We set $B_n = D_nN_n$ and $\bar{B}_n = D_n\bar{N}_n$. We extend a regular character χ of D_n to a character of B_n (*resp.* $\bar{B}_n$) by $\chi(hu) = \chi(vh) = \chi(h)$ for $h \in D_n$, $u \in N_n$ and $v \in \bar{N}_n$. A weight $\mu = \sum_{i=1}^n \mu_i\varepsilon_i$ of D_n is called *nonnegative* if $\mu_i \geq 0$ for all i. The weight μ is *dominant* if $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$.
When μ is dominant, we denote by (π_n^μ, F_n^μ) the irreducible representation of $\mathrm{GL}(n,\mathbb{C})$ with highest weight μ. If μ is dominant and nonnegative, we set

$$|\mu| = \sum \mu_i \qquad \text{(the } \textit{size} \text{ of } \mu\text{)}.$$

In this case it is convenient to extend μ to a dominant weight of D_l for all $l > n$ by setting $\mu_i = 0$ for all integers $i > n$. We define

$$\mathrm{depth}(\mu) = \min\{k : \mu_{k+1} = 0\}.$$

Thus we may view μ as a dominant integral weight of $\mathrm{GL}(l,\mathbb{C})$ for any $l \geq \mathrm{depth}(\mu)$. If μ is a nonnegative dominant weight of depth k, then

$$\mu = m_1\lambda_1 + \cdots + m_k\lambda_k$$

with $\lambda_i = \varepsilon_1 + \cdots + \varepsilon_i$ and $m_1, \ldots, m_k$ strictly positive integers.
The irreducible finite-dimensional regular representations of $G = \mathrm{GL}(p,\mathbb{C})\times\mathrm{GL}(q,\mathbb{C})$ are all given as outer tensor products $(\pi_p^\mu\widehat{\otimes}\pi_q^\nu,\ F_p^\mu\otimes F_q^\nu)$. For $i = 1,\ldots,\min\{p,q\}$ we denote by Δ_i the ith *principal minor* on $M_{p,q}$. We denote by $\mathcal{P}(M_{p,q})^{\bar{N}_p\times N_q}$ the subspace of polynomials on $M_{p,q}$ that are fixed by left translations by $\bar{N}_p$ and right translations by N_q.

Theorem 20.3 *The space of homogeneous polynomials on $M_{p\times q}$ of degree d decomposes under the representation ρ of $\mathrm{GL}(p,\mathbb{C})\times\mathrm{GL}(q,\mathbb{C})$ as a multiplicity-free sum*

$$\mathcal{P}^d(M_{p\times q}) \cong \bigoplus (F_p^\nu)^*\otimes F_q^\nu \tag{20.1}$$

with the sum over all nonnegative dominant weights ν of size d and depth$(\nu) \leq r$, where $r = \min\{p,q\}$. Furthermore,

$$\mathcal{P}(M_{p\times q})^{\bar{N}_p\times N_q} = \mathbb{C}[\Delta_1,\ldots,\Delta_r] \tag{20.2}$$

is a polynomial ring on r algebraically independent generators.

Decomposition of $S(S^2(V))$ under $\mathrm{GL}(V)$

Let $G = \mathrm{GL}(n,\mathbb{C})$ and let SM_n be the space of symmetric $n\times n$ complex matrices. We let G act on SM_n by $g, x \mapsto (g^t)^{-1}xg^{-1}$. Let ρ be the associated representation of G on $\mathcal{P}(SM_n)$:

$$\rho(g)f(x) = f(g^txg) \quad \text{for } f \in \mathcal{P}(SM_n).$$

Note that $SM_n \cong S^2(\mathbb{C}^n)^*$ (the symmetric bilinear forms on $\mathbb{C}^n$) as a G-module relative to this action, where a matrix $x \in SM_n$ corresponds to the symmetric bilinear form

$$\beta_x(u,v) = u^t x v \quad \text{for } u, v \in \mathbb{C}^n.$$

Thus

$$\mathcal{P}(SM_n) \cong \mathcal{P}(S^2(\mathbb{C}^n)^*) \cong S(S^2(\mathbb{C}^n))$$

as a G-module.

