CONTENTS

Preface ix

1 Classical Groups as Linear Algebraic Groups **1**

1.1 Linear Algebraic Groups 1

1.1.1 Definitions and Examples

1.1.2 Regular Functions

1.1.3 Representations

1.1.4 Connected Groups

1.1.5 Subgroups and Homomorphisms

1.1.6 Group Structures on Affine Varieties

1.1.7 Exercises

1.2 Lie Algebra of an Algebraic Group 17

1.2.1 Left-Invariant Vector Fields

1.2.2 Lie Algebras of the Classical Groups

1.2.3 Differential of a Representation

1.2.4 The Adjoint Representation

1.2.5 Exercises

1.3 Jordan Decomposition 34

1.3.1 Nilpotent and Unipotent Matrices

1.3.2 Semisimple One-Parameter Groups

1.3.3 Jordan–Chevalley Decomposition

1.3.4 Exercises

1.4 Real Forms of Classical Groups 41

1.4.1 Algebraic Groups as Lie Groups

1.4.2 Real Forms

1.4.3 Compact Forms

1.4.4 Quaternionic Unitary Group

1.4.5 Quaternionic General Linear Group

1.4.6 Exercises

1.5 Notes 49

2 Basic Structure of Classical Groups **50**
2.1 Semisimple and Unipotent Elements 50
2.1.1 Conjugacy of Maximal Tori
2.1.2 Unipotent Generators
2.1.3 Exercises
2.2 Irreducible Representations of SL(2, $\mathbb{C}$) 62
2.2.1 Representations of $\mathfrak{sl}(2, \mathbb{C})$
2.2.2 Representations of SL(2, $\mathbb{C}$)
2.2.3 Exercises
2.3 The Adjoint Representation 67
2.3.1 Roots with respect to a Maximal Torus
2.3.2 Commutation Relations of Root Spaces
2.3.3 Structure of Classical Root Systems
2.3.4 Irreducibility of the Adjoint Representation
2.3.5 Exercises
2.4 Reductivity of Classical Groups 84
2.4.1 Reductive Groups
2.4.2 Casimir Operator
2.4.3 Algebraic Proof of Complete Reducibility
2.4.4 The Unitarian Trick
2.4.5 Exercises
2.5 Weyl Group and Weight Lattice 92
2.5.1 Weyl Group
2.5.2 Root Reflections
2.5.3 Weight Lattice
2.5.4 Fundamental Weights and Dominant Weights
2.5.5 Exercises
2.6 Notes 109

3 Algebras and Representations **111**
3.1 Representations of Associative Algebras 111
3.1.1 Definitions and Examples
3.1.2 Schur's Lemma
3.1.3 Burnside's Theorem
3.1.4 Complete Reducibility
3.1.5 Exercises
3.2 Simple Associative Algebras 128
3.2.1 Wedderburn's Theorem
3.2.2 Representations of End(V)
3.2.3 Exercises
3.3 Commutants and Characters 133
3.3.1 Representations of Semisimple Algebras
3.3.2 Double Commutant Theorem

3.3.3 Characters
3.3.4 Exercises
3.4 Group Algebras of Finite Groups 147
3.4.1 Structure of Group Algebras
3.4.2 Schur Orthogonality Relations
3.4.3 Fourier Inversion Formula
3.4.4 The Algebra of Central Functions
3.4.5 Exercises
3.5 Representations of Finite Groups 155
3.5.1 Induced Representations
3.5.2 Characters of Induced Representations
3.5.3 Standard Representation of $\mathfrak{S}_n$
3.5.4 Representations of $\mathfrak{S}_k$ on Tensors
3.5.5 Exercises
3.6 Notes 167

