Probabilistic set theory, aka forcing

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0 Introduction

Forcing is probability theory. To illustrate this, here is a short proof of the consistency of $V \neq L$. The usual von Neumann hierarchy is generated by taking power sets:

$$V_0 = \varnothing,$$

$$V_{\alpha+1} = \mathcal{P}(V_\alpha),$$

$$V_\alpha = \bigcup_{\beta<\alpha} V_\beta \text{ if } \alpha \text{ is a limit},$$

$$V = \bigcup_{\alpha\in\mathrm{Ord}} V_\alpha.$$

To make things probabilistic, define a new hierarchy V^r_α as above except that we take the *random power set* at successor stages, whatever that means, so $V^r_{\alpha+1}$ consists of all random subsets of V^r_α. The idea is then to show that the resulting "probabilistic von Neumann universe" V^r models ZFC with probability 1. Now let G be a random subset of ω such that each natural number n belongs to G with probability 1/2. For every constructible set A of natural numbers (or indeed any "deterministic" set), the probability that G equals A is zero: say $A = \varnothing$, then $P(G = \varnothing) = \prod_{n=0}^{\infty} P(n \notin G) = \prod_{n=0}^{\infty} \frac{1}{2} = 0$. We conclude that a random subset of natural numbers is non-constructible, so $V \neq L$ holds in V^r with probability 1, in particular it is consistent.

Technically the above proof is nonsense, but it is not too far from the spirit of forcing, and in fact pretty close to the "Boolean-valued model" approach. Can we find a notion of "random subsets" that makes this proof rigorous? A first thought might be to mimic fuzzy set theory and define a fuzzy von Neumann hierarchy by letting $V^r_{\alpha+1}$ be the set of all functions $u : V^r_\alpha \to [0,1]$; we think of such a function u as a "random set", and $u(x)$ as the "probability" that x belongs to u. This doesn't quite work because of the incompatibility of fuzzy logic with classical logic. Instead, since classical logic and Boolean algebra are close friends, it is not unreasonable to let $V^r_{\alpha+1}$ consist of functions $u : V^r_\alpha \to B$ where B is some fixed Boolean algebra. A Boolean algebra is a structure $(B, \vee, \wedge, *, 0, 1)$ that behaves similar to an algebra of sets like $(\mathcal{P}(X), \cup, \cap, ^c, \varnothing, X)$; for precise definition see Section 2. It's harmless to think of B as a σ-algebra, its elements as events, and the operations in terms of Venn diagrams.

Having defined the probabilistic hierarchy, we next need to define the Boolean value, or probability, of a formula φ; it is going to be an element of B and denoted $[\![\varphi]\!]$. The propositional case is straightforward; for example, $[\![\varphi \wedge \psi]\!]$ is just $[\![\varphi]\!] \wedge [\![\psi]\!]$, where the second $\wedge$ means the meet operation in the Boolean algebra B. Quantifiers are also easy to handle once we add the requirement that B is a complete Boolean algebra. The most difficult case turns out to be atomic formulas, i.e., given random sets u, v how to define $[\![u = v]\!]$ and $[\![u \in v]\!]$. For a function $u : V^r_\alpha \to B$, we might be tempted to think $u(x)$ is the "probability" of $x \in u$ and hence coincides with $[\![x \in u]\!]$, but actually it's more complicated. Here is a simple example illustrating the subtlety. Suppose x, y, z are three unrelated random sets, u contains x with probability a and y with probability b, while v contains y with probability c and z with probability d, and they don't contain anything else. In symbol:

$$u = \{(x, a), (y, b)\}$$

$$v = \{(y, c), (z, d)\}$$

What should be $[\![u = v]\!]$, the probability that u equals v? There seem to be two ways for them to be equal: either $u = v = \{y\}$ or $u = v = \varnothing$. The probability of $u = \{y\}$ is $a^* \wedge b$, and $v = \{y\}$ is $c \wedge d^*$, so $u = v = \{y\}$ is $a^* \wedge b \wedge c \wedge d^*$. Similarly, $u = v = \varnothing$ has probability $a^* \wedge b^* \wedge c^* \wedge d^*$. Altogether, this suggests we define

$$[\![u = v]\!] = (a^* \wedge b \wedge c \wedge d^*) \vee (a^* \wedge b^* \wedge c^* \wedge d^*) =: r.$$

So far so good. Next consider

$$w = \{(u, p), (v, q)\}$$

What is $[\![u \in w]\!]$? Certainly it should be *at least* p, but:

- $[\![v \in w]\!]$ should be at least q;
- $[\![u = v]\!] = r$;
- we would like $u = v \wedge v \in w \to u \in w$ to be true with probability 1, so we probably want $[\![u = v]\!] \wedge [\![v \in w]\!] \le [\![u \in w]\!]$.

So $[\![u \in w]\!]$ should also be at least $r \wedge q$. Altogether, $[\![u \in w]\!]$ is at least $p \vee (r \wedge q)$, which will actually be our definition of $[\![u \in w]\!]$. In general, $u(x) = a$ should be interpreted as "x belongs to u with probability *at least* a", so $u(x) \le [\![x \in u]\!]$. As can be seen from this example, the calculation of probabilities of atomic formulas is complicated by the fact that random sets can in turn belong to other random sets. Fortunately, once we figure out the definition of $[\![u = v]\!]$ and $[\![u \in v]\!]$, the rest is relatively straightforward, including showing that the hierarchy satisfies ZFC with probability 1.

This concludes our sketchy overview of the Boolean-valued model approach. Section 1 will be an elaboration of the above discussion. A bit of history: Cohen originally used the ctm approach to forcing; he did not use either poset or Boolean algebra explicitly. It was noticed by several people that his method could be interpreted as building a Boolean-valued model. However, people soon realized that while Boolean-valued model might be more intuitive, ctm and poset are more convenient to work with in practice.

There is probably no single best reference for forcing. Many people just try to learn forcing from different sources multiple times and struggle with them, until one day they suddenly understand everything and write their own notes. Kunen [12] is the standard reference for the ctm approach to forcing. Standard references for the Boolean-valued model approach are Bell [3] and Jech [9, Chapter 14]. Jech gives a more complete account, including the relation between the two approaches, but Bell might be more beginner-friendly. See also writings by Matteo Viale such as [14–16]. Yet another way to interpret forcing is to use topoi and sheaves [8].

1 You could have invented forcing

This whole section is aimed at motivating forcing, so feel free to skip it if at some point it starts to create more confusion than motivation.

1.1 How not to do forcing

A basic application of forcing is the consistency of, e.g., $\mathsf{ZFC} + \neg\mathsf{CH}$; a more modest goal is to prove the consistency of $\mathsf{ZF} + V \neq L$, which will be our focus in this section. First let's see why it is impossible to achieve $V \neq L$ with inner model, as observed by Shepherdson and independently Cohen [4]. The discussion below is based on the beginning part of Kunen [12, Chapter IV].

What is an inner model? The basic example is the constructible universe L, which is used to show the consistency of AC and GCH. Let's recall the overall logic of consistency proof using L. We write down a formula $x \in L$, which is the abbreviation of $\exists\alpha(x \in L_\alpha)$, where $x \in L_\alpha$ is in turn the abbreviation of some complicated formula. Then we show the theorem scheme that "L is a class model of ZF" in the sense that for each ZF axiom φ, the relativization φ^L is a theorem of ZF; the relativization φ^L is defined inductively by $(\varphi \wedge \psi)^L := \varphi^L \wedge \psi^L$, $(\neg\varphi)^L := \neg(\varphi^L)$ and $(\forall x\varphi)^L := \forall x(x \in L \to \varphi^L)$.

Moreover, AC^L and GCH^L are also theorems of ZF. Even better, abbreviating the statement $\forall x\exists\alpha(x \in L_\alpha)$ ("all sets are constructible") as $V = L$, we can show that $(V = L)^L$ is a theorem of ZF, and $\mathsf{ZF} + V = L$ proves both AC and GCH. By an induction in the metatheory, we show that whenever $\mathsf{ZF} + V = L$ proves a sentence φ, the relativization φ^L is provable in ZF, so if $\mathsf{ZF} + V = L$ is inconsistent, namely it proves a contradiction $\varphi \wedge \neg\varphi$, then ZF is already inconsistent, since it proves $\varphi^L \wedge \neg\varphi^L$. Taking contrapositive, if ZF is consistent then so is $\mathsf{ZF} + V = L$, and hence $\mathsf{ZFC} + \mathsf{GCH}$.

In general, an inner model is essentially a formula $M(x)$, possibly with other free variables as parameters, such that for each ZF axiom φ, the relativization φ^M is a theorem of ZF, where φ^M is defined inductively by, e.g., $(\forall x\varphi)^M := \forall x(M(x) \to \varphi^M)$. Examples of inner models include the two types of relative constructible hierarchy $L(A)$ and $L[A]$, and the class of hereditarily ordinal definable sets HOD.

Now say we want to prove the consistency of $\mathsf{ZF} + V \neq L$. The inner model method cannot possibly work. More precisely, working in ZFC, one cannot find a formula $M(x)$ that provably defines an inner model M which violates CH or even $V = L$. Because if there were such a formula $M(x)$ that works in ZFC, then of course it would also work in $\mathsf{ZF} + V = L$, but if $V = L$ holds then the class defined by $M(x)$ must be the same as L, since L can be shown to be the smallest inner model (a class model of ZF containing all ordinals), so M satisfies $V = L$, a contradiction.

1.2 The ctm approach

Since inner model cannot work, a natural thought is to try the other direction: start with a ground model and expand it instead of shrinking it. We cannot let the ground model be the whole universe V, because V is already everything and there is nothing outside to add into V (so long as we stick to transitive models); we cannot let the ground model be L either, since it could be equal to V. So maybe instead of class model, let's start with a transitive set model M. An issue is that by Gödel's second incompleteness theorem, ZFC cannot prove the existence of such an M, but this can be circumvented in several ways, see for example Kunen [12, Chapter IV.5]; for now let's pretend there is such an M. Once we manage to construct a strictly larger transitive model $N \supsetneq M$ with the same ordinals, i.e., $M \cap \mathrm{Ord} = N \cap \mathrm{Ord}$, we get the consistency of $\mathsf{ZFC} + V \neq L$: by absoluteness of $\alpha \mapsto L_\alpha$, $L^N = L^M \subseteq M \subsetneq N$, so $N \models V \neq L$.

Now M shouldn't be a set that is too large either, such as V_κ, because there is no strictly bigger model N with the same ordinals (a strictly bigger N would contain something of rank at least κ, and if N satisfies any modest set theory it must contain κ). This suggests that M should be as small as possible, perhaps countable. So let M be a fixed *countable transitive model* (ctm); we also call it the *ground model*. M contains only countably many, say, subsets of ω, aka reals, so there are many reals outside of M that we potentially can add. The optimistic hope is that after throwing in a new real and perhaps closing it up under suitable operations, we do get a model of ZFC.

Consider some $G \subseteq \omega,\ G \notin M$; the letter G stands for *generic*, and the reason will become clear later. We want to throw G into M to get a new model denoted $M[G]$, called the *generic extension*. We will define $M[G]$ in a way that makes sense for any $G \subseteq \omega$ (even non-generic ones), and it will always be transitive and satisfy $M \cap \mathrm{Ord} = M[G] \cap \mathrm{Ord}$, but $M[G]$ may not satisfy any reasonable set theory if we choose a bad G. For example, since M is countable, $M \cap \mathrm{Ord}$ is a countable ordinal ρ, and there is a well-order $G \subseteq \omega \times \omega$ isomorphic to ρ. Using the bijection $f : \omega \times \omega \to \omega,\ (m, n) \mapsto (2m+1)2^n - 1$, we may also view G as a subset of ω. Then $M[G]$ cannot be a model of ZFC; in fact if $N \supseteq M \cup \{G\}$ satisfies any reasonable set theory, it should be able to "decode" G and therefore $\rho \in N$, so $M \cap \mathrm{Ord} \neq N \cap \mathrm{Ord}$. Bear in mind that our strategy to get $N \models V \neq L$ relies on $L^N = L^M$, which requires $M \cap \mathrm{Ord} = N \cap \mathrm{Ord}$. Can we fix this by choosing some G that does not code ρ? But then it might code ρ via a different bijection $f : \omega \times \omega \to \omega$, or code a countable cofinal sequence of ρ, or perhaps the countability of M itself. Thus it seems a daunting task to choose an appropriate $G \subseteq \omega$.

Surprisingly, it turns out if we choose a "generic" G it would most likely work. Here "generic" roughly means G belongs to all the relevant dense open sets. The power set $\mathcal{P}(\omega)$ is naturally identified with the space ${}^{\omega}2$ of infinite 0-1 sequences, aka the Cantor space; it is a compact metric space under the metric $d(x, y) = \sum_{i=0}^{\infty} \frac{|x_i - y_i|}{2^{i+1}}$. By the Baire Category Theorem, the intersection of any countably many dense open subsets of ${}^{\omega}2$ is dense, in fact comeager. Roughly speaking, we can choose any G from the intersection of all the dense open sets that "morally" belong to M, of which there are countably many; note that a dense open set D cannot literally belong to M, since M is countable transitive while D is uncountable, but as will be explained later an open subset of ${}^{\omega}2$ can be identified with a subset of the complete binary tree, which could belong to M. It takes some time to appreciate the importance of genericity, but here is a simple example: for any bijection $f : \omega \times \omega \to \omega$ that belongs to M, the set of all G that do not code a well-ordering under f is dense open, so choosing a generic G resolves the problem discussed in the previouse paragraph.

Digressing from how to choose the right G, let's now think about how to define the generic extension $M[G]$. Of course it's not $M \cup \{G\}$. Along with the set G we must also add all sets "generated by G over M", such as $\omega \setminus G$, $G \times (\omega \setminus G)$, $\{G, \omega \setminus G\}$, $\{(2m+1)2^n - 1 : m, n \in G\}$, etc. A natural candidate for $M[G]$ is the closure of $M \cup \{G\}$ under the Gödel operations (cf. [9, Definition 13.6]); indeed this was Cohen's original approach [5]. The modern definition is equivalent to Cohen's, but has a slightly different perspective.

The key idea is this: every set in $M[G]$ should have a "name" in M; although people living in M do not have access to G, for example they can't decide whether $1 \in G$ or $3 \in G$, they should nevertheless be able to "reason about" the new sets to some extent. This is a bit analogous to field extension (warning: this is only a loose analogy and not every logician likes it [10]). Say we want to adjoin square root of 2 to $\mathbb{Q}$. Pretend that we don't know anything about $\mathbb{C}$, so we have to construct the square root "from scratch" by taking the quotient $\mathbb{Q}[x]/(x^2 - 2) = \mathbb{Q}[\sqrt{2}]$. Every element in the extension has a name in $\mathbb{Q}[x]$, for example $2x + 1$ is a name for $2\sqrt{2} + 1$, while $x^3 + 1$ is another name for the same number. Although we as people living in $\mathbb{Q}$ cannot see $\sqrt{2}$, we know for sure that $2x + 1$ and $x^3 + 1$ name the same number in the extension $\mathbb{Q}[\sqrt{2}]$, and that number is, e.g., irrational.

Back to set theory. Imagine that we live in M, and we introduce a new symbol $\dot{G}$ as our "name" for the generic real $G \subseteq \omega$. Then we can reason about $\omega \setminus \dot{G}$, $\dot{G} \times (\omega \setminus \dot{G})$, $\{\dot{G}, \omega \setminus \dot{G}\}$, etc. For example, although we can't decide whether $(1, 3) \in \dot{G} \times (\omega \setminus \dot{G})$, we know that it holds just in case

$1 \in \dot{G}$ and $3 \notin \dot{G}$.

Formally, consider the set $\mathbb{P} = \mathrm{Fn}(\omega, 2)$ of finite partial functions from ω to $2 = \{0, 1\}$; elements of $\mathbb{P}$ are called *forcing conditions*, and are thought of as partial information about the generic real G. For example, the function $\{(1,1),(3,0),(5,1)\}$ intuitively means $1 \in G \wedge 3 \notin G \wedge 5 \in G$. If $p, q \in \mathbb{P}$ and $p \supseteq q$ (i.e., p as a function extends q), then intuitively p "carries more information" than q; for example $\{(1,1),(3,0),(5,1)\}$ tells us more about the generic real than $\{(1,1),(3,0)\}$. On the other hand, if $n \in \mathrm{dom}(p) \cap \mathrm{dom}(q)$ and $p(n) \neq q(n)$, then p and q carry "contradictory information". It is customory to partially order $\mathbb{P}$ by *reverse inclusion*, namely $p \leq q$ iff $p \supseteq q$, in which case we also say that p is "stronger than" q, so conditions that carry more information are stronger. The empty function $\varnothing$ carries no information, and hence is the maximum (hence weakest) element of $\mathbb{P}$; it is usually denoted $1_{\mathbb{P}}$.

It should be self-explanatory what it means for $G \subseteq \omega$ to *satisfy* a condition $p \in \mathbb{P}$. For example, if $p = \{(1,1),(3,0)\}$, then G satisfies p iff $1 \in G$ and $3 \notin G$. Put another way, if we consider the characteristic function $\chi_G \in {}^{\omega}2$ of G, then G satisfies p iff $\chi_G \supseteq p$.

The partial order $\mathbb{P}$ also allows us to give a more precise definition of genericity. There is a close relation between $\mathbb{P}$ and ${}^{\omega}2$; indeed, the sets $N_p := \{f \in {}^{\omega}2 : f \supseteq p\},\ p \in \mathbb{P}$ constitute a countable base for the topology on ${}^{\omega}2$. Call a set $D \subseteq \mathbb{P}$ *open* if $p \in D$ and $q \leq p$ imply $q \in D$, and *dense* if any $p \in \mathbb{P}$ has an extension q (namely $q \leq p$) that belongs to D. A dense open subset U of the space ${}^{\omega}2$ gives rise to a dense open subset D of the partial order $\mathbb{P}$, namely $D = \{p : N_p \subseteq U\}$; conversely, given D we can recover U by $U = \bigcup_{p \in D} N_p$. We thus define $G \subseteq \omega$ to be *generic for M* if:

for any dense open set $D \subseteq \mathbb{P}$ that is in M, there is some $p \in D$ such that G satisfies p.

Since M has only countably many dense open subsets of $\mathbb{P}$, and to say that G satisfies some $p \in D$ is the same as saying χ_G belongs to the corresponding dense open subset U of ${}^{\omega}2$, we see from Baire Category Theorem that generic reals G are abundant. We now summarize how the ctm approach to forcing works:

1. (everything has a name) Define the class $M^{\mathbb{P}}$ of $\mathbb{P}$-*names* in M. Devise an algorithm that, given $G \subseteq \omega$, *interprets* a name $\sigma \in M^{\mathbb{P}}$ into a set σ_G. Define the extension $M[G]$ to be $\{\sigma_G : \sigma \in M^{\mathbb{P}}\}$.

2. (names are meaningful) Show that for any formula $\varphi(x_1, \dots, x_n)$ and any names $\sigma_1, \dots, \sigma_n \in M^{\mathbb{P}}$, there is a set $C \subseteq \mathbb{P}$ of conditions in M, which might be called the "truth set" of φ, such that for any generic $G \subseteq \omega$, we have

$$M[G] \models \varphi((\sigma_1)_G, \dots, (\sigma_n)_G) \text{ iff } G \text{ satisfies some condition in } C.$$

   In fact we will show something stronger: for any fixed φ, the map that sends $\sigma_1, \dots, \sigma_n$ to the truth set C is definable in M.

3. (everything works out) Show that if $G \subseteq \omega$ is generic, then $M[G]$ is a model of ZFC.

For now let us not go into further detail about the ctm approach, except mentioning that Step 2 is the "real meat", and once it is figured out Step 3 will be relatively straightforward. Instead of $\mathrm{Fn}(\omega, 2)$, the theory works equally well for an arbitrary partial order (or even preorder) $\mathbb{P}$, often assumed to have a maximal element $1_{\mathbb{P}}$ for convenience; in practice $\mathbb{P}$ is often the set of partial approximations to the object we want to add by forcing.

Also, something used a lot (often implicitly) in the ctm approach is the *absoluteness* of various properties between the ctm M and the real world V, e.g., various transfinite recursions have the same result whether computed in M or in V, cf. Kunen [12, I.16, II.4]. For beginner this adds another layer of complexity.

1.3 The Boolean-valued model approach

The Boolean-valued model approach, though essentially the same as the ctm approach, is in my opinion more intuitive and beginner-friendly. It stems from two observations about the ctm approach:

- Replacing the partial order $\mathbb{P}$ by a complete Boolean algebra B simplifies the arithmetic of truth sets.

  For our basic example, we replace $\mathbb{P} = \mathrm{Fn}(\omega, 2)$ by its Boolean completion B. Roughly speaking, each element of B corresponds to a subset of $\mathbb{P}$. It is also possible to define B explicitly, say as the collection of regular open subsets of ${}^{\omega}2$.

- For the sake of consistency proof, there is really no need to choose an actual generic G and pass to the generic extension $M[G]$. We might as well reason entirely in the ground model M.

  The sole purpose of choosing M to be a ctm is to ensure generic G exist. Once we get rid of G we might as well choose V to be our ground model; this is called *forcing over the universe*.

The Boolean-valued model approach can then be summarized as follows:

1. (everything has a name) Fix a complete Boolean algebra $B = (B, \vee, \wedge, *, 0, 1)$. Define the class V^B of *B-names*.
2. (names are meaningful) To every formula $\varphi(x_1, \ldots, x_n)$ and any names $\sigma_1, \ldots, \sigma_n \in V^B$ assign an element $[\![\varphi(\sigma_1, \ldots, \sigma_n)]\!] \in B$, called the truth value of $\varphi(\sigma_1, \ldots, \sigma_n)$.
3. (soundness) Show that whenever φ is a ZFC axiom or logical axiom (such as $u = v \wedge v \in w \to u \in w$), it has truth value 1. Moreover, $[\![V \neq L]\!] = 1$ as long as B satisfies some mild condition. Then show by induction in the metatheory that whenever $\mathsf{ZF} + V \neq L$ proves φ, we have $[\![\varphi]\!] = 1$.

As before, the key is Step 2, and as mentioned in the introduction the most challenging case is when φ is atomic, i.e., $x \in y$ or $x = y$. Step 3 then tells us V^B is a Boolean-valued model of $\mathsf{ZF} + V \neq L$, and if a theory has a Boolean-valued model then it is consistent, thus establishing the consistency of $\mathsf{ZF} + V \neq L$. By varying B we can also manipulate the truth value of various statements in V^B, such as CH.

The class V^B of B-names, or "B-random sets", is defined by:

$V_0^B = \varnothing$,

$V_{\alpha+1}^B =$ the set of partial functions from V_α^B to B,

$V_\alpha^B = \bigcup_{\beta<\alpha} V_\beta^B$ if α is a limit,

$V^B = \bigcup_{\alpha\in\mathrm{Ord}} V_\alpha^B$.

The reason for using partial functions rather than total functions in the successor stage is to ensure $V_\alpha^B \subseteq V_{\alpha+1}^B$, although this is inessential. An element $b \in B$ might be called a Boolean value, event, or just probability. By Stone representation theorem, every Boolean algebra is isomorphic to an algebra of sets; complete Boolean algebras do *not* correspond to σ-algebra of sets, but they are similar in some ways, so the use of probabilistic language isn't unjustified. If $u, x \in V^B$ and $x \in \mathrm{dom}(u)$, we interpret $u(x) = b$ as "x belongs to u with probability at least b". There is a natural "embedding" $V \to V^B,\ x \mapsto \check{x}$, defined recursively by $\check{x} = \{(\check{y}, 1) : y \in x\}$.

The truth values $[\![u = v]\!]$ and $[\![u \in v]\!]$ are defined by a simultaneous induction on the "ranks" of the names u, v, similar to the sample calculation in Section 0. After handling the atomic formulas, it is straightforward to extend the Boolean value assignment to arbitrary formulas. The definition of truth values of atomoic formulas will ensure that V^B is indeed a Boolean-valued structure, namely it satisfies properties such as $[\![u = v]\!] \wedge [\![v = w]\!] \leq [\![u = w]\!]$. After that it will be relatively easy to show that $[\![\varphi]\!] = 1$ for every ZFC axiom φ.

Boolean-valued model is enough for consistency proofs, but if one really wants to work with transitive models, it is not hard to relativize all these to a ctm M, form the Boolean valued model M^B, pick a generic G and form the extension $M[G]$. Even in this approach the use of complete Boolean algebra makes the proofs somewhat cleaner compared to using partial order.

2 Boolean algebras

2.1 Definitions

An algebra of sets on X is a subset of $\mathcal{P}(X)$ that is closed under union, intersection and complement, and contains X and $\varnothing$. A Boolean algebra is roughly speaking a structure $(B, \vee, \wedge, *, 0, 1)$ that behaves like an algebra of sets. In fact, by the Stone Representation Theorem discussed in the next subsection, every Boolean algebra is isomorphic to an algebra of sets. However, the Boolean algebras most important to us are the complete ones, which most often arise as certain quotients, and it's not super helpful to think of them as algebras of sets. Hence the notion of an abstract Boolean algebra.

Definition 2.1. A Boolean algebra is a structure $(B, \vee, \wedge, *, 0, 1)$, consisting of a nonempty set B, two binary operations $\wedge$ and $\vee$, a unary operation $*$, and two constants 0 and 1, that satisfies the following axioms:

$$\begin{aligned} a \vee b = b \vee a, &\quad a \wedge b = b \wedge a \\ a \vee (b \vee c) = (a \vee b) \vee c, &\quad a \wedge (b \wedge c) = (a \wedge b) \wedge c \\ a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c), &\quad a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \\ a \vee a^* = 1, &\quad a \wedge a^* = 0 \\ (a \vee b) \wedge a = a, &\quad (a \wedge b) \vee a = a \end{aligned}$$

The most important axioms are the *absorption laws* $(a \vee b) \wedge a = a$ and $(a \wedge b) \vee a = a$, from which other important properties follow, such as:

$$\begin{gathered} a \vee a = a, \quad a \wedge a = a \\ a \vee 0 = a, \quad a \wedge 1 = a \\ a \vee 1 = 1, \quad a \wedge 0 = 0 \\ (a \vee b)^* = a^* \wedge b^*, \quad (a \wedge b)^* = a^* \vee b^* \\ a^{**} = a \end{gathered}$$

For example, $a \vee a = a \vee ((a \vee b) \wedge a) = a$, where we used both absorption laws. There is obviously a duality between $\vee$ and $\wedge$, 0 and 1, which means if $(B, \vee, \wedge, *, 0, 1)$ is a Boolean algebra then so is $(B, \wedge, \vee, *, 1, 0)$. It follows that, for example, if some sentence φ in the language $\{\vee, \wedge, *, 0, 1\}$ is provable from the axioms, then so is its dual φ', obtained by interchanging $\vee$ and $\wedge$, 0 and 1.

We use $a \leq b$ to mean $a \vee b = b$, or equivalently $a \wedge b = a$. It can be checked that $\leq$ is a partial order. Recall that for a partial order $(P, \leq)$ and $X \subseteq P$, a is called an upper bound of X if $a \geq x$ for all $x \in X$, and it is called the supremum if $a \leq b$ for any upper bound b; the definitions of lower bound and infimum are similar. It can then be checked that in a Boolean algebra, $a \vee b$ is the supremum of $\{a, b\}$, also called their join, and $a \wedge b$ is the infimum of $\{a, b\}$, also called their meet. A partial order that has joins and meets is called a lattice, and a Boolean algebra can be defined purely order-theoretically as a "complemented distributive lattice".

a, b are called *incompatible* if $a \wedge b = 0$. Denote by B^+ the set of nonzero elements of B. A set $A \subseteq B^+$ is an *antichain* if it consists of pairwise incompatible elements; be aware that some authors define an antichain to be a set of pairwise *incomparable* elements, meaning $a \not\leq b$ and $b \not\leq a$. A *maximal antichain* is an antichain that is not contained in a strictly larger antichain; for example if $a \neq 0, 1$, then $\{a, a^*\}$ is a maximal antichain. By Zorn's lemma, every antichain is contained in a maximal one.

We use $a \Rightarrow b$ to denote the *element* $a^* \vee b$. It can be checked that $a \Rightarrow b = 1$ iff $a \leq b$. Be aware that we use $\vee, \wedge$ both for Boolean operations and for logical connectives in our formal language, although we distinguish $\Rightarrow$ and $\rightarrow$.

The supremum of X, if it exists, is denoted $\bigvee X$, or sometimes $\bigvee^B X$ when we want to emphasize the dependence on B; if X is given as $\{a_i : i \in I\}$ we also write $\bigvee_{i \in I} a_i$ for $\bigvee X$. If the supremum of any $X \subseteq B$ exists, B is called a *complete Boolean algebra*; it follows that arbitrary meet exists, by considering the supremum of the set of lower bounds of X. Alternatively one can use the infinitary De Morgan's law:

$$\left(\bigvee_i a_i \right)^* = \bigwedge_i a_i^*$$

which can be deduced from the following distributive law that holds as long as $\bigvee_i a_i$ exists.

$$\left(\bigvee_i a_i \right) \wedge b = \bigvee_i (a_i \wedge b)$$

However, the followng infinitary distributive law does *not* hold in general:

$$\bigwedge_{j \in J} \bigvee_{i \in I} a_{ij} = \bigvee_{f : J \to I} \bigwedge_{j \in J} a_{f(j)j}$$

Infinitary distributive laws are closely related to forcing: a complete Boolean algebra satisfies the above law with $|J| \leq \kappa$ iff it does not create new sequences of ordinals of length $\leq \kappa$.

A *subalgebra* is a subset closed under $\vee$, $\wedge$ and $*$ and containing 0, 1. If B is a complete Boolean algebra, $B' \subseteq B$ is a subalgebra that is complete, and moreover $\bigvee^{B'} X = \bigvee^{B} X$ for any $X \subseteq B'$, then B' is called a *complete subalgebra*. Be aware that a subalgebra that is complete may not be a complete subalgebra; see the discussion below about Stone representation.

A Boolean homomorphism is a map that preserves the operations and 0, 1, and an embedding is an injective homomorphism. If $f : B \to C$ is an embedding between complete Boolean algebras such that $f(\bigvee X) = \bigvee f(X)$ for any $X \subseteq B$, then f is called a *complete embedding*, and $f(B)$ is a complete subalgebra of C. It can be checked that an embedding $f : B \to C$ is complete iff for any maximal antichain $A \subseteq B$, $f(A)$ is a maximal antichain in C.

A set $D \subseteq B^+$ is *dense* if it is dense in the order-theoretic sense, namely $\forall a \in B^+ \exists b \in D (b \leq a)$. We will later show that any Boolean algebra (in fact any poset) can be densely embedded into a complete one, called its Boolean completion.

2.2 Examples

We review some important examples and constructions of Boolean algebras.

The trivial Boolean algebra

The axioms of Boolean algebra do not exclude the possibility of $0 = 1$, but we are not interested in that case, so for us the simplest Boolean algebra is $2 = \{0, 1\}$ with the obvious operations. It is a complete subalgebra of any Boolean algebra.

Atomic algebra

For any set X, $\mathcal{P}(X)$ with the union, intersection and complementation operations form a complete Boolean algebra; supremum is simply union. Any finite Boolean algebra, or more generally any complete atomic Boolean algebra is isomorphic to some $\mathcal{P}(X)$; an atom is a minimally nonzero element a, namely $a \neq 0$ and if $b \leq a$, either $b = a$ or $b = 0$; B is atomic if below any nonzero element there is an atom.