Theorem 20.4 *The space of homogeneous polynomials on SM_n of degree r decomposes under* $\mathrm{GL}(n,\mathbb{C})$ *in a multiplicity-free sum*

$$\mathcal{P}^r(SM_n) \cong \bigoplus F_n^\mu \tag{20.3}$$

with the sum over all nonnegative dominant weights $\mu = \sum_i \mu_i \varepsilon_i$ of size r such that $\mu_i \in 2\mathbb{N}$ for all i. Furthermore,

$$\mathcal{P}(SM_n)^{N_n} = \mathbb{C}[\widetilde{\Delta}_1, \ldots, \widetilde{\Delta}_n], \tag{20.4}$$

where $\widetilde{\Delta}_i$ denotes the restriction of the ith *principal minor to the space of symmetric matrices. The functions $\widetilde{\Delta}_1, \ldots, \widetilde{\Delta}_n$ are algebraically independent.*

Decomposition of $S(\bigwedge^2(V))$ under $\mathrm{GL}(V)$

Let $G = \mathrm{GL}(n,\mathbb{C})$ and let AM_n be the space of skew-symmetric $n \times n$ matrices. Let G act on AM_n by $g, x \mapsto (g^t)^{-1} x g^{-1}$ and let

$$\rho(g)f(x) = f(g^t x g)$$

be the associated representation of G on $\mathcal{P}(AM_n)$. Note that $AM_n \cong \bigwedge^2((\mathbb{C}^n)^*)$ (the skew-symmetric bilinear forms on $\mathbb{C}^n$) as a G-module relative to this action, just as in the case of symmetric matrices and symmetric bilinear forms. Thus we have

$$\mathcal{P}(AM_n) \cong \mathcal{P}(\textstyle\bigwedge^2(\mathbb{C}^n)^*) \cong S(\textstyle\bigwedge^2 \mathbb{C}^n)$$

as a G-module. Let Pf_i be the ith principal Pfaffian on AM_n for $i = 1, \ldots, k$, where $k = [n/2]$.

Theorem 20.5 *The space of homogeneous polynomials on AM_n of degree r decomposes under* $\mathrm{GL}(n,\mathbb{C})$ *as a multiplicity-free sum*

$$\mathcal{P}^r(AM_n) \cong \bigoplus F_n^\mu$$

with the sum over all nonnegative dominant integral weights $\mu = \sum \mu_i \varepsilon_i$ such that $|\mu| = r$ and

$$\mu_{2i-1} = \mu_{2i} \quad \textit{for } i = 1, \ldots, k \quad \textit{and } \mu_{2k+1} = 0 \tag{20.5}$$

(the last equation only applies if n is odd). Furthermore,

$$\mathcal{P}(AM_n)^{N_n} = \mathbb{C}[\mathrm{Pf}_1, \ldots, \mathrm{Pf}_k]$$

and the functions $\mathrm{Pf}_1, \ldots, \mathrm{Pf}_k$ *are algebraically independent.*

Appendix: Linear and Associative Algebra for Lecture 20.

Gauss Decomposition

Let M_k be the space of $k \times k$ complex matrices, and $M_{k,n}$ the space of $k \times n$ complex matrices. Let N_n denote the group of upper triangular matrices $n \times n$ matrices with diagonal entries 1, $\bar{N}_k$ the group of lower triangular $k \times k$ matrices with diagonal entries 1, and $D_{k,n}$ the $k \times n$ matrices $x = [x_{ij}]$ with $x_{ij} = 0$ for $i \neq j$.
For $x \in M_{k,n}$ define the *principal minors*

$$\Delta_i(x) = \det \begin{pmatrix} x_{11} & \cdots & x_{1i} \\ \vdots & \ddots & \vdots \\ x_{i1} & \cdots & x_{ii} \end{pmatrix}$$

for $i = 1, \ldots, \min\{k, n\}$. It is also convenient to define $\Delta_0(x) = 1$.