4 Polynomial and Tensor Invariants 168
4.1 Polynomial Invariants 169
4.1.1 The Ring of Invariants
4.1.2 Invariant Polynomials for $\mathfrak{S}_n$
4.1.3 Exercises
4.2 Invariants for Classical Groups 180
4.2.1 First Fundamental Theorem
4.2.2 Proof of a Basic Case
4.2.3 Invariant Polynomials as Tensors
4.2.4 Exercises
4.3 Tensor Invariants 190
4.3.1 Tensor Invariants for $\mathrm{GL}(V)$
4.3.2 Tensor Invariants for $\mathrm{O}(V)$ and $\mathrm{Sp}(V)$
4.3.3 Exercises
4.4 Polynomial FFT for Classical Groups 198
4.4.1 Proof of Polynomial FFT for $\mathrm{GL}(V)$
4.4.2 Proof of Polynomial FFT for $\mathrm{O}(V)$ and $\mathrm{Sp}(V)$
4.5 Some Applications of the FFT 200
4.5.1 Skew Duality for Classical Groups
4.5.2 General Duality Theorem
4.5.3 A Duality Theorem for Weyl Algebras
4.5.4 $\mathrm{GL}(n)-\mathrm{GL}(k)$ Howe Duality
4.5.5 $\mathrm{O}(n)-\mathfrak{sp}(k)$ Howe Duality
4.5.6 $\mathrm{Sp}(n)-\mathfrak{so}(2k)$ Howe Duality
4.5.7 Capelli Identities
4.6 Notes 226

5 Highest Weight Theory **228**
5.1 Irreducible Representations of Classical Groups 228
5.1.1 Extreme Vectors and Highest Weights
5.1.2 Commuting Algebra and $\mathfrak{n}$-Invariant Vectors
5.1.3 Fundamental Representations
5.1.4 Cartan Product
5.1.5 Weights of Irreducible Representations
5.1.6 Lowest Weights and Dual Representations
5.1.7 Symplectic and Orthogonal Representations
5.1.8 Exercises
5.2 Some Applications 248
5.2.1 Irreducible Representations of $\mathrm{GL}(V)$
5.2.2 Irreducible Representations of $\mathrm{O}(V)$
5.2.3 Spherical Harmonics
5.2.4 $\mathrm{GL}(k)-\mathrm{GL}(n)$ Duality
5.2.5 Decomposition of $S(S^2(V))$ under $\mathrm{GL}(V)$
5.2.6 Decomposition of $S(\bigwedge^2(V))$ under $\mathrm{GL}(V)$
5.2.7 Second Fundamental Theorems
5.2.8 Exercises
5.3 Notes 268

6 Spinors **269**
6.1 Clifford Algebras 269
6.1.1 Construction of $\mathrm{Cliff}(V)$
6.1.2 Spaces of Spinors
6.1.3 Structure of $\mathrm{Cliff}(V)$
6.1.4 Exercises
6.2 Spin Representations of Orthogonal Lie Algebras 279
6.2.1 Embedding $\mathfrak{so}(V)$ in $\mathrm{Cliff}(V)$
6.2.2 Spin Representations
6.2.3 Exercises
6.3 Spin Groups 284
6.3.1 Action of $\mathrm{O}(V)$ on $\mathrm{Cliff}(V)$
6.3.2 Algebraically Simply Connected Groups
6.3.3 Exercises
6.4 Real Forms of $\mathrm{Spin}(n,\mathbb{C})$ 291
6.4.1 Real Forms of Vector Spaces and Algebras
6.4.2 Real Forms of Clifford Algebras
6.4.3 Real Forms of $\mathrm{Pin}(n)$ and $\mathrm{Spin}(n)$
6.4.4 Exercises
6.5 Notes 294

7 Cohomology and Characters **296**
7.1 Character and Dimension Formulas 296
7.1.1 Weyl Character Formula
7.1.2 Weyl Dimension Formula
7.1.3 Commutant Character Formulas
7.1.4 Exercises
7.2 Lie Algebra Cohomology 309
7.2.1 Cochain Complex
7.2.2 Cohomology Spaces
7.2.3 Cohomology Exact Sequences
7.2.4 The Koszul Complex
7.2.5 Cohomology of Enveloping Algebras
7.2.6 Exercises
7.3 Algebraic Approach to Weyl Character Formula 324
7.3.1 Casimir Identity on Cohomology
7.3.2 Weyl Group and Sets of Positive Roots
7.3.3 Expansion of an Invariant
7.3.4 Kostant's Lemma
7.3.5 Kostant's Theorem
7.3.6 Algebraic Proof of Weyl Character Formula
7.3.7 Exercises
7.4 Analytic Approach to Weyl Character Formula 337
7.4.1 Semisimple Conjugacy Classes
7.4.2 Maximal Compact Torus
7.4.3 Weyl Integral Formula
7.4.4 Fourier Expansions of Skew Functions
7.4.5 Analytic Proof of Weyl Character Formula
7.4.6 Exercises
7.5 Notes 347