B is called atomless if there is no atom, equivalently any nonzero element can be split into two nonzero elements. There exists a unique countable atomless Boolean algebra up to isomorphism. The uniqueness is proved by back-and-forth method, similar to the proof that there is a unique countable dense linear order without endpoints.

Algebra of sets

Any subset of $\mathcal{P}(X)$ closed under the set operations is also a Boolean algebra, called an algebra of sets. By Stone's Representation Theorem, this actually gives rise to all Boolean algebras, in other words any Boolean algebra B embeds into $\mathcal{P}(X)$ for some X. Thinking of a B as an algebra of set can be helpful, but not always; for example, when B is complete, the embedding given by the theorem is most often *not* complete, because a complete subalgebra of $\mathcal{P}(X)$ is easily seen to be atomic.

We briefly discuss Stone duality and how it implies the representation theorem. A *filter* on B is a subset G such that:

- $1 \in G$,

- $0 \notin G$,
- if $a \in G$ and $b \in G$ then $a \wedge b \in G$,
- if $a \in G$ and $a \leq b$ then $b \in G$.

A filter G is an *ultrafilter* if for any $a \in B$, at least (and therefore exactly) one of a and a^* is in G. The usual definition of ultrafilter on a set X is the special case of $B = \mathcal{P}(X)$. For any $a \neq 0$, $\{b \in B : a \leq b\}$ is a filter, and it is an ultrafilter iff a is an atom. Any filter can be extended to an ultrafilter using Zorn's lemma; in particular any nonzero a is contained in some ultrafilter.

Let $\mathrm{St}(B)$ be the set of all ultrafilters on B, and for each $b \in B$ let $[b] = \{G \in \mathrm{St}(B) : G \ni b\}$. We have $[b_1] \cap [b_2] = [b_1 \wedge b_2]$ since any G is closed under meet, so $\{[b] : b \in B\}$ is a basis for a topology on $\mathrm{St}(B)$; each $[b]$ is actually clopen—both closed and open, since $[b] \cup [b^*] = \mathrm{St}(B)$ and $[b] \cap [b^*] = \varnothing$. Under this topology $\mathrm{St}(B)$ is a compact Hausdorff space; compactness follows from the fact that any filter can be extended to an ultrafilter. It is also zero-dimensional, meaning having a basis consisting of clopen sets. Any clopen set is of form $[b]$ by compactness. Thus B is isomorphic to the algebra of clopen sets in $\mathrm{St}(B)$. This means:

- B is isomorphic to an algebra of set, because the clopen algebra is a subalgebra of $\mathcal{P}(\mathrm{St}(B))$,
- B can be recovered from $\mathrm{St}(B)$.

Moreover, Boolean homomorphisms induce continuous maps on Stone spaces in the opposite direction and vice versa. The Stone duality states that this is a contravariant equivalence between the category of Boolean algebras and the category of zero-dimensional compact Hausdorff spaces.

Regular open algebra

For a topological space X and $A \subseteq X$, denote the complement and closure of A by A' and A^- respectively; then $A^\circ := A'^{-\prime}$ is the interior. An open set U is called *regular* if $U^{-\circ} = U$; intuitively U does not contain "holes". $A^{-\circ}$ is regular open for any A, in other words $A^{-\circ-\circ} = A^{-\circ}$. $RO(X)$, the collection of all regular open sets in X, is a complete Boolean algebra under the operations $U \vee V = (U \cup V)^{-\circ}$, $U \wedge V = U \cap V$, and $U^* = X \setminus U^-$. The supremum of $(U_i)_i$ is $(\bigcup_i U_i)^{-\circ}$.

If X is a complete separable metric space such as $\mathbb{R}$ or the Cantor space ${}^\omega 2$, then $RO(X)$ can also be defined as the Boolean algebra of Borel subsets of X modulo the ideal of meager sets, since every Borel set differs from a regular open set by a meager set, and Baire Category Theorem implies different regular open sets are non equivalent.

The algebra $Cl(X)$ of clopen sets is a subalgebra of $RO(X)$. They are usually different, and if $RO(X) = Cl(X)$ (every regular open set is clopen, or equivalently the closure of open set is open), then X is called an extremally disconnected space. When $X = {}^\omega 2$, the clopen algebra $Cl({}^\omega 2)$ consists of finite unions of basic clopen sets $N_p = \{x \in {}^\omega 2 : x \supseteq p\}$ where p is a finite partial function from ω to 2. Clearly $Cl({}^\omega 2)$ is countable and atomless, so $RO({}^\omega 2)$ is the completion of the countable atomless Boolean algebra, also called the Cantor algebra or the Cohen algebra.

Random algebra

Let X be $[0, 1]$ or ${}^\omega 2$ with Lebesgue measure, and consider the Boolean algebra of measurable subsets modulo the ideal of null sets, denoted by $\mathrm{Mes}(X)$, also called the random algebra. $\mathrm{Mes}(X)$ has a countably additive measure inherited from the Lebesgue measure. $\mathrm{Mes}(X)$ is complete, as can be seen from the fact that it is countably complete and does not have uncountable antichain. Note that although $\mathrm{Mes}(X)$ satisfies the countable chain condition just like $RO(X)$, it does not have a countable dense set. Cohen forcing and random forcing are both similar and orthogonal in some ways.

Lindenbaum–Tarski algebra

Let T be an $\mathcal{L}$-theory. Two formulas φ, ψ are T-equivalent if $T \vdash \varphi \leftrightarrow \psi$. Formulas with free variables among $x_1, \ldots, x_n$ under T-equivalence form a Boolean algebra, whose Boolean operations are induced by the logical connectives. This is the Lindenbaum–Tarski algebra; its Stone space is known in model theory as the type space $S_n(T)$.

Algebra below an element

If B is a Boolean algebra and $b \in B$ is nonzero, then $b{\downarrow} := \{a \in B : a \leq b\}$ can be viewed as a Boolean algebra with maximal element b and complementation $a \mapsto b \wedge a^*$. If B is complete then so is $b{\downarrow}$. Note that this is not a subalgebra since the maximal element and complementation are different.

Quotient algebra

An *ideal* I on a Boolean algebra is dual to the notion of filter, namely it contains 0 but not 1, is closed under join, and is downward closed. A *prime ideal* P is dual to an ultrafilter, namely either $a \in P$ or $a^* \in P$.

Given an ideal I, we can form the quotient Boolean algebra B/I consisting of equivalence classes, where a and b are equivalent if their symmetric difference $(a \wedge b^*) \vee (b \wedge a^*)$ belongs to I, and the Boolean operations on B/I are defined in the natural way. An ideal P is prime iff the quotient B/P is the trivial Boolean algebra $\{0, 1\}$.

$b{\downarrow}$ is an ideal, called the principal ideal at b. The quotient of B by $b{\downarrow}$ is naturally isomorphic to $b^*{\downarrow}$.

Boolean ring

$\mathcal{P}(X)$ can be viewed as a ring, in fact an $\mathbb{F}_2$-algebra, with addition given by symmetric difference $A \triangle B = (A \setminus B) \cup (B \setminus A)$ and multiplication given by $A \cap B$; the additive unit is $\varnothing$ and multiplicative unit is X. This generalizes to any Boolean algebra, with symmetric difference defined as in the previous example. This results in a *Boolean ring*, namely a ring satisfying $x^2 = x$, which implies $x + x = 0$. A subset is an ideal in the Boolean algebra sense iff it is an ideal in the ring sense. Conversely, a Boolean ring $(B, +, \cdot, 0, 1)$ can be viewed as a Boolean algebra by letting $a \vee b$ be $a + b + a \cdot b$.

Cartesian and tensor product

If B and C are Boolean algebras, then the Cartesian product $B \times C$ is a Boolean algebra with operations defined pointwise. If B and C are complete then so is $B \times C$. Note that $b \mapsto (b, 0)$ is not a Boolean algebra embedding since it doesn't preserve 1. For any nonzero $b \in B$, there is a natural isomorphism between B and $b{\downarrow} \times b^*{\downarrow}$. More generally, if $X \subseteq B$ is a maximal antichain, then B is isomorphic to $\prod_{a \in X} a{\downarrow}$.

Caution: below we will talk about Boolean completions of posets. Denote the Boolean completion of $\mathbb{P}$ by $B(\mathbb{P})$; then $B(\mathbb{P}) \times B(\mathbb{Q})$ is *not* the Boolean completion of $\mathbb{P} \times \mathbb{Q}$, but rather the Boolean completion of the "disjoint sum" of $\mathbb{P}$ and $\mathbb{Q}$. The Boolean completion of $\mathbb{P} \times \mathbb{Q}$ is the *tensor product* $B(\mathbb{P}) \otimes B(\mathbb{Q})$. We omit the definition of tensor product since we don't strictly need it, but an instructive example to keep in mind is that $RO(\mathbb{R}) \otimes RO(\mathbb{R}) \simeq RO(\mathbb{R}^2)$.

3 Boolean-valued model

3.1 Definitions

We first define Boolean-valued structures in the language of set theory. It clearly generalizes to arbitrary language; later it will be useful to allow some unary predicates.

Definition 3.1. Let B be a fixed complete Boolean algebra. A B-valued structure of set theory is a set or class M, together with two maps $[\![\cdot = \cdot]\!] : M^2 \to B$ and $[\![\cdot \in \cdot]\!] : M^2 \to B$, such that for any $u, v, w \in M$, we have the following:

$$[\![u = u]\!] = 1$$

$$[\![u = v]\!] = [\![v = u]\!]$$

$$[\![u = v]\!] \wedge [\![v = w]\!] \leq [\![u = w]\!]$$

$$[\![u \in v]\!] \wedge [\![v = w]\!] \leq [\![u \in w]\!]$$

$$[\![u = v]\!] \wedge [\![u \in w]\!] \leq [\![v \in w]\!]$$

Note that they resemble the usual equality axioms in first order logic such as $x \in y \wedge y = z \to x \in z$. When $B = \{0, 1\}$ this almost recovers the usual notion of first order structure (a map $M^2 \to \{0, 1\}$ is basically a subset of M^2), the only difference being that $[\![u = v]\!]$ may be 1 while $u \neq v$. Aside: in “first order logic without equality”, it is allowed that $a = b$ while a, b are different elements of the structure, so under this convention first order structures coincide exactly with $\{0, 1\}$-valued structure.

Given a B-valued structure, we can define truth value of formulas recursively, using either of the two standard approaches in ordinary model theory: assignment or adding all $u \in M$ into the language as constant symbols.

$$[\![\phi \wedge \psi]\!] := [\![\phi]\!] \wedge [\![\psi]\!]$$

$$[\![\neg\phi]\!] := [\![\phi]\!]^*$$

$$[\![\forall x \phi(x)]\!] := \bigwedge_{u \in M} [\![\phi(u)]\!]$$

It follows that $[\![\exists x \phi(x)]\!] = \bigvee_{u \in M} [\![\phi(u)]\!]$, $[\![\phi \to \psi]\!] = [\![\phi]\!] \Rightarrow [\![\psi]\!]$, etc.

If φ is a sentence and $[\![\varphi]\!] = 1$, we say that M satisfies φ or $M \models \varphi$. We say M is a model of a theory T if $M \models \varphi$ for every sentence in T. It can be shown that if M is a model of T, then any logical consequence of T also has truth value 1; this is basically the Boolean version of soundness theorem, which will be discussed in more detail at the end of this section.

Here is a simple example of Boolean-valued model. Suppose X is any nonempty set; consider the class ${}^X V$ of all functions from X to V; it can be checked that this becomes a $\mathcal{P}(X)$-valued structure if we define $[\![f = g]\!] = \{x \in X : f(x) = g(x)\}$ and $[\![f \in g]\!] = \{x \in X : f(x) \in g(x)\}$. This model appears implicitly in the usual ultrapower construction: if U is an ultrafilter on $\mathcal{P}(X)$, we define the quotient ${}^X V/U$ by identifying f and g if $[\![f = g]\!] \in U$, and define the membership naturally. This generalizes to any B-valued model M: by picking an ultrafilter G on B we can form a quotient

M/G which is a classical structure, and there is an anologue of Łoś's Theorem; we will come back to this later.

Now we are finally ready to define the model we will use for consistency proof. Temporarily fix a complete Boolean algebra B. We will build a class V^B, which is a B-valued model of $\mathsf{ZFC} + V \neq L$, establishing the consistency of $\mathsf{ZFC} + V \neq L$ by Boolean soundness theorem. Although we work in ZFC, the basic theory goes through in $\mathsf{ZF} - \mathsf{P}$ or even weaker theories, though some specific results do require choice. As indicated in the introduction, we are going to build a "probabilistic von Neumann hierarchy", replacing the power set operation by the operation of taking "random subsets".

Definition 3.2. By transfinite recursion define V_α^B and V^B as follows:

$V_0^B = \varnothing$;

$V_{\alpha+1}^B$ is the set of all partial functions from V_α^B to B;

$V_\alpha^B = \bigcup_{\beta<\alpha} V_\beta^B$ if α is a limit;

$V^B = \bigcup_{\alpha\in\mathrm{Ord}} V_\alpha^B$.

An element $u \in V^B$ is called a B-*name*, or a B-random set.

In short, a B-name is a function from a set of B-names to B. We should interpret $u(v) = a$, in other words $(v, a) \in u$, as saying "v belongs to u with probability *at least* a".

Every $x \in V$ has a canonical name $\check{x}$, defined recursively by $\check{x} = \{(\check{y}, 1) : y \in x\}$. We may call such a $\check{x}$ a deterministic set, in contrast to B-random sets in general. Note that $\{0, 1\}$ is a complete Boolean subalgebra of B, and $\check{x}$ is actually a $\{0, 1\}$-name.

Next we define the probabilities for atomic formulas, which is the most difficult and important step, as in the poset approach.

Definition 3.3. Recursively define two functions $[\![\cdot = \cdot]\!]$ and $[\![\cdot \in \cdot]\!]$ from $V^B \times V^B$ to B as follows:

$$[\![u \in v]\!] = \bigvee_{y\in\mathrm{dom}(v)} [v(y) \wedge [\![u = y]\!]]$$

$$[\![u = v]\!] = \bigwedge_{x\in\mathrm{dom}(u)} [u(x) \Rightarrow [\![x \in v]\!]] \quad \wedge \quad \bigwedge_{y\in\mathrm{dom}(v)} [v(y) \Rightarrow [\![y \in u]\!]]$$

Recall that for $a, b \in B$, $a \Rightarrow b$ means the element $a^* \vee b$. When we want to emphasize the dependence on B, we use $[\![u \in v]\!]^B$ and $[\![u = v]\!]^B$ respectively.

Let us explain how exactly the recursion works. Define the B-*rank* of a B-name u as follows: the least α for which $u \in V_\alpha^B$ must be a successor $\beta + 1$, and we define the B-rank of u to be β, denoted $\mathrm{rk}^B(u)$, analogous to the usual rank of set. Note that if $u \in \mathrm{dom}(v)$ then $\mathrm{rk}^B(u) < \mathrm{rk}^B(v)$.

To define $[\![u \in v]\!]$, we need to know $[\![u = y]\!]$ for all $y \in \mathrm{dom}(v)$. To define $[\![u = v]\!]$, we need to know $[\![x \in v]\!]$ for $x \in \mathrm{dom}(u)$ and $[\![y \in u]\!]$ for $y \in \mathrm{dom}(v)$. It is therefore enough to show that the following relation on $V^B \times V^B$ is well-founded (it is obviously set-like),

$$(u', v') < (u, v) \text{ iff } u' = u \wedge v' \in \mathrm{dom}(v)$$

$$\text{or } v' = v \wedge u' \in \text{dom}(u)$$
$$\text{or } u' = v \wedge v' \in \text{dom}(u)$$
$$\text{or } v' = u \wedge u' \in \text{dom}(v)$$

where the third case is actually unnecessary, and we include it just for symmetry. Then we can define the two functions $[\![u = v]\!]$ and $[\![u \in v]\!]$ simultaneously by recursion on this well-founded and set-like relation.

The relation $(u', v') < (u, v)$ is indeed well-founded; this is intuitively clear, since to "decrease" (u, v), we either decrease one of the coordinates, or first switch the two names and then decrease one of the coordinates. Formally, if we let

$$\pi(u, v) = (\max\{\text{rk}^B(u), \text{rk}^B(v)\}, \min\{\text{rk}^B(u), \text{rk}^B(v)\})$$

then it is not hard to check that $(x, y) < (u, v)$ implies $\pi(x, y) <_{\text{lec}} \pi(u, v)$, where $<_{\text{lec}}$ is the lexicographical order on Ord $\times$ Ord. The idea is that if we "decrease" (u, v) then either the maximum or the minimum of $\{\text{rk}^B(u), \text{rk}^B(v)\}$ has to decrease.

The definitions of $[\![u \in v]\!]$ and $[\![u = v]\!]$ are partly motivated by the following facts in the usual set theory:

$$u \in v \leftrightarrow \exists y \in v(u = y)$$
$$u = v \leftrightarrow [\forall x \in u(x \in v)] \wedge [\forall y \in v(y \in u)]$$

The first formula is a logical tautology, and the second one is extensionality. We certainly cannot just define $[\![u \in v]\!]$ to be $\bigvee_{y \in V^B} [[\![y \in v]\!] \wedge [\![u = y]\!]]$, because this does not constitute a recursive definition. However, the RHS of the above formulas only contain bounded quantification; it is not unreasonable to expect that, e.g., to define $[\![u \in v]\!]$, it is enough to look at those names in the domain of v. Assuming this is true, together with the requirement that $u(x) \leq [\![x \in u]\!]$, one is led to the above definitions.

Remark 3.4. (a) One can alternatively define $[\![u = v]\!] = \bigwedge_{x \in \text{dom}(u) \cup \text{dom}(v)} [[\![x \in u]\!] \Leftrightarrow [\![x \in v]\!]]$, where of course $a \Leftrightarrow b$ means $(a \Rightarrow b) \wedge (b \Rightarrow a)$. This is closer to the standard definition of $p \Vdash u = v$ in the poset approach, such as the one in Kunen. It gives the same Boolean-valued model, as can be seen from the proof of Lemma 3.6 below. A slightly different argument is needed to show this definition is letigimate: after defining $[\![u \in v]\!]$ and $[\![u = v]\!]$ for all $u, v \in V^B_\alpha$, we first define $[\![u \in v]\!]$ for $u \in V^B_\alpha$ and $v \in V^B_{\alpha+1}$, then $[\![u = v]\!]$ for $u, v \in V^B_{\alpha+1}$, and finally $[\![u \in v]\!]$ for $u, v \in V^B_{\alpha+1}$.

(b) When $B = \{0, 1\}$, V^B is essentially just V. More precisely, for $x, y \in V$, $[\![\check{x} \in \check{y}]\!]$ is 1 if $x \in y$ and 0 otherwise (this is of course proved by induction), similarly for $[\![\check{x} = \check{y}]\!]$. Not all $u \in V^B$ are of form $\check{x}$. However, it is true that $[\![u = \check{x}]\!] = 1$ for some x.

(c) If X is a nonempty set, then ${}^X V$ and $V^{\mathcal{P}(X)}$ turn out to be equivalent as $\mathcal{P}(X)$-valued models. It might be easier to think in terms of generic filters: for the Boolean algebra $\mathcal{P}(X)$, generic filters are exactly the principal ones; for each name $u \in V^{\mathcal{P}(X)}$, the map that sends $x \in X$ to the interpretation of u under the principal filter at x provides the corresponding element in ${}^X V$.

(d) We have not proved that V^B is a B-valued structure, namely it satisfies the conditions in Definition 3.1. However we notice that if B' is a complete subalgebra of B, then for $u, v \in V^{B'}$,

$[\![u \in v]\!]^{B'} = [\![u \in v]\!]^B$ and $[\![u = v]\!]^{B'} = [\![u = v]\!]^B$, so it makes sense to say that $V^{B'}$ is a substructure of V^B. In particular, the trivial algebra $\{0, 1\}$ is a complete subalgebra of any B, so V is in some sense a substructure of V^B.

With the Boolean values of the atomic formulas defined, we can now proceed as in the remark after Definition 3.1 to define the Boolean value of all formulas by induction. *This is an induction in the metatheory*, since we are dealing with the class size B-valued model V^B; that is, for each particular formula $\varphi(x_1, \dots, x_n)$, we can write down a formula that defines the class function $f_\varphi : (V^B)^n \to B, (u_1, \dots, u_n) \mapsto [\![\varphi(u_1, \dots, u_n)]\!]$; the basic cases are the two atomic formulas, which we already handled. Of course, if we did not start with V but some set model M of ZFC, then we *could* define the Boolean value of all formulas at once.

3.2 V^B is a Boolean-valued model

Theorem 3.5. (i) $[\![u = u]\!] = 1$*;*

(ii) $u(x) \leq [\![x \in u]\!]$ *for* $x \in \mathrm{dom}(u)$*;*

(iii) $[\![u = v]\!] = [\![v = u]\!]$*;*

(iv) $[\![u = v]\!] \wedge [\![v = w]\!] \leq [\![u = w]\!]$*;*

(v) $[\![u \in v]\!] \wedge [\![v = w]\!] \leq [\![u \in w]\!]$*;*

(vi) $[\![u = v]\!] \wedge [\![u \in w]\!] \leq [\![v \in w]\!]$*;*

(vii) $[\![u = v]\!] \wedge [\![\varphi(u)]\!] \leq [\![\varphi(v)]\!]$ *for any formula* $\varphi(x)$ *possibly containing other names.*

Proof. (i) is proved by induction on names. (ii) follows from (i) and the definition of $[\![x \in u]\!]$; note that the inequality is strict in general. (iii) is true by symmetry in the definition. (iv), (v) and (vi) can be simultaneously proved using induction. We first present the proof and explain the induction details later. For (iv):

$$
\begin{aligned}
&[\![u = v]\!] \wedge [\![v = w]\!] \\
=&\left[[\![u = v]\!] \wedge \bigwedge_{x \in \mathrm{dom}(u)} [u(x) \Rightarrow [\![x \in v]\!]]\right] \wedge \left[[\![v = w]\!] \wedge \bigwedge_{z \in \mathrm{dom}(w)} [w(z) \Rightarrow [\![z \in v]\!]]\right] \\
=&\left[[\![v = w]\!] \wedge \bigwedge_{x \in \mathrm{dom}(u)} [u(x) \Rightarrow [\![x \in v]\!]]\right] \wedge \left[[\![u = v]\!] \wedge \bigwedge_{z \in \mathrm{dom}(w)} [w(z) \Rightarrow [\![z \in v]\!]]\right] \\
\leq&\left[\bigwedge_{x \in \mathrm{dom}(u)} [u(x) \Rightarrow [\![x \in v]\!] \wedge [\![v = w]\!]]\right] \wedge \left[\bigwedge_{z \in \mathrm{dom}(w)} [w(z) \Rightarrow [\![z \in v]\!] \wedge [\![u = v]\!]]\right] \\
\leq&\left[\bigwedge_{x \in \mathrm{dom}(u)} [u(x) \Rightarrow [\![x \in w]\!]]\right] \wedge \left[\bigwedge_{z \in \mathrm{dom}(w)} [w(z) \Rightarrow [\![z \in u]\!]]\right] \\
=&[\![u = w]\!]
\end{aligned}
$$

where in the second to last line, we assume by induction that (v) holds for (x, v, w), $\forall x \in \mathrm{dom}(u)$ and (z, v, u), $\forall z \in \mathrm{dom}(w)$. Similarly, (v) is proved using (vi) and (vi) is proved using (iv). For (v), note that for any $y \in \mathrm{dom}(v)$:

$$\begin{aligned}
&v(y) \wedge [\![u = y]\!] \wedge [\![v = w]\!] \\
\leq &v(y) \wedge [\![u = y]\!] \wedge (v(y) \Rightarrow [\![y \in w]\!]) \\
= &v(y) \wedge [\![u = y]\!] \wedge (v(y)^* \vee [\![y \in w]\!]) \\
= &v(y) \wedge [\![u = y]\!] \wedge [\![y \in w]\!] \\
\leq &[\![u \in w]\!]
\end{aligned}$$

Taking supremum over $y \in \mathrm{dom}(v)$, we get $[\![u \in v]\!] \wedge [\![v = w]\!] \leq [\![u \in w]\!]$. For (vi):

$$\begin{aligned}
&[\![u = v]\!] \wedge [\![u \in w]\!] \\
= &[\![u = v]\!] \wedge \bigvee_{z \in \mathrm{dom}(w)} [w(z) \wedge [\![u = z]\!]] \\
= &\bigvee_{z \in \mathrm{dom}(w)} [w(z) \wedge [\![u = z]\!] \wedge [\![u = v]\!]] \\
\leq &\bigvee_{z \in \mathrm{dom}(w)} [w(z) \wedge [\![v = z]\!]] \\
= &[\![v \in w]\!]
\end{aligned}$$

Now let's see why this is a legitimate induction. For brevity let x, y, z range over the domains of u, v, w respectively. To prove (iv) for (u, v, w) we need (v) for all tripes (x, v, w) and (z, v, u); to prove (v) we need (vi) for (u, y, w); to prove (vi) we need (iv) for (v, u, z). Define a map $\pi : (V^B)^3 \to (\mathrm{Ord})^3$ by $\pi(u, v, w) = (\alpha, \beta, \gamma)$, where (α, β, γ) lists $\mathrm{rk}^B(u), \mathrm{rk}^B(v), \mathrm{rk}^B(w)$ in non-increasing order. This maps witnesses that the induction is legitimate, similar to the recursive definition of $[\![u \in v]\!]$ and $[\![u = v]\!]$.

Finally (vii) follows by induction on complexity of φ, the base cases being (iv)-(vi). $\square$

A quick corollary is that, to calculate truth value of bounded quantification, it is enough to consider those names in the domain. This fact is very useful both in developing the basic theory of forcing and in concrete applications.

Lemma 3.6 (Bounded Quantification). *If $u \in V^B$ and $\varphi(x)$ is a formula with free variable x, possibly with other parameters, then*

$$[\![\forall x \in u\ \varphi(x)]\!] = \bigwedge_{x \in \mathrm{dom}(u)} [u(x) \Rightarrow [\![\varphi(x)]\!]] = \bigwedge_{x \in \mathrm{dom}(u)} [[\![x \in u]\!] \Rightarrow [\![\varphi(x)]\!]]$$

$$[\![\exists x \in u\ \varphi(x)]\!] = \bigvee_{x \in \mathrm{dom}(u)} [u(x) \wedge [\![\varphi(x)]\!]] = \bigvee_{x \in \mathrm{dom}(u)} [[\![x \in u]\!] \wedge [\![\varphi(x)]\!]]$$

Proof. We prove the universal case. For any $v \in V^B$:

$$\begin{aligned}
&[\![v \in u]\!] \Rightarrow [\![\varphi(v)]\!] \\
&= \left[\bigvee_{x\in \mathrm{dom}(u)} [u(x) \wedge [\![v = x]\!]]\right]^* \vee [\![\varphi(v)]\!] \\
&= \left[\bigwedge_{x\in \mathrm{dom}(u)} [u(x)^* \vee [\![v = x]\!]^*]\right] \vee [\![\varphi(v)]\!] \\
&= \left[\bigwedge_{x\in \mathrm{dom}(u)} [u(x)^* \vee [\![v = x]\!]^* \vee [\![\varphi(v)]\!]]\right] \\
&\geq \left[\bigwedge_{x\in \mathrm{dom}(u)} [u(x)^* \vee [\![\varphi(x)]\!]]\right] \\
&= \bigwedge_{x\in \mathrm{dom}(u)} [u(x) \Rightarrow [\![\varphi(x)]\!]] \\
&\geq \bigwedge_{x\in \mathrm{dom}(u)} [[\![x \in u]\!] \Rightarrow [\![\varphi(x)]\!]]
\end{aligned}$$

The first inequality uses that $[\![v = x]\!] \wedge [\![\varphi(x)]\!] \leq [\![\varphi(v)]\!]$, and therefore $[\![v = x]\!]^* \vee [\![\varphi(v)]\!] \geq [\![v = x]\!]^* \vee ([\![v = x]\!] \wedge [\![\varphi(x)]\!]) \geq [\![\varphi(x)]\!]$. The second inequality is because $u(x) \leq [\![x \in u]\!]$.

Now taking infimum over $v \in V^B$, we get

$$[\![\forall x(x \in u \to \varphi(x))]\!] \geq \bigwedge_{x\in \mathrm{dom}(u)} [u(x) \Rightarrow [\![\varphi(x)]\!]] \geq \bigwedge_{x\in \mathrm{dom}(u)} [[\![x \in u]\!] \Rightarrow [\![\varphi(x)]\!]]$$

and the other direction is clear since the infimum is taken over a smaller domain. □

Recall that for a sentence φ, we say that V^B is a model of φ if $[\![\varphi]\!] = 1$.

Lemma 3.7. *V^B is a model of extensionality, comprehension, and regularity.*

Proof. For extensionality, we want to show

$$[\![\forall a \forall b[a = b \leftrightarrow \forall x(x \in a \to x \in b) \wedge \forall y(x \in b \to x \in a)]]\!] = 1,$$

which is the same as showing for any $u, v \in V^B$,

$$[\![u = v]\!] = [\![\forall x \in u\ x \in v]\!] \wedge [\![\forall y \in v\ y \in u]\!].$$

This can be proved either directly or from the Bounded Quantification Lemma 3.6. Applying the lemma to the formula $x \in v$, we get $[\![\forall x \in u(x \in v)]\!] = \bigwedge_{x\in \mathrm{dom}(u)}[u(x) \Rightarrow [\![x \in v]\!]]$, which is half of the definition of $[\![u = v]\!]$; applying the lemma again to $y \in u$ gives the other half.

Comprehension is of course a theorem scheme. We want to show that

$$[\![\forall p_1 \cdots \forall p_n \forall x \exists y \forall z(z \in y \leftrightarrow z \in x \wedge \varphi(z, x, p_1, \dots, p_n))]\!] = 1$$

where $\varphi(z, x, p_1, \dots, p_n)$ is a formula in which y is not free. It suffices to show that for any $\tau_1, \dots, \tau_n, u \in V^B$, there exists $v \in V^B$ such that for all $w \in V^B$,

$$[\![w \in v]\!] = [\![w \in u]\!] \wedge [\![\varphi(w, u, \tau_1, \dots, \tau_n)]\!]$$

For brevity we suppress the names in φ other than w and write $\varphi(w)$. We simply throw in elements according to their probability of satisfying φ. That is, we let $\mathrm{dom}(v) = \mathrm{dom}(u)$ and $v(x) = u(x) \wedge [\![\varphi(x)]\!]$. Then for any $w \in V^B$,

$$\begin{aligned}
[\![w \in v]\!] &= \bigvee_{x \in \mathrm{dom}(v)} v(x) \wedge [\![w = x]\!] \\
&= \bigvee_{x \in \mathrm{dom}(u)} u(x) \wedge [\![\varphi(x)]\!] \wedge [\![w = x]\!] \\
&= \bigvee_{x \in \mathrm{dom}(u)} u(x) \wedge [\![w = x]\!] \wedge [\![\varphi(w)]\!] \\
&= \left[\bigvee_{x \in \mathrm{dom}(u)} u(x) \wedge [\![w = x]\!] \right] \wedge [\![\varphi(w)]\!] \\
&= [\![w \in u]\!] \wedge [\![\varphi(w)]\!]
\end{aligned}$$

For regularity, we want that for every $u \in V^B$, $[\![\exists x (x \in u) \to \exists x \in u (\forall y \in x\ y \notin u)]\!] = 1$. If this were false, then

$$[\![\exists x (x \in u) \wedge \forall x \in u (\exists y \in x\ y \in u)]\!] =: b \neq 0$$

Consider the class $C = \{x \in V^B : b \wedge [\![x \in u]\!] \neq 0\}$, which is nonempty since $b = b \wedge [\![\exists x (x \in u)]\!]$; choose an $x \in C$ of minimal B-rank. By definition of b we have

$$b \leq [\![x \in u]\!] \Rightarrow [\![\exists y \in x\ y \in u]\!],$$

which means

$$b \wedge [\![x \in u]\!] \leq [\![\exists y \in x\ y \in u]\!].$$

So by the Bounded Quantification Lemma, there exists $y \in \mathrm{dom}(x)$ such that $b \wedge [\![y \in u]\!] \neq 0$, contradicting the choice of x. □

Theorem 3.8. $V^B \models \mathsf{ZFC}$

Proof. The remaining axioms are pairing, union, infinity, power set, replacement and choice. The proof of their validity in V^B is relatively simple thanks to comprehension.

