The Representation Theory of Riemannian Curvature Tensors

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Outline

H. Weyl: Reine Infinitesimalgeometrie (1918)
Das gruppentheoretische Fundament der Tensorrechnung (1924)
R. S. Kulkani: On the Bianchi Identities (1972)
A. Besse: Einstein Manifolds (1987)
R. S. Strichartz: Linear Algebra of Curvature Tensors and their Covariant Derivatives (1988)

Riemannian Connection and Curvature Tensor

(M, g) – smooth (pseudo) Riemannian manifold:
nondegenerate bilinear form g_p on tangent space $T_p(M)$
(Riemannian if g_p positive definite)
$\mathcal{C}(M)$ – smooth functions $\quad \mathcal{T}(M)$ – smooth vector fields
Riemannian connection:
$X \in \mathcal{T}(M)$ acts as covariant derivative ∇_X on tensor fields:

- $\nabla_{\varphi X} Y = \varphi \nabla_X Y \qquad \nabla_X(\varphi Y) = X(\varphi)Y + \nabla_X Y$

for $\varphi \in \mathcal{C}(M)$ and $Y \in \mathcal{T}(M)$
∇ uniquely determined from g by requiring

- covariant constant metric tensor:
  $X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$
- zero torsion: $\quad \nabla_X Y - \nabla_Y X = [X, Y]$

Curvature tensor field $R_p(x, y) \in \mathrm{End}(T_pM) = T_pM \otimes (T_pM)^*$:

$$R_p(x, y)z = \left(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z\right)_p$$

- Only depends on $X_p = x$, $Y_p = y$, and $Z_p = z$ (tensorial) (unlike Lie derivative $\theta(X)$ – not tensorial)
- Measures failure of $X \mapsto \nabla_X$ to be Lie algebra homomorphism

Algebraic Symmetries of the curvature tensor:

(C1) $R_p(x, y) = -R_p(y, x)$, so $R_p : \bigwedge^2 T_p(M) \to \mathrm{End}(T_p(M))$

(C2) $R_p(x, y)^* = -R_p(x, y)$ ($*$ via g_p), so $R_p(x, y) \in \mathrm{Lie}(\mathrm{O}(g_p))$

(C3) Jacobi identity for $\mathcal{T}(M)$ + zero torsion $\Longrightarrow$ Bianchi identity: $R_p(x, y)z + R_p(y, z)x + R_p(z, x)y = 0$

The Space of Curvature Tensors

Fix $p \in M$. Let $E = (T_pM)_{\mathbb{C}} \cong E^*$ (via $Q = (g_p)_{\mathbb{C}}$)
Define Riemann-Christoffel curvature tensor $R \in \bigotimes^4 E \cong \bigotimes^4 E^*$:

$$R(v, w, x, y) = Q(R_p(v, w)x, y) \quad \text{for } v, w, x, y \in E$$

Program: Study subspace $\mathrm{Curv}(E) \subset \bigotimes^4 E$ of tensors with algebraic symmetries (C1) + (C2) + (C3).
Let $R \in \mathrm{Curv}(E)$, σ = permutation representation of $\mathfrak{S}_4$ on $\bigotimes^4 E$

- (C1) + (C2) + (C3) $\implies \sigma(12)R = -R, \quad \sigma(34)R = -R,$ $\sigma(13)\sigma(24)R = R \implies R \in S^2(\bigwedge^2 E)$
- $S^2(\bigwedge^2 E)$ is invariant under the Bianchi operator $b = \frac{1}{3}(I+\sigma(123)+\sigma(123)^2) = \frac{1}{3}(I+\sigma(13)\sigma(12)+\sigma(23)\sigma(12))$

