Alice through Looking Glass after Looking Glass: The Mathematics of Mirrors and Kaleidoscopes

Roe Goodman

Rutgers Math Department – U Seminar

May 1, 2008

Alice Through Looking Glass After Looking Glass:

Alice finds in one corner of her chamber a cone-shaped arrangement of three *magic mirrors*. She steps through one of the mirrors and discovers that she is in a new *virtual chamber*.

Looking more closely, she sees that her books are all written backwards in this new chamber and there is another set of *magic mirrors* in the corner.

She steps through another magic mirror and continues her trip through *all* the different virtual chambers. She is very relieved when she finally gets back in her own real chamber just in time for tea.

Alice wonders *how many different ways* the three mirrors in the corner could be arranged so that she could have other trips through the looking glasses and still get back in time for tea.

Alice's Kaleidoscope Problem:

- *Kaleidoscope:* Arrangement of n mirrors (hyperplanes through 0) in n-dimensional Euclidean space

- *Kaleidoscope Condition:* Multiple reflections of a real object through the mirrors generates a finite number of images (virtual objects).

Solution: H.S.M. Coxeter's classification (*Annals of Math 1934*) of all finite reflection groups

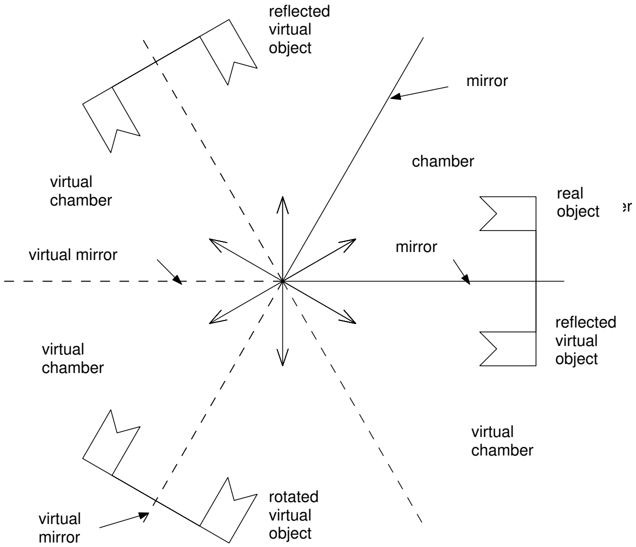
- Infinite number of 2-dimensional kaleidoscopes
  $\longleftrightarrow$ Regular polygons with even number of sides

- Three types of 3-dimensional kaleidoscopes
  $\longleftrightarrow$ dual pairs of Platonic solids

- Finite number of n-dimensional kaleidoscopes
  $\longleftrightarrow$ Root systems in n dimensions ($n > 3$)

Two-Mirror Kaleidoscope (Brewster, 1819)

Example: Mirror angle $60^\circ = \pi/3$



Reflections in the two mirrors generate *virtual* mirrors, chambers, and objects

Linear Algebra Description of Reflections in n Dimensions

Mirror in Euclidean space $\mathbf{R}^n$

$\longleftrightarrow$ $(n-1)$-dimensional subspace $M \subset \mathbf{R}^n$

$\longleftrightarrow$ *root vectors* $\pm\alpha \perp M$ (column vectors in $\mathbf{R}^n$, length one)

Reflection matrix $R_\alpha = I - 2\alpha\,\alpha^T$ ($\pm$ gives same $n \times n$ matrix)

$$R_\alpha \mathbf{v} = \begin{cases} -\mathbf{v} & \text{if } \mathbf{v} \perp M \\ \mathbf{v} & \text{if } \mathbf{v} \in M \end{cases}$$

Algebraic Properties

- $R_\alpha^2 = I$ (identity matrix)
- $\det R_\alpha = -1$ (reverses orientation)
- $R_\alpha^T = R_\alpha$ (symmetric and orthogonal matrix)

Theorem (Finite Reflection Groups in Two Dimensions)

Let R_α and R_β be 2×2 reflection matrices corresponding to mirror root vectors α and β. Let θ $(\leq \pi/2)$ be the dihedral angle between the mirrors.