Pairing: Let $w = \{(u, 1), (v, 1)\}$. It can be checked that if $\varphi(x, y, z)$ is the formula expressing "z is the unordered pair of x and y", then $[\![\varphi(u, v, w)]\!] = 1$. The exact choice of the formula φ doesn't matter, since if $\mathsf{ZFC} \vdash \varphi \to \psi$ and $V^B \models \varphi$ then $V^B \models \psi$. This follows from the soundness theorem discussed below.

Union: We can cheat by proving the weak union axiom $\forall u \exists v (\forall x \in u \forall y \in x\ y \in v)$, since this together with comprehension implies the usual union axiom. The proof of the weak union axiom is easy; we can simply let $v = \bigcup_{x \in u} \mathrm{dom}(x) \times \{1\}$. It is also possible to directly write down a name that is the union of u with probability 1.

Infinity: One can directly check that $[\![\check{\omega} \text{ is an inductive set}]\!] = 1$, or use the absoluteness result in the next section that $V \models \varphi(a) \leftrightarrow V^B \models \varphi(\check{a})$ for a Δ_1^{ZF} formula $\varphi(x)$ and $a \in V$.

Power set: Again it suffices to prove the weak power axiom $\forall u \exists v \forall w (w \subseteq u \to w \in v)$. Let W be the set of all partial functions from $\mathrm{dom}(u)$ to B and $v = W \times \{1\}$.

Replacement: It suffices to prove the collection scheme:

$$(\forall x \exists y\ \varphi(x, y)) \to (\forall u \exists v \forall x \in u \exists y \in v\ \varphi(x, y))$$

Using collection in V, there is a set Y such that for every $x \in \mathrm{dom}(u)$ and every $b \in B$, if there exists $y \in V^B$ such that $[\![\varphi(x, y)]\!] = b$ then there is such a y in Y. Let $v = Y \times \{1\}$.

Choice: It suffices to prove the well-ordering principle. Here it might be easier to think in terms of actual generic extension $M[G]$. The idea is that for any name u, the evaluation map $f : \mathrm{dom}(u) \to M[G], x \mapsto x_G$ satisfies $\mathrm{ran}(f) \supseteq u$, and $\mathrm{dom}(u)$ is well-orderable in the ground model, hence in the generic extension also. Of course here we are working with Boolean-valued model and don't have the G. So we define a name

$$f = \{\mathrm{op}(\check{x}, x) : x \in \mathrm{dom}(u)\} \times \{1\}$$

where $\mathrm{op}(u, v)$ is the natural name such that $V^B \models$"$\mathrm{op}(u, v)$ is the ordered pair with coordinates u, v", similar to the unordered pair $\{(u, 1), (v, 1)\}$; note that any $x \in \mathrm{dom}(u)$ is in particular a set, and we are considering its canonical name $\check{x}$. Let $X = \mathrm{dom}(u)$. It can be checked that

$$V^B \models \text{"}f \text{ is a function with domain } \check{X} \text{ and its range contains } u\text{"}$$

In V there is a bijection $g : \alpha \to X$ for some ordinal α, and we have $V^B \models$"$\check{g}$ is a bijection between $\check{\alpha}$ and $\check{X}$". Thus $V^B \models$"$f \circ \check{g}$ is a surjection from $\check{\alpha}$ onto u". □

To be fair, we have implicitly used the Boolean soundness and absoluteness results that have not been discussed yet. For example, we used soundness and the fact that $V^B \models \mathsf{ZF}$ to conclude that V^B also satisfies things like "if α surjects onto u then u is well-orderable". We used absoluteness to show that if g is a bijection in V then $\check{g}$ is a bijection in V^B. We shall repay our debts shortly.

Recall that the collection scheme is equivalent to the replacement scheme under Zermelo set theory, and stronger than replacement in the absence of power set axio. The well-ordering principle is also stronger than axiom of choice when there is no power set. Denote ZF without power set and with replacement strengthened to collection by $\mathsf{ZF} - \mathsf{P}$, and $\mathsf{ZF} - \mathsf{P}$ plus well-ordering principle by $\mathsf{ZFC} - \mathsf{P}$. Our proof shows that if V satisfies $\mathsf{ZF} - \mathsf{P}$ then so does V^B, and the same holds for $\mathsf{ZFC} - \mathsf{P}$. Without power set axiom, the definition of V^B might seem problematic at first, but we can recursively define a name to be a function from a set of names to B. Thus forcing works over theories weaker than ZF. Actually Kripke–Platek set theory is more than enough for the basic development of forcing.

However, without power set one cannot show that every poset $\mathbb{P}$ has a Boolean completion, so for maximal generality one has to give up the niceties of Boolean-valued model. Moreover, the poset approach seems necessary when it comes to class forcing (when $\mathbb{P}$ is a class instead of set).

3.3 Boolean soundness theorem

In the next section we will show that $V^B \models V \neq L$ as long as B is atomless, and choosing appropriate B gives $V^B \models \neg\mathsf{CH}$. To conclude the consistency of $V \neq L$, we still need one more step which is often glossed over in textbooks.

Theorem 3.9. *If $V^B \models T$ then the theory T is consistent.*

Proof sketch. This is basically the soundness theorem of first order logic, that if a theory has a model then it is consistent. So if you have seen a detailed proof of the usual soundness theorem before, such as [11, Lemma II.10.5], you know the proof is going to be routine but tedious. There are just two essentially minor differences with the usual soundness theorem: (1) this is a Boolean version; (2) it is for the class model V^B, which means it is really a metatheorem.

Recall how we prove the relative consistency of $\mathsf{ZF} + V = L$. We can't just say "L is a model of $\mathsf{ZF} + V = L$ so it is consistent", because of the difference between set model and class model. It is not even possible to express "L is a model of $\mathsf{ZF} + V = L$" using a single sentence in the language of set theory. What we show is the scheme that for each ZF axiom φ, the relativization φ^L is a theorem of ZF, and so is $(V = L)^L$. Then we prove by induction in the metatheory that, whenever $\mathsf{ZF} + V = L$ proves a theorem φ, the relativization φ^L is a theorem of ZF. Therefore if ZF is consistent then so is $\mathsf{ZF} + V = L$, because if $\mathsf{ZF} + V = L$ proves $\varphi \wedge \neg\varphi$ then ZF already proves $\varphi^L \wedge \neg\varphi^L$.

Similarly, our job here is to show that if $V^B \models T$ and T proves a sentence φ, then $V^B \models \varphi$, and hence we can conclude the consistency of T in the metatheory. Of course we prove this by induction on proofs; to facilitate the inudction, we actually show that:

(∗) if T proves a formula $\varphi(x_1, \dots, x_n)$ then $V^B \models \varphi(u_1, \dots, u_n)$ for any $u_1, \dots, u_n \in V^B$.

The argument also depends somewhat on the particular proof system we use. Assume we work in a Hilbert style deductive system, where the logical axioms consist of the propositional axioms, the equality axioms, together with:

$\forall x\varphi \to \varphi[t/x]$, where t is a term substitutable for x in φ,

$\forall x(\varphi \to \psi) \to (\varphi \to \forall x\psi)$, where x is not free in φ.

Recall that $\varphi[t/x]$ is the formula obtained by replacing every free occurence of x in φ by t, and t is called substitutable for x in φ if no variable in t becomes bounded in $\varphi[t/x]$. The inference rules are.

Modus ponens: From φ and $\varphi \to \psi$, deduce ψ,

Universal generalization: From φ, deduce $\forall x\varphi$.

A proof from T is a finite sequence of formulas, such that each of them is either an axiom in T, or a logical axiom, or obtained from previous formulas by one of the inference rules. Axioms in T hold in V^B by assumption. The propositional axioms hold basically by definition; the equality axioms were handled in Theorem 3.5; the second quantification axiom is also easy.

The first quantification axiom $\forall x\varphi \to \varphi[t/x]$ takes more effort. It is slightly simplified by the fact that we are considering the language of set theory, so the only terms are variables. We need to show by induction that if $\varphi = \varphi(x, y, z_1, \ldots, z_n)$ is a formula with no free occurence of x in the scope of $\forall y$ or $\exists y$, then letting $\psi(y, z_1, \ldots, z_n) = \varphi(y, y, z_1, \ldots, z_n)$, for any $v, w_1, \ldots, w_n \in V^B$ we have

$$[\![\varphi(v, v, w_1, \ldots, w_n)]\!] = [\![\psi(v, w_1, \ldots, w_n)]\!]$$

whose proof we omit; it is similar to [11, Lemma II.8.10]. Consequently $[\![\forall x \varphi(x, v, w_1, \ldots, w_n)]\!] \leq [\![\psi(v, w_1, \ldots, w_n)]\!]$.

It remains to check that inference rules preserve $(*)$, e.g., if T satisfy the universal closures of φ and $\varphi \to \psi$, then it satisfies the universal closure of ψ. This is straightforward. □

We remark that although we only defined Boolean-valued structures in the language of set theory, it is straightforward to extend the definition to any language, and prove the more general soundness theorem that if a theory T has a set or class model then it is consistent.

4 Independence results

4.1 Absoluteness and the consistency of $V \neq L$

A formula in the language of set theory is called bounded if all quantifications are of form $\forall x \in y$ or $\exists x \in y$. Σ_1 and Π_1 formulas are, respectively, those of form $\exists x_1 \cdots \exists x_n \varphi$ or $\forall x_1 \cdots \forall x_n \varphi$ where φ is bounded. A formula φ is Δ_1^{ZF} if there are Σ_1 formula τ and Π_1 formula η s.t. $\mathsf{ZF} \vdash \varphi \leftrightarrow \tau$ and $\mathsf{ZF} \vdash \varphi \leftrightarrow \eta$. Bounded formulas are absolute between any transitive sets or classes, Σ_1 formulas are upward absolute, Π_1 formulas are downward absolute, and Δ_1^{ZF} are absolute between transitive models of ZF. For more on absoluteness see Kunen [12, I.16, II.4].

Basically the same proof works for Boolean-valued models. Recall that if B' is a complete subalgebra of B, then atomic formulas involving B'-names have the same Boolean value whether calculated in B' or B, so $V^{B'}$ can be viewed as a Boolean substructure of V^B.

Lemma 4.1. *Suppose B is a complete Boolean algebra and B' is a complete subalgebra of B. For any $u_1, \ldots, u_n \in V^{B'}$, if the formula $\varphi(x_1, \ldots, x_n)$ is bounded (resp. Σ_1, Π_1, Δ_1^{ZF}), then $[\![\varphi(u_1, \ldots, u_n)]\!]^{B'}$ is equal to (resp. at most, at least, equal to) $[\![\varphi(u_1, \ldots, u_n)]\!]^B$.*

V can more or less be identified with $V^{\{0,1\}}$. More precisely, one can prove the scheme $\varphi(x_1, \ldots, x_n) \leftrightarrow V^{\{0,1\}} \models \varphi(\check{x}_1, \ldots, \check{x}_n)$ by induction. Since $\{0, 1\}$ is a complete subalgebra of any B, the previous lemma tells us the following: whenever $x_1, \ldots, x_n \in V$ and φ is a Δ_1^{ZF} formula, then $\varphi(x_1, \ldots, x_n)$ iff $V^B \models \varphi(\check{x}_1, \ldots, \check{x}_n)$. We already used this when proving V^B satisfies choice, using that, e.g.,"g is a bijection between α and X" is a bounded formula.

We want to show $V^B \models V \neq L$. For that we need to understand what are the ordinals and constructible sets of V^B; it turns out they are simply random combinations of ordinals and constructible sets of V. Let $\mathrm{Ord}(x)$ be the formula that says "x is an ordinal", more precisely "x is a transitive set of transitive sets", so it is bounded.

Lemma 4.2. (i) $[\![\mathrm{Ord}(u)]\!] = \bigvee_{\alpha \in \mathrm{Ord}} [\![u = \check{\alpha}]\!]$ *for any $u \in V^B$.*

(ii) *For any formula $\varphi(x)$, $[\![\exists \alpha \varphi(\alpha)]\!] = \bigvee_{\alpha \in \mathrm{Ord}} [\![\varphi(\check{\alpha})]\!]$.*

Proof. (i) By absoluteness $[\![\mathrm{Ord}(\check{\alpha})]\!]^B = 1$ for any ordinal α in V. Let $u \in V^B$ be any name. On one hand,

$$\bigvee_{\alpha\in\mathrm{Ord}} [\![u = \check{\alpha}]\!] = \bigvee_{\alpha\in\mathrm{Ord}} [[\![u = \check{\alpha}]\!] \wedge [\![\mathrm{Ord}(\check{\alpha})]\!]] \leq \bigvee_{\alpha\in\mathrm{Ord}} [\![\mathrm{Ord}(u)]\!] = [\![\mathrm{Ord}(u)]\!]$$

On the other hand, $[\![u = \check{\alpha}]\!]$ and $[\![u = \check{\beta}]\!]$ are incompatible for $\alpha \neq \beta$, namely

$$[\![u = \check{\alpha}]\!] \wedge [\![u = \check{\beta}]\!] \leq [\![\check{\alpha} = \check{\beta}]\!] = 0.$$

In particular $[\![u = \check{\alpha}]\!] \neq [\![u = \check{\beta}]\!]$ if they are both nonzero, so $\{\alpha : [\![u = \check{\alpha}]\!] \neq 0\}$ is a set. Choose γ large enough so that whenever $[\![u = \check{\alpha}]\!]$ is nonzero or $[\![x = \check{\alpha}]\!]$ is nonzero for some $x \in \mathrm{dom}(u)$, we have $\alpha < \gamma$. It follows that $[\![\check{\gamma} \in u]\!] = 0$ and $[\![\check{\gamma} = u]\!] = 0$. We have $[\![\mathrm{Ord}(u) \to \check{\gamma} \in u \vee \check{\gamma} = u \vee u \in \check{\gamma}]\!] = 1$ by the Boolean Soundness Theorem 3.9. Therefore $[\![\mathrm{Ord}(u)]\!] \Rightarrow [\![u \in \check{\gamma}]\!] = 1$, i.e.,

$$[\![\mathrm{Ord}(u)]\!] \leq [\![u \in \check{\gamma}]\!] = \textstyle\bigvee_{\alpha\in\gamma} [\![u = \check{\alpha}]\!] \leq \bigvee_{\alpha\in\mathrm{Ord}} [\![u = \check{\alpha}]\!].$$

(ii) We show the $\leq$ direction.

$$[\![\exists\alpha\varphi(\alpha)]\!] = \bigvee_{u\in V^B} [\![\mathrm{Ord}(u)]\!] \wedge [\![\varphi(u)]\!] = \bigvee_{u\in V^B} \left[\bigvee_{\alpha\in\mathrm{Ord}} [\![u = \check{\alpha}]\!]\right] \wedge [\![\varphi(u)]\!]$$
$$= \bigvee_{u\in V^B} \bigvee_{\alpha\in\mathrm{Ord}} [\![u = \check{\alpha}]\!] \wedge [\![\varphi(\check{\alpha})]\!] \leq \bigvee_{u\in V^B} \bigvee_{\alpha\in\mathrm{Ord}} [\![\varphi(\check{\alpha})]\!] = \bigvee_{\alpha\in\mathrm{Ord}} [\![\varphi(\check{\alpha})]\!]$$

The other direction is clear. □

Lemma 4.3. $[\![u \in L]\!] = \bigvee_{x\in L} [\![u = \check{x}]\!]$ *for any* $u \in V^B$.

Proof. Let $\psi(z, \alpha)$ be the formula that expresses "z belongs to the α-th level of L". We need the "obvious" fact that for any name $u \in V^B$, $[\![\psi(u, \check{\alpha})]\!] = [\![u \in \check{L_\alpha}]\!]$. First, let $\varphi(x, \alpha)$ be the formula that expresses "x is the α-th level of L", which is Δ_1^{ZF}. Since $\varphi(L_\alpha, \alpha)$ is true in V, by absoluteness $V^B \models \varphi(\check{L_\alpha}, \check{\alpha})$. A more confusing way to say this is $V^B \models \check{L_\alpha} = L_{\check{\alpha}}$.

Next, $\varphi(x, \alpha) \to \forall z(z \in x \leftrightarrow \psi(z, \alpha))$ is a theorem of ZFC, hence true in V^B by soundness. Since $\varphi(\check{L_\alpha}, \check{\alpha})$ has truth value 1, so does $\forall z(z \in \check{L_\alpha} \leftrightarrow \psi(z, \check{\alpha}))$; in other words $[\![u \in \check{L_\alpha}]\!] = [\![\psi(u, \check{\alpha})]\!]$ for any $u \in V^B$. Finally,

$$[\![u \in L]\!] = [\![\exists\alpha\psi(u, \alpha)]\!] = \bigvee_{\alpha\in\mathrm{Ord}} [\![\psi(u, \check{\alpha})]\!] = \bigvee_{\alpha\in\mathrm{Ord}} [\![u \in \check{L_\alpha}]\!] = \bigvee_{\alpha\in\mathrm{Ord}} \bigvee_{x\in L_\alpha} [\![u = x]\!] = \bigvee_{x\in L} [\![u = \check{x}]\!]$$

□

Theorem 4.4. $\mathsf{ZFC} + V \neq L$ *is consistent.*

Proof. Let $B = RO(2^\omega)$ be the Cohen algebra, and denote the basic clopen set $\{x \in 2^\omega : x(n) = 1\}$ by p_n, whose complement (either in the sense of set or Boolean algebra) is $p_n^* = \{x \in 2^\omega : x(n) = 0\}$. Consider the name $\dot{G} := \{(\check{n}, p_n) : n \in \omega\}$. It is easy to calculate that $[\![\check{n} \in \dot{G}]\!] = p_n$. Moreoever, for $x \in V$, $x \nsubseteq \omega$ we have $[\![\check{x} = \dot{G}]\!] = 0$, and if $x \subseteq \omega$, then

$$[\![\check{x} = \dot{G}]\!] = \left[\bigwedge_{n\in x} [\![\check{n} \in \dot{G}]\!]\right] \wedge \left[\bigwedge_{n\in\omega} (p_n \Rightarrow [\![\check{n} \in \check{x}]\!])\right]$$

$$= \left[\bigwedge_{n\in x} p_n\right] \wedge \left[\bigwedge_{n\notin x} p_n^*\right]$$
$$= 0$$

where at the last step we note that if x is infinite then $\bigwedge_{n\in x} p_n = (\bigcap_{n\in x} p_n)^{-\circ} = (\bigcap_{n\in x} p_n)^\circ = \varnothing$, since each p_n is clopen; the case when $\omega \setminus x$ is infinite is similar.

So $[\![\check{x} = \dot{G}]\!] = 0$ for any $x \in V$, in particular for any $x \in L$. Thus $[\![\dot{G} \in L]\!] = 0$ by the previous lemma. Consequently $V^B \models \exists x(x \notin L)$. □

4.2 Delta system lemma and the consistency of $\neg\mathsf{CH}$

Using $B = RO(2^\kappa)$ for some large κ instead of $RO(2^\omega)$, we can obtain $V^B \models \neg\mathsf{CH}$. The idea is that $\kappa \simeq \kappa \times \omega$, so $RO(2^\kappa) \simeq RO(2^{\kappa\times\omega})$. Let $p_{\alpha,n}$ be the basic clopen set $\{x \in 2^{\kappa\times\omega} : x(\alpha, n) = 1\}$. If we define $\dot{G}_\alpha = \{(\check{n}, p_{\alpha,n}) : n \in \omega\}$, then similar to the above proof, $[\![\dot{G}_\alpha \neq x]\!] = 1$ for any $x \in V$. Moreover, it's not hard to show $[\![\dot{G}_\alpha \neq \dot{G}_\beta]\!] = 1$ for $\alpha \neq \beta$, and thus in V^B there is an injection from $\check{\kappa}$ to $\mathcal{P}(\omega)$.

It may seem like we have proved that $V^B \models \neg\mathsf{CH}$, but there is a caveat: how do we know that κ remains large in V^B? Say $\kappa = \omega_2$. How do we know that $V^B \models$"$\check{\kappa}$ is the second uncountable cardinal"? A priori there might exist a name $\dot{f}$, such that $V^B \models$"$\dot{f}$ is a surjection from $\check{\omega}_1$ to $\check{\kappa}$", or even from $\check{\omega}$ to $\check{\kappa}$. In fact this does happen for some forcings, but fortunately not in the case of adding Cohen reals. We shall show that B has the so called countable chain condition, which ensures that whenever κ is a cardinal, $V^B \models \check{\kappa}$ is a cardinal.

First a combinatorial lemma extremely useful in forcing. A collection of sets $(x_i : i \in I)$ is called a *delta system* if there exists a *root* R, meaning $x_i \cap x_j = R$ for any different $i, j \in I$.

Lemma 4.5 (Delta system lemma)**.** *If $(x_i : i < \omega_1)$ is a collection of finite sets, then there exists an uncountable $I \subseteq \omega_1$ such that $(x_i : i \in I)$ is a delta system.*

Proof. There exists an uncountable $I \subseteq \omega_1$ such that all the $(x_i : i \in I)$ have the same size, so without loss of generality we might assume all the $(x_i : i < \omega_1)$ have the same size n, and prove the lemma by induction on n. The case $n = 0$ is obvious since all of them are empty.

Suppose the lemma holds for n, and we want to show that it holds for $n + 1$. If there exists $a \in \bigcup_{i<\omega_1} x_i$ such that $I_1 := \{i < \omega_1 : a \in x_i\}$ is uncountable, then we apply the induction hypothesis to $(x_i \setminus \{a\} : i \in I_1)$, obtaining an uncountable $I_2 \subseteq I_1$ such that $(x_i \setminus \{a\} : i \in I_2)$ is a delta system with some root R. It is clear that $(x_i : i \in I_2)$ is a delta system with root $R \cup \{s\}$.

Otherwise, for any $a \in \bigcup_{i<\omega_1} x_i$, the set $\{i < \omega_1 : a \in x_i\}$ is countable; it follows that for any countable set S, we have $S \cap x_i = \varnothing$ for all large enough i. Inductively define an increasing sequence of countable ordinals $(i_\alpha : \alpha < \omega_1)$ as follows: let i_0 be arbitrary, and when i_β has been defined for every $\beta < \alpha$, consider the countable set $S = \bigcup_{\beta<\alpha} x_{i_\beta}$ and let α be such that $S \cap x_{i_\gamma} = \varnothing$ for all $\gamma \geq \alpha$. Then $(x_{i_\alpha} : \alpha < \omega_1)$ is a sequence of disjoint sets, i.e., a delta system with empty root. □

Recall that B^+ is the set of nonzero elements, $a, b \in B^+$ are called incompatible if $a \wedge b = 0$, and $A \subseteq B^+$ is called an antichain if its elements are pairwise incompatible. We say that B has *countable chain condition* or ccc, if any antichain $A \subseteq B^+$ is countable. It is easy to see that B is ccc iff whenever $(b_i : i < \omega_1)$ are (not necessarily distinct) elements of B^+, there exist $i < j < \omega_1$ such that $b_i \wedge b_j \neq 0$.

Lemma 4.6. *$B = RO(2^\kappa)$ is ccc for any infinite cardinal κ.*

Proof. Suppose $(b_i : i < \omega_1)$ are nonempty regular open sets in 2^κ; recall that the meet operation of $RO(2^\kappa)$ is simply intersection, so we want to show that $b_i \cap b_j \neq \varnothing$ for some $i \neq j$. For every finite partial function $p : \kappa \rightharpoonup \{0, 1\}$, let

$$u_p = \{x \in 2^\kappa : x \supseteq p\} = \{x \in 2^\kappa : \forall \alpha \in \mathrm{dom}(p)\ x(\alpha) = p(\alpha)\}$$

By definition of product topology, the collection of all u_p is a basis for the topology on 2^κ. Since each b_i is a nonempty open set, there exists a partial function $p_i : \kappa \rightharpoonup \{0, 1\}$ such that $b_i \supseteq u_{p_i}$.

It suffices to show that for some $i < j < \omega_1$ we have $u_{p_i} \cap u_{p_j} \neq \varnothing$. This is the same as saying p_i and p_j are compatible as functions, i.e., agree on the intersection of their domains, in which case $p_i \cup p_j$ is also a partial function and $u_{p_i} \cap u_{p_j} = u_{p_i \cup p_j}$. Now we apply delta system lemma to $(\mathrm{dom}(p_i) : i < \omega_1)$ to obtain an uncountable $I_1 \subseteq \omega_1$ and a root $R \subseteq \kappa$ such that $\mathrm{dom}(p_i) \cap \mathrm{dom}(p_j) = R$ for any different $i, j \in I_1$. Since there are only finitely many functions from R to $\{0, 1\}$, there is an uncountable $I_2 \subseteq I_1$ such that all the $p_i,\ i \in I_2$ restricted to R are the same. We have found not just two, but uncountably many p_i's that are pairwise compatible, thus finishing the proof. □

Theorem 4.7. *If B is ccc, then for any cardinal κ, $V^B \models \check{\kappa}$ is a cardinal.*

Proof. This can be checked directly for $\kappa = \omega$, so suppose κ is an uncountable cardinal and $\lambda < \kappa$ is infinite. Fix a name $\dot{f}$, and denote $[\![\dot{f}$ is a function from λ to $\kappa]\!]$ by b. It might be easier to think about the case $b = 1$, and in fact there is no loss of generality in considering this case because of Lemma 5.8, the maximal principle. Since V^B satisfies "if f is a function, $f(x) = y$ and $f(x) = z$ then $y = z$", for any different $\alpha_1, \alpha_2 < \kappa$ and any $\beta < \lambda$ we have:

$$b \wedge [\![\dot{f}(\check{\beta}) = \check{\alpha}_1]\!] \wedge [\![\dot{f}(\check{\beta}) = \check{\alpha}_2]\!] \leq [\![\check{\alpha}_1 = \check{\alpha}_2]\!] = 0,$$

so for each $\beta < \lambda$, $\{b \wedge [\![\dot{f}(\check{\beta}) = \check{\alpha}]\!] : \alpha < \kappa\}$ is an antichain; since B is ccc the antichain must be countable, and therefore $A_\beta := \{\alpha < \kappa : b \wedge [\![\dot{f}(\check{\beta}) = \check{\alpha}]\!] \neq 0\}$ is countable.

Since κ is a cardinal (in V), the union of all the $A_\beta,\ \beta < \lambda$ has size at most $\omega \times \lambda = \lambda < \kappa$; choose some $\alpha < \kappa$ not in the union, so that $b \wedge [\![\dot{f}(\check{\beta}) = \check{\alpha}]\!] = 0$ for any $\beta < \lambda$. Using the Bounded Quantification Lemma 3.6, it is not hard to calculate that $[\![\dot{f}$ is surjective$]\!] = \bigwedge_{\alpha<\kappa} \bigvee_{\beta<\lambda} [\![\dot{f}(\check{\beta}) = \check{\alpha}]\!]$, so $b \wedge [\![\dot{f}$ is surjective$]\!] = 0$. Therefore V^B satisfies that no function from $\check{\lambda}$ to $\check{\kappa}$ is surjective, in other words $\check{\kappa}$ is a cardinal. □

Basically the same proof shows that if B is ccc and κ is regular, then $V^B \models \check{\kappa}$ is regular.

Theorem 4.8. *Let $B = RO(2^\kappa)$ where $\kappa \geq \omega_2$ is an infinite cardinal. Then $V^B \models |\mathcal{P}(\omega)| \geq \kappa$, so in particular $V^B \models \neg\mathsf{CH}$. Moreover, if $\kappa^\omega = \kappa$ in V then $V^B \models |\mathcal{P}(\omega)| = \kappa$.*

Proof. As mentioned before, we might view B as $RO(2^{\kappa\times\omega})$, and define $\dot{G}_\alpha = \{(\check{n}, p_{\alpha,n}) : n \in \omega\}$ where $p_{\alpha,n}$ is the basic clopen set $\{x \in 2^{\kappa\times\omega} : x(\alpha, n) = 1\}$. It is not difficult to calculate that $[\![\dot{G}_\alpha = \dot{G}_\beta]\!] = \bigwedge_{n\in\omega}(p_{\alpha,n} \Leftrightarrow p_{\beta,n}) = 0$, since any open subset of $2^{\kappa\times\omega}$ contains some x and y such that $x(\alpha, n) \neq y(\beta, n)$ for some n. Then one can cook up a name $\dot{f}$ such that $V^B \models$"$\dot{f}$ is an injection from κ to $\mathcal{P}(\omega)$". Since B is ccc, all cardinals of V remain cardinals in V^B, so $V^B \models$"$\check{\kappa} \geq$ the second uncountable cardinal".

To estimate the size of $\mathcal{P}(\omega)$, we note that for any name u, if we define $A_u = \{(\check{n}, [\![\check{n} \in u]\!]) : n \in \omega\}$, then $[\![u \subseteq \check{\omega}]\!] = [\![u = A_u]\!]$. So if we let W be the set of all functions from $\{\check{n} : n \in \omega\}$ to B and $Z = W \times \{1\}$, then $V^B \models$"Z is the power set of $\check{\omega}$".

It remains to count the number of such functions, for which we first calculate the size of B. Everything happens in V until further notice. Since there are κ many basic clopen sets and κ many finite Boolean combinations of them, the topology on 2^κ has a basis P of size κ. Clearly P is a dense subset of the poset B^+. For any nonzero $b \in B$, by Zorn's lemma let A be an antichain that is maximal among all antichains satisfying: (i) $A \subseteq P$, (ii) $\forall p \in A\ p \leq b$. Since B is ccc, A is countable. Clearly $\bigvee A \leq b$, and we claim that $\bigvee A = b$; otherwise $(\bigvee A)^* \wedge b \neq 0$, so there exists $p \in P$ such that $p \leq (\bigvee A)^* \wedge b$, and $A \cup \{p\}$ satisfies the above requirements, contradicting the choice of A.

Therefore any $b \in B$ can be written as $\bigvee A$ for some countable $A \subseteq P$, so $B \leq \kappa^\omega$. If $\kappa^\omega = \kappa$ then $B = \kappa$, and moreover the number of functions from $\{\check{n} : n \in \omega\}$ to B is $\kappa^\omega = \kappa$. Finally, let

$$f = \{\mathrm{op}(\check{x}, x) : x \in W\} \times \{1\}$$

where $\mathrm{op}(\check{x}, x)$ is the name for the ordered pair, and let $g : \kappa \to W$ be a bijection in V. Similar to the proof of choice, we have $V^B \models$"$f \circ \check{g}$ is a surjection from $\check{\kappa}$ to $Z = \mathcal{P}(\omega)$". □

5 Translation to ctm

The Boolean-valued models $V^{RO(2^\omega)}$ or $V^{RO(2^\kappa)}$ are enough to show the consistency of $V \neq L$ or $\neg\mathsf{CH}$ by the Boolean Soundness Theorem, but in practice it is somewhat more convenient to work with actual transitive models. In this section we explain how to relativize everything we have done so far to a countable transitive model M. We can then choose a generic filter G and form either the generic extension $M[G]$ or the quotient M^B/G, which turn out to be the same. We also discuss some metamathematical issues surrounding the Boolean-valued model approach and the ctm approach.

