Harmonic Analysis on Compact Symmetric Spaces: the Legacy of Cartan and Weyl

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Theme: Harmonic analysis from algebraic group/complex manifold viewpoint

0. Introduction: Weyl, Schur, and Cartan

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- Isotypic Decomposition of $\mathcal{O}[X]$, X affine G-space
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0. Weyl, Schur, and Cartan

Weyl (1949): "Frobenius and Issai Schur's spadework on finite and compact groups and Cartan's early work on semi-simple Lie groups and their representations had nothing to do with it [relativity theory]. But for myself I can say that the wish to understand what really is the mathematical substance behind the formal apparatus of relativity theory led me to the study of representations and invariants of groups, and my experience in this regard is probably not unique."

Weyl's book *Raum, Zeit, Materie* **(4th ed. – 1921)**:

- Helmholtz-Lie space problem *(Weyl interacts with Cartan)*
- Decompose $\otimes^k \mathbb{C}^n$ under commuting actions of general linear group and symmetric group *(Weyl interacts with Schur)*

Weyl (1949): This decomposition is "an epistemological principle basic for all theoretical science, that of projecting the actual upon the background of the possible."

Schur (1905): Found subspaces of tensor space that are invariant and irreducible under all transformations that commute with $\mathfrak{S}_k$.

- Use minimal projections in the group algebra of $\mathfrak{S}_k$
- These subspaces give all irreducible representations of $\mathrm{SL}(n, \mathbb{C})$

Cartan (1913): finite-dimensional irreducible representations of a simply-connected simple Lie group – constructs the *fundamental representations* (case-by-case) and then uses tensor products.

Weyl (1926): "The correct starting point for building representations does not lie in the adjoint group, but rather in the *regular representation*, which through its reduction yields *in one blow* all irreducible representations."

$U =$ compact real form of G (complex semisimple group)

Peter–Weyl (1927): Normalized matrix entries of the irreducible unitary representations of U furnish an orthonormal basis for $L^2(U)$.

Cartan (1929): The use of integral equations by Peter & Weyl is a "*transcendental* solution to a problem of an *algebraic* nature."

(determination of all finite-dimensional irreducible representations)

Cartan's Goal: Decompose $L^2(X)$ (X homogeneous U-space) "to give an *algebraic* solution to a problem of a *transcendental* nature, more general than that treated by Weyl."

Weyl (1934): "the systematic exposition by which I should like to replace the two papers Peter–Weyl and Cartan."

- Finds irreducible subspaces of $C(X)$ – *harmonic sets*
- Constructs intertwining operators between $C(X)$ and the left regular representation of U on $C(U)$

I. Algebraic Group Version of Peter–Weyl

1. Isotypic Decomposition of $\mathcal{O}[X]$

Cartan–Weyl–Chevalley ($\mathfrak{g}$ – complex semisimple Lie algebra):

Lie Algebra $\rightarrow$ Lie Group: There is a simply-connected complex linear algebraic group G with Lie algebra $\mathfrak{g}$.

Infinitesimal $\longleftrightarrow$ Global: Finite-dimensional representations of $\mathfrak{g}$ $\longleftrightarrow$ rational representations of G.

Compact Real Form: There is a real form $\mathfrak{u}$ of $\mathfrak{g}$ and a simply-connected compact Lie group $U \subset G$ with Lie algebra $\mathfrak{u}$.

Unitary Trick: Finite-dimensional unitary representations of $U \longleftrightarrow$ rational representations of G; U-invariant subspaces $\longleftrightarrow$ G-invariant subspaces ($\Longrightarrow G$ is reductive).

Highest Weight: Irreducible rational representations of $G \longleftrightarrow$ cone in a lattice of rank l.