Lemma 20.6 *Suppose $x \in M_{k,n}$ satisfies*

$$\Delta_i(x) \neq 0 \quad \text{for } i = 1, \ldots, \min\{k, n\}.$$

Then there are matrices $\bar{u} \in \bar{N}_k$, $u \in N_n$ and $h \in D_{k,n}$ such that

$$x = \bar{u} h u. \tag{20.6}$$

The matrix h is uniquely determined by x and its nonzero entries are $h_{ii} = \Delta_i(x)/\Delta_{i-1}(x)$. If $k = n$ then the matrices $\bar{u}$ and u are also uniquely determined by x.

Factorization of Symmetric Matrices

Lemma 20.7 *Suppose $x \in M_n$ is a symmetric matrix and $\Delta_i(x) \neq 0$ for $i = 1, \ldots, n$. Then there exists an upper-triangular matrix $b \in M_n$ such that $x = b^t b$. The matrix b is uniquely determined by x up to left multiplication by a diagonal matrix with entries ± 1.*

Factorization of Skew-symmetric Matrices

Let $A = [a_{ij}]$ be a skew-symmetric $2n \times 2n$ matrix. Given $2n$ vectors $x_1, \ldots, x_{2n} \in \mathbb{C}^{2n}$, define

$$F_A(x_1, \ldots, x_{2n}) = \frac{1}{n! 2^n} \sum_{s \in \mathfrak{S}_{2n}} \operatorname{sgn}(s) \prod_{i=1}^{n} (x_{s(2i-1)}, A x_{s(2i)}),$$

where $(x, Ay) = x^t A y$ is the skew-symmetric bilinear form associated to A. Then F_A is a skew-symmetric multilinear function of $x_1, \ldots, x_{2n}$. Hence there is a complex number $\mathrm{Pfaff}(A)$ (called the *Pfaffian* of A) such that

$$F_A(x_1, \ldots, x_{2n}) = \mathrm{Pfaff}(A) \det[x_1, \ldots, x_{2n}]. \tag{20.7}$$

In particular, taking $x_i = e_i$, the standard basis for $\mathbb{C}^{2n}$, we have

$$\mathrm{Pfaff}(A) = \frac{1}{n! 2^n} \sum_{s \in \mathfrak{S}_{2n}} \operatorname{sgn}(s) \prod_{i=1}^{n} a_{s(2i-1), s(2i)}, \tag{20.8}$$

since $\det[e_1, \ldots, e_{2n}] = 1$.
Let $g \in \mathrm{GL}(2n, \mathbb{C})$. Then

$$\mathrm{Pfaff}(g^t A g) = \det g \mathrm{Pfaff}(A). \tag{20.9}$$

Let A and B be a skew-symmetric matrices of sizes $2k \times 2k$ and $2n \times 2n$, respectively. Then

$$\mathrm{Pfaff}(\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}) = \mathrm{Pfaff}(A)\mathrm{Pfaff}(B). \tag{20.10}$$

Let $A = [a_{ij}]$ be skew-symmetric $n \times n$ matrix. For $k = 1, \ldots, [n/2]$ define the truncated matrix $A_{(k)}$ to be the $2k \times 2k$ matrix $[a_{ij}]_{1 \le i,j \le 2k}$. Set

$$\mathrm{Pf}_k(A) = \mathrm{Pfaff}(A_{(k)}). \tag{20.11}$$

Then Pf_k is a homogeneous polynomial of degree k in the variables a_{ij}, for $1 \le i < j \le 2k$, that we will call the kth *principal Pfaffian* of A.
Let $B_n \subset \mathrm{GL}(n, \mathbb{C})$ be the subgroup of upper-triangular matrices $b = [b_{ij}]$ (so $b_{ij} = 0$ for $i > j$). For $b \in B_n$ and A any $n \times n$ matrix, one has