5.1 Generic extension

The discussion in previous sections happened in V, and showed the scheme that if B is a complete Boolean algebra, then $V^B \models \varphi$ for every ZFC axiom φ. Now let M be a countable transitive model of ZFC, which does not provably exist in ZFC, but we shall discuss this hypothesis later. Suppose $B \in M$ and $M \models$ "B is a complete Boolean algebra". Then B is also a Boolean algebra in V, but most likely not complete. So let us extend the definition of Boolean-valued models as follows.

Definition 5.1. Suppose B is a Boolean algebra and $[\![\cdot = \cdot]\!] : M^2 \to B$ and $[\![\cdot \in \cdot]\!] : M^2 \to B$ are functions. We call $(M, [\![\cdot = \cdot]\!], [\![\cdot \in \cdot]\!])$ a B-valued structure if it satisfies the same equality axioms as in Definition 3.1, and moreover B has all the "relevant" suprema and infima, namely those that appear in the recursive definition $[\![\exists x \varphi(x)]\!] = \bigvee_{u \in M} [\![\varphi(u)]\!]$ and $[\![\forall x \varphi(x)]\!] = \bigwedge_{u \in M} [\![\varphi(u)]\!]$.

If we carry out the construction of Boolean-valued model inside M, we get a B-valued structure $((V^B)^M, [\![\cdot = \cdot]\!]^M, [\![\cdot \in \cdot]\!]^M)$. Recall that instead of defining V^B by a hierarchy, we can also define a B-name as a function from a set of B-names to B. This is absolute, as with many definitions by transfinite recursion, so we have $(V^B)^M = V^B \cap M$, which we also denote by M^B. Then

$(M^B, [\![\cdot = \cdot]\!]^M, [\![\cdot \in \cdot]\!]^M)$ is a B-valued model of ZFC. This is a single statement in the language of set theory instead of a scheme, since M^B is a set-size Boolean-valued model unlike the case of V^B.

Recall that a filter G on B is a set $G \subseteq B$ such that:

- $1 \in G$ and $0 \notin G$,
- $a \in G \wedge a \leq b \to b \in G$,
- $a \in G \wedge b \in G \to a \wedge b \in G$.

A set $D \subseteq B^+$ is dense if $\forall a \in B^+ \exists b \in D (b \leq a)$.

Definition 5.2. Let M, B be as above. A filter G on B is called (M, B)*-generic* if $G \cap D \neq \varnothing$ for every dense set $D \subseteq B^+$ such that $D \in M$.

Sometimes we refer to G as M-generic or simply generic. In Section 1 we tried to motivate genericity (for the special case of $RO({}^{\omega}2)$) using topology. We shall see that it also naturally pops up when we try to show the generic extension $M[G]$ satisfies ZFC.

Note that "D is a dense subset of B^+" is a bounded formula and hence absolute. Since M is countable, it contains only countably many dense sets, while V usually contain continuum many. We also note that if G is generic then it is an "M-complete ultrafilter".

Lemma 5.3. *If G is (M, B)-generic, then:*

(i) *it is an ultrafilter, i.e., $b \in G$ or $b^* \in G$ for any $b \in B$;*

(ii) *it is M-complete in the sense that if if $X \subseteq G$ and $X \in M$, then $\bigwedge X \in G$.*

Moreover, a filter G is (M, B)-generic iff for any maximal antichain $A \subseteq B$ such that $A \in M$ we have $G \cap A \neq \varnothing$.

Proof. For (1), note that $\{a \in B^+ : a \leq b \vee a \leq b^*\}$ is a dense set that belongs to M. For (2), first note that the notion of infimum is absolute, and $\bigwedge X$ exists by the assumption that $M \models$"B is complete". Now consider the set $D = \{a \in B^+ : (\exists b \in X,\ a \wedge b = 0) \vee (a \leq \bigwedge X)\}$, which belongs to M; it is dense because if $a \wedge (\bigwedge X) = 0$ then $a \wedge (\bigwedge X)^* \neq 0$, so $a \wedge b^* \neq 0$ for some $b \in X$.

For the moreover part, use that if A is a maximal antichain then $\{b \in B^+ : \exists a \in A,\ b \leq a\}$ is dense, and conversely any dense set contains a maximal antichain. □

Taking contrapositive of (2), we see that if $\bigvee X \in G$ and $X \in M$ then $X \cap G \neq \varnothing$.

The existence of generic filters follows from the countability of M and a simple diagonalization argument resembling the Baire Category Theorem.

Lemma 5.4 (Rasiowa–Sikorski)**.** *Any $b \in B^+$ is contained in an (M, B)-generic filter G.*

Proof. List the dense subsets of B^+ that are in M as $D_1, D_2, \dots$ Inductively define a decreasing sequence $(b_0, b_1, \dots)$ as follows. $b_0 = b$, $b_{n+1} \leq b_n$ and $b_{n+1} \in D_{n+1}$, which is possible by the denseness of D_{n+1}. Let G be the upward closure of $\{b_0, b_1, \dots\}$. □

Definition 5.5 (Generic extension)**.** For $u \in M^B$, define the *interpretation* of u with respect to G recursively by $u_G = \{x_G : x \in \mathrm{dom}(u),\ u(x) \in G\}$. Let $M[G] = \{u_G : u \in M^B\}$.

The hard work we have done so far pays off to give a smooth proof of $M[G] \models \mathsf{ZFC}$. The following theorem is the counterpart of truth and definability lemmas in the ctm approach. It is the same as Jech [9, Theorem 14.29].

Theorem 5.6. *Suppose G is (M, B)-generic. For any formula $\varphi(x_1, \ldots, x_n)$ and $u_1, \ldots, u_n \in M^B$, $M[G] \models \varphi((u_1)_G, \ldots, (u_n)_G)$ iff $[\![\varphi(u_1, \ldots, u_n)]\!] \in G$.*

Proof. Needless to say we prove by induction on complexity of formulas, and start with the atomic ones. This is yet another induction on the well-founded relation we used to define their truth values:

$$\begin{aligned} u_G \in v_G \Leftrightarrow & \exists y \in \mathrm{dom}(v)(u_G = y_G \wedge v(y) \in G) \\ \Leftrightarrow & \exists y \in \mathrm{dom}(v)([\![u = y]\!] \in G \wedge v(y) \in G) \\ \Leftrightarrow & \exists y \in \mathrm{dom}(v)(v(y) \wedge [\![u = y]\!] \in G) \\ \Leftrightarrow & [\![u \in v]\!] \in G \end{aligned}$$

where we use induction hypothesis for $u = y$ at the third step, and at the last step we use that the set $X = \{v(y) \wedge [\![u = y]\!] : y \in \mathrm{dom}(v)\}$ belongs to M, since the Boolean values are defined relativized to M, so $X \cap G \neq \varnothing$ iff $\bigvee X \in G$. Similar arguments are used below.

$$\begin{aligned} u_G \subseteq v_G & \Leftrightarrow \forall x \in \mathrm{dom}(u)[u(x) \in G \to x_G \in v_G] \\ & \Leftrightarrow \forall x \in \mathrm{dom}(u)[u(x) \in G \to [\![x \in v]\!] \in G] \\ & \Leftrightarrow \forall x \in \mathrm{dom}(u)[u(x) \notin G \vee [\![x \in v]\!] \in G] \\ & \Leftrightarrow \forall x \in \mathrm{dom}(u)[u(x) \Rightarrow [\![x \in v]\!] \in G] \\ & \Leftrightarrow \bigwedge_{x \in \mathrm{dom}(u)} [u(x) \Rightarrow [\![x \in v]\!]] \in G \end{aligned}$$

It follows that $u_G = v_G$ iff $[\![u = v]\!] \in G$. The induction steps for propositional connectives $\varphi \wedge \psi$ and $\neg\varphi$ are easy. For quantifier,

$$\begin{aligned} M[G] \models \forall x \varphi(x) & \Leftrightarrow \forall u \in M^B, M[G] \models \varphi(u_G) \\ & \Leftrightarrow \forall u \in M^B, [\![\varphi(u)]\!] \in G \\ & \Leftrightarrow \bigwedge_{u \in M^B} [\![\varphi(u)]\!] \in G \\ & \Leftrightarrow [\![\forall x \varphi(x)]\!] \in G \end{aligned}$$

□

5.2 Maximal principle, canonical names and cores

In this subsection we record some technical results that are needed for defining the quotient M^B/G as well as later material such as iterated forcing. Let B be a complete Boolean algebra.

Lemma 5.7 (Mixing lemma)**.** *If $A = \{a_i : i \in I\}$ is an antichain and $u_i \in V^B$, $i \in I$, then there exists $u \in V^B$ such that $[\![u = u_i]\!] \geq a_i$ for every i.*

Proof. Let $\text{dom}(u) = \bigcup_{i\in I} \text{dom}(u_i)$ and $u(x) = \bigvee_{i\in I}[a_i \wedge [\![x \in u_i]\!]]$. $\square$

Lemma 5.8 (Maximal principle)**.** *For any formula* $\varphi = \varphi(x, y_1, \ldots, y_n)$ *and* $v_1, \ldots, v_n \in V^B$*, there exists* $u \in V^B$ *such that* $[\![\exists x\varphi(x, v_1, \ldots, v_n)]\!] = [\![\varphi(u, v_1, \ldots, v_n)]\!]$.

Proof. For brevity suppress the variables besides x. Let $b = [\![\exists x\varphi(x)]\!]$, and $A = \{a_i : i \in I\}$ be an antichain maximal below $\{[\![\varphi(u)]\!] : u \in V^B\}$; then $\bigvee A = b$. For each a_i pick a u_i such that $[\![\varphi(u_i)]\!] \geq a_i$. Let u be their mix. $\square$

Note that the maximal principle requires AC.

Call $u, v \in V^B$ equivalent if $[\![u = v]\!] = 1$. For later use we would like to choose a canonical representative from each equivalence class. There are several ways for $u \neq v$ and yet $[\![u = v]\!] = 1$ to happen:

- $u(x) = 0$ for some $x \notin \text{dom}(v)$,
- $[\![x = y]\!] = 1$ for some $x, y \in \text{dom}(u)$,
- $u(x) < [\![x \in u]\!]$.

We shall see that if we make sure these don't happen then we indeed get a representative.

Definition 5.9. Inductively define $cV^B = \bigcup_{\alpha\in\mathsf{Ord}} cV^B_\alpha$, the collection of *canonical names*, as follows. Let $cV^B_0 = \varnothing$, at limit stage take union, and at successor stage let $cV^B_{\alpha+1}$ be the union of cV^B_α together with the set of all partial functions u from cV^B_α to B such that:

- $[\![u = x]\!] \neq 1$ for any $x \in cV^B_\alpha$;
- for any $x \in cV^B_\alpha$, $x \in \text{dom}(u)$ iff $[\![x \in u]\!] > 0$, in which case $u(x) = [\![x \in u]\!]$.

Clearly we have $cV^B_\alpha \subseteq V^B_\alpha$ by induction; in particular $[\![u = x]\!]$ and $[\![x \in u]\!]$ in the above definition make sense. Our definition of canonical names is related to but a bit different from the notion of "full names" in Kunen [12, Definition V.5.5], and also different from that of Cummings [6, Definition 7.1].

Also note that the terminology clashes slightly with calling $\check{x}$ the canonical name for $x \in V$, and $\dot{G} = \{(\check{b}, b) : b \in B\}$ the canonical name for the generic filter, etc. From now on we shall try to call these the "natural names".

Lemma 5.10. *For every* $u \in V^B$*, there exists a unique* $u' \in cV^B$ *such that* $[\![u = u']\!] = 1$.

Proof. We show by induction on α that any $u \in V^B_\alpha$ is equivalent to a name in cV^B_α, and names in cV^B_α are mutually non-equivalent. It suffices to show this at successor stage. Let $\text{dom}(u') = \{x \in cV^B_\alpha : [\![x \in u]\!] > 0\}$ and $u'(x) = [\![x \in u]\!]$. Using induction hypothesis it's easy to show that $[\![u = u']\!] = 1$. Either u' is already equivalent to some name in cV^B_α, or $u' \in cV^B_{\alpha+1}$, both of which imply u is equivalent to some name in $cV^B_{\alpha+1}$.

If $u, v \in cV^B_{\alpha+1} \setminus cV^B_\alpha$, then $[\![u = v]\!] = 1$ implies $\text{dom}(u) = \text{dom}(v)$ and $u(x) = v(x)$, and thus $u = v$. $\square$

In general $\{v \in V^B : [\![v \in u]\!] \in 1\}$ is a proper class; for example if $[\![u = \check{\alpha}]\!] = b$, $[\![u = \check{0}]\!] = b^*$ and $v = \{(u, b^*)\}$, then $[\![v \in \check{2}]\!] = 1$. However, we will show that $\{v \in cV^B : [\![v \in u]\!] = 1\}$ is a set, called the *core* of u, and that if $u \in cV^B$ then its core is a subset of $\mathrm{dom}(u)$. An application of the mixing lemma shows that u can more or less be identified with its core, provided that the core is nonempty.

Lemma 5.11. (i) *If $u, v \in cV^B$ and $[\![v \in u]\!] = 1$ then $v \in \mathrm{dom}(u)$. Thus for any $u \in V^B$,* $\mathrm{Core}(u) := \{v \in cV^B : [\![v \in u]\!] = 1\}$ *is a set.*

(ii) *If $[\![u \neq \varnothing]\!] = 1$, then for any $w \in V^B$, there exists $v \in \mathrm{Core}(u)$ s.t. $[\![w \in u]\!] = [\![w = v]\!]$.*

Proof. (i) Let α be the smallest s.t. $\mathrm{dom}(u) \subseteq cV^B_\alpha$, and $Y = \bigcup_{x \in \mathrm{dom}(u)} \mathrm{dom}(x)$. If $v \in V^B$ is such that $[\![v \in u]\!] = 1$, let $v' = \{(y, [\![y \in v]\!]) : y \in Y\}$. It can be checked that $[\![v = v']\!] = 1$ and $v' \in V^B_\alpha$, so the conventional name for v' (equivalently for v) is in cV^B_α. In particular, if $v \in cV^B$ then $v \in cV^B_\alpha$, and hence $v \in \mathrm{dom}(u)$.

For the "thus" part, if u' is the conventional name for u then $\mathrm{Core}(u) = \mathrm{Core}(u')$.

(ii) By maximal principle (which requires AC) $\mathrm{Core}(u) \neq \varnothing$. Choose $v_0 \in \mathrm{Core}(u)$. For any $w \in V^B$, consider the antichain $\{[\![w \in u]\!], [\![w \notin u]\!]\}$ and use the mixing lemma to get a name v_0' such that $[\![v_0' = w]\!] \geq [\![w \in u]\!]$ and $[\![v_0' = v_0]\!] \geq [\![w \notin u]\!]$. It follows that $[\![v_0' \in u]\!] = 1$ and $[\![v_0' = w]\!] = [\![w \in u]\!]$. Let v be the conventional name for v_0'.

Note that v is of course not unique, but if we already have a v_0, then there is also a canonical choice of v since the proof of mixing lemma is constructive. This is somewhat relevant in the proof of factor lemma for iterated forcing. □

A consequence of (ii) when we consider ctm is that $u_G = \{v_G : v \in \mathrm{Core}(u)\}$ for any G. Note that (ii) is analogous to maximal principle: by definition $[\![w \in u]\!] = \bigvee_{v \in \mathrm{dom}(u)} u(v) \wedge [\![w = v]\!]$. If $u \in cV^B$, then (ii) tells us the supremum is achieved by some $v \in \mathrm{dom}(u)$, and moreover $u(v) = 1$.

The argument used to prove (ii) is quite common, so we isolate it as a lemma.

Lemma 5.12 (Existential completeness)**.** *If $[\![\exists x \varphi(x)]\!] = 1$ and $w \in V^B$, there exists $v \in V^B$ such that $[\![\varphi(v)]\!] = 1$ and $[\![\varphi(w)]\!] = [\![w = v]\!]$.*

Proof. Same as above, replacing $x \in u$ by $\varphi(x)$. □

5.3 Quotient and Boolean Łoś's Theorem

We are going to use the quotient structure M^B/G and Boolean Łoś's Theorem to give a different (and longer) proof of Theorem 5.6. We will not need this material later, but it is related to the fascinating topic of Boolean ultrapower [7], and thus deserve some attention in our opinion.

Suppose M is either V or a ctm, B is a complete Boolean algebra in M, and M^B is the corresponding Boolean-valued model. Let G be *any* ultrafilter on B.

Definition 5.13. Define $u, v \in M^B$ to be equivalent if $[\![u = v]\!] \in G$, and denote the equivalence class of u by $[u]$. Let M^B/G be the (ordinary) first order structure with underlying set $\{[u] : u \in M^B\}$, and the membership relation defined by $[u] \in [v]$ iff $[\![u \in v]\!] \in G$.

That $[\![u = v]\!] \in G$ is indeed an equivalence relation and the membership relation is well-defined follow from the equality axioms satisfied by M^B and properties of ultrafilter.

In the case of V^B, each equivalence class is in fact a proper class, so strictly speaking we need to employ Scott's trick and define $[u]$ to be the set of names equivalent to u and of minimal rank.

Lemma 5.14. *In the case $M = V$, the membership relation is set-like.*

Proof. Fix $u \in V^B$; we want to show there are up to equivalence only set many v's for which $[\![v \in u]\!] \in G$. By existential completeness, there exists u_0 such that $[\![u_0 \neq \varnothing]\!] = 1$ and $[\![u \neq \varnothing]\!] = [\![u = u_0]\!]$. Note that $[\![v \in u]\!] \leq [\![u \neq \varnothing]\!]$, so if $[\![v \in u]\!] \in G$ then $[\![u \neq \varnothing]\!] = [\![u = u_0]\!] \in G$.

Thus replacing u by u_0 we may assume $[\![u \neq \varnothing]\!] = 1$. We can then talk about $\mathrm{Core}(u)$. By Lemma 5.11, for any v there exists $w \in \mathrm{Core}(u)$ such that $[\![v \in u]\!] = [\![v = w]\!]$, so if $[\![v \in u]\!] \in G$ then $[v] = [w]$ for some $w \in \mathrm{Core}(u)$. □

Theorem 5.15 (Boolean Łoś's Theorem)**.** $M^B/G \models \varphi([u_1], \ldots, [u_n])$ *iff* $[\![\varphi(u_1, \ldots, u_n)]\!] \in G$

Proof. This is again an induction on φ, and most cases are straightforward. Let us discuss the induction step for universal quantifier. As usual we suppress the irrelevant names. If $[\![\forall x \varphi(x)]\!] \in G$, then $[\![\varphi(u)]\!] \in G$ for any $u \in M^B$, so by induction hypothesis we have $M^B/G \models \varphi([u])$ for any $[u]$.

Now suppose $[\![\forall x \varphi(x)]\!] \notin G$, so $[\![\exists x \neg\varphi(x)]\!] \in G$. By maximal principle, there exists $u \in M^B$ for which $[\![\neg\varphi(u)]\!] = [\![\exists x \neg\varphi(x)]\!] \in G$, and by induction hypothesis (applied to $\neg\varphi$ rather than φ, but that's fine) we have $M^B/G \models \neg\varphi([u])$. □

More generlly the Boolean Łoś's Theorem holds for B-valued structures that satisfy the maximal principle, also called *fullness*.

So far G is just any ultrafilter on B. In particular we can form V^B/G, which is a class-size first order structure. The *Boolean ultrapower* $\check{V}_G$ is the substructure of V^B/G consisting of those $[u]$ for which $[\![u \in V]\!] \in G$, where $[\![u \in V]\!] := \bigvee_{x \in V} [\![u = x]\!]$. It can be shown that $\check{V}_G$ is an elementary extension of V, and that V^B/G is a forcing extension of $\check{V}_G$ by B [7]. Thus in some sense we can use forcing to literally extend V to a larger structure.

The drawback is that the structure V^B/G is always ill-founded unless B has atoms; if B is atomic, then basically $B = \mathcal{P}(X)$ for some X, which reduces to the usual ultrapower construction. On the other hand, if M is a ctm and G is generic then M^B/G turns out to be well-founded, in fact isomorphic to $M[G]$. This is yet another proof that genericity is a natural condition.

Theorem 5.16. *Suppose M is a ctm, B is a complete Boolean algebra in M, and G is (M, B)-generic. Then M^B/G is well-founded, and its transitive collapse is $M[G]$.*

Proof. In general a model of ZFC is well-founded iff its ordinals are, so it suffices to show the ordinals of M^B/G are well-founded. Suppose $u \in M^B$ and $[\![\mathrm{Ord}(u)]\!] \in G$. Since $[\![\mathrm{Ord}(u)]\!] = \bigvee_{\alpha \in M} [\![u = \check{\alpha}]\!]$, by M-completeness of G (cf. Lemma 5.3) we have $[\![u = \check{\alpha}]\!] \in G$ for some $\alpha \in M$, and the result is clear.

By definition, the collapsing map π on M^B/G satisfies $\pi([u]) = \{\pi([v]) : [v] \in [u]\}$. A routine induction plus M-completeness of G show that $\pi([u]) = u_G$. □

Well-foundedness of M^B/G is not equivalent to genericity of G: consider the case $B = \mathcal{P}(X)$ and G is a σ-complete ultrafilter. Bell [3, Theorem 4.6] shows that genericity is equivalent to the statement that M^B/G and M have the same ordinals.

5.4 Some metamathematical nonsense

Are the Boolean-valued model approach and the ctm approach equivalent? Of course not, at least not literally. The former approach starts with the class model V and produces a Boolean-valued class model V^B of ZFC, which can be upgraded to an ordinary (but ill-founded) class model V^B/G if you wish; the latter approach starts with a ctm M of ZFC and produces a larger ctm $M[G]$.

In fact, it is not accurate to say that M or $M[G]$ is a ctm of ZFC. It is really a ctm of $\ulcorner\mathsf{ZFC}\urcorner$—we have totally ignored the difference between what we call formal theory and coded theory. Let us fix some terminology.

Formal theory refers to the theory we work in. For us this is usually ZFC. For others it could be Z, ZF, ZFC plus large cardinals, or ETCS (elementary theory of category of sets).

After fixing a formal theory, say ZFC, we can start doing mathematics and prove theorems, such as Cantor's theorem, Bolzano–Weierstrass theorem, Fermat's last theorem, etc. These are what we call *formal theorems*. If we follow a Hilbert style deductive system, then a formal proof is a sequence of formulas each of which is either a logical axiom, a ZFC axiom, or obtained from previous formulas using one of the inference rules.

Of course, the everyday proofs that we write on paper or type in LaTeX do not follow such strict grammatical rules, and are full of abbreviations and details left to readers, but in principle they can be turned into formal proofs. As people who have worked with proof assistants like Lean would know, this is not always a straightforward task.

Metatheory is what we use to study and compare various formal theories. Below are some examples of metatheorems, in contrast with formal theorems.

1. The unique readability theorem, that any formula in any formal language can be read in an unambiguous way.
2. Gödel's first incompleteness theorem, an instance of which is "if T is formal theory in the language of set theory that is consistent, recursively enumerable and contains ZF, then there is a formal statement φ such that T proves neither φ nor $\neg\varphi$".
3. The reflection theorem, that for any formula $\varphi(x)$ in the language of set theory, ZF proves the formal theorem $\exists\kappa\forall x(\varphi(x)\leftrightarrow\varphi(x)^{V_\alpha})$.
4. The statement "L is a class model of ZFC", or more precisely, for any ZFC axiom φ, the formal theory ZF proves φ^L.
5. The statement "ZF and $\mathsf{ZFC}+\mathsf{GCH}$ are equiconsistent", which is proven using the method of inner model. Note that the proof is finitistic: we described an algorithm that transforms a proof of φ from $\mathsf{ZFC}+\mathsf{GCH}$ into a proof of φ^L from ZF.

Different people might have different opinion on what "metatheory" is exactly, but it should include finitistic reasoning such as natural numbers and induction that is needed for proving the above metatheorems. My viewpoint is that metatheory is just common sense; essentially the foundation of mathematics is common sense, since we need to take it for granted that everybody understands natural numbers and induction the same way.

Coded theory, a non-standard term, is what we call the theories that are coded as natural numbers or sets when we are doing model theory inside a formal theory like ZFC. We set up first order logic again, this time not using common sense but within the formal theory ZFC. We define languages, formulas, theories, etc. as certain natural numbers or (usually hereditarily finite) sets, except that to distinguish them from the formal counterparts we call them coded langauges, coded formulas and coded theories. In particular, there is the coded version of the language of set theory and the coded version of ZFC, usually denoted $\ulcorner\mathsf{ZFC}\urcorner$.

We can then happily do model theory and prove Gödel's completeness theorem, which is indeed a theorem rather than metatheorem since it involves the notion of model, as well as Łoś's theorem, Tarski–Vaught criterion, Morley's categoricity theorem, etc. We can also prove Gödel's second incompleteness theorem, which is really a metatheorem, that if the formal theory ZFC is consistent then it does not prove $\mathrm{Con}(\ulcorner\mathsf{ZFC}\urcorner)$.

Back to forcing. One of the missions of set theory is to compare the consistency strength of various theories, and to us it seems more natural to interpret "theories" as formal theories instead of coded theories. Now in the Boolean-valued model approach, we use that V^B is a Boolean-valued class model of ZFC (which is a metatheorem) plus the Boolean Soundness Theorem 3.9 (also a metatheorem) to conclude that if the formal theory ZFC is consistent, then so is the formal theory $\mathsf{ZFC} + \neg\mathsf{CH}$. This is exactly what we need for relative consistency proof.

In contrast, what the ctm approach seems to show is the following theorem in ZFC: if there exists a ctm of $\ulcorner\mathsf{ZFC}\urcorner$, then there also exists a ctm of $\ulcorner\mathsf{ZFC} + \neg\mathsf{CH}\urcorner$. This is a bit ad hoc and on its face not so related to relative consistency proof. However, the ctm approach can be modified to provide an honest consistency proof, as was already observed by Cohen.

In fact it is unnecessary to mention coded theories. Assume ZFC is consistent. To prove the consistency of $\mathsf{ZFC} + \neg\mathsf{CH}$, it suffices to prove the consistency of $\Gamma + \mathsf{CH}$ for any fintie fragment Γ of ZFC. If we go over the whole forcing construction very carefully, we see that it actually shows that for any such Γ, there is a possibly larger fragment Γ' of ZFC such if M is a ctm such that the relativization φ^M holds for every φ in Γ' and G is generic, then $\varphi^{M[G]}$ holds for every φ in Γ. By reflection theorem, there is a cardinal κ such that φ^{V_κ} holds for every φ in Γ'. Take a countable elementary submodel of V_κ and let M be its transitive collapse. Then the forcing construction together with soundness shows the consistency of $\Gamma + \mathsf{CH}$ as desired.

So is the Boolean-valued model approach or the ctm approach more popular? The answer is neither. In most research papers you will see people write "the generic extension $V[G]$", as if we literally step outside V, find a generic filter G and form a generic extension. It is left to the reader to justify this using either the Boolean-valued model approach or the ctm approach. We will say a bit more about this in Subsection 6.4.

6 Posets and maps between them

Although Boolean-valued model is intuitive and elegant, in practice posets are often more convenient to work with. In this section we translate everything into the poset language.

6.1 Forcing posets

For a poset $\mathbb{P}$, we want to define the class $V^{\mathbb{P}}$ of $\mathbb{P}$-names, as well as the forcing relation $p \Vdash \varphi(\sigma_1, \dots, \sigma_n)$ for any $p \in \mathbb{P}$, formula φ in the language of set theory, and $\sigma_1, \dots, \sigma_n \in V^{\mathbb{P}}$. When we relativize everything to a ctm M, we will have $(p \Vdash \varphi(\sigma_1, \ldots, \sigma_n))^M$ iff $M[G] \models \varphi((\sigma_1)_G, \ldots, (\sigma_n)_G)$ for all generic filters G on $\mathbb{P}$ that contain p, where the interpretation σ_G is defined in the same way as before. The forcing relation is defined inductively, similar to truth values in Boolean-valued models. We have two options:

1. Develop poset forcing from scratch.
2. Take a Boolean completion $\iota : \mathbb{P} \to B$, define a name translation map $V^{\mathbb{P}} \to V^B$, $\sigma \mapsto \bar{\sigma}$, and then define $p \Vdash \varphi(\sigma)$ iff $\iota(p) \le [\![\varphi(\bar{\sigma})]\!]$. We then get a recursive definition of $p \Vdash \varphi(\sigma)$ by unraveling what $\iota(p) \le [\![\varphi(\bar{\sigma})]\!]$ means.

The second option is not really much simpler than the first, but as a byproduct it shows that poset forcing and Boolean-valued model are essentially equivalent, so we will go with the second option.

Definition 6.1. A *forcing poset*, or just poset, is a triple $(\mathbb{P}, \le, 1_{\mathbb{P}})$ where $\le$ is a reflexive and transitive relation on $\mathbb{P}$ and $1_{\mathbb{P}}$ is a maximal element under $\le$.

Having a distinguished maximal element $1_{\mathbb{P}}$ makes it a bit easier to define check names $\check{x}$ and iterated forcing, though it is not really essenial. Note that there could exist other maximal elements. Also a poset may not be antisymmetric—we distinguish between "poset" and "partial order".

In practice $\mathbb{P}$ often consists of partial approximations to an object we want to add to the ground model, and the order is defined so that $p \le q$ when p is a larger piece of approximation. For this reason, when $p \le q$ we often say "p extends q" or "p is stronger than q".

Let us review some notions about posets. A Boolean algebra B is in particular a partial order and has a unique maximal element. But it also has a minimal element, and as we will see a forcing poset with a minimal element is uninteresting. So when we view a Boolean algebra B as a forcing poset we always mean $(B^+, \le)$; then notions like compatibility or antichain have the same meaning for posets and Boolean algebras.

Let $\mathbb{P}$ be a poset. Unless otherwise stated, p, q, r, etc. range over elements of $\mathbb{P}$, and A, D, etc. range over subsets. We say that $p, q \in \mathbb{P}$ are *compatible*, denoted $p \not\perp q$, if there exists r such that $r \le p$ and $r \le q$; otherwise they are *incompatible*, denoted $p \perp q$. A is an *antichain* if its elements are pairwise incompatible; A is a *maximal antichain* if it is not contained in any strictly larger antichain. D is *dense* if $\forall p \exists q \in D\ q \le p$. D is *open* if it is downward closed, namely $p \in D \land q \le p \to q \in D$. For $X \subseteq \mathbb{P}$, let $X{\downarrow}$ denote $\{p \in \mathbb{P} : \exists q \in X,\ p \le q\}$, which is clearly open; when $X = \{p\}$ we just write $p{\downarrow}$. X is *predense* if $\forall p \exists q \in X (p \not\perp q)$; clearly this is equivalent to saying $X{\downarrow}$ is dense. A dense set is of course predense, and an antichain is predense iff it is maximal. If B is a complete Boolean algebra, $X \subseteq B^+$ is predense iff $\bigvee E = 1$.

p is an *atom* if any $q, r \le p$ are compatible; this happen for example when p is a minimal element. $\mathbb{P}$ is *trivial* if $1_{\mathbb{P}}$ is an atom, in other words all elements are compatible. $\mathbb{P}$ is *separative* if for any p and q, if $p \not\le q$ then there exists $r \le p$ such that $r \perp q$.