Hence $\mathrm{Curv}(E) = \mathrm{Ker}(b) \cap S^2(\bigwedge^2 E)$

$\mathrm{Curv}(E)$ as $\mathrm{GL}(E)$ module

- $\mathrm{Range}(b) \cap S^2(\bigwedge^2 E) = \bigwedge^4 E$ (irreducible for $\mathrm{GL}(E)$)
- $b^2 = b \implies S^2(\bigwedge^2 E) = \mathrm{Curv}(E) \oplus \left(\bigwedge^4 E\right)$
  (second summand zero if $\dim E < 4$)
- $\dim \mathrm{Curv}(\mathbb{C}^n) = \frac{1}{12} n^2(n+1)(n-1)$ $(\star)$

Fix Q-orthonormal basis $e_1, \ldots, e_n$ for $E \cong \mathbb{C}^n$ $(n \geq 2)$
Let $\lambda = [\lambda_1, \ldots, \lambda_k] \in \mathbb{N}^k$, $\lambda_1 \geq \cdots \geq \lambda_k > 0$ $(k \leq n)$
F_n^λ = irreducible $\mathrm{GL}(n, \mathbb{C})$ representation, highest weight λ

Theorem

$\mathrm{Curv}(\mathbb{C}^n) \cong F_n^{[2,2]}$ is an irreducible $\mathrm{GL}(n, \mathbb{C})$ module.

Proof.

High wt vector $R = (e_1 \wedge e_2) \otimes (e_1 \wedge e_2) \implies F_n^{[2,2]} \subset \mathrm{Curv}(\mathbb{C}^n)$.
Weyl dim. formula + $(\star) \implies \dim F_n^{[2,2]} = \dim \mathrm{Curv}(\mathbb{C}^n)$ □

Curvature and Young Symmetrizers

Young tableau A (k boxes) has row group and column group

Young symmetrizer: $\mathbf{p}_A \in \mathrm{End}_{\mathrm{GL}(n,\mathbb{C})}(\bigotimes^k \mathbb{C}^n)$

(alternate over column group)·(symmetrize over row group)

Weyl module: Range $\mathbf{p}_A$ is an irreducible $\mathrm{GL}(n,\mathbb{C})$ module, highest weight $\lambda = \mathrm{shape}(A)$

Corollary

$$\mathrm{Curv}(\mathbb{C}^n) = \mathrm{Range}\,\mathbf{p}_A \quad \textit{where} \quad A = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}$$

Proof.

$$\mathbf{p}_A = \tfrac{1}{12}\left\{\big(1-\sigma(12)\big)\big(1-\sigma(34)\big)\right\}\cdot\left\{\big(1+\sigma(13)\big)\big(1+\sigma(24)\big)\right\}$$

- $\mathrm{Range}(\mathbf{p}_A) \cong F_n^{[2,2]}$ $\quad$ ($\mathrm{shape}(A) = [2,2]$)
- $\mathrm{Range}(\mathbf{p}_A) \subset S^2(\bigwedge^2 \mathbb{C}^n)$

Orthogonal Decomposition of Curvature Tensors

Let $R \in \mathrm{Curv}(\mathbb{C}^n)$.

Ricci curvature:

$\mathrm{Ric}_Q(R)(v, w) = \sum_{i=1}^n R(e_i, v, e_i, w)$

- Ricci contraction operator $\mathrm{Ric}_Q : S^2(\bigwedge^2 \mathbb{C}^n) \to S^2(\mathbb{C}^n)$
- Ric_Q intertwines $\mathrm{O}(Q)$ actions on $\mathrm{Curv}(\mathbb{C}^n)$ and $S^2(\mathbb{C}^n)$
- Ric_Q is the only nonzero contraction operator on $S^2(\bigwedge^2 \mathbb{C}^n)$

Scalar curvature:

$s_Q(R) = \mathrm{tr}_Q(\mathrm{Ric}_Q(R)) = \sum_{i,j=1}^n R(e_i, e_j, e_i, e_j)$

$R \mapsto s_Q(R)$ gives $\mathrm{O}(Q)$ intertwining operator $s_Q : \mathrm{Curv}(\mathbb{C}^n) \to \mathbb{C}$ (trivial $\mathrm{O}(Q)$ module)