Highest Weight Details:

Borel subgroup $B = HN^+ \subset G$ (upper triangular matrices)
$H \cong (\mathbb{C}^\times)^l$ – maximal algebraic torus in G (diagonal matrices)
N^+ – unipotent radical of B ($\longleftrightarrow$ positive roots of H on $\mathfrak{g}$)
$\bar{B} = HN^-$ – opposite Borel subgroup (lower triangular matrices)

$\mathfrak{h} = \mathrm{Lie}(H)$ $\quad \Phi \subset \mathfrak{h}^*$ – roots of $\mathfrak{h}$ on $\mathfrak{g}$
$P(\Phi) \subset \mathfrak{h}^*$ – *weight lattice* of H $\quad P_{++} \subset P(\Phi)$ – *dominant weights*
$\lambda \in P(\Phi)$ determines character $hn \mapsto h^\lambda$ of B

Models for Irreducible Representations:

$\lambda \in P_{++} \longleftrightarrow (\pi_\lambda, E_\lambda)$ – irreducible rational representation of G

$\dim E_\lambda^{N^+} = 1$ unique H weight space for weight λ (*highest*)

$\dim E_\lambda^{N^-} = 1$ unique H weight space for weight $w_0\lambda$ (*lowest*)

$w_0 \in \mathrm{Norm}_G(H)/H$ – interchanges positive and negative roots

$(\pi_{\lambda^*}, E_\lambda^*)$ = dual representation (highest weight $\lambda^* = -w_0\lambda$)

Highest/Lowest weight vectors: $e_\lambda \in E_\lambda^{N^+}$ $f_{\lambda^*} \in (E_\lambda^*)^{N^-}$

Normalization: $\langle e_\lambda, f_{\lambda^*}\rangle = 1$ (tautological form on $E_\lambda \times E_\lambda^*$)

Algebraic Setting for Cartan–Weyl Decomposition:

X – irreducible affine algebraic G space

$\mathcal{O}[X]$ – regular functions on X Representation ρ of G on $\mathcal{O}[X]$:

$\rho(g)f(x) = f(g^{-1}x)$ for $f \in \mathcal{O}[X]$ and $g \in G$

$\mathrm{Span}\{\rho(G)f\}$ is a rational G-module for all $f \in \mathcal{O}[X]$

$$\begin{aligned}\mathrm{Hom}_G(E_\lambda, \mathcal{O}[X]) &\cong \mathcal{O}[X]^{N^+}(\lambda) \\ &= \{f \in \mathcal{O}[X] : \rho(hn)f = h^\lambda f\}\end{aligned}$$

isomorphism: $T \in \mathrm{Hom}_G(E_\lambda, \mathcal{O}[X]) \longleftrightarrow Te_\lambda \in \mathcal{O}[X]^{N^+}(\lambda)$

$\mathrm{Spec}(X) = \{\lambda \in P_{++} : \mathcal{O}[X]^{N^+}(\lambda) \neq 0\}$ (G ***spectrum*** of X)

Tautological G-intertwining map:

$$E_\lambda \otimes \mathrm{Hom}_G(E_\lambda, \mathcal{O}[X]) \hookrightarrow \mathcal{O}[X] \qquad v \otimes T \mapsto Tv$$

Theorem 1. [Isotypic Decomposition]

$$\mathcal{O}[X] \cong \bigoplus_{\lambda \in \mathrm{Spec}(X)} E_\lambda \otimes \mathcal{O}[X]^{N^+}(\lambda) \quad \text{(algebraic direct sum)}$$

as a G-module, with action $\pi_\lambda(g) \otimes 1$ on the λ summand.

Corollary. The *multiplicity* of π_λ in $\mathcal{O}[X]$ is $\dim \mathcal{O}[X]^{N^+}(\lambda)$.

Cartan product: G acts by algebra automorphisms of $\mathcal{O}[X] \Longrightarrow \mathrm{Spec}(X)$ is an additive semigroup of P_{++}.

Cartan's program: Determine the decomposition of $\mathcal{O}[X]$ when G acts transitively on X – especially when X is a symmetric space.

2. Multiplicity Free Spaces

Definition. X is *multiplicity free* if $\dim \mathcal{O}[X]^{N^+}(\lambda) \leq 1$ $(\forall\lambda)$.
Assume Borel subgroup has an *open orbit* $B \cdot x_0$ on X.
Let $H_{x_0} = \{h \in H \,:\, h \cdot x_0 = x_0\}$ (isotropy group at x_0).