If $\mathbb{P}$ and $\mathbb{Q}$ are posets, their *product* is the poset with underlying set $\mathbb{P} \times \mathbb{Q}$ and distinguished element $(1_{\mathbb{P}}, 1_{\mathbb{Q}})$, and the product order defined by $(p, q) \le (p', q')$ iff $p \le p'$ and $q \le q'$.

A map $i : \mathbb{P} \to \mathbb{Q}$ between posets is a *complete embedding* if:

1. $i(1_{\mathbb{P}}) = 1_{\mathbb{Q}}$,

2. $p_1 \le p_2 \to i(p_1) \le i(p_2)$,

3. $p_1 \perp p_2 \leftrightarrow i(p_1) \perp i(p_2)$,

4. if $A \subseteq \mathbb{P}$ is a maximal antichain then so is $f(A) \subseteq \mathbb{Q}$.

As explained below, this is consistent with our definition of complete embedding between Boolean algebras in Section 2. Incidentally, 3 is in fact implied by 4 since if $p_1 \perp p_2$, we can enlarge $\{p_1, p_2\}$ to a maximal antichain.

A typical example of complete embedding is $i : \mathbb{P} \to \mathbb{P} \times \mathbb{Q},\ p \mapsto (p, 1_{\mathbb{Q}})$; this example is generalized by iterated forcing. We will see that if $i : \mathbb{P} \to \mathbb{Q}$ is a complete embedding, then roughly speaking, forcing with $\mathbb{Q}$ "does more than" forcing with $\mathbb{P}$. The converse is also true in some sense: if forcing with $\mathbb{Q}$ always does more than $\mathbb{P}$, then there exists a complete embedding $i : \mathbb{P} \to \mathbb{Q}$. A priori the notion of completeness is Π_1 so may not be absolute. We will show later that in fact it is Δ_1^{ZF}; however, this seldom matters in practice since it is often enough to know that i is a complete embedding in the ground model.

A map $i : \mathbb{P} \to \mathbb{Q}$ is a *dense embedding* if it satisfies 1-3 as in the definition of complete embedding and $i(\mathbb{P})$ is dense in $\mathbb{Q}$. We will see that if i is a dense embedding then $\mathbb{P}$ and $\mathbb{Q}$ are exactly the same for the purpose of forcing. Note that despite the name, neither a complete embedding nor a dense embedding is required to be injective. We will see that any poset $\mathbb{P}$ densely embed into B^+ for some complete Boolean algebra B, called its Boolean completion.

We list some simple and useful facts, which are also good exercises for getting used to posets.

Lemma 6.2. (i) *A dense embedding $i : \mathbb{P} \to \mathbb{Q}$ is complete.*

(ii) *If $i : \mathbb{P} \to \mathbb{Q}$ is a dense embedding and $D \subseteq \mathbb{Q}$ is a dense open set, then the preimage $i^{-1}(D)$ is dense in $\mathbb{P}$.*

(iii) *Condition 4 in the definition of a complete embedding $i : \mathbb{P} \to \mathbb{Q}$ can be replaced by the following: the image of any dense set is predense; equivalently, the image of a predense set is predense.*

(iv) *If $i : \mathbb{P} \to \mathbb{Q}$ is a complete embedding, so is the restriction $i : p{\downarrow} \to i(p){\downarrow}$ for any p.*

(v) *If B and C are complete Boolean algebras, a map $i : B \to C$ is a complete embedding in the Boolean algebra sense iff $i(0_B) = 0_C$, $i(B^+) \subseteq C^+$ and the restriction $f : B^+ \to C^+$ is a complete embedding in the poset sense.*

Proof. (i) If $A \subseteq \mathbb{P}$ is a maximal antichain, then $A{\downarrow}$ is dense open, and $i(A{\downarrow})$ is easily seen to be dense and contained in $i(A){\downarrow}$, so $i(A)$ is maximal.

(ii) For any p there is $q \in D$ such that $q \le i(p)$. Let p_1 be such that $i(p_1) \le q$. In particular $i(p_1) \not\perp i(p)$, so $p_1 \not\perp p$, say $p_2 \le p_1, p$. Then $i(p_2) \le i(p_2)$ so $i(p_2) \in D$ by openness.

(iii) Suppose i is complete. If $D \subseteq \mathbb{P}$ is dense, consider $\{A \subseteq D : A \text{ is an antichain in } \mathbb{P}\}$; by Zorn's lemma it has a maximal element A. We claim that A must be a maximal antichain; otherwise there exists p that is incompatible with everything in A, but there exists $q \in D$ such that $q \in p$, and $A \cup \{q\}$ contradicts the maximality of A. We conclude that A and therefore $i(A)$ is a maximal antichain, and $i(D) \supseteq i(A)$ is predense.

Conversely, suppose the image of any dense set is predense. Then the image of a predense set X is also predense, since $i(X{\downarrow}) \subseteq i(X){\downarrow}$. Since an antichain is maximal iff it is predense, the result is clear. This also establishes the "equivalently" part.

(iv) Let A be a maximal antichain in the subposet $p{\downarrow}$. Suppose for contradiction that $i(A)$ is not maximal in $i(p){\downarrow}$, so there exists $r \in \mathbb{P}$ such that $i(r) \leq i(p)$ and $\forall a \in A(i(r) \perp i(a))$, which implies $r \not\perp p$ and $\forall a \in A(r \perp a)$. Let s be a common extension of r, p, and $A \cup \{s\}$ contradicts the maximality of A.

(v) Recall that a complete embedding between complete Boolean algebras means $i(\bigvee X) = \bigvee i(X)$, and that this is equivalent to saying i preserves maximal antichain, so the forward direction is clear. For the backward direction, let $b_1, b_2 \in B$. If $X := \{b_1 \wedge b_2, b_1^* \wedge b_2, b_1 \wedge b_2^*, b_1^* \wedge b_2^*\}$ does not contain 0_B then it is a maximal antichain, and so is $Y := \{i(b_1 \wedge b_2), i(b_1^* \wedge b_2), i(b_1 \wedge b_2^*), i(b_1^* \wedge b_2^*)\}$. Since i preserves order we have, e.g., $i(b_1 \wedge b_2) \leq i(b_1) \wedge i(b_2)$, and the fact that Y is a maximal antichain implies equality must hold. The case when X contains 0_B is similar, as with the preservation of other operations. □

We turn to the Boolean completion of a poset and its uniqueness. The idea is to view $\mathbb{P}$ as a topological space. Recall from Section 2 that whenever X is a topological space, the set $RO(X)$ of regular open subsets is a complete Boolean algebra. We then define a dense embedding from $\mathbb{P}$ to $RO(\mathbb{P})$ by sending p to $(p{\downarrow})^{-\circ}$.

Recall that a subset of $\mathbb{P}$ is called open if it is downward closed. The collection of all open sets is a topology; in fact it is closed under *arbitrary* intersection. Moreover, each point $p \in \mathbb{P}$ has a smallest open neighborhood, namely $p{\downarrow}$, so $\mathbb{P}$ is highly non-Hausdorff. A topological space with the property that every point has a smallest open neighborhood is called Alexandroff.

It is not hard to see that $(p{\downarrow})^{-}$ consists of all q that are compatible with p, so $(p{\downarrow})^{-\circ}$ consists of all q such that any $r \leq q$ is compatible with p.

Lemma 6.3. *Let $\mathbb{P}$ be a poset and $B = RO(\mathbb{P})$. The map from $\mathbb{P}$ to B^+ sending p to $(p{\downarrow})^{-\circ}$ is a dense embedding.*

Proof. Clearly the map preserves maximal element and order and has dense image. To show that it preserves incompatibility, note that:

$$(q{\downarrow})^{-\circ} \subseteq (p{\downarrow})^{-\circ} \text{ iff } q \in (p{\downarrow})^{-\circ} \text{ iff any } r \leq q \text{ is compatible with } p,$$

so if $(q{\downarrow})^{-\circ} \subseteq (p_1{\downarrow})^{-\circ}$ and $(q{\downarrow})^{-\circ} \subseteq (p_2{\downarrow})^{-\circ}$, then we can first choose a common extension r of q and p_1, then a common extension s of r and p_2, showing $p_1 \not\perp p_2$. □

Remark 6.4. (a) If $\mathbb{P} = B^+$ for some complete Boolean algebra B, then $RO(\mathbb{P})$ is isomorphic to B.

(b) A dense embedding of posets is not required to be injective. However, it is easily seen that the Boolean completion map is injective when $\mathbb{P}$ is a separative partial order, which happens quite often (but not always, iterated forcing being a counterexample). The converse does not hold.

(c) The Boolean completion of the product poset $\mathbb{P} \times \mathbb{Q}$ is *not* $RO(\mathbb{P}) \times RO(\mathbb{Q})$, but (the completion of) the tensor product $RO(\mathbb{P}) \otimes RO(\mathbb{Q})$; an instructive example is $RO(\mathbb{R}) \otimes RO(\mathbb{R}) \simeq RO(\mathbb{R}^2)$.

Now we show that the Boolean completion is unique. For later use we prove a slightly more general result.

Lemma 6.5. (i) *Suppose $i : \mathbb{P} \to \mathbb{Q}$ is a complete embedding between posets, and $\iota : \mathbb{P} \to B^+$, $\eta : \mathbb{Q} \to C^+$ are dense embeddings into complete Boolean algebras. Then there exists a unique complete embedding $f : B \to C$ such that $f \circ \iota = \eta \circ i$.*

(ii) *Suppose $i : \mathbb{P} \to \mathbb{Q}$ is a dense embedding, and $\iota : \mathbb{P} \to B^+$ and $\eta : \mathbb{P} \to C^+$ are their Boolean completions. Then $B \simeq C$. Taking $i : \mathbb{P} \to \mathbb{P}$ to be identity, this shows the Boolean completion is unique.*

Proof. (i) Define $f(b) = \bigvee\{\eta(i(p)) : \iota(p) \le b\}$. It can be checked that $\iota(p_1) \le \iota(p_2)$ iff $p_2{\downarrow}$ is dense below p_1, in which case $i(p_2){\downarrow}$ is dense below $i(p_1)$ and thus $\eta(i(p_1)) \le \eta(i(p_2))$. Thus $f(\iota(p_0)) = \bigvee\{\eta(i(p)) : \iota(p) \le \iota(p_0)\} = \eta(i(p_0))$.

Clearly f is order-preserving; it also preserves incompatibility since if $b_1 \perp b_2$, $\iota(p_1) \le b_1$ and $\iota(p_2) \le b_2$ then $p_1 \perp p_2$. To show that it is a complete embedding, it suffices to show that if $A \subseteq B^+$ is a maximal antichain then $f(A) \subseteq C^+$ is maximal. It is not hard to see that $\iota^{-1}(A) \subseteq \mathbb{P}$ is predense; let $A' \subseteq \iota^{-1}(A){\downarrow}$ be a maximal antichain. By assumption $i(A') \subseteq \mathbb{Q}$ is a maximal antichain, and the same is true of $\eta(i(A')) \subseteq C^+$; on the other hand $\eta(i(A')) = f(\iota(A'))$ is below $f(A)$, and thus $f(A)$ is also maximal.

A complete Boolean algebra embedding is determined by its values on a dense set, hence the uniqueness.

(ii) By (i) there is a complete embedding $f : B \to C$ with dense image, which must be an isomorphism. □

6.2 Poset forcing

Define the hierarchy $V^{\mathbb{P}}$ inductively by:

- $V^{\mathbb{P}}_0 = \varnothing$,
- $V^{\mathbb{P}}_{\alpha+1} = \mathcal{P}(V^{\mathbb{P}}_\alpha \times \mathbb{P})$ is the set of relations on $V^{\mathbb{P}}_\alpha \times \mathbb{P}$,
- $V^{\mathbb{P}}_\alpha = \bigcup_{\beta<\alpha} V^{\mathbb{P}}_\beta$ if α is a limit ordinal,
- $V^{\mathbb{P}} = \bigcup_{\alpha\in\mathrm{Ord}} V^{\mathbb{P}}_\alpha$.

There is a small discrepancy since a Boolean algebra B is in particular a poset, in which case the above definition doesn't coincide with the previous definition of V^B. However, although we will sometimes consider the case $\mathbb{P} = B^+$, we will never consider $\mathbb{P} = B$, so V^B always denotes the good old Boolean-valued model. Technically V^B is different from V^{B^+} in that the former consists of "hereditary functions" and the latter of "hereditary relations", although they are essentially the same, as we will see. We use Greek letters σ, π, σ etc. to denote elements in $V^{\mathbb{P}}$.

We need to translate between $\mathbb{P}$-names and B-names. For the forward direction, let $\iota : \mathbb{P} \to B$ be the Boolean completion. We define a map $\iota : V^{\mathbb{P}} \to V^B$, $\sigma \mapsto \bar{\sigma}$ (using the same letter as the Boolean completion for convenience) recursively by:

$$\mathrm{dom}(\bar{\sigma}) = \{\bar{\tau} : \tau \in \mathrm{dom}(\sigma)\} \quad \text{and} \quad \bar{\sigma}(x) = \bigvee\{\iota(q) : \exists(\tau, q) \in \sigma\ \bar{\tau} = x\}$$

We can think of this as two steps: first replace each $\tau \in \mathrm{dom}(\sigma)$ by $\bar{\tau}$, and then for each x in the new domain take the supremum on the second coordinate. The *forcing relation* is defined by:

$$p \Vdash \varphi(\sigma_1, \ldots, \sigma_n) \text{ iff } \iota(p) \leq [\![\varphi(\bar{\sigma}_1, \ldots, \bar{\sigma}_n)]\!].$$

We wrtie $p \Vdash_{\mathbb{P}} \varphi$ when there is need to emphasize the dependence on poset. $\Vdash \varphi$ is the same as $1_{\mathbb{P}} \Vdash \varphi$.

In texts such as Kunen [12] that use poset instead of Boolean algebra, the following is not a theorem but rather the recursive definition of forcing relation.

Theorem 6.6. (i) *$p \Vdash \pi \in \sigma$ iff $\{q \in \mathbb{P} : \exists(\tau, r) \in \sigma\ q \leq r \wedge q \Vdash \pi = \tau\}$ is dense below p;*

(ii) *$p \Vdash \pi = \sigma$ iff $\forall \tau \in \mathrm{dom}(\pi) \cup \mathrm{dom}(\sigma) \forall q \leq p\ q \Vdash \tau \in \pi \leftrightarrow q \Vdash \tau \in \sigma$;*

(iii) *$p \Vdash \varphi \wedge \psi$ iff $p \Vdash \varphi \wedge p \Vdash \psi$;*

(iv) *$p \Vdash \neg\varphi$ iff $\forall q \leq p\ q \nVdash \varphi$;*

(v) *$p \Vdash \forall x \varphi(x)$ iff $\forall \sigma \in V^{\mathbb{P}}\ p \Vdash \varphi(\sigma)$.*

Proof. We prove by induction on complexity of formula and rank of name; for justification of this induction see Remark 3.4. As usual, atomic formulas are the heart of the proof. We say that $X \subseteq \mathbb{P}$ is *dense below* p if $\forall q \leq p \exists r \in X (r \leq q)$; it is not necessary that $X \subseteq p\downarrow$.

(i)

$$\begin{aligned}
p \Vdash \pi \in \sigma &\text{ iff } \iota(p) \leq [\![\bar{\pi} \in \bar{\sigma}]\!] \\
&\text{ iff } \iota(p) \leq \bigvee_{x \in \mathrm{dom}(\bar{\sigma})} [\bar{\sigma}(x) \wedge [\![\bar{\pi} = x]\!]] \\
&\text{ iff } \iota(p) \leq \bigvee_{x \in \mathrm{dom}(\bar{\sigma})} \bigvee_{\substack{(\tau,q) \in \sigma \\ \bar{\tau} = x}} \iota(q) \wedge [\![\bar{\pi} = x]\!] \\
&\text{ iff } \iota(p) \leq \bigvee_{(\tau,q) \in \sigma} \iota(q) \wedge [\![\bar{\pi} = \bar{\tau}]\!] \\
&\text{ iff } \{b \in B^+ : \exists(\tau, q) \in \sigma\ b \leq \iota(q) \wedge b \leq [\![\bar{\pi} = \bar{\tau}]\!]\} \text{ is dense below } \iota(p) \\
&\text{ iff } \{r \in \mathbb{P} : \exists(\tau, q) \in \sigma\ \iota(r) \leq \iota(q) \wedge \iota(r) \leq [\![\bar{\pi} = \bar{\tau}]\!]\} \text{ is dense below } p \\
&\text{ iff } \{r \in \mathbb{P} : \exists(\tau, q) \in \sigma\ r \leq q \wedge \iota(r) \leq [\![\bar{\pi} = \bar{\tau}]\!]\} \text{ is dense below } p \\
&\text{ iff } \{r \in \mathbb{P} : \exists(\tau, q) \in \sigma\ r \leq q \wedge r \Vdash \pi = \tau\} \text{ is dense below } p
\end{aligned}$$

The fourth-to-last equivalence is because in a complete Boolean algebra, $a \leq \bigvee_i b_i$ iff for all nonzero $a' \leq a$, there exists some nonzero $a'' \leq a$ and some i such that $a'' \leq b_i$.

The third-to-last equivalence is because the set in the previous line is open; since ι is a dense embedding, if D is dense open below $\iota(p)$ then the preimage $i^{-1}(D)$ is dense below p.

The second-to-last equivalence is because $\iota(r) \leq \iota(q)$ implies $r \not\perp q$.

(ii) Recall that for $a, b \in B$, $a \Leftrightarrow b$ means the element $(a \Rightarrow b) \wedge (b \Rightarrow a)$.

$$\begin{aligned}
p \Vdash \pi = \sigma &\text{ iff } \iota(p) \leq [\![\bar{\pi} = \bar{\sigma}]\!] \\
&\text{ iff } \iota(p) \leq \bigwedge_{x \in \mathrm{dom}(\bar{\pi}) \cup \mathrm{dom}(\bar{\sigma})} [[\![x \in \bar{\pi}]\!] \Leftrightarrow [\![x \in \bar{\sigma}]\!]] \\
&\text{ iff } \forall x \in \mathrm{dom}(\bar{\pi}) \cup \mathrm{dom}(\bar{\sigma}),\ \iota(p) \leq [[\![x \in \bar{\pi}]\!] \Leftrightarrow [\![x \in \bar{\sigma}]\!]]
\end{aligned}$$

$$\begin{aligned}&\text{iff } \forall\tau \in \mathrm{dom}(\pi)\cup\mathrm{dom}(\sigma),\ \iota(p) \leq [[\![\bar\tau \in \bar\pi]\!] \Leftrightarrow [\![\bar\tau \in \bar\sigma]\!]]\\ &\text{iff } \forall\tau \in \mathrm{dom}(\pi)\cup\mathrm{dom}(\sigma)\forall q \leq p\ \iota(q) \leq [\![\bar\tau \in \bar\pi]\!] \leftrightarrow \iota(q) \leq [\![\bar\tau \in \bar\sigma]\!]\\ &\text{iff } \forall\tau \in \mathrm{dom}(\pi)\cup\mathrm{dom}(\sigma)\forall q \leq p\ q \Vdash \tau\in\pi \leftrightarrow q \Vdash \tau \in \sigma\end{aligned}$$

The second-to-last equivalence is because $a \leq b \Leftrightarrow c$ iff $\forall a' \leq a(a' \leq b \leftrightarrow a' \leq c)$ iff $\{a' : a' \leq b \leftrightarrow a' \leq c\}$ is dense below a.

(iii) is clear.

(iv) Note that $p \Vdash \neg\varphi$ iff $\iota(p) \perp [\![\varphi]\!]$. If $\iota(p) \not\perp [\![\varphi]\!]$, since ι is a dense embedding, there exists p' such that $\iota(p') \leq \iota(p)$ and $\iota(p') \leq [\![\varphi]\!]$. In particular $\iota(p') \not\perp \iota(p)$, so $p' \not\perp p$ and there exists p'' such that $p'' \leq p$ and $p'' \leq p'$; the latter implies $\iota(p'') \leq [\![\varphi]\!]$, namely $p'' \Vdash \varphi$.

We frequently use this in the following form: $p \not\Vdash \varphi$ iff $\exists q \leq p\ q \Vdash \neg\varphi$.

(v) The quantifier case follows from the fact that the map $\sigma \mapsto \bar\sigma$ is "essentially surjective", namely for any $u \in V^B$ there is $\sigma \in V^{\mathbb{P}}$ such that $[\![u = \bar\sigma]\!] = 1$. Consider the subclass $V^B_+ \subseteq V^B$ of names that "hereditarily don't take the value 0 "; in other words we define a hierarchy $(V^B_+)_\alpha$ in the same way as V^B_α except that $(V^B_+)_{\alpha+1}$ consists of partial functions from $(V^B_+)_\alpha$ to B^+. Clearly for every $u \in V^B$ there exists $u' \in V^B_+$ s.t. $[\![u = u']\!] = 1$. We claim that for any $u \in V^B_+$, there exists $u^\circ \in V^{\mathbb{P}}$ such that $\bar u^\circ = u$. Just inductively let $u^\circ = \{(x^\circ, p) : x \in \mathrm{dom}(u) \wedge \iota(p) \leq u(x)\}$ and check that it works. □

$u \mapsto u^\circ$ is the other direction of name translation. It follows immediately from definition that $p \Vdash \varphi(u^\circ)$ iff $\iota(p) \leq [\![\varphi(u)]\!]$.

We can now translate many results about Boolean-valued model to posets. They can be proven either directly or indirectly using the Boolean counterparts and the name translation maps. We start with the bounded quantification lemma, which will be frequently used in passing.

Lemma 6.7 (Bounded Quantification). *If $\sigma \in V^{\mathbb{P}}$ and $\varphi(x)$ is a formula with free variable x, possibly with other parameters, then:*

$$\begin{aligned}p \Vdash \forall x \in \sigma\ \varphi(x) &\Leftrightarrow \forall(\sigma, s) \in \sigma \forall r \leq p[r \leq s \to r \Vdash \varphi(\sigma)]\\ &\Leftrightarrow \forall \sigma \in dom(\sigma)\forall r \leq p[r \Vdash \sigma \in \sigma \to r \Vdash \varphi(\sigma)]\end{aligned}$$

$$\begin{aligned}p \Vdash \exists x \in \sigma\ \varphi(x) &\Leftrightarrow \{r : \exists(\sigma, s) \in \sigma[r \leq s \wedge r \Vdash \varphi(\sigma)]\} \textit{ is dense below } p\\ &\Leftrightarrow \{r : \exists \sigma \in dom(\sigma)[r \Vdash \sigma \in \sigma \wedge r \Vdash \varphi(\sigma)]\} \textit{ is dense below } p\end{aligned}$$

In particular, when $\sigma = \check y$ for some y we have:

$$\begin{aligned}&p \Vdash \forall x \in \check y\ \varphi(x) \Leftrightarrow \forall x \in y[p \Vdash \varphi(\check x)]\\ &p \Vdash \exists x \in \check y\ \varphi(x) \Leftrightarrow \{r : \exists x \in y[r \Vdash \varphi(\check x)]\} \textit{ is dense below } p\end{aligned}$$

Lemma 6.8 (Maximal principle). *If $p \Vdash \exists x\varphi(x)$, then there exists $\sigma \in V^{\mathbb{P}}$ such that $p \Vdash \varphi(\sigma)$.*

Proof. $\iota(p) \leq [\![\exists x\varphi(x)]\!] = [\![\varphi(u)]\!]$ for some $u \in V^B$, and thus for some $u \in V^B_+$; then $p \Vdash \varphi(u^\circ)$. □

Results about canonical names and cores can also be generalized to arbitrary posets.

Definition 6.9. Inductively define $cV^{\mathbb{P}} = \bigcup_{\alpha\in\mathrm{Ord}} cV^{\mathbb{P}}_\alpha$, the collection of *canonical* names, as follows. Let $cV^{\mathbb{P}}_0 = \varnothing$, at limit stage take union, and at successor stage let $cV^{\mathbb{P}}_{\alpha+1}$ be the union of $cV^{\mathbb{P}}_\alpha$ together with all $\sigma \subseteq cV^{\mathbb{P}}_\alpha \times \mathbb{P}$ such that:

- $1_{\mathbb{P}} \not\Vdash \tau = \sigma$ for any $\tau \in cV^{\mathbb{P}}_\alpha$;
- for any $\tau \in cV^{\mathbb{P}}_\alpha$ and $p \in \mathbb{P}$, $(\tau, p) \in \sigma$ iff $p \Vdash \tau \in \sigma$.

Lemma 6.10. (i) *$\sigma \mapsto \bar{\sigma}$ and $u \mapsto u^\circ$ are bijections between $cV^{\mathbb{P}}$ and cV^B.*

(ii) *For every $\sigma \in V^{\mathbb{P}}$ there is a unique $\tau \in cV^{\mathbb{P}}$ such that $1_{\mathbb{P}} \Vdash \sigma = \tau$.*

Proof. (i) Recall that $\bar{u}^\circ = u$ for any $u \in V^B_+$; clearly $cV^B \subseteq V^B_+$, and this gives one direction. Next we show $\bar{\sigma}^\circ = \sigma$ by induction on the rank of σ.

$$\begin{aligned}
\bar{\sigma}^\circ &= \{(x^\circ, p) : x \in \mathrm{dom}(\bar{\sigma}),\ \iota(p) \le \bar{\sigma}(x)\} \\
&= \{(\bar{\tau}^\circ, p) : \tau \in \mathrm{dom}(\sigma),\ \iota(p) \le \textstyle\bigvee\{\iota(q) : \exists(\pi, q) \in \sigma,\ \bar{\pi} = \bar{\tau}\}\} \\
&= \{(\tau, p) : \tau \in \mathrm{dom}(\sigma),\ \iota(p) \le \textstyle\bigvee\{\iota(q) : \exists(\pi, q) \in \sigma,\ \bar{\pi} = \bar{\tau}\}\} \\
&= \{(\tau, p) : \tau \in \mathrm{dom}(\sigma),\ \iota(p) \le \textstyle\bigvee\{\iota(q) : (\tau, q) \in \sigma\}\} \\
&= \{(\tau, p) : \tau \in \mathrm{dom}(\sigma),\ p \Vdash \tau \in \sigma\} \\
&= \sigma
\end{aligned}$$

The third and fourth equalities follow from induction hypothesis. To see the fifth equality, we claim that if $\iota(p) \le \bigvee\{\iota(q) : (\tau, q) \in \sigma\}$ then $p \Vdash \tau \in \sigma$. Otherwise there exists $p' \le p$ such that $p' \Vdash \tau \notin \sigma$, but $\iota(p')$ must be compatible with one of those $\iota(q)$, so p' is compatible with some q for which $(\tau, q) \in \sigma$, leading to a contradiction.

(ii) Let $u \in cV^B$ be the unique name such that $[\![\bar{\sigma} = u]\!] = 1$, so that u° is as desired. To show uniqueness, if $\tau, \pi \in cV^{\mathbb{P}}$, $1_{\mathbb{P}} \Vdash \sigma = \tau$ and $1_{\mathbb{P}} \Vdash \sigma = \pi$ then $1_{\mathbb{P}} \Vdash \tau = \pi$, so $[\![\bar{\tau} = \bar{\pi}]\!] = 1$, which means $\bar{\tau} = \bar{\pi}$ and finally $\tau = \pi$. □

Lemma 6.11. (i) *If $\sigma, \tau \in cV^{\mathbb{P}}$ and $1_{\mathbb{P}} \Vdash \tau \in \sigma$ then $\tau \in \mathrm{dom}(\sigma)$. Thus for any $\sigma \in V^{\mathbb{P}}$, $\mathrm{Core}(\sigma) := \{\tau \in cV^{\mathbb{P}} : 1_{\mathbb{P}} \Vdash \tau \in \sigma\}$ is a set.*

(ii) *If $1_{\mathbb{P}} \Vdash \sigma \neq \varnothing$, then for any $\pi \in V^{\mathbb{P}}$, there exists $\tau \in \mathrm{Core}(\sigma)$ such that $q \Vdash \pi \in \sigma$ iff $q \Vdash \pi = \tau$ for any q.*

Lemma 6.12 (Existential completeness)**.** *If $1_{\mathbb{P}} \Vdash \exists x\varphi(x)$, and $p \Vdash \varphi(\sigma)$, then there exists π s.t. $1_{\mathbb{P}} \Vdash \varphi(\pi)$ and $p \Vdash \sigma = \pi$.*

6.3 Translation to ctm

Just like we can do consistency proof with V^B without the need of a ctm, we can also do consistency proof with $V^{\mathbb{P}}$. Forget about all the Boolean algebra stuff. Imagine that we start from scratch and take Theorem 6.6 as the definition of the forcing relation $\Vdash$, so we recursively define what it means for p to force $\varphi(\sigma_1, \dots, \sigma_n)$, and prove theorems analogous to Theorem 3.5 and Theorem 3.9. Then we show that $1_{\mathbb{P}} \Vdash \varphi$ for any ZFC axiom φ, and that choosing an appropriate $\mathbb{P}$

we have $1_{\mathbb{P}} \Vdash \neg\mathsf{CH}$. This is probably the least popular way of doing forcing, since it lacks both the elegance of Boolean-valued model and the comfort of working with actual transitive models.

Our forcing relation is actually what texts like Kunen [12] call $\Vdash^*$. They reserve $\Vdash$ for the following:

$$p \Vdash \varphi(\sigma_1, \ldots, \sigma_n) \text{ iff for any generic } G \text{ that contains } p, \text{ we have } M[G] \models \varphi((\sigma_1)_G, \ldots, (\sigma_n)).$$

Of course this depends on a fixed ctm M; since this definition quantifies over all generic filters, a priori it is far from being definable in M. Kunen then considers an auxiliary relation $\Vdash^*$ (which is our $\Vdash$), relativize it to M and show that $\Vdash$ and $(\Vdash^*)^M$ are equivalent, so for any particular φ, the relation $p \Vdash \varphi(\sigma_1, \ldots, \sigma_n)$ is a definable class in M. This is the Definability Lemma, which is of course proven by induction on formulas, and to facilitate the induction one proves at the same time the Truth Lemma, which says if G is generic and $M[G] \models \varphi((\sigma_1)_G, \ldots, (\sigma_n))$ then G must contain some p such that $p \Vdash \varphi(\sigma_1, \ldots, \sigma_n)$; in other words, anything true in $M[G]$ is forced by some condition.

Since we already did the hard work in the Boolean-valued model setting, the Truth and Definability Lemmas come basically for free. We just need to show that we get the same generic extension $M[G]$ whether we use names in $M^{\mathbb{P}}$ or M^B. Let M be a ctm, $\mathbb{P}$ be a poset in M, and $M^{\mathbb{P}} = (V^{\mathbb{P}})^M = M \cap V^{\mathbb{P}}$. An $(M, \mathbb{P})$-generic filter is a set $G \subseteq \mathbb{P}$ that is upward closed, downward directed and meets all the dense subsets of $\mathbb{P}$ that are in M. Note that being a poset and being a dense set are both Δ_1 hence absolute. It is not hard to check that "dense" can equivalently be replaced by dense open, predense, or maximal antichain.

Let $\iota : \mathbb{P} \to B$ be the Boolean completion of $\mathbb{P}$ *in* M, namely $B = (RO(\mathbb{P}))^M$. We first show that $(M, \mathbb{P})$-generic filters and (M, B)-generic filters are in bijection with each other. For later use we prove something slightly more general.