Construct $\mathrm{O}(Q)$ intertwining operator $S^2(\mathbb{C}^n) \to S^2(\bigwedge^2 \mathbb{C}^n)$:

- $S^2(\mathbb{C}^n) \cong Q$-symmetric linear maps of $\mathbb{C}^n$ $\quad (Q \leftrightarrow I)$
- $S^2(\bigwedge^2 \mathbb{C}^n) \cong Q \otimes Q$ -symmetric linear maps of $\bigwedge^2 \mathbb{C}^n$

Given $A, B \in S^2(\mathbb{C}^n)$, define linear map $A \owedge B$ on $\bigwedge^2 \mathbb{C}^n$:

$$(A \owedge B)(v \wedge w) = Av \wedge Bw + Bv \wedge Aw \quad \text{for } v, w \in \mathbb{C}^n$$

- $A \owedge B \in S^2(\bigwedge^2 \mathbb{C}^n)$
- $A \owedge B$ satisfies Bianchi identity, so $A \owedge B \in \mathrm{Curv}(\mathbb{C}^n)$
- $\mathrm{Ric}_Q(A \owedge Q) = \mathrm{tr}_Q(A)Q + (n-2)A \qquad (\star\star)$

$n = 2$:

- $\dim \mathrm{Curv}(\mathbb{C}^2) = 1$
- $R = \frac{1}{4} s_Q(R)\, Q \owedge Q$
- $\mathrm{Ric}_Q(R) = \frac{1}{2} s_Q(R) Q$

Assume $n \geq 3$. Given $R \in \mathrm{Curv}(\mathbb{C}^n)$, set

$$A = \frac{1}{n-2}\Big\{ \mathrm{Ric}_Q(R) - \frac{1}{n}s_Q(R)Q \Big\} \in S^2(\mathbb{C}^n)$$

$$C = R - A \owedge Q - \gamma s_Q(R) Q \owedge Q \in \mathrm{Curv}(\mathbb{C}^n) \quad (\gamma = \frac{1}{n(2n-1)})$$

The normalizing constants are chosen so that

- $\mathrm{tr}_Q(A) = 0$
- $\mathrm{Ric}_Q(C) = 0$ by $(\star\star)$

$$R = \gamma s_Q(R) Q \owedge Q + A \owedge Q + C$$

$$= \text{scalar part} + \text{traceless Ricci part} + \text{Weyl part} \quad (\star\star\star)$$

Representation-theoretic description of decomposition $(\star\star\star)$

Define Weyl conformal curvature tensors:

$$\mathrm{Weyl}_Q(\mathbb{C}^n) = \{C \in \mathrm{Curv}(\mathbb{C}^n) \,:\, \mathrm{Ric}_Q(C) = 0\}$$

Q-harmonic (traceless) symmetric two-tensors:

$$\mathcal{H}^2_{\mathrm{sym}}(\mathbb{C}^n, Q) = \{A \in S^2(\mathbb{C}^n) \,:\, \mathrm{tr}_Q(A) = 0\}$$

Properties of $\mathcal{H}^2_{\mathrm{sym}}(\mathbb{C}^n, Q)$:

- Irreducible under $\mathrm{SO}(Q)$ (Cartan component of $\mathbb{C}^n \otimes \mathbb{C}^n$)
- Highest weight $2\varpi_1$ ($n \neq 4$) or $2(\varpi_1 + \varpi_2)$ ($n = 4$)
- Dimension $= \frac{1}{2}n(n+1) - 1 = \frac{1}{2}(n+2)(n-1)$
- $\mathcal{H}^2_{\mathrm{sym}}(\mathbb{C}^n, Q) \hookrightarrow \mathrm{Curv}(C^n)$ by $A \mapsto A \owedge Q$

Theorem

The space of curvature tensors decomposes under $\mathrm{O}(Q)$ *as*

$$\mathrm{Curv}(\mathbb{C}^n) = \mathbb{C}(Q \owedge Q) \oplus \left(\mathcal{H}^2_{\mathrm{sym}}(\mathbb{C}^n, Q) \owedge Q\right) \oplus \mathrm{Weyl}_Q(\mathbb{C}^n)$$

Hence $\dim \mathrm{Weyl}_Q(\mathbb{C}^n) = \frac{1}{12}n(n+1)(n+2)(n-3)$.
In particular, Ricci curvature determines curvature when $n = 3$.