Assume the interior of the wedge (chamber) C between the mirrors does not contain any virtual mirror generated by multiple reflections in the two mirrors.

Let G be the group of 2×2 orthogonal matrices generated by R_α and R_β

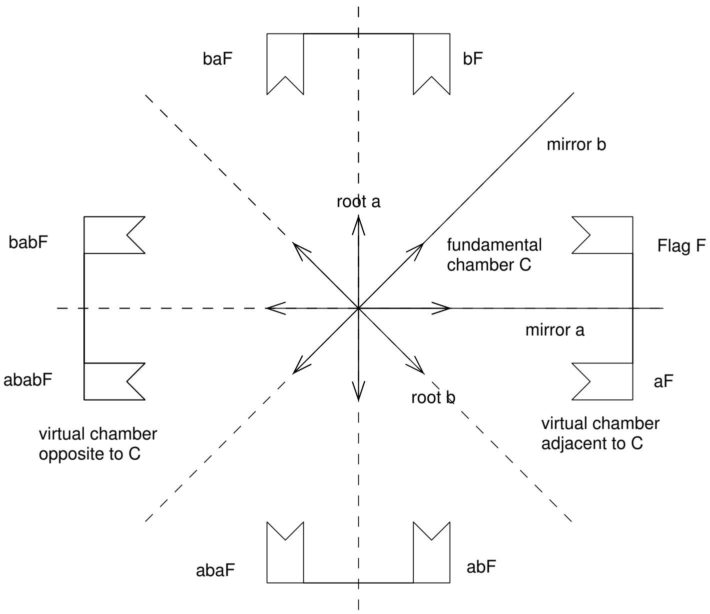
(a) *G is finite $\Longleftrightarrow \theta = \pi/m$ for some integer $m \geq 2$.*

(b) *Let $\theta = \pi/m$ for an integer $m \geq 2$.*

- *The reflected chambers $g \cdot C$ for $g \in G$ have mutually disjoint interiors and fill up 2-space.*
- *Each chamber corresponds to an element of G, so $|G| = 2m$.*
- *G is generated by $a = R_\alpha$ and $b = R_\beta$ with relations $a^2 = b^2 = (ab)^m = 1$. (quotient of free group)*

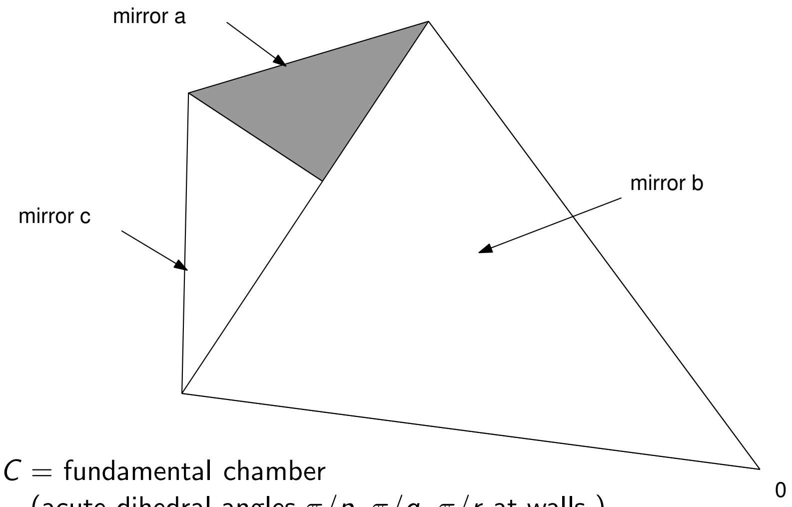
Example: Mirror angle $45° = \pi/4$ (Pizza with 8 slices)

$G = \{1, a, b, ab, ba, aba, bab, abab\}$ Relation $abab = baba$



Finite Reflection Groups in Three Dimensions

Three mirrors in $\mathbf{R}^3$ through 0



C = fundamental chamber

(acute dihedral angles π/p, π/q, π/r at walls)

Assume $2 \leq p \leq q \leq r$ and $q > 2$

(otherwise we are really in $2+1$ dimensions)

Theorem (Finite Reflection Groups in Three Dimensions)

Let G be the group of orthogonal matrices generated by reflections in the three faces of the cone C.