8 Branching Laws **349**
8.1 Branching for Classical Groups 349
8.1.1 Statement of Branching Laws
8.1.2 Branching Patterns and Weight Multiplicities
8.1.3 Exercises
8.2 Branching Laws from Weyl Character Formula 356
8.2.1 Partition Functions
8.2.2 Kostant Multiplicity Formulas
8.2.3 Exercises
8.3 Proofs of Classical Branching Laws 359
8.3.1 Restriction from $\mathrm{GL}(n)$ to $\mathrm{GL}(n-1)$

8.3.2 Restriction from $\mathrm{Spin}(2n+1)$ to $\mathrm{Spin}(2n)$
8.3.3 Restriction from $\mathrm{Spin}(2n)$ to $\mathrm{Spin}(2n-1)$
8.3.4 Restriction from $\mathrm{Sp}(n)$ to $\mathrm{Sp}(n-1)$
8.4 Notes 370

9 Tensor Representations of GL(V) 372
9.1 Schur Duality 372
9.1.1 Duality between $\mathrm{GL}(n)$ and $\mathfrak{S}_k$
9.1.2 Characters of $\mathfrak{S}_k$
9.1.3 Frobenius Formula
9.1.4 Exercises
9.2 Dual Reductive Pairs 384
9.2.1 Seesaw Pairs
9.2.2 Reciprocity Laws
9.2.3 Schur Duality and $\mathrm{GL}(k)$–$\mathrm{GL}(n)$ Duality
9.2.4 Exercises
9.3 Young Symmetrizers and Weyl Modules 392
9.3.1 Tableaux and Symmetrizers
9.3.2 Weyl Modules
9.3.3 Standard Tableaux
9.3.4 Projections onto Isotypic Components
9.3.5 Exercises
9.4 Notes 404

10 Tensor Representations of O(V) and Sp(V) 406
10.1 Commuting Algebras on Tensor Spaces 406
10.1.1 Centralizer Algebra
10.1.2 Generators and Relations
10.1.3 Exercises
10.2 Decomposition of Harmonic Tensors 416
10.2.1 Harmonic Tensors
10.2.2 Harmonic Extreme Tensors
10.2.3 Decomposition of Harmonics for Sp(V)
10.2.4 Decomposition of Harmonics for $\mathrm{O}(2l+1)$
10.2.5 Decomposition of Harmonics for $\mathrm{O}(2l)$
10.2.6 Exercises
10.3 Decomposition of Tensor Spaces 433
10.3.1 Partially Harmonic Tensors
10.3.2 Proof of Partial Harmonic Decomposition
10.3.3 Decomposition in the Stable Range
10.3.4 Exercises
10.4 Invariant Theory and Knot Polynomials 446
10.4.1 The Braid Relations

10.4.2 Orthogonal Invariants and the Yang–Baxter Equation
10.4.3 The Braid Group
10.4.4 The Jones Polynomial
10.4.5 Exercises
10.5 Notes 461

11 Algebraic Groups and Homogeneous Spaces 464
11.1 Structure of Algebraic Groups 465
11.1.1 Quotient Groups
11.1.2 Commutative Algebraic Groups
11.1.3 Solvable and Semisimple Lie Algebras
11.1.4 Levi Decomposition of Lie Algebras
11.1.5 Unipotent Radical
11.1.6 Connected Algebraic Groups and Lie Groups
11.2 Homogeneous Spaces 481
11.2.1 G-Spaces and Orbits
11.2.2 Flag Manifolds
11.2.3 Involutions and Symmetric Spaces
11.2.4 Involutions of Classical Groups
11.2.5 Classical Symmetric Spaces
11.2.6 Exercises
11.3 Borel Subgroups 499
11.3.1 Solvable Groups
11.3.2 Lie–Kolchin Theorem
11.3.3 Structure of Connected Solvable Groups
11.3.4 Conjugacy of Borel Subgroups
11.3.5 Centralizer of a Torus
11.3.6 Exercises
11.4 Further Properties of Real Forms 506
11.4.1 Groups with a Compact Real Form
11.4.2 Polar Decomposition by a Compact Form
11.5 Gauss Decomposition 512
11.5.1 Gauss Decomposition of $\mathrm{GL}(n, \mathbb{C})$
11.5.2 Gauss Decomposition of an Algebraic Group
11.5.3 Gauss Decomposition for Real Forms
11.5.4 Exercises
11.6 Notes 517

