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Using Technology Offstage in Linear Algebra Courses

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I have created MATLAB-based instructional labs for three linear algebra courses at Rutgers:

- An introductory undergraduate course (required for math, computer science, and chemistry students and taken as a technical elective by upper-level engineering students)
- A course for first-year graduate engineering students
- An upper-level undergraduate course emphasizing linear transformations, finite Fourier transform, and finite wavelet transforms (math, computer science, and engineering students)

The labs for the first and second courses use some of the T-codes developed for the M.I.T. linear algebra course. They have been revised several times over the last six years and have been used by many faculty members besides myself. I created the third course last year; in the current semester I am teaching it again and revising the lab assignments and accompanying lecture notes.

The labs use the MATLAB computational environment to illustrate all the major topics in the courses. They are written so that the lecturer does not need to devote class time to the labs. Because of this, faculty having no experience or particular interest in the computational aspects of linear algebra have been willing to teach the MATLAB sections of the introductory course and the graduate course.

A significant part of each lab assignment is the computational exploration (using randomly-generated matrices) of the linear algebra concepts presented in the lectures. At the beginning of the interactive lab session the student enters a personal identification number to seed the MATLAB random number generator; this prevents sharing of lab reports. Students create diaries of their MATLAB sessions and write comments to explain, from a theoretical standpoint, the numerical results that they have obtained. Each assignment also includes applied problems, so that students can see the power and range of the linear algebra results that they have been learning in class.

In each course the MATLAB assignments are done out of class and count for 25% of the course grade (5 or 6 assignments). Details of MATLAB programming (such as m-files) are included so that students with no prior MATLAB experience can complete the assignments with a minimum of outside help. Graders (upper-level undergraduates) are hired for each section of the two undergraduate courses; with the help of a detailed grading key and some initial assistance by the course instructors they grade the lab writeups. Electronic submission of assignments is not used because a large part of the grading is based on free-form answers to questions about the MATLAB output, and graders write comments and queries on the student's paper. These answers often involve (algebraic) hand calculation of the type the students use on homework and exams.

Although students work on the MATLAB assignments "offstage" (outside of class time), the lab experience reinforces the ideas presented in class and helps to bridge the gap between theory and computation. Student comments in the course evaluations consistently mention the MATLAB components of the courses as outstanding features.

The instructional materials are posted on the course web pages, which can be accessed via `http://math.rutgers.edu/~goodman`. The courses and the topics covered in the labs are as follows:

Math 250: Introduction to Linear Algebra

Text: Spence, Insel & Friedberg *Elementary Linear Algebra: A Matrix Approach*

Enrollment: Roughly 50-50 split between Freshman-Sophomore Math, Statistics, Computer Science, Chemistry majors (required course) and Junior-Senior engineering students (technical elective). Annual enrollment (including summer sessions) is 750-800 students in 31 small sections (this is the only large-enrollment math course that is still taught in small sections at Rutgers—New Brunswick). Students choose between 26 sections not using MATLAB or other technology, and 5 sections using MATLAB. Both modes have the same number of class meetings (28 eighty-minute periods), the same syllabus, and carry 3 credit hours. Two 80-minute midterm exams and a three-hour final exam.

LAB 1: Matrix and Vector Computations in Matlab

- Creating matrices and vectors
- Adding matrices and multiplying a vector by a matrix
- Finding the reduced row echelon form of a matrix
- Using matrices in the Leontief input-output economic model

LAB 2: Linear Equations and Matrix Algebra

- Solving a system of linear equations by the reduced row echelon form
- Forming linear combinations of a set of vectors and testing for linear independence
- Matrix multiplication and its properties
- Application of Adjacency Matrices to Communication Networks

LAB 3: *LU* Decomposition and Determinants

- LU decomposition of an invertible matrix A
- Solving $A\mathbf{x} = \mathbf{b}$ using the LU decomposition
- Computation time to solve $A\mathbf{x} = \mathbf{b}$ by Gaussian elimination and by LU decomposition
- The determinant of a matrix: effects of row operations and matrix multiplication, and efficient calculation by the LU decomposition
- The geometric properties of matrices for rotations, dilations, and shears

LAB 4: General Solution to $A\mathbf{x} = \mathbf{b}$

- The column space $\mathrm{Col}(A)$
- The null space $\mathrm{Null}(A)$
- Particular solutions to $A\mathbf{x} = \mathbf{b}$
- The complete solution of $A\mathbf{x} = \mathbf{b}$
- Application of the theory of inhomogeneous linear equations to a traffic flow problem