Proof.

Formula $(\star\star) \Longrightarrow$ the sum $(\star\star\star)$ is direct $\Longrightarrow$ dimension formula

□

The Space of Weyl Curvature Tensors

Goal:
Show $\mathrm{Weyl}_Q(\mathbb{C}^n)$ is irreducible under $\mathrm{O}(Q)$ ($n \geq 4$)
Method:
Find a highest weight vector and use Weyl dimension formula.

Take Q-isotropic basis for $\mathbb{C}^n$:

$$f_k = \frac{1}{\sqrt{2}}\big(e_k + \sqrt{-1}e_{n+1-k}\big), \quad f_{-k} = \frac{1}{\sqrt{2}}\big(e_k - \sqrt{-1}e_{n+1-k}\big)$$

for $k = 1, \ldots, l$ ($l = \lfloor \frac{n}{2} \rfloor$) (For n odd: $f_0 = e_{l+1}$)
Let $F_k = \mathrm{Span}\{f_1, \ldots, f_k\}$ for $k = 1, \ldots, l$ (Q-isotropic subspace)
Take Borel subgroup $B \subset \mathrm{SO}(Q)$ as stabilizer of isotropic flag

$$F_1 \subset F_2 \subset \cdots \subset F_l$$

(include $F_{l+1} = \mathrm{Span}\{f_0, f_1, \ldots, f_l\}$ when n odd)
$\varpi_1, \ldots, \varpi_l$ fundamental highest weights for $\mathfrak{so}(Q)$

Set $C_0 = (f_1 \wedge f_2) \otimes (f_1 \wedge f_2) \in S^2(\bigwedge^2 \mathbb{C}^n)$

- $C_0 \in \mathrm{Weyl}_Q(\mathbb{C}^n)$
- C_0 is a B eigenvector of weight $2\varpi_2$ (if $n > 4$) or $4\varpi_1$ (if $n = 4$)
- $V_n = \mathrm{Span}\, \mathrm{SO}(Q) \cdot C_0$ is irreducible under $\mathrm{SO}(Q)$ (theorem of the highest weight)

Theorem

Assume $n > 4$. $\mathrm{Weyl}_Q(\mathbb{C}^n)$ is irreducible under $\mathrm{SO}(Q)$ and has highest weight $2\varpi_2$.

Proof.

Have $V_n \subset \mathrm{Weyl}_Q(\mathbb{C}^n)$, and Weyl dimension formula $\Longrightarrow$

$$\dim V_n = \tfrac{1}{12} n(n+1)(n+2)(n-3) = \dim \mathrm{Weyl}_Q(\mathbb{C}^n) \qquad \square$$

Assume $n = 4$:

Special feature: $\mathfrak{so}_4 \cong \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ (semisimple)

Let $\tau \in \mathrm{O}(\mathbb{C}^4, Q)$ fix $f_{\pm 1}$ and interchange $f_2 \leftrightarrow f_{-2}$ $\quad(\det \tau = -1)$

Set $\overline{C_0} = \tau \cdot w = (f_1 \wedge f_{-2}) \otimes (f_1 \wedge f_{-2}) \in \mathrm{Weyl}_Q(\mathbb{C}^4)$

- $\overline{C_0}$ eigenvector for B of weight $4\varpi_2$
- $\overline{V}_4 = \tau \cdot V_4 \subset \mathrm{Weyl}_Q(\mathbb{C}^4)$ is irreducible under $\mathrm{SO}(Q)$

Theorem

$\mathrm{Weyl}_Q(\mathbb{C}^4) = V_4 \oplus \overline{V}_4$ *and is irreducible under* $\mathrm{O}(\mathbb{C}^4, Q)$.