Theorem 2. [Vinberg–Kimelfeld]
(1) *X is multiplicity free.*
(2) *If $\lambda \in \mathrm{Spec}(X)$, then $h^\lambda = 1$ for all $h \in H_{x_0}$.*

Example. Two-sided Regular Representation
$G \times G$ acting on $X = G$ by left and right translations:

$$\rho(y, z)f(x) = f(y^{-1}xz) \quad \text{for} \quad f \in \mathcal{O}[G] \ \text{ and } \ x, y, z \in G$$

Cartan subgroup: $H \times H$ Borel subgroup: $\bar{B} \times B$

Borel Orbit of $x_0 = e$ is $N^- H N^+$ (dense in G), so
(1) $\Longrightarrow$ X is multiplicity-free for $G \times G$

Let $(w_0\mu,\, \lambda) \in \mathrm{Spec}(X)$. $(H \times H)_{x_0} = \{(h, h) \,:\, h \in H\}$, so
(2) $\Longrightarrow$ $\mu = \lambda^*$

Generating function: Define $\psi_\lambda(g) = \langle \pi_\lambda(g)e_\lambda,\ f_{\lambda^*}\rangle$
Then ψ_λ is a $\bar{B} \times B$ highest weight vector for $G \times G$
$\Longrightarrow \mathrm{Spec}(X) = \{\, (w_0\lambda^*, \lambda)\,\}_{\lambda \in P_{++}}$

Theorem 3. [Algebraic Peter–Weyl]

(1) $V_\lambda = \mathrm{Span}\{\rho(G \times G)\psi_\lambda\} \cong E_{\lambda^*} \otimes E_\lambda$ *as a* $G \times G$ *module.*

(2) $\mathcal{O}[G] = \bigoplus_{\lambda \in P_{++}} V_\lambda$

(3) $\mathcal{O}[G]$ *is multiplicity free as a* $G \times G$ *representation.*

Let $\lambda_1, \ldots, \lambda_l$ be the *fundamental weights* and let $\psi_i(g) = \psi_{\lambda_i}(g)$.

Product Formula: *Let* $\lambda = m_1\lambda_1 + \cdots + m_l\lambda_l$ *with* $m_i \in \mathbb{Z}_+$*. Then*

$$\psi_\lambda(g) = \psi_1(g)^{m_1} \cdots \psi_l(g)^{m_l} \quad \text{for } g \in G$$

Example. $G = \mathrm{SL}(n, \mathbb{C})$, $B =$ upper-triangular matrices in G
Weights: For $\lambda \in \mathbb{Z}^n$ let $h^\lambda = x_1^{\lambda_1} \cdots x_n^{\lambda_n}$ ($h = \mathrm{diag}[x_1, \ldots, x_n]$)
Dominant weights: $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n = 0$
Fundamental weights: $\lambda_i = [\underbrace{1, \ldots, 1}_{i}, 0, \ldots, 0]$ for $i = 1, \ldots, n-1$
Fundamental representations: $E_{\lambda_i} = \bigwedge^i \mathbb{C}^n$
Generating function: $\psi_i(g) = \det_i(g)$ (ith principal minor of g)

$$N^- H N^+ = \{g \in \mathrm{SL}(n, \mathbb{C}) \,:\, \psi_i(g) \neq 0 \quad \text{for } i = 1, \ldots, n-1\,\}$$

(LDU matrix factorization)

Let $K \subset G$ be an algebraic subgroup.

$\mathcal{O}[G]^{R(K)} =$ *right* K-fixed functions, $\quad E_\lambda^K = K$-fixed vectors in E_λ

Corollary. Under the left G action,

$$\mathcal{O}[G]^{R(K)} \cong \bigoplus_{\lambda \in P_{++}} E_\lambda \otimes E_{\lambda^*}^K$$

with G *acting by* $\pi_\lambda \otimes 1$ *on the* λ*-isotypic summand.*

- G/K is a complex manifold on which G acts holomorphically
- $\mathcal{O}[G]^{R(K)} \hookrightarrow$ holomorphic functions on G/K
- K *reductive* algebraic subgroup $\iff X = G/K$ is an affine algebraic G-space with $\mathcal{O}[X] = \mathcal{O}[G]^{R(K)}$ **[Matsushima]**

Definition. K is a *spherical* subgroup if $\dim E_\lambda^K \leq 1 \quad (\forall \lambda \in P_{++})$.