Lemma 6.13. (i) *Suppose $M \models i : \mathbb{P} \to \mathbb{Q}$ is a complete embedding. If H is $(M, \mathbb{Q})$-generic, then $G := i^{-1}(H)$ is $(M, \mathbb{P})$-generic.*

(ii) *In case i is a dense embedding, then for any G that is $(M, \mathbb{P})$-generic, the filter on $\mathbb{Q}$ generated by $i(G)$ (namely the upward closure of $i(G)$) is $(M, \mathbb{Q})$-generic. Denote this filter by $i_*(G)$. The operations $G \mapsto i_*(G)$ and $H \mapsto i^{-1}(H)$ are inverse to each other.*

Proof. (i) G is downward directed because for any $p_1, p_2 \in G$, $\{p : (p \le p_1 \wedge p \le p_2) \vee p \perp p_1 \vee p \perp p_2\}$ is a dense set in M. G is generic because for any maximal antichain $A \subseteq \mathbb{P}$ such that $A \in M$, $i(A) \subseteq \mathbb{Q}$ is also maximal and in M.

(ii) $i_*(G)$ is generic because preimage of dense open set is dense.

It is clear that $i^{-1}(i_*(G)) \supseteq G$ and $i_*(i^{-1}(H)) \subseteq H$. To show equality, note that there cannot be two generic filters G_1 and G_2 such that $G_1 \subsetneq G_2$. □

For $\sigma \in M^{\mathbb{P}}$, define σ_G recursively by $\sigma_G = \{\tau_G : \exists(\tau, p) \in \sigma,\ p \in G\}$. Define the generic extension by $M[G] = \{\sigma_G : \sigma \in M^{\mathbb{P}}\}$.

Theorem 6.14. *Let M be a ctm, $\mathbb{P} \in M$ be a poset and $\iota : \mathbb{P} \to B$ be its Boolean completion in M.*

(i) *If G is $(M, \mathbb{P})$-generic and H is $\iota_*(G)$, then for any $\sigma \in M^{\mathbb{P}}$, $\sigma_G = \bar{\sigma}_H$.*

(ii) *(Definability Lemma)* $p \Vdash \varphi(\sigma_1, \dots, \sigma_n)$ *iff* $M[G] \models \varphi((\sigma_1)_G, \dots, (\sigma_n)_G)$ *for all* $(M, \mathbb{P})$*-generic* G *that contains* p.

(iii) *(Truth Lemma) If* $M[G] \models \varphi((\sigma_1)_G, \dots, (\sigma_n)_G)$ *then there exists* $p \in G$ *such that* $p \Vdash \varphi(\sigma_1, \dots, \sigma_n)$.

Proof. (i) Note that $\bar{\sigma}_H$ refers to the interpretation of the B-name $\bar{\sigma}$ as defined in Subsection 5.1. We prove this by induction on rank of names. Trivially we have $\sigma_G \subseteq \{\tau_G : \tau \in \mathrm{dom}(\sigma)\}$ and $\bar{\sigma}_H \subseteq \{\bar{\tau}_H : \tau \in \mathrm{dom}(\sigma)\}$, and by induction $\{\tau_G : \tau \in \mathrm{dom}(\sigma)\} = \{\bar{\tau}_H : \tau \in \mathrm{dom}(\sigma)\}$. Now for any $\pi \in \mathrm{dom}(\sigma)$,

$$\begin{aligned}
&\bar{\pi}_H \in \bar{\sigma}_H \\
\Leftrightarrow &\exists \eta \in \mathrm{dom}(\sigma)\ \bar{\eta}_H = \bar{\pi}_H \text{ and } \bar{\sigma}(\bar{\eta}) \in H \\
\Leftrightarrow &\exists \eta \in \mathrm{dom}(\sigma)\ \bar{\eta}_H = \bar{\pi}_H \text{ and } \bigvee \{\iota(q) : \exists (\tau, q) \in \sigma\ \bar{\tau} = \bar{\eta}\} \in H \\
\Leftrightarrow &\exists \eta \in \mathrm{dom}(\sigma) \exists (\tau, q) \in \sigma\ \bar{\eta}_H = \bar{\pi}_H \text{ and } \bar{\tau} = \bar{\eta} \text{ and } \iota(q) \in H \\
\Leftrightarrow &\exists \eta \in \mathrm{dom}(\sigma) \exists (\tau, q) \in \sigma\ \bar{\tau} = \bar{\eta} \text{ and } \iota(q) \wedge [\![\bar{\eta} = \bar{\pi}]\!] \in H \\
\Leftrightarrow &\exists \eta \in \mathrm{dom}(\sigma) \exists (\tau, q) \in \sigma\ \bar{\tau} = \bar{\eta} \text{ and } \iota(q) \wedge [\![\bar{\tau} = \bar{\pi}]\!] \in H \\
\Leftrightarrow &\exists (\tau, q) \in \sigma\ \iota(q) \wedge [\![\bar{\tau} = \bar{\pi}]\!] \in H \\
\Leftrightarrow &\exists (\tau, q) \in \sigma\ \iota(q) \in H \text{ and } \bar{\tau}_H = \bar{\pi}_H \\
\Leftrightarrow &\exists (\tau, q) \in \sigma\ q \in G \text{ and } \tau_G = \pi_G \\
\Leftrightarrow &\pi_G \in \sigma_G
\end{aligned}$$

where we used Theorem 5.6, the Boolean algebra version of this theorem, for the fourth equivalence and the third-to-last equivalence; the second-to-last equivalence uses induction hypothesis.

(ii) It follows from (i) and the essential surjectivity of the map $\sigma \mapsto \bar{\sigma}$ that $M[G] = M[H]$. Using everything we have proved,

$$M[G] \models \varphi(\sigma_G) \Leftrightarrow M[H] \models \varphi(\bar{\sigma}_H) \Leftrightarrow [\![\varphi(\bar{\sigma})]\!] \in H \Leftrightarrow \exists p \in G\ \iota(p) \leq [\![\varphi(\bar{\sigma})]\!] \Leftrightarrow \exists p \in G\ p \Vdash \varphi(\sigma)$$

Therefore, if $p \Vdash \varphi(\sigma)$ then for any generic $G \ni p$ we have $M[G] \models \varphi(\sigma_G)$. Conversely, if $p \nVdash \varphi(\sigma)$, then $\iota(p) \not\leq [\![\neg\varphi(\sigma)]\!]$, so there exists $q \leq p$ s.t. $q \Vdash \neg\varphi(\sigma)$. Consider any generic $G \ni q$.

(iii) The set of p such that either $p \Vdash \varphi(\sigma_1, \dots, \sigma_n)$ or $p \Vdash \neg\varphi(\sigma_1, \dots, \sigma_n)$ is dense. □

Lemma 6.15 (Rasiowa–Sikorski)**.** *If M is a countable transitive model of* ZFC *and* $(\mathbb{P}, \leq) \in M$ *is a poset, then for any* $p \in \mathbb{P}$*, there exists an* $(M, \mathbb{P})$*-generic filter* G *containing* p.

Proof. By the same diagonal argument as in Lemma 5.4. □

It is not necessary that M be countable; all we need is that M contains only countably many dense sets of $\mathbb{P}$. For example, if $0^\sharp$ exists, then any set definable in L without parameter is countable, and if a poset is definable so is the collection of its dense subsets, so we can literally force over L using any definable poset.

The above lemma can be generalized as follows. Say $\mathbb{P}$ is countably closed (any countable decreasing sequence has a lower bound), then we can produce a G that meets any $\aleph_1$ many dense sets. Similar arguments are sometimes used to lift elementary embedding in the study of large cardinals and forcing.

Below is the ctm version of the Existential Completeness Lemma 6.12, and is the one that is often used in practice.

Lemma 6.16 (Existential completeness)**.** *If* $1_{\mathbb{P}} \Vdash \exists x\varphi(x)$ *and* $M[G] \models \varphi(\sigma_G)$*, then there exists* π *s.t.* $1_{\mathbb{P}} \Vdash \varphi(\pi)$ *and* $\sigma_G = \pi_G$*.*

Proof. By the Truth Lemma there exists $p \in G$ such that $p \Vdash \varphi(\sigma)$. By Lemma 6.12 there exists π such that $1_{\mathbb{P}} \Vdash \varphi(\pi)$ and $p \Vdash \sigma = \pi$, which implies $\sigma_G = \pi_G$ by Definability Lemma. □

A useful consequence is that if $1_{\mathbb{P}} \Vdash \exists x\varphi(x)$ and we want to show that $M[G] \models \forall x(\varphi(x) \to \psi(x))$, it suffices to show that if $1_{\mathbb{P}} \Vdash \varphi(\sigma)$ then $1_{\mathbb{P}} \Vdash \psi(\sigma)$.

6.4 How people actually write forcing

There are at least $2 \times 2 = 4$ ways to present forcing: we can either use Boolean algebra or use poset; we can either stick with V^B or $V^{\mathbb{P}}$, or relativize them a ctm M and form the generic extension $M[G]$. We also mentioned earlier that nowadays nobody gives a [vulgarity] about these logical details. Most people just use the notation $V[G]$ freely, as if we are literally expanding V to a larger transitive class model $V[G]$. Whether to interpret the reasoning about $V[G]$ using the Boolean-valued model V^B or the ctm $M[G]$ is up to the reader, which can make it intimidating for beginner. I hope these notes are an exception; if you think otherwise, please let me know how they can be improved!

As an example, Theorem 4.7 would be written and proven in the following style:

Theorem. *If* $\mathbb{P}$ *is ccc and* G *is* $(V, \mathbb{P})$*-generic, then every cardinal* κ *in* V *remains a cardinal in* $V[G]$*.*

Proof. This is immediate if $\kappa = \omega$, so assume κ is uncountable and $\lambda < \kappa$. Suppose $f \in V[G]$ is a function from λ to κ; we want to show that it is not surjective. Let $\dot{f} \in V^{\mathbb{P}}$ be a name such that $\dot{f}_G = f$. By the Truth Lemma, there exists $p \in G$ such that:

$$p \Vdash \text{“}\dot{f} \text{ is a function from } \check{\lambda} \text{ to } \check{\kappa}\text{”},$$

or if we unravel the definition of a function,

$$p \Vdash \forall y \in \check{\lambda} \exists! x \in \check{\kappa}\ \dot{f}(y) = x.$$

Let us reason in V. Let $A_\beta = \{\alpha < \kappa : \exists r \leq p[r \Vdash \dot{f}(\check{\beta}) = \check{\alpha}]\}$. We claim that $|A_\beta| \leq \omega$. Indeed, for each $\alpha \in A_\beta$ let r_α be such that $r_\alpha \Vdash \dot{f}(\check{\beta}) = \check{\alpha}$. Then $\{r_\alpha : \alpha \in A_\beta\}$ is an antichain, since if had $s \leq r_\alpha$, $s \leq r_{\alpha'}$ for distinct α, α' then $s \Vdash \dot{f}(\check{\beta}) = \check{\alpha}$ and $s \Vdash \dot{f}(\check{\beta}) = \check{\alpha}'$, contradicting the definition of a function. Thus $|A_\beta| \leq \omega$ by the assumption that $\mathbb{P}$ is ccc.

Now take any $\alpha \in \kappa \setminus \bigcup_{\beta<\lambda} A_\beta$. We claim that $p \Vdash \forall y \in \lambda\ \dot{f}(y) \neq \check{\alpha}$, which would imply α is not in the range of f by the Definability Lemma. By the Bounded Quantification Lemma 6.7, it suffices to show $p \Vdash \dot{f}(\check{\beta}) \neq \check{\alpha}$ for any $\beta < \lambda$. If not, then by Theorem 6.6 there exists $r \leq p$ such that $r \Vdash \dot{f}(\check{\beta}) = \check{\alpha}$, but this contradicts out exact assumption. □

One way to make the "generic extension" $V[G]$ less weird is to work in a formal theory with a unary predicate $\check{V}$ in addition to the membership symbol $\in$. The axioms are the usual ZFC axioms, except that in separation and replacement schema the formula φ is allowed to contain the symbol $\check{V}$. We also add an axiom that says, e.g., "the universe is a generic extension of the class model $\check{V}$ by the Cohen forcing".

This theory can be shown to be consistent. Indeed it is true in V^B for $B = RO({}^{\omega}2)$ if we define the truth value of the predicate $\check{V}$ by

$$[\![\check{V}(u)]\!] = \bigvee_{x \in V} [\![u = x]\!].$$

Then it can be checked that, in addition to the equlity axioms about membership, we also have:

$$[\![u = v]\!] \wedge [\![\check{V}(u)]\!] \leq \check{V}(v),$$

and that in the proof of Theorem 3.8, that V^B satisfies ZFC, we may allow formulas in the expanded language in separation and replacement schema. This is more or less the "naturalist account of forcing" in [7].

Having a predicate for the ground model V allows us to reason about the relation between V and $V[G]$. Incidentally, it turns out the ground model is always a class that is definable (with parameters) in the generic extension, so it is automatic that we can reason about V in $V[G]$, but this is a quite non-trivial result due independently to Laver [13] and Woodin [17], fourty years after the discovery of forcing.

Alternatively, one can work exclusively with ctm if they prefer. Of course, existence of ctm is not provable in ZFC, but it follows from, e.g., the existence of an inaccessible cardinal (by downward Löwenheim–Skolem), which is a quite moderate large cardinal assumption compared to all the other things set theorists today take for granted, like Woodin cardinals or supercompact cardinals. Or one can use ctm of finite fragment of ZFC and carefully keep track of that throughout all arguments. The ctm counterpart of the ground model predicate would be the statement that $x \cap M \in M[G]$ for every $x \in M[G]$, which can be proven by cooking up a concrete name for $x \cap M$.

We make some remarks on notations. The above proof that ccc forcing preserves cardinals is not exactly what you would see in papers—the actual proofs are much more concise, without mentioning the Bounded Quantification or Definability Lemmas, and people tend to drop the checks on $\check{x}$ for $x \in V$, especially when x is an ordinal. For more on this see Kunen [12, IV.6].

From now on, we will adopt the $V[G]$ notation and gradually phase out the check names. We will mostly work with posets, although Boolean algebra may resurface from time to time. As a general rule of thumb, Boolean algebras are useful when we prove theorems *about* forcing, while posets are useful when we prove theorems *using* forcing.

6.5 Induced generic

If $i : \mathbb{P} \to \mathbb{Q}$ is a complete embdding in M and H is $(V, \mathbb{Q})$-generic, then $V[H]$ contains an $(V, \mathbb{P})$-generic filter, namely $i^{-1}(H)$, so forcing with $\mathbb{Q}$ "does more" than with $\mathbb{P}$. Actually it is sufficient that $p{\downarrow}$ completely embeds into $\mathbb{Q}$ for some $p \in \mathbb{P}$, since a $p{\downarrow}$-generic filter easily extends to a $\mathbb{P}$-generic one.

We shall show that the converse is also true, at least in the realm of complete Boolean algebras. The idea is roughly this: suppose B and C are complete Boolean algebras, and $V^C \models$ "there exists a $(V, \check{B})$-generic filter"; by the maximal principle (which still holds with the ground model predicate added), there is a C-name $\dot{G}$ such that $V^C \models$ "$\dot{G}$ is a $(V, \check{B})$-generic filter"; then we shall see that the map $b \mapsto [\![\check{b} \in \dot{G}]\!]^C$ is complete; it's not necessarily an embedding, but the kernel is some principal ideal $b_0\downarrow$, so $b_0^*\downarrow$ completely embeds into C.

$\pi : \mathbb{Q} \to \mathbb{P}$ is called a *projection* if it is order preserving, $\pi(1_\mathbb{Q}) = 1_\mathbb{P}$, and for any q and $p \leq \pi(q)$, there exists $q' \leq q$ such that $\pi(q') \leq p$. The typical example is the projection of $\mathbb{P} \times \mathbb{Q}$ onto one of its coordinate. Later we shall show that complete embedding, projection and two-step iterated forcing are just different aspects of the same thing.

If $i : \mathbb{P} \to \mathbb{Q}$ is any map, $\tilde{q} \in \mathbb{P}$ is called a *reduct* of q if $i(p) \perp q \to p \perp \tilde{q}$. $\pi : \mathbb{Q} \to \mathbb{P}$ is called a *reduction* if $\pi(q)$ is a reduct of q. If $\tilde{q}$ is a reduct of q, then so is any strengthening of $\tilde{q}$, so a reduct is not unique; however, in the case of complete Boolean algebra there is a canonical reduct.

Lemma 6.17. (i) *If $\pi : \mathbb{Q} \to \mathbb{P}$ is a projection, and H is $(V, \mathbb{Q})$-generic, then $\pi_*(H)$ is $(V, \mathbb{P})$-generic.*

(ii) *An order preserving map $i : \mathbb{P} \to \mathbb{Q}$ is a complete embedding iff any q has a reduct; note that under* AC *this is the same as the existence a reduction $\pi : \mathbb{Q} \to \mathbb{P}$. In particular the notion of complete embedding is absolute.*

(iii) *If $\mathbb{P}$ is separative, $i : \mathbb{P} \to \mathbb{Q}$ is order and compatibility preserving, and $\pi : \mathbb{Q} \to \mathbb{P}$ is an order preserving reduction satisfying $\pi(1_\mathbb{Q}) = 1_\mathbb{P}$, then π is a projection.*

(iv) *If $i : B \to C$ is a complete embedding between complete Boolean algebras, then any $c \in C$ has a greatest reduct defined by $\pi(c) = \bigwedge\{b \in B : i(b) \geq c\}$.*

Proof. (i) If π is a projection, then the preimage of a dense open set is dense. Order preservation is needed to show that $\pi(H)$ is directed.

(ii) Suppose $\pi : \mathbb{Q} \to \mathbb{P}$ is a reduction, $A \subseteq \mathbb{P}$ is a maximal antichain and yet $i(A)$ is not maximal in $\mathbb{Q}$. Then there exists $q \in \mathbb{Q}$ such that $i(p) \perp q$, and thus $p \perp \pi(q)$ for every $p \in A$, a contradiction.

Suppose $i : \mathbb{P} \to \mathbb{Q}$ is a complete embedding. Fix $q \in \mathbb{Q}$, consider $E = \{p : i(p) \perp q\}$ and let A be an antichain maximal below E. Then $i(A)$ is an antichain that is not maximal, since $i(A) \cup \{q\}$ is an antichain. Thus A is not maximal, and there exists $\tilde{q}$ s.t. $\tilde{q} \perp r$ for all $r \in A$; it follows that $\tilde{q} \perp p$ for all $p \in E$, since if $s \leq \tilde{q}$ and $s \leq p$ then $A \cup \{s\}$ is an antichain below E.

(iii) Suppose $p \leq \pi(q)$, then in particular $p \not\perp \pi(q)$, so there exists q' witnessing $i(p) \not\perp q$. We claim that $\pi(q') \leq p$. Otherwise, there exists $r \leq \pi(q')$ such that $r \perp p$, and thus $i(r) \perp i(p)$. Since $q' \leq i(p)$ we have $i(r) \perp q'$, and by definition of reduction $r \perp \pi(q')$, contradiction.

(iv) If $\tilde{c}$ is a reduct of c, and $i(b) \geq c$, then $i(b^*) \perp c$, so $b^* \perp \tilde{c}$ and $b \geq \tilde{c}$. It follows that $\tilde{c} \leq \pi(c)$. The proof that $\pi(c)$ is a reduct is similar. □

By the completeness of i we have $i(\pi(c)) = \bigwedge\{i(b) : i(b) \geq c\} \geq c$, so $i(\pi(c)) \geq c$ and $\pi(c)$ is the smallest $b \in B$ such that $i(b) \geq c$. It is also easy to see that $\pi(i(b)) = b$, and that π is order preserving, so by (iii) it is a projection.

An illustrating example is $i : RO(\mathbb{R}) \to RO(\mathbb{R}^2)$, $U \mapsto U \times \mathbb{R}$, for which $\pi(V)$ is literally the projection onto x-axis (followed by regularization).

Theorem 6.18. *For complete Boolean algebras B and C, the following are equivalent:*

(i) $V^C \models$ *there exists a* $(V, \check{B})$*-generic filter.*

(ii) *There exists a complete embedding* $i : b\!\downarrow \to C$ *for some* $b \in B$.

(iii) *There exists a projection* $\pi : C \to b\!\downarrow$ *for some* $b \in B$.

Proof. (iii)⇒(i) and (ii)⇒(i) are essentially already proved, although we stated them in the ctm language, namely $M[H]$ always contains an (M, B)-generic filter.

(ii)⇒(iii): As observed above, the canonical reduction map $\pi : C \to b\!\downarrow$ is a projection.

(i)⇒(ii): By maximal principle there is a C-name $\dot{G}$ s.t. $V^C \models \dot{G}$ is $(V, \check{B})$-generic. We claim that the map $i : B \to C,\ b \mapsto [\![\check{b} \in \dot{G}]\!]^C$ is a Boolean homomorphism and complete, namely it preserves arbitrary join. First we present a somewhat non-rigorous argument. For example, let $a = b_1 \wedge b_2$, and we want to show $[\![\check{b_1} \in \dot{G}]\!]^C \wedge [\![\check{b_2} \in \dot{G}]\!]^C = [\![\check{a} \in \dot{G}]\!]^C$. Note that because a Boolean algebra is separative, two elements are equal iff they belong to exactly the same generic filters. Now for any (V, C)-generic filter H, denoting $G = \dot{G}_H$, we have $[\![\check{b_1} \in \dot{G}]\!]^C \wedge [\![\check{b_2} \in \dot{G}]\!]^C \in H$ iff $b_1 \in G \wedge b_2 \in G$ iff $a \in G$ iff $[\![\check{a} \in \dot{G}]\!]^C \in H$, so i preserves meet. Similarly, if $a = \bigwedge_{i \in I} b_i$, since H and G are V-complete ultrafiters, $\bigwedge_{i \in I} [\![\check{b_i} \in \dot{G}]\!]^C \in H$ iff $\forall i \in I\ [\![\check{b_i} \in \dot{G}]\!]^C \in H$ iff $\forall i \in I\ b_i \in G$ iff $a \in G$ iff $[\![\check{a} \in \dot{G}]\!]^C \in H$.

Now i may not be injective, but if we let $X = \{b \in B : i(b) = 0\}$, then $i(\bigvee X) = \bigvee i(X) = 0$, so the kernel of i is the principal ideal generated by $a := \bigvee X$, and i induces a complete embedding from $B/(a\!\downarrow) \simeq b\!\downarrow$ to C, where $b = a^*$. It cannot be that $a = 1_B$ since $i(1_B) = 1_C$.

Taken literally, the above proof is absurd since the generic filter existence lemma only works for ctm, not the universe V. It can be made fully rigorous in two ways. First, by reflection theorem there exists some V_α that contains all the relevant sets and satisfies as much ZFC as we want. Let $M \prec V_\alpha$ be a countable elementary submodel and $\pi : M \to \overline{M}$ be the transitive collapse. The above argument does show that in $\overline{M}$ there is a complete homomorphism from $\pi(B)$ to $\pi(C)$, so by elementarity there is in V_α (and thus in V) a complete homomorphism from B to C. Alternatively, we can reason in V^C as follows. Suppose $a = b_1 \wedge b_2$, so by absoluteness $V^C \models \check{a} = \check{b_1} \wedge \check{b_2}$; since V^C satisfies "$\dot{G}$ is $(V, \check{B})$-generic", it also satisfies "$\check{a} \in \dot{G}$ iff $\check{b_1} \in \dot{G}$ and $\check{b_2} \in \dot{G}$", and thus $[\![\check{b_1} \in \dot{G}]\!]^C \wedge [\![\check{b_2} \in \dot{G}]\!]^C = [\![\check{a} \in \dot{G}]\!]^C$. □

6.6 Forcing equivalence

When are two complete Boolean algebras B and C "equivalent" for the purpose of forcing? By the above theorem there should exist complete embeddings in both directions. It seems reasonable to add the requirement that they give rise to the same forcing extensions; this implies something stronger than mere complete embeddings. As before, everything can be made fully rigorous by reasoning in the Boolean world, but for simplicity let's pretend we can literally form forcing extensions of V.

Theorem 6.19. *For complete Boolean algebras* B *and* C*, the following are equivalent:*

(i) *For every* (V, C)*-generic filter* H*,* $V[H]$ *contains a* (V, B)*-generic filter* G *such that* $V[G] = V[H]$.

(ii) $\{c \in C : \exists b \in B^+\ c\!\downarrow \simeq b\!\downarrow\}$ *is dense in* C^+.

Proof. The backward direction is clear, so let us assume (i) and prove (ii). For any $c \in C$, consider some (V, C)-generic filter H containing c. By assumption, $V[H]$ contains a (V, B)-generic filter G s.t. $V[G] = V[H]$, or equivalently $H \in V[G]$, so there exists $u \in V^B$ s.t. $u_G = H$. By Truth Lemma there exists a C-name $\dot{G}$, some $u \in V^B$ and some $c_0 \in H$ such that $c_0 \Vdash_C$"$\dot{G}$ is $(V, \check{B})$-generic and $\check{u}_{\dot{G}} = \dot{H}$". We may assume $c_0 \leq c$. It suffices to show that $c_0{\downarrow}$ is isomorphic to some $b{\downarrow}$.

Consider the map $i : B \to c_0{\downarrow},\ b \mapsto c_0 \wedge [\![\check{b} \in \dot{G}]\!]^C$. As before this is a homomorphism between complete Boolean algebras that preserves arbitrary joins, and thus is an embedding when restricted to $b_0{\downarrow}$ for some b_0. We will be done if i is surjective, since then it is an isomorphism when restricted to $b_0{\downarrow}$. Define $j : c_0{\downarrow} \to b_0{\downarrow},\ c \mapsto b_0 \wedge [\![\check{c} \in u]\!]^B$. It is enough to show $i(j(c)) = c$ for any $c \leq c_0$.

Again, this is easiest done by showing $i(j(c)) \in H$ iff $c \in H$ for any (V, C)-generic filter H that contains c_0 (since $i(j(c)) \leq c_0$ and $c \leq c_0$). First notice that $\dot{G}_H = i^{-1}(H)$, because $i(b) \in H \Leftrightarrow c_0 \wedge [\![\check{b} \in \dot{G}]\!]^C \in H \Leftrightarrow b \in \dot{G}_H$. It follows that $b_0 \in \dot{G}_H$, since $i(b_0) = c_0$.

Finally, denote $b = j(c)$, then $i(j(c)) \in H \Leftrightarrow c_0 \wedge [\![\check{b} \in \dot{G}]\!]^C \in H \Leftrightarrow [\![\check{b} \in \dot{G}]\!]^C \in H \Leftrightarrow b \in \dot{G}_H \Leftrightarrow b_0 \wedge [\![\check{c} \in u]\!]^B \in \dot{G}_H \Leftrightarrow [\![\check{c} \in u]\!]^B \in \dot{G}_H \Leftrightarrow c \in u_{\dot{G}_H} \Leftrightarrow c \in H$. The last equivalence is due to the assumption that $c_0 \in H$ and the definition of c_0. □

For the Boolean completion $\iota : \mathbb{P} \to B$, we already showed how to transform $\mathbb{P}$-names to B-names and vice versa. Now we generalize this to a complete embedding $i : \mathbb{P} \to \mathbb{Q}$. Define a map from $V^{\mathbb{P}}$ to $V^{\mathbb{Q}}$, still denoted i, inductively by $i(\sigma) = \{(i(\tau), i(p)) : (\tau, p) \in \sigma\}$. We can make the same definition for $j : B \to C$ a complete embedding of Boolean algebras. It is easily checked that $\sigma_G = i(\sigma)_H$ for any $(V, \mathbb{P})$-generic G and $(V, \mathbb{Q})$-generic H such that $i^{-1}(H) = G$. Moreover, if $\iota : \mathbb{P} \to B$ and $\eta : \mathbb{Q} \to C$ are Boolean completions, then we have seen that there is a unique embedding $j : B \to C$ such that $j \circ \iota = \eta \circ i$. It can be checked that $j \circ \iota = \eta \circ i$ still holds when we view them as name translation maps.

Lemma 6.20. *(i) If $j : B \to C$ is a complete embedding between complete Boolean algebras and $\varphi(x_1, \ldots, x_n)$ is a Δ_1 formula, then $j([\![\varphi(u_1, \ldots, u_n)]\!]^B) = [\![\varphi(j(u_1), \ldots, j(u_n))]\!]^C$ for any $u_1, \ldots, u_n \in V^B$.*

(ii) If $i : \mathbb{P} \to \mathbb{Q}$ is a complete embedding between posets and $\varphi(x_1, \ldots, x_n)$ is a Δ_1 formula, then $p \Vdash \varphi(\sigma_1, \ldots, \sigma_n)$ iff $i(p) \Vdash \varphi(i(\sigma_1), \ldots, i(\sigma_n))$. If "complete" is changed to "dense" then this holds for arbitrary formula.

Proof. (i) For atomic formulas prove by induction on rank. The extension to Δ_1 formula is as usual.

(ii) Let $\iota : \mathbb{P} \to B$ and $\eta : \mathbb{Q} \to C$ be Boolean completions. $p \Vdash \varphi(\sigma)$ iff $\iota(p) \leq [\![\varphi(\iota(\sigma))]\!]$ iff $j(\iota(p)) \leq [\![\varphi(j(\iota(\sigma)))]\!]$ iff $\eta(i(p)) \leq [\![\varphi(\eta(i(\sigma)))]\!]$ iff $i(p) \Vdash \varphi(i(\sigma))$. If i is dense then $j : B \to C$ is an isomorphism, and thus $j : V^B \to V^C$ is an isomorphism of Boolean-valued models. □

Name translations will show up frequently when we discuss iterated forcing.

7 More examples of forcing

7.1 Higher Cohen forcing

We already showed the consistency of $\neg\mathsf{CH}$. Next we want to show the consistency of statements such as $2^{\aleph_0} = \aleph_1 \wedge 2^{\aleph_1} = \aleph_3$. Let κ be an infinite cardinal, $|I| \geq \kappa$ and $|J| \geq 2$; denote by $\mathrm{Fn}_\kappa(I, J)$

the collection of all partial functions $p : I \rightharpoonup J$ such that $|p| < \kappa$, ordered by reverse inclusion, so this is a partial order with maximal element $\varnothing$. As we will see, this poset isn't very useful when κ is singular, so usually it's assumed to be regular. The special case $\mathrm{Fn}_\omega(I, 2)$ has $RO(2^I)$ as its Boolean completion, and we used the latter to show the consistency of $\neg$CH. An important ingredient was that delta system lemma implies $RO(2^\kappa)$ has ccc property and thus preserves all cardinals. To study $\mathrm{Fn}_\kappa(I, J)$, we need a general delta system lemma.