12 Representations on Spaces of Regular Functions 518
12.1 Some General Results 518
12.1.1 Isotypic Decomposition of $\mathrm{Aff}(X)$
12.1.2 Decomposition of $\mathrm{Aff}(G)$
12.1.3 Frobenius Reciprocity

12.1.4 Models for Irreducible Representations on Function Spaces
12.1.5 Exercises
12.2 Multiplicity-Free Spaces 526
12.2.1 Multiplicity and B-Orbits
12.2.2 B-Eigenfunctions for Linear Actions
12.2.3 Branching from $\mathrm{GL}(n)$ to $\mathrm{GL}(n-1)$
12.2.4 Exercises
12.3 Regular Functions on Symmetric Spaces 534
12.3.1 Iwasawa Decomposition for Symmetric Spaces
12.3.2 Examples of Iwasawa Decompositions
12.3.3 Spherical Representations
12.3.4 Exercises
12.4 Separation of Variables for Isotropy Representations 553
12.4.1 A Theorem of Kostant and Rallis
12.4.2 Some Theorems of Chevalley
12.4.3 Classical Examples
12.4.4 Some Results from Algebraic Geometry
12.4.5 Proof of the Kostant–Rallis Theorem
12.4.6 Some Remarks on the Proof
12.4.7 Exercises
12.5 Notes 576

A Algebraic Geometry **579**
A.1 Affine Algebraic Sets 579
A.1.1 Basic Properties
A.1.2 Zariski Topology
A.1.3 Products of Affine Sets
A.1.4 Principal Open Sets
A.1.5 Irreducible Components
A.1.6 Transcendence Degree and Dimension
A.1.7 Exercises
A.2 Maps of Algebraic Sets 591
A.2.1 Rational Maps
A.2.2 Extensions of Homomorphisms
A.2.3 Image of a Dominant Map
A.2.4 Factorization of a Regular Map
A.2.5 Exercises
A.3 Tangent Spaces 597
A.3.1 Tangent Space and Differentials of Maps
A.3.2 Vector Fields
A.3.3 Dimension

A.3.4 Differential Criterion for Dominance
A.3.5 Exercises
A.4 Projective and Quasiprojective Sets 604
A.4.1 Basic Definitions
A.4.2 Products of Projective Sets
A.4.3 Regular Functions and Maps

B Linear and Multilinear Algebra 612
B.1 Jordan Decomposition 612
B.1.1 Primary Projections
B.1.2 Additive Jordan Decomposition
B.1.3 Multiplicative Jordan Decomposition
B.2 Multilinear Algebra 615
B.2.1 Bilinear Forms
B.2.2 Tensor Products
B.2.3 Symmetric Tensors
B.2.4 Alternating Tensors
B.2.5 Determinants and Gauss Decomposition
B.2.6 Pfaffians and Skew-Symmetric Matrices
B.2.7 Irreducibility of Determinants and Pfaffians

C Associative Algebras and Lie Algebras 632
C.1 Some Associative Algebras 632
C.1.1 Filtered and Graded Algebras
C.1.2 Tensor Algebra
C.1.3 Symmetric Algebra
C.1.4 Exterior Algebra
C.1.5 Exercises
C.2 Universal Enveloping Algebras 639
C.2.1 Lie Algebras
C.2.2 Universal Cyclic Module
C.2.3 Poincaré–Birkhoff–Witt Theorem
C.2.4 Adjoint Representation of Enveloping Algebra
C.2.5 Exercises

D Manifolds and Lie Groups 648
D.1 C^∞ Manifolds 648
D.1.1 Basic Definitions
D.1.2 Tangent Space
D.1.3 Differential Forms and Integration
D.1.4 Exercises

D.2 Lie Groups 660
D.2.1 Basic Definitions
D.2.2 Lie Algebra of a Lie Group
D.2.3 Homogeneous Spaces
D.2.4 Integration on Lie Groups and Homogeneous Spaces
D.2.5 Exercises

Bibliography 673

Index 679