Assume the interior of C does not contain any virtual mirror generated by multiple reflections in the faces of C.

(a) *If the orbit $G \cdot x$ is finite for some point x inside C, then p, q, r are positive integers that satisfy*

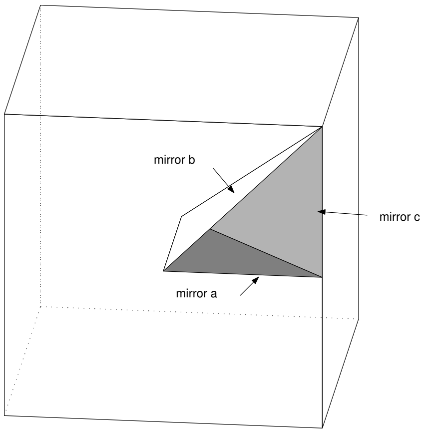
$$2 \leq p \leq q \leq r, \quad \frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1. \qquad (\star)$$

(b) *The integer solutions to $(\star)$ with $q > 2$ are*
$(2, 3, 3)$, $(2, 3, 4)$, $(2, 3, 5)$.
Let (p, q, r) be one of these triples and let C be a cone with the corresponding dihedral angles.

- *The reflected chambers $g \cdot C$ for $g \in G$ have mutually disjoint interiors and fill 3-space.*
- *reflected chambers $\longleftrightarrow$ elements of G, so $|G| = \#$ chambers.*

Proof of (b): For each triple (p, q, r) construct

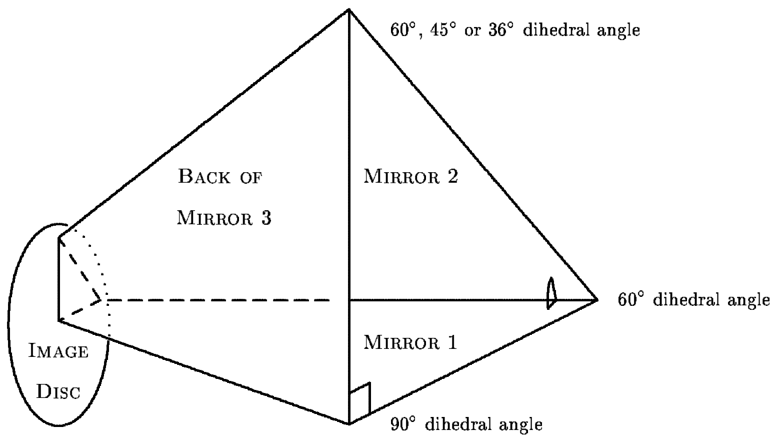
- Regular polyhedron (tetrahedron, cube, icosahedron), center at 0.



- Triangulation of the faces of the polyhedron by congruent triangles
- Cone C from 0 through one of the triangles with dihedral angles π/p, π/q, π/r.

Kaleidoscopes in Three Dimensions

"These groups can be made vividly comprehensible by using actual mirrors for the generating reflections. It is found that a candle makes an excellent object to reflect. By hinging two vertical mirrors at an angle π/k we easily see $2k$ candle flames, in accordance with the group $[k]$. To illustrate the groups $[k_1, k_2]$, we hold a third mirror in the appropriate positions." (H.S.M. Coxeter, 1934)



Kaleidoscopes in $\mathbf{R}^n$

Φ = unit root vectors for finite set of mirrors (hyperplanes) in $\mathbf{R}^n$

Call Φ a *Root System* if it satisfies the *Kaleidoscope Condition:*

(KC) For every $\alpha, \beta \in \Phi$, the reflected root vector $R_\alpha \beta \in \Phi$.

Lemma
We can choose a set $\{\alpha_1, \ldots, \alpha_n\}$ of simple roots and a fundamental chamber $C = \{\mathbf{v} \in \mathbf{R}^n : \alpha_i \cdot \mathbf{v} \geq 0 \text{ for } i = 1, \ldots, n\}$ so that no mirror hyperplane passes through the interior of C.
Then $\alpha_i \cdot \alpha_j < 0$ for $i \neq j$.