$$(b^t A b)_{(k)} = b^t_{(k)} A_{(k)} b_{(k)},$$

where $b_{(k)} = [b_{ij}]_{1 \le i,j \le 2k}$. Hence if A is skew-symmetric, (20.9) gives

$$\mathrm{Pf}_k(b^t A b) = \Delta_{2k}(b)\mathrm{Pf}_k(A), \tag{20.12}$$

where $\Delta_{2k}(b) = \det(b_{(k)})$ is the principal minor of b of order $2k$.
We have the following analog of Lemma 20.7 for skew-symmetric matrices. For $n = 2k$ even, define the $n \times n$ skew-symmetric matrix $J_n = J \oplus \cdots \oplus J$ (k summands), where

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

For $n = 2k + 1$ odd define the $n \times n$ skew-symmetric matrix $J_n = J \oplus \cdots \oplus J \oplus 0$ (k copies of J).

Lemma 20.8 *Let A be a skew-symmetric $n \times n$ matrix. Assume that $\mathrm{Pf}_k(A) \neq 0$ for $k = 1, \ldots, [n/2]$. Then there exists $b \in B_n$ so that $A = b^t J_n b$.*

Corollary 20.9 *Let A be a skew-symmetric $2n \times 2n$ matrix. Then*

$$(\mathrm{Pfaff}(A))^2 = \det A.$$

Exercises for Lecture 20.

1. Let $\mathfrak{g}$ be a semisimple Lie algebra with Cartan subalgebra $\mathfrak{h}$, root system Φ. Fix positive roots Φ^+. Let $\Delta \subset \Phi^+$ be the simple roots, and for $\alpha \in \Delta$ let $h_\alpha \in \mathfrak{h}$ be the coroot to α. Fix $\lambda \in P_{++}(\mathfrak{g})$ and define $\Phi_0 = \{\alpha \in \Phi : \langle \lambda, h_\alpha \rangle = 0\}$ and $S = \Phi_0 \cap \Delta$.

   (a) Write $\lambda = n_1 \varpi_1 + \cdots + n_l \varpi_l$, where ϖ_i is the ith fundamental weight and $n_i \in \mathbb{N}$. Show that $S = \{\alpha_i : n_i = 0\}$.

(b) Set $\Psi = \{\alpha \in \Phi^+ : \langle \lambda, h_\alpha \rangle > 0 \text{ for all } \alpha \in S_\lambda\}$, $\mathfrak{h}_0 = \mathrm{Span}\{h_\alpha : \alpha \in \Phi_0\}$, and $\mathfrak{a} = \{h \in \mathfrak{h} : \langle \alpha, h \rangle = 0 \text{ for all } \alpha \in S\}$. Let

$$\mathfrak{m} = \mathfrak{h}_0 + \sum_{\alpha \in \Phi_0} \mathfrak{g}_\alpha, \quad \mathfrak{u} = \sum_{\beta \in \Psi} \mathfrak{g}_\beta, \quad \bar{\mathfrak{u}} = \sum_{\beta \in \Psi} \mathfrak{g}_{-\beta}.$$

Thus $\mathfrak{g} = \bar{\mathfrak{u}} + \mathfrak{m} + \mathfrak{a} + \mathfrak{u}$. Show that $\mathfrak{m} + \mathfrak{a}$ normalizes $\mathfrak{u}$ and $\bar{\mathfrak{u}}$, that $\mathfrak{a}$ is the center of $\mathfrak{m} + \mathfrak{a}$, and that $\mathfrak{m}$ is a semisimple Lie algebra with Dynkin diagram corresponding to S. Thus $\mathfrak{p}_\lambda = \mathfrak{m} + \mathfrak{a} + \mathfrak{u}$ is the parabolic subalgebra of $\mathfrak{g}$ corresponding to the subset $\Delta \setminus S$ of simple roots. In particular, $\mathfrak{p}_{\varpi_i} = \mathfrak{m} + \mathfrak{a} + \mathfrak{u}$ is the maximal parabolic subalgebra corresponding to $\{\alpha_i\}$.