If K reductive algebraic:

K is a spherical subgroup $\iff G/K$ is multiplicity free

II. Representations on Symmetric Spaces

3. Complexified Iwasawa Decomposition

G_0 noncompact real form of G with maximal compact subgroup K_0
$\mathfrak{g}_0 = \mathrm{Lie}(G_0)$ (real semisimple Lie algebra)

Cartan decomposition: $\theta =$ Cartan involution: $(G_0)^\theta = K_0$
θ eigenspaces: $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ $G_0 = K_0 \exp \mathfrak{p}_0$

Iwasawa decomposition: $G_0 = K_0 A_0 N_0$ (analytic isomorphism)
$A_0 = \exp \mathfrak{a}_0$ ($\mathfrak{a}_0$ maximal abelian subspace of $\mathfrak{p}_0$)
$N_0 =$ nilpotent subgroup normalized by A_0

Complexifications:

$K =$ complexification of K_0 in G (reductive algebraic group)
$A =$ complexification of A_0 in G (algebraic torus of rank l)
$N =$ complexification of N_0 in G (unipotent subgroup)

Properties:

- KAN is Zariski-dense in G.
- $M = \mathrm{Cent}_K(A)$ is reductive and normalizes N.
- Cartan subgroup $H = AT$ where $T = H \cap K$ and $A \cap T$ is finite
- Borel subgroup B with $HN \subset B \subset MAN$

4. Spherical Representations

Proposition 1.

(1) **[Cartan]** *K is a spherical subgroup of G.*

(2) *If $\lambda \in \mathrm{Spec}(G/K)$ then*

$$t^{\lambda} = 1 \quad \text{for all } t \in T. \qquad (\star)$$

Proof: (1) KAN dense $\Longrightarrow B \cdot K$ open in G/K.
(2) Let $x_0 = K \in G/K$. Then $T = H_{x_0} \Longrightarrow (\star)$ by Theorem 2. □

Definition. λ is θ-*admissible* if it satisfies $(\star)$

Example. $G = \mathrm{SL}(n, \mathbf{C}) \quad \theta(g) = (g^t)^{-1} \quad K = \mathrm{SO}(n, \mathbb{C})$
$A = H$ (diagonal matrices in G)
$N =$ upper-triangular unipotent matrices
$M = T = \{\mathrm{diag}[\delta_1, \ldots, \delta_n] : \delta_i = \pm 1, \ \ \delta_1 \cdots \delta_n = 1\}$
$\lambda = [\lambda_1, \ldots, \lambda_{n-1}, 0]$ is θ-admissible $\Longleftrightarrow \lambda_i$ is even for all i.

Theorem 4. [Helgason] *Let (π_λ, E_λ) be an irreducible rational representation of G with highest weight λ (relative to B). The following are equivalent:*

(1) *λ is K-spherical $\quad (E_\lambda^K \neq 0)$.*

(2) *M fixes the B-highest weight vector in E_λ.*

(3) *λ is θ-admissible.*

Proof that (3) $\Longrightarrow$ (1): Define $v_0 = \int_{K_0} \pi^\lambda(k) e_\lambda \, dk \in E_\lambda^K$

Claim: $v_0 \neq 0$. By definition

$$\langle v_0, \, f_{\lambda^*} \rangle = \int_{K_0} \psi_\lambda(k) \, dk \qquad (\star\star)$$

(i) λ admissible $\Longrightarrow h^\lambda > 0$ for all $h \in H \cap G_0 = (T \cap G_0) \exp \mathfrak{a}_0$

(ii) $\psi_\lambda(g) \geq 0$ for $g \in G_0$ by (i)

$K_0 \subset G_0$ and property (ii) $\Longrightarrow$ the integral in ($\star\star$) is nonzero. $\square$