Lemma 7.1. *Suppose $\omega \leq \kappa < \lambda$ are regular cardinals and $(x_\alpha)_{\alpha<\lambda}$ is a family of sets such that $|x_\alpha| < \kappa$, and moreover $\theta^{<\kappa} < \lambda$ for any $\theta < \lambda$. Then there is an unbounded set $A \subseteq \lambda$ (hence of size λ) such that $(x_\alpha)_{\alpha\in A}$ is a delta system.*

Proof. The set $E^\lambda_\kappa = \{\alpha < \lambda : \mathrm{cf}(\alpha) = \kappa\}$ is stationary. For any $\alpha \in E^\lambda_\kappa$ we have $|x_\alpha \cap \alpha| < \kappa$, and thus $\sup(x_\alpha \cap \alpha) < \alpha$ since κ is regular. By Fodor's lemma there exists a stationary $A \subseteq E^\lambda_\kappa$ and some $\beta < \lambda$ such that $\sup(x_\alpha \cap \alpha) < \beta$ for all $\alpha \in A$. If $\sup x_\alpha,\ \alpha \in A$ are bounded in λ, say bounded by γ, then since $\gamma^{<\kappa} < \lambda$ and λ is regular, λ many of the x_α are the same, which certainly form a delta system. Otherwise, we can inductively pick a sequence $(\alpha_i)_{i<\lambda}$ such that $\alpha_i \in A$ and $\alpha_i > \sup x_{\alpha_j}$ for all $j < i$; then $x_{\alpha_i} \cap x_{\alpha_j} \subseteq \beta$ for any $i < j < \lambda$. Since $\beta^{<\kappa} < \lambda$, we can refine the sequence so that $x_{\alpha_i} \cap \beta$ are the same for all i. □

For an uncountable cardinal κ, we say that a poset $\mathbb{P}$ satisfies *κ-chain condition* (κ-cc) if any antichain $A \subseteq \mathbb{P}$ has size strictly less than κ; thus ccc is the same as ω_1-cc. For every poset $\mathbb{P}$ there is a smallest cardinal κ such that $\mathbb{P}$ is κ-cc, and it can be shown that this κ must be regular; put another way, if κ is singular and $\mathbb{P}$ has antichains of size arbitrarily large below κ, then it has an antichain of size κ. See Jech [9, Theorem 7.15] or Kunen [12, Exercise III.3.94]. Jech only proves it for complete Boolean algebras but the proof can be modified to also work for poset; alternatively, use the fact that $\mathbb{P}$ is κ-cc iff its Boolean completion B is.

Lemma 7.2. *If $\mathbb{P}$ is κ-cc, and $\lambda \geq \kappa$ is a regular cardinal, then λ remains a regular cardinal in $V[G]$. It follows that forcing with $\mathbb{P}$ preserves all cardinals and cofinalities above κ, i.e., if $\lambda \geq \kappa$ is a cardinal in V then it remains a cardinal in $V[G]$, and if $\mathrm{cf}(\lambda) = \sigma \geq \kappa$ in V then the same is true in $V[G]$.*

Proof. Suppose $\tau < \lambda$ and $f : \tau \to \lambda$ belongs to $V[G]$. Let $\dot{f} \in V^{\mathbb{P}}$ be such that $\dot{f}_G = f$. By Truth Lemma there exists $p_0 \in G$ such that $p_0 \Vdash$"$\dot{f}$ is a map from τ to λ". By Bounded Quantification Lemma, for each $\alpha < \tau$, any $p \leq p_0$ has an extension q such that q decides $f(\alpha)$, namely $q \Vdash \dot{f}(\alpha) = \beta$ for some $\beta < \lambda$. Thus for each $\alpha < \tau$ the set $E_\alpha = \{p \leq p_0 : \exists\beta < \lambda\ p \Vdash \dot{f}(\alpha) = \beta\}$ is open dense below p_0; choose an antichain $A_\alpha \subseteq E_\alpha$ that is maximal below p_0; for each $p \in A_\alpha$ there is by definition some $\beta < \lambda$ such that $p \Vdash \dot{f}(\alpha) = \beta$, and we let B_α be the set of all such β as p varies. Note that A_α and B_α are in V. We have $|B_\alpha| \leq |A_\alpha| < \kappa \leq \lambda$, so $\sup\left(\bigcup_{\alpha<\tau} B_\alpha\right) < \lambda$ by regularity in V. Since $p_0 \in G$ and A_α is maximal below p_0, we have $f(\alpha) \in B_\alpha$ by construction, so the image of f is bounded.

We have shown that every regular cardinal $\lambda \geq \kappa$ remains regular in $V[G]$. Suppose $\mu > \kappa$ is a singular cardinal with $\mathrm{cf}(\mu) = \lambda \geq \kappa$ in V; first note that μ remains a cardinal in $V[G]$ because all regular cardinals large enough below μ remain cardinals, and a limit of cardinals is a cardinal. Next, it is still true in $V[G]$ that $\mathrm{cf}(\mu) = \mathrm{cf}(\lambda)$, and since λ is regular in V, by the first paragraph we have $\mathrm{cf}(\mu) = \mathrm{cf}(\lambda) = \lambda$ in $V[G]$. □

Lemma 7.3. *$\mathrm{Fn}_\kappa(I, J)$ is $(|J|^{<\kappa})^+$-cc for any infinite cardinal κ. If κ is regular and $|J| \leq 2^{<\kappa}$ then $\mathrm{Fn}_\kappa(I, J)$ is $(2^{<\kappa})^+$-cc.*

Proof. Let $\mu = |J|^{<\kappa}$. Suppose $(p_\alpha)_{\alpha<\mu^+}$ are conditions in $\mathrm{Fn}_\kappa(I, J)$, so $|p_\alpha| < \kappa$. We will show that there exists $X \subseteq \mu^+$ such that $|X| = \mu^+$ and $p_\alpha \not\perp p_\beta$ for any $\alpha, \beta \in X$, and thus $\mathrm{Fn}_\kappa(I, J)$ doesn't have antichain of size μ^+. We may assume κ is regular, since if it's singular then there exists a regular $\kappa' < \kappa$ such that μ^+ many of those partial functions p_α have size less than κ'.

Since κ is regular, using the induction formula for cardinal arithmetic it can be calculated that $\mu^{<\kappa} = \mu$. Apply delta system lemma to $\kappa < \mu^+$ and the collection $(\mathrm{dom}(p_\alpha) : \alpha < \mu^+)$, we get a subset $X \subseteq \mu^+$ of size μ^+ such that $\mathrm{dom}(p_\alpha) \cap \mathrm{dom}(p_\beta) = R$ for all different $\alpha, \beta \in X$, where $R \subseteq \mu^+$ has size less than κ. Since there are only $|J|^R < \mu^+$ many functions from R to J, by refining X we may assume $p_\alpha \restriction R$ are all the same, and thus $p_\alpha \not\perp p_\beta$ for $\alpha, \beta \in X$.

If κ is regular, then by the induction formula $(2^{<\kappa})^{<\kappa} = 2^{<\kappa}$. □

We have shown that $\mathrm{Fn}_\kappa(I, J)$ preserves all large enough cardinals. Now we show cardinals up to κ are also preserved. For an uncountable cardinal κ, we say that a poset $\mathbb{P}$ is *κ-closed* if for any $\lambda < \kappa$ and any sequence of conditions $(p_i)_{i<\lambda}$ such that $p_i \le p_j$ whenever $i > j$, there exists a lower bound p, namely $p \le p_i$ for all $i < \lambda$. By convention $\aleph_1$-closed is often denoted σ-closed. Note that it doesn't matter whether we let λ range over ordinals or cardinals below κ. Clearly if κ is singular and $\mathbb{P}$ is κ-closed then it is actually κ^+-closed.

Lemma 7.4. *If κ is uncountable regular and $\mathbb{P}$ is κ-closed, then for any $\lambda < \kappa$, any function $f : \lambda \to V$ that is in $V[G]$ is already in V. In particular, all cardinals $\lambda \le \kappa$ remain cardinals in $V[G]$ and have the same cofinality.*

Proof. Working in V, we shall show that whenever $\dot{f} \in V^{\mathbb{P}}$ and $p \in \mathbb{P}$ are such that $p \Vdash$"$\dot{f}$ is a function with domain λ", there exists $q \le p$ and some $g : \lambda \to V$ such that $q \Vdash \dot{f} = g$. This implies that if $p \Vdash$"$\dot{f}$ is a function with domain λ" then $D = \{q \le p : \exists g\; q \Vdash \dot{f} = g\}$ is dense below p, and thus if $G \ni p$ then $G \cap D \ne \varnothing$, which means $\dot{f}_G = g$ for some $g \in V$.

Define a decreasing sequence of conditions $(p_i)_{i\le\lambda}$ as follows. Let $p_0 = p$, and $p_{i+1} \le p_i$ be some condition that decides $\dot{f}(i)$, namely there exists $x \in V$ such that $p_{i+1} \Vdash \dot{f}(i) = x$. If $i \le \lambda$ is a limit, by assumption we may let p_i be a lower bound of $(p_j)_{j<i}$. Then p_λ forces $\dot{f}$ to equal the ground model function g, defined by $g(i) =$ the unique x such that $p_\lambda \Vdash \dot{f}(i) = x$.

Consequently, any $\lambda \le \kappa$ that is regular in V remains so in $V[G]$, because there is no new function from smaller cardinal to λ. Also, if $\mu < \kappa$ and $\mathrm{cf}(\mu) = \lambda$ in V then $\mathrm{cf}(\mu) = \mathrm{cf}(\lambda) = \lambda$ in $V[G]$. □

It is pretty clear that $\mathrm{Fn}_\kappa(I, J)$ is κ-closed as long as κ is regular, so it preserves cardinals and cofinalities up to κ. If $J = \{0, 1\}$ and $2^{<\kappa} = \kappa$ then $\mathrm{Fn}_\kappa(I, J)$ is κ^+-cc, so it preserves cardinals and cofinalities starting from κ^+, which means all cardinals and cofinalities are preserved. In particular, if we start with $V \models \mathsf{GCH}$ and force with $\mathrm{Fn}_{\aleph_1}(\aleph_3, 2)$, then $V[G]$ has the same cardinals as V. It also has the same $\mathcal{P}(\omega)$, since the poset is σ-closed so it doesn't add ω-sequences. A standard density argument shows $2^{\aleph_1} \ge \aleph_3$ in $V[G]$. To show equality we need one more lemma, whose proof would be a bit more elegant if we use Boolean completion.

Lemma 7.5. *In V, suppose $\mathbb{P}$ is κ-cc and $\mu = (|\mathbb{P}|^{<\kappa})^\lambda$. Then in $V[G]$ we have $|\mathcal{P}(\lambda)| \le \mu$.*

Proof. Define a nice name for a subset of λ to be a name of form $\bigcup_{\alpha<\lambda}\{\check{\alpha}\} \times A_\alpha$ where $A_\alpha \subseteq \mathbb{P}$ is an antichain. It's routine to count in V that there are at most μ many nice names. If p and $\dot{x}$ are such that $p \Vdash$"$\dot{x}$ is a subset of λ", choose for each $\alpha < \lambda$ an antichain A_α that is maximal among subsets of $\{p : p \Vdash \alpha \in \dot{x}\}$. Let $\dot{y} = \bigcup_{\alpha<\lambda}\{\check{\alpha}\} \times A_\alpha$; then $\dot{y}$ is a nice name and it can be checked that $p \Vdash \dot{x} = \dot{y}$. Thus in $V[G]$, any subset of λ is the interpretation of some nice name. □

$|\mathrm{Fn}_\kappa(I,J)| = \sup_{\sigma<\kappa} |I|^\sigma \cdot |J|^\sigma = (|I| \cdot |J|)^{<\kappa}$. Therefore, if $V \models \mathsf{GCH}$ and we force with $\mathrm{Fn}_{\aleph_1}(\aleph_3, 2)$, then in $V[G]$ we have $2^{\aleph_1} = 2^{\aleph_2} = \aleph_3$ and $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for all other α.

7.2 Preserving and collapsing cardinals

In general, we say that a cardinal $\kappa \in V$ is *preserved* in a forcing extension $V[G]$ if $V[G] \models$"κ is a cardinal"; otherwise it is *collapsed*.

Suppose κ is regular. The poset $\mathrm{Fn}_\kappa(\kappa \times \lambda, 2)$, also denoted $\mathrm{Add}(\kappa, \lambda)$, is called the forcing that adds λ many Cohen subsets of κ; note that if $\lambda > \kappa$ then $\mathrm{Fn}_\kappa(\kappa \times \lambda, 2) \simeq \mathrm{Fn}_\kappa(\lambda, 2)$ and if $\lambda \le \kappa$ then $\mathrm{Fn}_\kappa(\kappa \times \lambda, 2) \simeq \mathrm{Fn}_\kappa(\kappa, 2)$. This forcing preserves all cardinals (and cofinalities) as long as $2^{<\kappa} = \kappa$. On the other hand, the poset $\mathrm{Fn}_\kappa(\kappa, \lambda)$, also denoted $\mathrm{Col}(\kappa, \lambda)$, adds a surjection from κ to λ by a simple density argment and thus collapses λ if $\kappa < \lambda$. In fact, all cardinals θ such that $\kappa < \theta \le \lambda$ become ordinals of cofinality κ.

$\mathrm{Col}(\kappa, \lambda)$ is sometimes referred to as the Lévy collapse, but we reserve that name for the forcing $\mathrm{Col}(\kappa, <\lambda)$ where λ is an inaccessible cardinal. A condition in $\mathrm{Col}(\kappa, <\lambda)$ is a partial function $p : \kappa \times \lambda \rightharpoonup \lambda$ such that $p(\alpha, \beta) < \beta$; a density argument shows that if $f = \bigcup G$ then $f(\cdot, \beta)$ is a surjection from κ to β, and thus all cardinals in the interval (κ, λ) are collapsed and have size κ in $V[G]$. On the other hand, the Delta System Lemma 7.1 and the inaccessibility of λ show that $\mathrm{Col}(\kappa, <\lambda)$ is λ-cc, so λ is preserved and is the new successor of κ, in symbol $\lambda = (\kappa^+)^{V[G]}$. Some authors say "$\lambda$ is collapsed to κ^+", but this conflicts with our definition of collapse, so it might be better to say "λ is turned into κ^+".

Lévy collapse is one of the most useful forcing posets in set theory. Solovay showed that if we do the Lévy collapse $\mathrm{Col}(\omega, <\lambda)$ and consider $N = L(\mathbb{R})$, the class of all sets constructible from reals, as defined in the generic extension, then N is a model of "$\mathsf{ZF} + \mathsf{DC}$+all subsets of $\mathbb{R}$ are Lebesgue measurable" and much more; see e.g. Jech [9, Chapter 26].

After one works with forcing for a while, it is natural to ask: what are the possible patterns of cardinal preservation in a forcing extension? $\mathrm{Col}(\omega, \omega_1)$ collapses ω_1 and preserves all other cardinals. Similarly, if CH holds then $\mathrm{Col}(\omega_1, \omega_2)$ collapses ω_2 while preserving all other cardinals. Is the assumption CH necessary? Or does ZFC prove the existence of a poset that collapses ω_2 and preserves everything else? A positive answer was provided by Abraham [1], but his method does not seem to generalize to cardinals other than ω_2. Asperó [2] showed how to do this for ω_3; again the method does not seem to address the case of ω_4.

As another example, it is a notorious open question whether $\aleph_{\omega+1}^V = \aleph_2^{V[G]}$ is possible, or more generally whether a successor of singular cardinal can become a successor of (uncountable) regular cardinal. Note that $\mathrm{Col}(\omega, \aleph_\omega)$ is $\aleph_{\omega+1}$-cc and achieves $\aleph_{\omega+1}^V = \aleph_1^{V[G]}$, but the naive generalization $\mathrm{Col}(\aleph_1, \aleph_\omega)$ doesn't work, because it is equivalent to $\mathrm{Col}(\aleph_1, \aleph_\omega^{\aleph_0})$ and thus collapses $\aleph_{\omega+1}$. To see why, by definition $\mathrm{Col}(\aleph_1, \aleph_\omega) = \mathrm{Fn}_{\aleph_1}(\aleph_1, \aleph_\omega) \simeq \mathrm{Fn}_{\aleph_1}(\aleph_1 \times \aleph_0, \aleph_\omega)$; a countable partial function from $\aleph_1$ to $\aleph_\omega^{\aleph_0}$ can be viewed as a countable partial function from $\aleph_1 \times \aleph_0$ to $\aleph_\omega$ via currying, which gives an embedding of $\mathrm{Col}(\aleph_1, \aleph_\omega^{\aleph_0})$ into $\mathrm{Fn}_{\aleph_1}(\aleph_1 \times \aleph_0, \aleph_\omega)$; it's not hard to see that this embedding is dense.

A singular cardinal in V either stays singular in $V[G]$ or is collapsed, and a successor cardinal either remains a successor or is collapsed. Can a regular cardinal κ become a singular cardinal? Note that κ must be a limit cardinal in V, and hence an inaccessible. As we are going to explain next, the naive way to singularize κ doesn't work, and an affirmative answer turns out to require a measurable cardinal. It is tempting to consider the poset $\mathbb{P}$ of all finite increasing sequences of ordinals below κ. Unfortunately this collapses tons of things. Say the generic sequence is $(\alpha_n : n < \omega)$, then for any $\alpha < \kappa$, by a density argument there exists n such that α_n is of form $\beta + \alpha$, where $\beta > \alpha$ is some indecomposable ordinal (so that each n corresponds to at most one α), so in the extension there is a surjection from ω to κ. We may try to modify the poset by only considering sequences of indecomposable ordinals, or even cardinals, but then the same issue occurs: by density, for every $\alpha < \kappa$ there exists n for which $\alpha_n = \aleph_{\beta+\alpha}$ where $\beta > \alpha$ is indecomposable. So maybe we want the sequence to eventually consist of limit cardinals, and also cardinals that are fixed points of $\aleph$ function, cardinals that are fixed points of fixed points, etc.

Consider the forcing consisting of conditions (s, C) where s is a finite increasing sequence of ordinals below κ and C belongs to the club filter; to extend (s, C) we are allowed to extend s using finitely many elements from C and also shrinking C; this is starting to look similar to Prikry forcing. However this still doesn't work, because for every regular cardinal $\sigma < \kappa$ and every club C, there are many ordinals in C of cofinality σ, so by density $\alpha_n \mapsto \mathrm{cf}^V(\alpha_n)$ surjects onto V-regular cardinals below κ. To fix this we would want the sequence to eventually consist of regular cardinals, which suggests κ should be Mahlo. Continuing this line of thought, we are eventually led to the standard Prikry forcing. It is a forcing that requires a measurble cardinal κ, turns the cofinality of κ into ω and preserves all cardinals (including κ).

The *Singular Cardinal Hypothesis* says if κ is a strong limit cardinal then $2^\kappa = \kappa^+$. We have seen how $\mathrm{Add}(\kappa, \lambda)$ can be used to show that the Generalized Continuum Hypothesis fails at a regular cardinal κ; we also mentioned that $\mathrm{Fn}_\kappa(I, J)$ for singular κ isn't very useful. For example, consider the poset that "adds a Cohen subset of $\aleph_\omega$", namely $\mathrm{Fn}_{\aleph_\omega}(\aleph_\omega, 2)$. We claim that it collapses $\aleph_\omega$. We may equivalently consider $\mathrm{Fn}_{\aleph_\omega}(X, 2)$ where $X = \bigsqcup_n \aleph_n \times \aleph_n$, so the forcing adds a subset G_n of $\aleph_n \times \aleph_n$ for each n; by a density argument, for each $\alpha < \aleph_\omega$ there exists n such that G_n is up to a small difference just a line $\aleph_n \times \{\alpha\}$; this defines a surjection from ω to $\aleph_\omega^V$. By a so-called "absorption theorem", this shows $\mathrm{Fn}_{\aleph_\omega}(\aleph_\omega, 2)$ is actually equivalent to $\mathrm{Col}(\omega, \aleph_\omega)$; see Jech [9, Lemma 26.7].

This shows the naive way to force "$\aleph_\omega$ is a strong limit and $2^{\aleph_\omega} > \aleph_{\omega+1}$" (namely the failure of SCH at $\aleph_\omega$) doesn't work. In fact, forcing $\neg\mathsf{SCH}$ requires a measurable cardinal of Mitchell order κ^{++}—something quite a bit stronger than a mere measurable. We outline how to get the consistency of $\neg\mathsf{SCH}$. First we need to get a measurable κ satisfying $2^\kappa = \kappa^{++}$; this is easiest done by starting with a κ that is supercompact, and then use the Silver's method of *Easton iteration* to blow up 2^κ in a way that κ remains at least measurable. Then we use Prikry forcing to change the cofinality of κ to ω while preserving all cardinals, thus obtaining $\neg\mathsf{SCH}$ at κ; to bring this down to $\aleph_\omega$ requires "Prikry forcing interleaved with collapse".

8 Iteration

This section currently does not contain any application of iterated forcing. I just wrote it to make sure I understand the details of factor lemma, which roughly says if $\beta < \alpha$, then a length α iteration can be viewed as a length β iteration followed by a length $\alpha - \beta$ iteration.

We mostly stick to posets. For the presentation using Boolean algebra and how it can help in the study of semiproper iteration, see [15, 16].

8.1 Two-step iteration

If we can force once we can force any finitely many times. For example, to show the consistency of $2^{\aleph_0} = \aleph_2 \wedge 2^{\aleph_1} = \aleph_3$, we start with a ground model V that satisfies GCH, force with $\mathrm{Add}(\aleph_1, \aleph_3)$ so that in $V[G]$ we have $2^{\aleph_0} = \aleph_1 \wedge 2^{\aleph_1} = \aleph_3$, and then over $V[G]$ we force with $\mathrm{Add}(\aleph_0, \aleph_2)$. Note that we cannot do $\mathrm{Add}(\aleph_0, \aleph_2)$ first, since then CH fails in the extension and $\mathrm{Add}(\aleph_1, \aleph_3)$ would collapse cardinal. Similarly, to get $2^{\aleph_0} = \aleph_2 \wedge 2^{\aleph_1} = \aleph_3 \wedge 2^{\aleph_2} = \aleph_5$ we first add Cohen subsets to $\aleph_2$, then $\aleph_1$ and finally $\aleph_0$.

However, it is not clear how to do this infinitely many times, in order to, e.g., violate GCH at all $\aleph_n$. Suppose we do it in the ctm approach, so we force over M_n to get M_{n+1}; the union $\bigcup_n M_n$ is most often not a model of power set axiom. If we use the Boolean-valued model approach, it is not even clear what "union" should mean.

The key is to combine two steps into one: if G is $(V, \mathbb{P})$-generic, $\mathbb{Q}$ is a forcing poset in $M[G]$ and H is $(V[G], \mathbb{Q})$-generic, we will find a poset $\mathbb{R} \in V$ and a $(V, \mathbb{R})$-generic filter K such that $V[K] = V[G][H]$. If we can combine two-step iteration into one, it is not hard to do it for finitely many steps. Now at limit stage, instead of taking the limit of models, we take the limit of posets.

The simplest case of iteration is when $\mathbb{Q}$ in fact belongs to V, in which case we are dealing with a product $\mathbb{P} \times \mathbb{Q}$. It is easy to show that a filter K on $\mathbb{P} \times \mathbb{Q}$ is the same as a product of filters $G \times H$. Using the complete embedding $i : \mathbb{P} \to \mathbb{P} \times \mathbb{Q},\ p \mapsto (p, 1_{\mathbb{Q}})$, or alternatively the projection $\pi : \mathbb{P} \times \mathbb{Q} \to \mathbb{P},\ (p, q) \mapsto p$, we see that if K is generic over M then so is G, similarly for H. But the converse doesn't hold: G and H being generic over V doesn't imply that $G \times H$ is generic. For example, $G \times G$ is almost never generic, essentially for the same reason that the diagonal $D = \{(x, y) \in \mathbb{R}^2 : x = y\}$ is nowhere dense. It turns out we need H to be generic over $V[G]$.

Lemma 8.1 (Factor lemma for product)**.** *$K = G \times H$ is $(V, \mathbb{P} \times \mathbb{Q})$-generic iff G is $(V, \mathbb{P})$-generic and H is $(V[G], \mathbb{Q})$-generic.*

Proof. Suppose $G \times H$ is $\mathbb{P} \times \mathbb{Q}$ generic over V. That G is $(V, \mathbb{P})$-generic is immediate since $\mathbb{P}$ completely embeds into $\mathbb{P} \times \mathbb{Q}$. Suppose $D \subseteq \mathbb{Q}$ is a dense set in $V[G]$; we want to show that $H \cap D \neq \varnothing$. Note that:

$$\begin{aligned} H \cap D \neq \varnothing &\Leftrightarrow \exists q \in H (q \in D) \\ &\Leftrightarrow \exists q \in H \exists p \in G (p \Vdash q \in \dot{D}) \\ &\Leftrightarrow \exists (p, q) \in G \times H\ (p, q) \in E \end{aligned}$$

So it suffices to show that E is dense below $(p_0, 1)$. By Truth Lemma there exists $p_0 \in G$ such that $p_0 \Vdash$"$\dot{D}$ is a dense set in $\mathbb{Q}$". Let $E = \{(p, q) : p \Vdash q \in \dot{D}\}$. Note that:

$$\begin{aligned} & p_0 \Vdash \dot{D} \text{ is dense in } \mathbb{Q} \\ \Leftrightarrow\ & p_0 \Vdash \forall q \exists q' \leq q (q' \in \dot{D}) \\ \Leftrightarrow\ & \forall q\ p_0 \Vdash \exists q' \leq q (q' \in \dot{D}) \\ \Leftrightarrow\ & \forall q \forall p \leq p_0 \exists p' \leq p \exists q' \leq q (p' \Vdash q' \in \dot{D}) \\ \Leftrightarrow\ & E \text{ is dense below } (p_0, 1) \end{aligned}$$

where for example, $\forall q$ abbreviates $\forall q \in \mathbb{Q}$, to which we apply the Bounded Quantification Lemma. Thus $H \cap D \neq \varnothing$, and H is $(V[G], \mathbb{Q})$-generic.

Conversely suppose G is $(V, \mathbb{Q})$-generic and H is $(V[G], \mathbb{Q})$-generic, and $E \subseteq \mathbb{P} \times \mathbb{Q}$ is a dense set in M.

$$
\begin{aligned}
& (G \times H) \cap E \neq \varnothing \\
\Leftrightarrow\ & \exists q \in H \exists p \in G, (p, q) \in E \\
\Leftarrow\ & \{q : \exists p \in G, (p, q) \in E\} \text{ is dense in } \mathbb{Q} \\
\Leftrightarrow\ & \forall q \exists q' \leq q \exists p \in G, (p, q') \in E \\
\Leftrightarrow\ & \forall q \exists p \in G \exists q' \leq q, (p, q') \in E \\
\Leftarrow\ & \forall q\ \{p : \exists q' \leq q, (p, q') \in E\} \text{ is dense in } \mathbb{P} \\
\Leftrightarrow\ & \forall q \forall p \exists p' \leq p \exists q' \leq q, (p', q') \in E
\end{aligned}
$$

So $G \times H$ is generic. □

When both $\mathbb{P}$ and $\mathbb{Q}$ are Cohen forcing, this is related to the Kuratowski–Ulam theorem. We remark that once we understand product of two posets we understand arbitrary product, and this is enough to get any desired power function on the $\aleph_n$'s, or even on all regular cardinals. This is the infamous Easton's Theorem; see Kunen [12, V.2]. As mentioned earlier, the situation for singular cardinals is much more subtle.

A simple example of two step iteration that is not a product is when $\mathbb{Q} \in V[G]$ is a subset of some $(\mathbb{Q}_0, \leq) \in V$ with the induced order. Let $\dot{\mathbb{Q}}$ be such that $\dot{\mathbb{Q}}_G = \mathbb{Q}$. A reasonable guess of the iteration poset is $\mathbb{P} * \dot{\mathbb{Q}} := \{(p, q) \in \mathbb{P} \times \mathbb{Q}_0 : p \Vdash q \in \dot{\mathbb{Q}}\}$, ordered as a subposet of the product poset. This turns out to work, and it inspires the general definition of iterated forcing.

Definition 8.2. Suppose $\dot{\mathbb{Q}}$, $\dot{\leq}_{\mathbb{Q}}$, and $\dot{1}_{\mathbb{Q}}$ are three $\mathbb{P}$-names such that $1_{\mathbb{P}} \Vdash$"$\dot{\mathbb{Q}}$ equipped with $\dot{\leq}_{\mathbb{Q}}$ is a poset with maximal element $\dot{1}_{\mathbb{Q}}$". Then the poset $\mathbb{P} * \dot{\mathbb{Q}}$ has underlying set:

$$\{(p, \dot{q}) \in \mathbb{P} \times \mathrm{dom}(\dot{\mathbb{Q}}) : p \Vdash q \in \dot{\mathbb{Q}}\},$$

and the order is defined by:

$$(p_1, \dot{q}_1) \leq (p_2, \dot{q}_2) \text{ iff } p_1 \leq p_2 \text{ and } p_1 \Vdash \dot{q}_1 \dot{\leq}_{\mathbb{Q}} \dot{q}_2.$$

To be unambiguous we should write, e.g., $(p_1, \dot{q}_1) \leq_{\mathbb{P} * \dot{\mathbb{Q}}} (p_2, \dot{q}_2)$ iff $p_1 \leq_{\mathbb{P}} p_2$ and $p_1 \Vdash_{\mathbb{P}} \dot{q}_1 \dot{\leq}_{\mathbb{Q}} \dot{q}_2$, but we try to stick with the convention that p range over elements of $\mathbb{P}$ and $\dot{q}$ ranges over names for elements of $\dot{\mathbb{Q}}$, so hopefully the intended order is clear from context.

It can be checked that $\mathbb{P} * \dot{\mathbb{Q}}$ is indeed a poset (usually not a partial order) with maximal element $(1_{\mathbb{P}}, \dot{1}_{\mathbb{Q}})$. It's natural that if $p_1 \leq p_2$ then we should have $(p_1, \dot{q}) \leq (p_2, \dot{q})$, and if $p \Vdash \dot{q}_1 \dot{\leq}_{\mathbb{Q}} \dot{q}_2$ we should have $(p, \dot{q}_1) \leq (p, \dot{q}_2)$, which lead to the above definition of the order. The map $p \mapsto (p, \dot{1}_{\mathbb{Q}})$ is a complete embedding of $\mathbb{P}$ into $\mathbb{P} * \dot{\mathbb{Q}}$, and $(p, \dot{q}) \mapsto p$ is a projection; we will see that complete embedding, projection and two-step iteration are essentially the same thing. The set $\{(p, \dot{q}) : \exists p'[p \leq p' \wedge (\dot{q}, p') \in \dot{\mathbb{Q}}]\}$ is dense in $\mathbb{P} * \dot{\mathbb{Q}}$, which is sometimes useful.

Lemma 8.3 (Factor lemma for two-step iteration)**.** (i) *If K is $(V, \mathbb{P} * \dot{\mathbb{Q}})$-generic, and G is the $(V, \mathbb{P})$-generic filter induced by K, then $H := \{\dot{q}_G : \exists p\ (p, \dot{q}) \in K\}$ is $(V[G], \dot{\mathbb{Q}}_G)$-generic.*

(ii) *If G is $(V, \mathbb{P})$-generic and H is $(V[G], \dot{\mathbb{Q}}_G)$-generic, then $K = G * H := \{(p, \dot{q}) \in \mathbb{P} * \dot{\mathbb{Q}} : p \in G \wedge \dot{q}_G \in H\}$ is $(V, \mathbb{P} * \dot{\mathbb{Q}})$-generic. It follows that we have $K = G * H$ in (i).*

Proof. (i) H is a filter because if $(p_1, \dot{q}_1)$ and $(p_2, \dot{q}_2)$ are in K, then there exists $(p_3, \dot{q}_3) \in K$ s.t. $p_3 \leq p_1$, $p_3 \leq p_2$, $p_3 \Vdash \dot{q}_3 \leq \dot{q}_1$ and $p_3 \Vdash \dot{q}_3 \leq \dot{q}_2$; since $p_3 \in G$ we have $(\dot{q}_3)_G \leq (\dot{q}_1)_G$ and $(\dot{q}_3)_G \leq (\dot{q}_2)_G$. Upward closure is similar.