2. (*Same notation as previous exercise*). Suppose V^λ is the irreducible $\mathfrak{g}$ module with highest weight λ. Let v_λ be a highest weight vector in V^λ.

   (a) Prove that $\mathfrak{p}_\lambda$ is the stabilizer of the line $[v_\lambda]$ in $\mathbb{P}(V^\lambda)$. (*Hint:* First check that $\mathfrak{p}_\lambda$ stabilizes $[v_\lambda]$. Then use the representation theory of $\mathfrak{sl}_2$ to show that $\mathfrak{g}_{-\beta} v_\lambda \neq 0$ if $\beta \in \Psi$, and hence $\mathfrak{p}_\lambda$ is the full stabilizer of $[v_\lambda]$.)

   (b) Let G be a connected group with Lie algebra $\mathfrak{g}$ and Borel subgroup B corresponding to the choice Φ^+ of positive roots. Assume that $\lambda \in P_{++}(G)$ so that V^λ is a G module. Let $P \subset G$ be the stabilizer of $[v_\lambda]$ in $\mathbb{P}(V^\lambda)$. Prove that $\mathrm{Lie}(P) = \mathfrak{p}_\lambda$ and that the G orbit of $[v_\lambda]$ is closed in $\mathbb{P}(V^\lambda)$. (*Hint: P contains B, so G/P is a projective variety.*)

   (c) Let X be the Zariski-closure of the orbit $G \cdot v_\lambda$. Then X is a G-invariant affine variety in V^λ, called a *highest vector* variety. Show that $X = G \cdot v_\lambda \cup \{0\}$ and that X is invariant under multiplication by $\mathbb{C}^\times$. (*Hint*: Use (b) to show that X is the cone over a closed subset of $\mathbb{P}(V^\lambda)$.)

3. Let G, V^λ and X be as in the previous exercise.

   (a) Show that X is a multiplicity-free G-space. (*Hint*: Let $\bar{B} = H\bar{N}$ be the Borel subgroup opposite to B. Show that $\bar{B}$ has an open orbit on X.)

   (b) Let $\mathrm{Aff}(X)^{(n)}$ be the restrictions to X of the homogeneous polynomials of degree n on V^λ. Show that the isotypic decomposition of $\mathrm{Aff}(X)$ as a G module is

$$\mathrm{Aff}(X) = \bigoplus_{n \in \mathbb{N}} \mathrm{Aff}(X)^{(n)}$$

   and $\mathrm{Aff}(X_\lambda)^{(n)}$ is an irreducible G-module isomorphic to $(V^{n\lambda})^*$. (*Hint*: Let $f_\lambda(x) = \langle v_\lambda^*, x \rangle$ for $x \in X_\lambda$, where v_λ^* is the lowest weight vector in $(V^\lambda)^*$. Show that f_λ^n is a $\bar{B}$ eigenfunction of weight $-n\lambda$ for the representation ρ_X, and hence $(V^{n\lambda})^* \subseteq \mathrm{Aff}(X)^{(n)}$ for all positive integers n. Now use Theorem 19.4 to show that if μ occurs as a $\bar{B}$-extreme weight in $\mathrm{Aff}(X_\lambda)$, then μ is proportional to λ.)

4. Let $G = \mathrm{SO}(\mathbb{C}^n, \omega)$, $n \geq 3$ (take the matrix for ω with 1 on antidiagonal, 0 elsewhere). Let $X = \{x \in \mathbb{C}^n : \omega(x, x) = 0\}$ be the set of ω-isotropic vectors (the *nullcone*).

   (a) Show that X is the Zariski closure of the orbit $G \cdot e_1$.

   (b) Show that X is multiplicity free as a G space. (*Hint:* The vector e_1 is the highest weight vector for G.)

   (c) Find the decomposition of $\mathrm{Aff}(X)$ as a G-module. (*Hint:* Use the previous exercise.)