Example. $G_0 = \mathrm{SL}(n, \mathbb{R}) \quad \theta(g) = (g^t)^{-1} \quad K_0 = \mathrm{SO}(n)$

$$\psi_\lambda(g) = \det_1(g)^{m_1} \cdots \det_{n-1}(g)^{m_{n-1}} \qquad (m_i = \lambda_i - \lambda_{i+1})$$

λ is θ-admissible $\Longrightarrow$ all λ_i are even $\Longrightarrow \psi_\lambda(g) \geq 0$ on G_0. $\square$

There exist *fundamental* K-spherical highest weights

$$\mu_1, \ldots, \mu_l \quad \text{(linearly independent)}$$

so that

$$\Lambda = \{ m_1\mu_1 + \cdots + m_l\mu_l \, : \, m_i \in \mathbb{Z}_+ \}$$

is the set of K-spherical highest weights.

Corollary. *As a G-module* $\mathcal{O}[G/K] \cong \bigoplus_{\mu \in \Lambda} E_\mu$

5. Zonal Spherical and Horospherical Functions

For $\mu \in \Lambda$ choose K-fixed spherical vector: $e_\mu^K \in E_\mu$
MN-fixed conical vector: $e_\mu \in E_\mu$

Normalization: $\langle e_\mu, e_{\mu^*}^K \rangle = 1, \quad \langle e_\mu^K, e_{\mu^*}^K \rangle = 1$

Zonal spherical function: $\varphi_\mu \in \mathcal{O}[G]^{R(K)}$

$$\varphi_\mu(g) = \langle \pi_\mu(g) e_\mu^K, e_{\mu^*}^K \rangle \qquad (\varphi_\mu(1) = 1)$$

Transformation properties:

$$\varphi_\mu(kgk') = \varphi_\mu(g) \quad \text{for } k, k' \in K \text{ and } g \in G$$

Zonal horospherical function: $\Delta^\mu \in \mathcal{O}[G]^{R(MN)}$

$$\Delta^\mu(g) = \langle \pi_\mu(g) e_\mu, e_{\mu^*}^K \rangle \qquad (\Delta^\mu(1) = 1)$$

Transformation properties:

$$\Delta^\mu(kgman) = a^\mu \Delta^\mu(g) \quad \text{for } k \in K,\ g \in G,\ man \in MAN$$

Sylvester functions: $\Delta_j(g) = \Delta^{\mu_j}(g)$
(generalization of principal minors)

Product formula: For $\mu = m_1\mu_1 + \cdots + m_l\mu_l \in \Lambda$

$$\Delta^\mu(g) = \Delta_1(g)^{m_1} \cdots \Delta_l(g)^{m_l}$$

Set $\Omega = \{g \in G : \Delta_j(g) \neq 0 \quad \text{for } j = 1, \ldots, l\} \supset KAN$

Proposition 2. [Clerc] $\Omega = KAN$

Theorem 5. [Clerc] *Let $g = k(g)a(g)n(g)$ be the Iwasawa factorization in G_0.*

(1) *The function $g \mapsto n(g)$ extends holomorphically to a map from Ω to N.*

(2) *The functions $g \mapsto k(g)$ and $g \mapsto a(g)$ extend to multivalent holomorphic functions on Ω, with values in K and A, resp. The branches are related by elements of the finite group $A \cap K$.*

(3) *Let $g \mapsto \mathcal{H}(g)$ be the multivalent $\mathfrak{a}$-valued function on Ω such that $a(g) = \exp \mathcal{H}(g)$. Then $\Delta^{\mu}(g) = e^{\langle \mathcal{H}(g), \mu \rangle}$ for $g \in \Omega$.*

Corollary. [Clerc's Integral Formula] *For $g \in G$ let*

$$K_g = \{k \in K_0 \,:\, gk \in \Omega\}$$

Then K_g is an open set in K_0 whose complement has measure zero, and

$$\varphi_\mu(g) = \int_{K_g} e^{\langle \mathcal{H}(gk), \mu \rangle} \, dk$$

Application: [Clerc] Asymptotic behavior of $\varphi_\mu(u)$ as $\mu \to \infty$ in a cone (u a regular element of U) – use method of complex stationary phase