Proof.

$\tau : V_4 \leftrightarrow \overline{V}_4$ and $\dim V_4 = \dim \overline{V}_4 = 5$ while $\dim \mathrm{Weyl}_Q(\mathbb{C}^4) = 10$

$V_4 \cap \overline{V}_4 = 0$ (inequivalent for $\mathrm{SO}(Q)$) □

Note: Likewise, $\bigwedge^2 \mathbb{C}^4$ is irreducible under $\mathrm{O}(Q)$, but decomposes under $\mathrm{SO}(Q)$ with highest weights $2\varpi_1$ and $2\varpi_2$.

Conformal Change of Metric Tensor

Replace g by $\widetilde{g} = e^{2f} g$ where $f \in \mathcal{C}(M)$.
Orthogonal group $\mathrm{O}(g) = \mathrm{O}(\widetilde{g})$ is unchanged.

Problem: Determine the change in the Weyl, traceless Ricci, and scalar parts of the Riemann curvature tensor.

New Riemannian connection:

$$\widetilde{\nabla}_X Y = \nabla_X Y + \Phi(X, Y) \quad \text{with } \Phi(X, Y) = \Phi(Y, X) \in \mathcal{T}(M)$$

Explicit formula:

$$\Phi(X, Y) = df(X)Y + df(Y)X - g(X, Y)Df \quad (Df = \mathrm{grad}_g f)$$

[Follows from $e^{-2f} X(e^{2f} g(Y, Z)) = g(\widetilde{\nabla}_X Y, Z) + g(Y, \widetilde{\nabla}_X Z)$ and cyclic permutation of X, Y, Z.]

New Curvature Tensor:

$\widetilde{R} = e^{2f} R - e^{2f} Q \owedge A$ with $A \in S^2(\mathbb{C}^n) \cong S^2(\mathbb{C}^n)^*$ via Q

Explicit formula:

$A = D^2 f - df \otimes df + \frac{1}{2}|Df|^2 Q$ (long calculation)

where $D^2 f(X, Y) = XY(f) - (\nabla_X Y)(f)$ (Q-Hessian of f)

$|Df|^2 = Q(Df, Df)$

Conclusion: Under conformal change of metric $g \to e^{2f} g$

- Weyl part of the curvature is multiplied by e^{2f}.
- Traceless Ricci curvature is modified by adding term
  $(n-2)D^2 f - (n-2)(df \otimes df) + \frac{n-2}{n}\{\Delta f + |Df|^2\}Q$
  where $\Delta f = -\operatorname{tr}_Q(D^2 f)$ is the Q-Laplacian.
- Scalar curvature is multiplied by e^{2f} plus term
  $e^{2f}\{2(n-1)\Delta f - (n-1)(n-2)|Df|^2\}$

Further Topics

- Spaces of Covariant Derivatives $\nabla R, \nabla^2 R, \ldots$
  Decomposition as representation spaces for orthogonal group (Kulkarni, Strichartz)
- Structure of Weyl Tensors
  - $n = 4$: Petrov (1954) $n > 4$: Coley et al. (2004)
    Important for study of gravity waves
  - Problem: Describe classification in terms of structure of orbits of $G = \mathrm{SO}(n, Q)$ on $V = \mathrm{Weyl}_Q(\mathbb{C}^n)$.
  - Highest weight vector orbit: $X = \overline{G \cdot C_0} = G \cdot C_0 \cup \{0\} \subset V$
    X is a multiplicity-free G space (Vinberg–Popov)
    $X/\mathbb{C}^\times = G/P$ with $P =$ maximal parabolic $\longleftrightarrow \varpi_2$
  - Find other G orbits on V and orbit invariants (Strichartz)
    $n = 4$: Use classical invariant theory of binary quartics

Hermann Weyl:

“The wish to understand what really is the mathematical substance behind the formal apparatus of relativity theory led me to the study of representations and invariants of groups.”

Relativity theory as a stimulus in mathematical research (1949)