LAB 5: Eigenvalues and Eigenvectors

- Geometric meaning of eigenvalues and eigenvectors
- Determination of eigenvalues and eigenvectors using the characteristic polynomial
- Use of eigenvectors to transform a matrix to diagonal form
- Steady-state eigenvector for a transition matrix
- Applications of eigenvalues and eigenvectors to Markov chains

LAB 6: Orthonormal Bases, Orthogonal Projections, and Least Squares

- Geometric aspects of vectors — norm, dot product, and orthogonal projection onto a line
- Gram-Schmidt Algorithm

- Orthogonal projection of a vector onto a subspace
- Best approximate solution to an inconsistent linear system $A\mathbf{x} = \mathbf{b}$
- Method of least squares for fitting straight lines and parabolas to data points

Math 550: Linear Algebra and Applications

Text: Gilbert Strang, *Linear Algebra and its Applications*

Enrollment: First-year graduate engineering students (predominantly computer & electrical engineering). It has 28 eighty-minute classes and 3 credit hours. One 80-minute midterm exam and a three-hour final exam.

There are five MATLAB assignments in this course:

LAB 1: Gaussian Elimination, LU Factorization, and Solving $A\mathbf{x} = \mathbf{b}$

LAB 2: Orthogonal Projections, the Four Fundamental Subspaces, QR Factorization, and Inconsistent Linear Systems

LAB 3: Determinants, Eigenvalues, and Eigenvectors

These three labs are similar to the Math 250 labs covering the listed topics—additional items are:

- The four fundamental subspaces associated with a matrix
- QR matrix factorization
- Determinant formula for the inverse of a matrix

The fourth and fifth labs cover topics not in the Math 250 course:

LAB 4: Unitary Diagonalization of Matrices, QR Algorithm, Finite Fourier Transform, and Fast Fourier Transform

- Diagonalization of hermitian and normal matrices by unitary matrices
- QR algorithm for fast computation of eigenvalues
- Fourier matrix and Fourier basis for $\mathbf{C}^n$
- Diagonalization of circulant matrices by the Fourier matrix
- Fast Fourier transform

LAB 5: Positive-Definite Matrices, Cholesky Factorization, Singular Value Decomposition, and Digital Image Processing

- Tests for positive-definiteness of a real symmetric matrix
- Cholesky factorization of a positive-definite matrix
- Singular value decomposition (SVD)
- Digital image processing using the SVD

Math 357: Topics in Applied Algebra

Texts:
S. Leon: *Linear Algebra with Applications* (chapters 3-5)
A. Jensen & A. la Cour-Harbo: *Ripples in Mathematics: The Discrete Wavelet Transform*
R. Goodman: *Discrete Fourier Transform and Wavelet Transforms* (lecture notes)

Enrollment: Junior-senior math, computer-science, and engineering students. The prerequisites are the Math 250 course and multivariable calculus. No advanced calculus or real analysis is required. This course is a technical elective for engineering students (as an alternative to the theoretically-oriented second linear algebra course offered by the department). It has 28 eighty-minute classes and 3 credit hours. Two 80-minute midterm exams and a three-hour final exam.

There are five MATLAB assignments in the course, and more emphasis is placed on the applications to signal processing and image analysis.

LAB 1: Visualizing Linear Transformations

- Encoding two-dimensional polygonal figures as matrices
- Rotations, dilations, and shearing transformations
- Homogeneous coordinates and affine transformations

LAB 2: Convolution and Finite Fourier Transform

- Fourier matrix and Fourier basis for $\mathbf{C}^n$
- Applications of the finite Fourier transform—touch-tone dialing and the spectrum of a train whistle
- Discrete periodic signals and convolution—diagonalization of circulant matrices
- Fast Fourier transform

LAB 3: Haar Wavelet Transform

- Haar wavelet basis, the Haar analysis matrix, and the Haar synthesis matrix
- Fast Haar transform implemented by lifting
- Applications of the Haar transform—analysis of synthetic signals, filtering and compressing a noisy signal

LAB 4: Implementation of Wavelet Transforms

- The CDF(2,2) wavelet transform (matrix version)
- The Daub4 wavelet transform (matrix version)
- Fast Daub4 wavelet transform
- Multiresolution analysis using the CDF(2,2) wavelet transform—analysis of synthetic signals, filtering and compressing a noisy signal

LAB 5: Wavelet Analysis of Two-Dimensional Images

- Two-dimensional discrete wavelet transform
- Multiscale analysis of images
- Fast two-dimensional wavelet transform
- Denoising and compressing images by wavelet methods