6. Horospherical Cauchy–Radon Transform

Let $\quad f \in \mathcal{O}[G]^{R(K)} \longleftrightarrow \{v_\mu \in E_\mu \,:\, \mu \in \Lambda\}$

Peter–Weyl expansion: $\quad$ ($v_\mu = 0$ except for finitely many μ)

$$f(g) = \sum_{\mu\in\Lambda} d(\mu)\langle v_\mu,\, \pi_{\mu^*}(g)e^K_{\mu^*}\rangle \qquad (d(\mu) = \dim E_\mu)$$

Definition. The *horospherical Cauchy–Radon transform* $\hat{f}$ of f is

$$\hat{f}(g) = \sum_{\mu\in\Lambda} \langle v_\mu,\, \pi_{\mu^*}(g)e_{\mu^*}\rangle$$

Algebraic description:

- Replace spherical vector by conical vector for each μ
- Divide by dimension factor (Plancherel measure) for each μ

Analytic description:

Theorem 6. [Gindikin] *The map $f \mapsto \hat{f}$ is a G isomorphism between $\mathcal{O}[G]^{R(K)}$ and $\mathcal{O}[G]^{R(MN)}$. One has*

$$\hat{f}(g) = \sum_{\mu\in\Lambda} \int_X f(u)\Delta^\mu(u^{-1}g)\,du \qquad \textit{for} \quad g \in G$$

(integrals over compact symmetric space $X = U/K_0$)

Proof: $\quad \mathcal{O}[G]^{R(K)} \cong \bigoplus_{\mu\in\Lambda} E_\mu \cong \mathcal{O}[G]^{R(MN)} \quad$ (Helgason)

Intertwining maps $\quad e^K_{\mu^*} \mapsto d(\mu)^{-1}e_{\mu^*}$ (use Schur orthogonality) $\quad \square$

Double fibration: $(\dim Z = \dim \Xi, \quad \dim N = \dim K/M)$

$$
\begin{array}{ccccc}
 & & G & & \\
(g\cdot MN)K \cong N & \swarrow & & \searrow & (g\cdot K)MN \cong K/M \\
\text{(horospheres)} \quad Z = G/K & & & & G/MN = \Xi \quad \text{(pseudospheres)}
\end{array}
$$

Inversion of Cauchy–Radon Transform:

Invariant differential operator $\phi(D)$ on $A \longleftrightarrow$ symbol $\phi(\mu) \in \mathcal{P}(\mathfrak{a}^*)$:

$$\phi(D)a^\mu = \phi(\mu)a^\mu \quad \text{for } a \in A$$

Weyl dimension formula: $d(\mu) = \prod_{\alpha>0} \dfrac{(\mu+\delta,\alpha)}{(\delta,\alpha)} \qquad (\delta = \tfrac{1}{2}\sum_{\alpha>0}\alpha)$

Weyl operator:

$W(D) =$ invariant differential operator on A with symbol $d(\mu)$

Fiber bundle:

$$
\begin{array}{lcl}
\text{(quasi-projective)} & G/MN = \Xi & (\text{right } A \text{ action}) \\
 & \downarrow & \\
\text{(projective)} & G/MAN = F & \text{(flag manifold)}
\end{array}
$$

$W(D) : \mathcal{O}[G]^{R(MN)} \longrightarrow \mathcal{O}[G]^{R(MN)}$ (differentiation along the fibers)

$W(D)$ commutes with left translations by G

Theorem 7. [Gindikin] *Let* $f \in \mathcal{O}[G]^{R(K)}$. *Then*

$$f(g) = \int_{K_0/M_0} (W(D)\hat{f})(gk)\,dk \qquad \text{for } g \in G$$

(integral over the flag manifold for noncompact dual symmetric space)