Notation:
M_i = mirror for α_i, R_i = reflection in M_i

$\theta_{ij} =$ dihedral angle$(M_i, M_j) = \dfrac{\pi}{p_{ij}}$ $(p_{ij} \geq 2$ and $\cos(\theta_{ij}) = -\alpha_i \cdot \alpha_j)$

Lemma
(KC) $\implies$ for each pair (i, j) of mirrors

$$p_{ij} \in \{2, 3, 4, \ldots\} \quad \text{and} \quad (R_i R_j)^{p_{ij}} = 1$$

Coxeter Graph of a Root System

- vertices $\longleftrightarrow$ simple roots $\alpha_1, \ldots, \alpha_n$
- edge between vertex i and vertex j if and only if $p_{ij} > 2$
  ($\alpha_i \cdot \alpha_j \neq 0$)
- label the edge with integer p_{ij} if $p_{ij} > 3$
  ($p_{ij} = 3 \longleftrightarrow$ dihedral angle $\pi/3$)

Coxeter Matrix of the root system: $A = [\alpha_i \cdot \alpha_j] = A^t$
(inner products of simple roots)

Geometry becomes Graph Theory:
$A \longleftrightarrow$ Coxeter graph of root system

Theorem

The Coxeter matrix of a root system is positive definite.

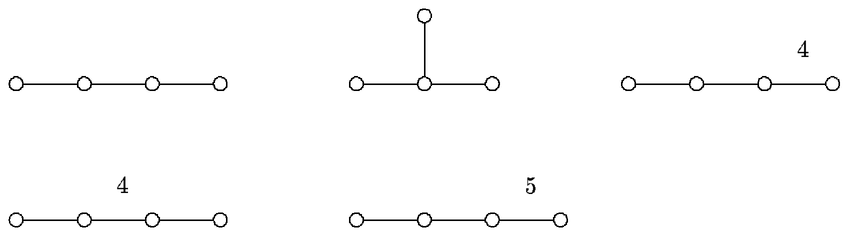
This means that $A\mathbf{v} \cdot \mathbf{v} > 0$ for all vectors $\mathbf{v} \in \mathbf{R}^n$.

- all principal minors of A are positive
- all eigenvalues of A are positive

Classification Problem

Determine all Coxeter graphs whose matrix is positive definite.

Solution to Classification Problem ($n = 4$):



Coxeter's 1934 results:

- For every positive-definite Coxeter matrix there is a root system.
- The reflection group G for the root system is finite.
- The relations in G are $R_i^2 = (R_i R_j)^{p_{ij}} = 1$.

Finite Reflection Groups in N dimensions with connected Coxeter Graph:

N	# GROUPS	# MIRRORS	# CHAMBERS
4	5	10, 12, 16 24 60	$5 \cdot 4!$, $2^3 \cdot 4!$, $2^4 \cdot 4!$ $2 \cdot 6 \cdot 8 \cdot 12$ $2 \cdot 12 \cdot 20 \cdot 30$
5	3	15, 20, 25	$6 \cdot 5!$, $2^4 \cdot 5!$, $2^5 \cdot 5!$
6	4	21, 30, 36 36	$7 \cdot 6!$, $2^5 \cdot 6!$, $2^6 \cdot 6!$ $2 \cdot 5 \cdot 6 \cdot 8 \cdot 9 \cdot 12$
7	4	28, 42, 49 63	$8 \cdot 7!$, $2^7 \cdot 7!$, $2^6 \cdot 7!$ $2 \cdot 6 \cdot 8 \cdot 10 \cdot 12 \cdot 14 \cdot 18$
8	4	36, 56, 64 120	$9 \cdot 8!$, $2^7 \cdot 8!$, $2^8 \cdot 8!$ $2 \cdot 8 \cdot 12 \cdot 14 \cdot 18 \cdot 20 \cdot 24 \cdot 30$
$N > 8$	3	$N(N+1)/2$ $N(N-1)$ N^2	$(N+1) \cdot N!$ $2^{N-1} \cdot N!$ $2^N \cdot N!$

Alice wants to go through the looking glasses in four dimensions!