Proof. Integral over K_0 takes conical vectors $\to$ spherical vectors

7. Cauchy–Radon Transform as a Singular Integral

Complex symmetric space: origin $x_0 = K \in Z = G/K$

Complex horospheric manifold: origin $\zeta_0 = MN \in \Xi = G/MN$

For $z = y \cdot x_0 \in Z$ ($y \in G$) and $\zeta = g \cdot \zeta_0 \in \Xi$ ($g \in G$) let

$$\Delta_j(z \mid \zeta) = \Delta_j(y^{-1}g) \quad \text{for } 1 \le j \le l$$

Define a meromorphic function on $Z \times \Xi$ by

$$K(z \mid \zeta) = \prod_{1 \le j \le l} \frac{1}{1 - \Delta_j(z \mid \zeta)}$$

- G invariance: $K(g \cdot z \mid g \cdot \zeta) = K(z \mid \zeta)$ for $g \in G$
- Singular set of $K(z \mid \zeta)$ in $Z \times \Xi$ is $\bigcup_{1 \le j \le l} \{\Delta_j(z \mid \zeta) = 1\}$

Let $X = U/K_0$ (compact symmetric space). Define

$$\Xi(0) = \{\zeta \in \Xi \, : \, |\Delta_j(x \mid \zeta)| < 1 \quad \text{for } 1 \le j \le l \text{ and } x \in X\}$$

- $U \cdot \Xi(0) = \Xi(0)$ (by definition)
- Product formula for Δ^μ $\Longrightarrow$

$$K(x \mid \zeta) = \sum_{\mu \in \Lambda} \Delta^\mu(u^{-1}g) \quad \text{(absolutely convergent series)}$$

for $x = u \cdot x_0 \in X$ and $\zeta = g \cdot \zeta_0 \in \Xi(0)$

Lemma

(1) $(U/M_0) \times A \longrightarrow \Xi$ *by* $(u, a) \mapsto u \cdot \zeta_0 \cdot a$ *is regular and surjective.*

(2) **[Clerc]** *Let* $A_+ = \{a \in A \, : \, |a^{\mu_j}| < 1$ *for* $j = 1, \ldots, l\}$. *Then* $U \cdot \zeta_0 \cdot A_+ \subset \Xi(0)$. *Hence* $\Xi(0)$ *is a nonempty open set in* Ξ.

Let $\nu = gU \in G/U$ (the space of compact real forms of G). Set

$$\begin{aligned} X(\nu) &= g \cdot X \quad \text{(compact totally-real cycle in } Z\text{)} \\ \Xi(\nu) &= g \cdot \Xi(0) \quad \text{(nonempty open set in } \Xi\text{)} \end{aligned}$$

Then

$$\Xi = \bigcup_{\nu \in G/U} \Xi(\nu) \qquad \text{(parameter space } G/U \text{ is contractible)}$$

Theorem 8. [Gindikin] *For $f \in \mathcal{O}[Z]$ the horospherical Cauchy–Radon transform is given on each set of the covering $\{\Xi(\nu)\}$ by the Cauchy-type singular integral*

$$\hat{f}(\zeta) = \int_{X(\nu)} f(x) K(x \mid \zeta)\, dx \quad \text{for} \quad \zeta \in \Xi(\nu)$$

(the integrand is continuous on $X(\nu)$).

Proof. Use series formula for $K(x \mid \zeta)$ when $\zeta \in \Xi(0)$. Then translate by $g \in G$. □

Concluding Remarks

- We have carried out the harmonic analysis of finite functions on a compact symmetric space using algebraic group and Lie group methods, extending the results of Cartan and Weyl.
- A compact symmetric space has a canonical dual object that is a complex manifold.
- The integral formulas for the direct and inverse horospherical Cauchy–Radon transform hold for all holomorphic functions on X and Ξ (not just the G-finite functions), and for hyperfunctions (Gindikin).
- Gindikin develops this transform using complex analysis and integral geometry (Japanese J. of Math. 2006).
- *Another problem:* holomorphic extension of real analytic functions on a compact symmetric space. These functions extend holomorphically to complex neighborhoods of the space.
- The geometric and analytic properties of these neighborhoods were studied by B. Beers and A. Dragt, L. Frota-Mattos, and M. Lasalle in the 1970's.