Suppose $p_0 \in G$ forces that $\dot{D}$ is dense in $\dot{\mathbb{Q}}$.

$$\begin{aligned} H \cap \dot{D}_G \neq \varnothing &\Leftrightarrow \exists (p, \dot{q}) \in K \ \dot{q}_G \in \dot{D}_G \\ &\Leftrightarrow \exists (p, \dot{q}) \in K \ \exists (p', \dot{1}) \in K \ p' \Vdash \dot{q} \in \dot{D} \\ &\Leftrightarrow \exists (p, \dot{q}) \in K \ p \Vdash \dot{q} \in \dot{D} \end{aligned}$$

so it suffices to show that $E = \{(p, \dot{q}) : p \Vdash \dot{q} \in \dot{D}\}$ is dense below $(p_0, \dot{1})$.

$$\begin{aligned} &p_0 \Vdash \dot{D} \text{ is dense} \\ \Leftrightarrow &p_0 \Vdash \forall q \in \dot{\mathbb{Q}} \exists q' \in \dot{\mathbb{Q}} \ q' \leq q \wedge q' \in \dot{D} \\ \Leftrightarrow &\forall \dot{q} \in \mathrm{dom}(\dot{\mathbb{Q}}) \forall p \leq p_0, \ p \Vdash \dot{q} \in \dot{\mathbb{Q}} \to p \Vdash \exists q' \in \dot{\mathbb{Q}} \ q' \leq \dot{q} \wedge q' \in \dot{D} \\ \Leftrightarrow &\forall (p, \dot{q}) \leq (p_0, \dot{1}) \ p \Vdash \exists q' \in \dot{\mathbb{Q}} \ q' \leq \dot{q} \wedge q' \in \dot{D} \\ \Leftrightarrow &\forall (p, \dot{q}) \leq (p_0, \dot{1}) \ \{p' : \exists \dot{q}' \in \mathrm{dom}(\dot{\mathbb{Q}})[p' \Vdash \dot{q}' \in \dot{\mathbb{Q}} \wedge p' \Vdash \dot{q}' \leq \dot{q} \wedge \dot{q}' \in \dot{D}]\} \text{ is dense below } p \\ \Leftrightarrow &\forall (p, \dot{q}) \leq (p_0, \dot{1}) \forall r \leq p \exists p' \leq r \exists \dot{q}' \in \mathrm{dom}(\dot{\mathbb{Q}})[p' \Vdash \dot{q}' \in \dot{\mathbb{Q}} \wedge p' \Vdash \dot{q}' \leq \dot{q} \wedge \dot{q}' \in \dot{D}] \\ \Leftrightarrow &\forall (p, \dot{q}) \leq (p_0, \dot{1}) \forall r \leq p \exists (p', \dot{q}') \leq (r, \dot{q}) \ p' \Vdash \dot{q}' \in \dot{D} \\ \Leftrightarrow &\forall (p, \dot{q}) \leq (p_0, \dot{1}) \exists (p', \dot{q}') \leq (p, \dot{q}) \ p' \Vdash \dot{q}' \in \dot{D} \\ \Leftrightarrow &E \text{ is dense below } (p_0, \mathbb{1}) \end{aligned}$$

(ii) Let $E \subseteq \mathbb{P} * \dot{\mathbb{Q}}$ be a dense *open* set.

$$\begin{aligned} &(G * H) \cap E \neq \varnothing \\ \Leftarrow &\{\dot{q}_G : \exists p \in G \ (p, \dot{q}) \in E\} \text{ is dense in } \dot{\mathbb{Q}}_G \\ \Leftrightarrow &\forall \dot{q}_0 \in \mathrm{dom}(\dot{\mathbb{Q}})[\exists p_0 \in G \ p_0 \Vdash \dot{q}_0 \in \dot{\mathbb{Q}} \to \exists p \in G \exists \dot{q} \in \mathrm{dom}(\dot{\mathbb{Q}}) \ \dot{q}_G \leq (\dot{q}_0)_G \wedge (p, \dot{q}) \in E] \\ \Leftrightarrow &\forall \dot{q}_0 \in \mathrm{dom}(\dot{\mathbb{Q}})[\exists p_0 \in G \ p_0 \Vdash \dot{q}_0 \in \dot{\mathbb{Q}} \to \exists p \in G \exists \dot{q} \in \mathrm{dom}(\dot{\mathbb{Q}}) \ p \Vdash \dot{q} \leq \dot{q}_0 \wedge (p, \dot{q}) \in E] \\ \Leftarrow &\forall \dot{q}_0 \in \mathrm{dom}(\dot{\mathbb{Q}})\{p : [p \Vdash \dot{q}_0 \notin \dot{\mathbb{Q}}] \vee [\exists \dot{q} \in \mathrm{dom}(\dot{\mathbb{Q}}) \ p \Vdash \dot{q} \leq \dot{q}_0 \wedge (p, \dot{q}) \in E]\} \text{ is dense in } \mathbb{P} \\ \Leftrightarrow &\forall \dot{q}_0 \in \mathrm{dom}(\dot{\mathbb{Q}}) \forall p_0 \ p_0 \Vdash \dot{q}_0 \in \mathbb{Q} \to \exists p \exists \dot{q} \in \mathrm{dom}(\dot{\mathbb{Q}})[p \leq p_0 \wedge p \Vdash \dot{q} \leq \dot{q}_0 \wedge (p, \dot{q}) \in E] \\ \Leftrightarrow &\forall (p_0, \dot{q}_0) \exists (p, \dot{q}) \leq (p_0, \dot{q}_0) \ (p, \dot{q}) \in E \\ \Leftrightarrow &E \text{ is dense} \end{aligned}$$

so $G * H$ is generic. In (i) it can be checked that $K \supseteq G * H$, and since they are both generic they are equal. □

8.2 Two-step iteration, again

The above definition of iteration is as in Kunen [12]. Although it's quite natural, the requirement $p \Vdash q \in \dot{Q}$ in the definition of $\mathbb{P} * \dot{\mathbb{Q}}$ makes the proof quite messy compared to the case of product. More problematic is the restriction $\dot{q} \in \mathrm{dom}(\dot{\mathbb{Q}})$; this works well for finite iteration or more generally finite support iteration (which is enough for showing consistency of Martin's Axiom), but is badly behaved in general. Roughly speaking, under the "correct" definition, a countable support iteration of countably closed forcings is again countably closed, but under Kunen's definition there exist counterexamples; see Kunen [12, Example V.5.4].

Jech defines the two-step iteration as $\mathbb{P} * \dot{\mathbb{Q}} := \{(p, \dot{q}) : p \in \mathbb{P} \wedge 1_{\mathbb{P}} \Vdash q \in \dot{\mathbb{Q}}\}$. This is a terrible definition since it defines a proper class, but we can remedy this by using core. Note that in the definition of two-step iteration we require $1_{\mathbb{P}} \Vdash \dot{1}_{\mathbb{Q}} \in \dot{\mathbb{Q}}$, which in particular means $1_{\mathbb{P}} \Vdash \dot{\mathbb{Q}} \neq \varnothing$, so $\mathrm{Core}(\dot{\mathbb{Q}}) \neq \varnothing$ by Lemma 6.11.

We are going to give a new definition of $\mathbb{P} * \dot{\mathbb{Q}}$, so to avoid ambiguity, from now on we refer to the poset in Definition 8.2 by $\mathbb{P} *_K \dot{\mathbb{Q}}$.

Definition 8.4. With the same assumptions as Definition 8.2, $\mathbb{P} * \dot{\mathbb{Q}}$ is the poset with underlying set $\mathbb{P} \times \mathrm{Core}(\dot{\mathbb{Q}})$ and the same order as before.

It can be checkd that the factor lemma still holds. Moreover our new definition is equivalent to the previous one: by Existential Completeness Lemma 6.12, if $p \Vdash \dot{q} \in \dot{\mathbb{Q}}$ then there exists $\dot{x} \in \mathrm{Core}(\dot{\mathbb{Q}})$ s.t. $p \Vdash \dot{q} = \dot{x}$. Define a map $(p, \dot{q}) \mapsto (p, \dot{x})$; it can be checked that this is a dense embedding from $\mathbb{P} *_K \dot{\mathbb{Q}}$ to $\mathbb{P} * \dot{\mathbb{Q}}$.

8.3 General iteration

We can then define three step iteration $(\mathbb{P} * \dot{\mathbb{Q}}) * \dot{\mathbb{R}}$, where $\dot{\mathbb{Q}}$ is a $\mathbb{P}$-name for a poset and $\dot{\mathbb{R}}$ is a $\mathbb{P} * \dot{\mathbb{Q}}$-name, and then $((\mathbb{P} * \dot{\mathbb{Q}}) * \dot{\mathbb{R}}) * \dot{\mathbb{S}}$, etc. If we have a sequence $\mathbb{P}_0$, $\dot{\mathbb{Q}}_0$, $\dot{\mathbb{Q}}_1$, $\dot{\mathbb{Q}}_2 \ldots$ where $\dot{\mathbb{Q}}_n$ is a $\mathbb{P}_n$-name for a poset, $\mathbb{P}_n = \mathbb{P}_0 * \dot{\mathbb{Q}}_0 * \cdots * \dot{\mathbb{Q}}_{n-1}$ (left associative), then there is both a complete embedding from $\mathbb{P}_n$ to $\mathbb{P}_{n+1}$ and a projection from $\mathbb{P}_{n+1}$ to $\mathbb{P}_n$. At the ω-th stage we can either form the direct limit of the complete embeddings, or the inverse limit of the projections; the direct limit is called finite support iteration, and the inverse limit is called full support iteration, which in this case is the same as countable support iteration.

To avoid handling ugly expressions like $(((p, \dot{q}), \dot{r}), \dot{s})$, we replace them by tuples $(p, \dot{q}, \dot{r}, \dot{s})$. Thus a general transfinite iteration is defined as follows.

Definition 8.5. An α-stage iterated forcing consists of two sequences $((\mathbb{P}_\xi, \leq_\xi, 1_\xi) : \xi \leq \alpha)$ and $((\dot{\mathbb{Q}}_\xi, \dot{\leq}_{\mathbb{Q}_\xi}, \dot{1}_{\mathbb{Q}_\xi}) : \xi < \alpha)$ such that:

1. $\mathbb{P}_\xi$ is a set of sequences of length ξ, and $(\mathbb{P}_\xi, \leq_\xi, 1_\xi)$ is a poset with maximal element 1_ξ; in particular $\mathbb{P}_0 = \{\varnothing\}$ and $1_\xi = \varnothing$.
2. $\dot{\mathbb{Q}}_\xi$, $\dot{\leq}_{\mathbb{Q}_\xi}$, and $\dot{1}_{\mathbb{Q}_\xi}$ are $\mathbb{P}_\xi$-names such that $\Vdash_{\mathbb{P}_\xi}$"$\dot{\mathbb{Q}}_\xi$ equipped with $\dot{\leq}_{\mathbb{Q}_\xi}$ is a poset with maximal element $\dot{1}_{\mathbb{Q}_\xi}$", and $\dot{1}_{\mathbb{Q}_\xi} \in \mathrm{Core}(\dot{\mathbb{Q}}_\xi)$.
3. A sequence p of length $\xi + 1$ belongs to $\mathbb{P}_{\xi+1}$ iff $p \upharpoonright \xi \in \mathbb{P}_\xi$ and $p(\xi) \in \mathrm{Core}(\dot{\mathbb{Q}}_\xi)$.

   The order $\leq_{\xi+1}$ is defined by $p_1 \leq_{\xi+1} p_2$ iff $p_1 \upharpoonright \xi \leq_\xi p_2 \upharpoonright \xi$ and $p_1 \upharpoonright \xi \Vdash_{\mathbb{P}_\xi} p_1(\xi) \dot{\leq}_{\mathbb{Q}_\xi} p_2(\xi)$.
4. If $\gamma \leq \alpha$ is a limit, then $\mathbb{P}_\gamma$ is a subset of $\{p : \mathrm{dom}(p) = \gamma \wedge \forall \xi < \gamma [p \upharpoonright \xi \in \mathbb{P}_\xi]\}$. If equality holds, we say the iteration takes inverse limit at γ.
   (a) $\mathbb{P}_\gamma$ must also contain the set $\{p : \mathrm{dom}(p) = \gamma \wedge \exists \xi < \gamma [p \upharpoonright \xi \in \mathbb{P}_\xi \wedge \forall \xi \leq \eta < \gamma \; p(\eta) = \dot{1}_{\mathbb{Q}_\eta}]\}$. If equality holds, we say the iteration takes direct limit at γ.
   (b) We further require that if $p \in \mathbb{P}_\gamma$, $\xi < \gamma$ and p_1 is a sequence of length γ such that $p_1 \upharpoonright \xi \in \mathbb{P}_\xi$, $p_1 \upharpoonright \xi \leq_\xi p \upharpoonright \xi$ and $p_1 \upharpoonright (\gamma \setminus \xi) = p \upharpoonright (\gamma \setminus \xi)$, then $p_1 \in \mathbb{P}_\gamma$.

The order $\leq_\gamma$ is defined by $p_1 \leq_\gamma p_2$ iff $p_1 \restriction \xi \leq_\xi p_2 \restriction \xi$ for all $\xi < \gamma$.

5. It follows by induction that the sequence $(\dot{1}_{\mathbb{Q}_\eta} : \eta < \xi)$ is in $\mathbb{P}_\xi$ and is a maximal element. We require that 1_ξ is equal to this element.

Note that $\mathbb{P}_{\xi+1}$ is isomorphic to $\mathbb{P}_\xi * \dot{\mathbb{Q}}_\xi$ via the map $p^\frown \dot{q} \mapsto (p, \dot{q})$. The clause 4(b) ensures the restriction map from $\mathbb{P}_\gamma$ to $\mathbb{P}_\xi$ is a projection, and the map from $\mathbb{P}_\xi$ to $\mathbb{P}_\gamma$ defined by concatenating $p \in \mathbb{P}_\xi$ with $\dot{1}_{\mathbb{Q}_\eta}$, $\xi \leq \eta < \gamma$ is a complete embedding; by induction this is true for all γ, not just limits.

For brevity we often simply write, e.g., $(\dot{\mathbb{Q}}_\xi : \xi < \alpha)$ instead of $((\dot{\mathbb{Q}}_\xi, \dot{\leq}_{\mathbb{Q}_\xi}, \dot{1}_{\mathbb{Q}_\xi}) : \xi < \alpha)$. The iteration is completely determined by $((\dot{\mathbb{Q}}_\xi, \dot{\leq}_{\mathbb{Q}_\xi}, \dot{1}_{\mathbb{Q}_\xi}) : \xi < \alpha)$ and what happens at limit stages.

If $G = G_\alpha$ is $(V, \mathbb{P}_\alpha)$-generic, then it induces a G_ξ that is $(V, \mathbb{P}_\xi)$-generic, and $G_{\xi+1}$ induces an H_ξ that is $(V[G_\xi], (\dot{\mathbb{Q}}_\xi)_{G_\xi})$-generic. This is enough for many applications of iterated forcing.

A natural question is whether $*$ is associative, i.e., whether $(\mathbb{P} * \dot{\mathbb{Q}}) * \dot{\mathbb{R}}$ is the same as $\mathbb{P} * (\dot{\mathbb{Q}} * \dot{\mathbb{R}})$. It is not immediately clear what $\dot{\mathbb{Q}} * \dot{\mathbb{R}}$ even means. Essentially we need to prove a factor lemma for general iteration, namely for any $\beta < \alpha$, $\mathbb{P}_\alpha$ can be viewed as the iteration $\mathbb{P}_\beta$ followed by another iteration of length $\alpha - \beta$ (the unique ordinal γ s.t. $\beta + \gamma = \alpha$). Here is the plan: first we show that for any complete embedding $i : \mathbb{P} \to \mathbb{R}$ between posets, there is a $\mathbb{P}$-name $\dot{\mathbb{Q}}$ for a poset so that $\mathbb{R}$ is equivalent to $\mathbb{P} * \dot{\mathbb{Q}}$; so complete embedding and iteration are actually the same thing. In particular, if $\mathbb{P}_\alpha$ is an iterated forcing and $\beta < \alpha$, the complete embedding $\mathbb{P}_\beta \to \mathbb{P}_\alpha$ induces an equivalence between $\mathbb{P}_\alpha$ and some $\mathbb{P}_\beta * \dot{\mathbb{R}}_{\beta\alpha}$, where $\dot{\mathbb{R}}_{\beta\alpha}$ is thought of as the "remainder". Then we show that $\mathbb{R}_{\beta\alpha}$ can be viewed as an iteration in $V[G_\beta]$.

8.4 Two-step iteration, yet again

We wish to motivate this using Boolean-valued model, so let us digress for a moment and redo the basic two step iteration using V^B. This part will not be needed for the factor lemma.

Suppose B is a complete Boolean algebra and $V^B \models$"$\dot{C}$ is a complete Boolean algebra". Let $D = \mathrm{Core}(\dot{C})$. Then D is naturally endowed with a Boolean algebra structure. For example, if $d_1, d_2 \in D$ then $V^B \models \exists d(d \in \dot{C}$ and $d = d_1 \vee_{\dot{C}} d_2)$, so by maximal principle and properties of core, there exists a unique $d \in D$ for which $V^B \models d = d_1 \vee_{\dot{C}} d_2$, which we define as $d_1 \vee d_2$. Also it can be checked that $d_1 \leq d_2$ iff $V^B \models d_1 \leq_{\dot{C}} d_2$. This Boolean algebra D is complete, since if $X \subseteq D$ then $V^B \models \exists d(d \in \dot{C}$ and $d = \bigvee^{\dot{C}} \dot{X})$, where $\dot{X} = \{(d, 1) : d \in X\}$.

Let us write $D = B *_b \dot{C}$ to distinguish this from the poset iteration. We want to show that they are somehow the same. Firstly, there is a complete embedding $i : B \to D$ defined as follows: let $\dot{0}, \dot{1} \in D$ be the minimal and maximal element of D respectively; for $b \in B$, let u_b be a B-name such that $[\![u_b = \dot{1}]\!] = b$ and $[\![u_b = \dot{0}]\!] = b^*$ (using the mixing lemma); observe that $[\![u_b \in \dot{C}]\!] = 1$, so u_b is equivalent to some unique element of D, which is defined as $i(b)$.

Now what is the name for the positive part of $\dot{C}$? It is not $D \setminus \{\dot{0}\}$. It can be shown that $b \Vdash d \neq \dot{0}$ iff b is a reduct of d w.r.t. the complete embedding i. Thus a natural name for the positive part of $\dot{C}$ is $\dot{C}^+ := \{(d, b) \in D \times B^+ : b$ is a reduct of $d\}$. Then the Kunen iteration $B^+ *_K \dot{C}^+$ contains $\{(b, d) \in B \times D : b$ is a reduct of $d\}$ as a dense subset, and is ordered by $(b_1, d_1) \leq (b_2, d_2)$ iff $b_1 \leq b_2$ and $b_1 \Vdash d_1 \leq_{\dot{C}} d_2$. The map from $B^+ * \dot{C}^+$ to D^+ that sends (b, d) to $i(b) \wedge d$ is a dense embedding. This shows the Boolean iteration $D = B *_b \dot{C}$ is equivalent to the poset iteration.

Conversely, if $i : B \to D$ is a complete embedding between complete Boolean algebras, then we shall show that there exists a B-name $\dot{C}$ such that $V^B \models$"$\dot{C}$ is a complete Boolean algebra" and $D \simeq B *_b \dot{C}$. If G is (V, B)-generic, then $i(G)$ generates a filter on D in $V[G]$.

Lemma 8.6. *The quotient Boolean algebra $D/i(G)$ defined in $V[G]$ is complete. Equivalently, in the preorder $\leq_G$ defined on D by $d_1 \leq d_2$ iff $d_1 \wedge i(b) \leq d_2 \wedge i(b)$ for some $b \in G$, any $X \subseteq D$ has a supremum.*

Proof. This is Jech [9, Exercise 16.4]; we write down full details mainly for practice. As a warm up let's show if $X \in M$ then it has a supremum. We claim that $e = \bigvee^D X$ is the supremum. If f is an upper bound of X, then for any $d \in X$ there exists $b \in G$ such that $d \wedge f^* \wedge i(b) = 0$. Recall the reduction map $\pi : D \to B$ defined by $\pi(d) = \bigwedge\{b \in B : i(b) \geq d\}$; it has the property that $\pi(d)$ is the smallest b s.t. $i(b) \geq d$; thus $\pi(\bigvee_i d_i) = \bigvee_i \pi(d_i)$. Since $d \wedge f^* \perp i(b) = 0$, by definition of π we have $\pi(d \wedge f^*) \perp b$, namely $\pi(d \wedge f^*)^* \in G$. By V-completeness of G, $\bigwedge_{d\in X} \pi(d \wedge f^*)^* \in G$, so $\pi(e \wedge f^*)^* \in G$, $\pi(e \wedge f^*) \perp b$ for some $b \in G$, and $i\pi(e \wedge f^*) \perp i(b) = 0$. Since $i\pi(e \wedge f^*) \geq e \wedge f^*$ we have $e \wedge f^* \perp i(b) = 0$, so $e \leq_G f$.

In general, if $[\![\dot{X} \subseteq D]\!] = 1$, in V let $e = \bigvee\{d \wedge i([\![d \in \dot{X}]\!]) : d \in D\}$. Suppose f is an upper bound of $\dot{X}_G$. For every $d \in D$, if $[\![d \in \dot{X}]\!] \in G$ then $d \leq_G f$, which by the same argument as above implies $\pi(d \wedge f^*)^* \in G$. Thus for arbitrary $d \in D$, $[\![d \in \dot{X}]\!]^* \vee \pi(d \wedge f^*)^*$. By V-completeness we have $G \ni \bigwedge_{d\in D}[[\![d \in \dot{X}]\!]^* \vee \pi(d \wedge f^*)^*] = \left[\bigvee_{d\in D}[\![d \in \dot{X}]\!] \wedge \pi(d \wedge f^*)\right]^*$, so there exists $b \in G$ s.t. $b \perp \bigvee_{d\in D}[\![d \in \dot{X}]\!] \wedge \pi(d \wedge f^*)$, and therefore $i(b) \perp \bigvee_{d\in D} i([\![d \in \dot{X}]\!]) \wedge d \wedge f^*$, or $i(b) \perp e \wedge f^*$, recalling that i is a complete embedding, and also $i\pi(d) \geq d$. It follows that $e \leq_G f$, thus finishing the proof that $D/i(G)$ is a complete Boolean algebra. □

Now we actually need a B-name for the Boolean algebra $D/i(G)$, but it's a bit tedious to write down directly so we omit it. Instead note that the poset $(\{d \in D^+ : d \not\leq i(b), \forall b \in G\}, \leq)$ densely embeds into $((D/i(G))^+, \leq_G)$, and it has a simple name $\dot{\mathbb{R}} = \{(\check{d}, \pi(d)) : d \in D\}$. Then the Kunen iteration $B^+ *_K \dot{\mathbb{R}}$ is equivalent to D^+, and thus $B *_b \dot{C} \simeq D$. Indeed, $B^+ *_K \dot{\mathbb{R}}$ contains the dense subset $\{(b, \check{d}) : b \leq \pi(d)\}$, which is isomorphic to $\{(b, d) \in B^+ \times D^+ : b \leq \pi(d)\}$ viewed as a subposet of $B^+ \times D^+$. The map $(b, d) \mapsto i(b) \wedge d$ is a dense embedding into D; compatibility is preserved because if $(b_1, d_1) \perp (b_2, d_2)$ then $b_1 \wedge b_2 \perp \pi(d_1 \wedge d_2)$, so $i(b_1) \wedge i(b_2) \wedge d_1 \wedge d_2 = 0$.

Here is how to generalize the above to posets. Suppose $i : \mathbb{P} \to \mathbb{R}$ is a complete embedding. Let $\dot{\mathbb{Q}} = \{(\check{r}, p) : r \in \mathbb{R} \wedge p \in \mathbb{P} \wedge p \text{ is a reduct of } r\}$, ordered (in the extension by $\mathbb{P}$) as subposet of $\mathbb{R}$, then both the iteration $\mathbb{P} * \dot{\mathbb{Q}}$ and $\mathbb{R}$ densely embed into the Boolean completion of $\mathbb{R}$, and are therefore equivalent for forcing. See also Kunen [12, Lemma V.4.44].

8.5 Factor lemma

Back to general iteration. If $\mathbb{P}_\alpha$ is an α-stage iteration and $\beta < \alpha$, then there is a complete embedding $i : \mathbb{P}_\beta \to \mathbb{P}_\alpha$ defined by filling in $\dot{1}_{\mathbb{Q}_\xi}$'s. Applying the above definition literally may not be the best option, since although $q \in \mathbb{P}_\beta$ is a reduct of $p \in \mathbb{P}_\alpha$ if $q \leq_\beta p{\restriction}\beta$, the converse may not hold if $\mathbb{P}_\beta$ is not separative. So we simply define the $\mathbb{P}_\beta$-name $\dot{\mathbb{R}}_{\beta\alpha}$ by $\dot{\mathbb{R}}_{\beta\alpha} = \{(\check{p}, p{\restriction}\beta) : p \in \mathbb{P}_\alpha\}$. Then $\mathbb{P}_\beta * \dot{\mathbb{R}}_{\beta\alpha}$ is forcing equivalent to $\mathbb{P}_\alpha$. Indeed, the Kunen iteration $\mathbb{P}_\beta *_K \dot{\mathbb{R}}_{\beta\alpha}$ has $\{(q, \check{p}) : q \leq_\beta p{\restriction}\beta\}$ as a dense subset, which in turn has a dense subset $\{(q, \check{p}) : q = p{\restriction}\beta\}$ that is isomorphic to $\mathbb{P}_\alpha$; recall that if $q \leq_\beta p{\restriction}\beta$ then $q^\frown p{\restriction}[\beta, \alpha) \in \mathbb{P}_\alpha$. So $\mathbb{P}_\alpha$ densely embeds into $\mathbb{P}_\beta * \dot{\mathbb{R}}_{\beta\alpha}$.

Recall our convention that G is $\mathbb{P}_\alpha$-generic and G_β is the induced $\mathbb{P}_\beta$-generic filter. It is clear that $\mathbb{R}_{\beta\alpha} := (\dot{\mathbb{R}}_{\beta\alpha})_{G_\beta} = \{p \in \mathbb{P}_\alpha : p\restriction\beta \in G_\beta\}$. Now we want to show that in $V[G_\beta]$, $\mathbb{R}_{\beta\alpha}$ looks like an iteration $(\mathbb{P}^{(\beta)}_\xi : \xi \leq \alpha - \beta)$ built from some $(\dot{\mathbb{Q}}^{(\beta)}_\xi : \xi < \alpha - \beta)$. Informally, $\mathbb{P}^{(\beta)}_\xi = \mathbb{R}_{\beta,\beta+\xi}$ and $\dot{\mathbb{Q}}^{(\beta)}_\xi = \dot{\mathbb{Q}}_{\beta+\xi}$; the problem is that $\dot{\mathbb{Q}}_{\beta+\xi}$ is a $\mathbb{P}_{\beta+\xi}$-name, but $\dot{\mathbb{Q}}^{(\beta)}_\xi$ is supposed to be a $\mathbb{P}^{(\beta)}_\xi$-name for a poset.

We need a map in the intermediate model $V[G_\beta]$ that "partially interprets a name". In general, given a two step iteration $\mathbb{P} * \dot{\mathbb{Q}}$ and a $(V, \mathbb{P})$-generic G, there is a map in $V[G]$ that transforms a $\mathbb{P} * \dot{\mathbb{Q}}$-name σ to a $\dot{\mathbb{Q}}_G$-name σ_G, such that for any $\dot{\mathbb{Q}}_G$-generic H, $(\sigma_G)_H = \sigma_K$ where $K = G * H = \{(p, \dot{q}) : p \in G \wedge \dot{q}_G \in H\}$. Inductively let $\sigma_G = \{(\sigma_G, \dot{q}_G) : \exists p \in G\ (\sigma, (p, \dot{q})) \in \sigma\}$. Hopefully it will be clear from context whether σ_G means the interpretation of a $\mathbb{P}$-name or the partial interpretation of a $\mathbb{P} * \dot{\mathbb{Q}}$-name. Recall that $\dot{q} \in \mathrm{Core}(\dot{\mathbb{Q}})$ in our definition of iteration, so $\dot{q}_G \in \dot{\mathbb{Q}}_G =: \mathbb{Q}$, and inductively $\sigma_G \in V[G]^{\mathbb{Q}}$.

Lemma 8.7. (i) *$(\sigma_G)_H = \sigma_K$ for any $(V, \mathbb{P})$-generic G and $(V[G], \mathbb{Q})$-generic H, where $K = G * H = \{(p, \dot{q}) : p \in G \wedge \dot{q}_G \in H\}$.*

(ii) *If $p \in G$ and $(p, \dot{q}) \Vdash \varphi(\sigma)$ in V, then $\dot{q}_G \Vdash \varphi(\sigma_G)$ in $V[G]$. In particular, if $(1_{\mathbb{P}}, \dot{1}_{\mathbb{Q}}) \Vdash \varphi(\sigma)$ in V then $1_{\mathbb{Q}} \Vdash \varphi(\sigma_G)$ in $V[G]$.*

(iii) *If $\dot{q}_G \Vdash \varphi(\sigma_G)$ in $V[G]$, then there is $p \in G$ such that $(p, \dot{q}) \Vdash_{\mathbb{P}*\dot{\mathbb{Q}}} \varphi(\sigma)$.*

(iv) *The map $V^{\mathbb{P}*\dot{\mathbb{Q}}} \to V[G]^{\mathbb{Q}}$, $\sigma \mapsto \sigma_G$ is surjective.*

Proof. (i) By induction on names,

$$\begin{aligned}(\sigma_G)_H &= \{(\sigma_G)_H : \exists \dot{q}_G \in H\ (\sigma_G, \dot{q}_G) \in \sigma_G\} \\ &= \{(\sigma_G)_H : \exists \dot{q}_G \in H \exists p \in G\ (\sigma, (p, \dot{q})) \in \sigma\} \\ &= \{\sigma_K : \exists (p, \dot{q}) \in K\ (\sigma, (p, \dot{q})) \in \sigma\} \\ &= \sigma_K\end{aligned}$$

(ii) $H \ni \dot{q}_G \to K \ni (p, \dot{q}) \to V[K] \models \varphi(\sigma_K) \to V[G][H] \models \varphi((\sigma_G)_H)$, which is the same as saying $\dot{q}_G \Vdash \varphi(\sigma_G)$.

(iii) Since the statement $\dot{q}_G \Vdash \varphi(\sigma_G)$ is true in $V[G]$, it is forced by some $p \in G$. For any $K \ni (p, \dot{q})$ we have $p \in G$ and $\dot{q}_G \in H$. The former tells us $\dot{q}_G \Vdash_{\mathbb{Q}} \varphi(\sigma_G)$ in $V[G]$; then by the latter we have $V[G][H] \models \varphi((\sigma_G)_H)$, namely $V[K] \models \varphi(\sigma_K)$.

(iv) First we devise $\mathbb{P}$-names for names in $V[G]^{\mathbb{Q}}$. Inductively define $X_\alpha \subseteq V^{\mathbb{P}}$ such that $\{\sigma_G : \sigma \in X_\alpha\} = V[G]^{\mathbb{Q}}_\alpha$, as follows. $X_0 = \varnothing$ and $X_\alpha = \bigcup_{\beta<\alpha} X_\beta$ if α is a limit. An element in $V[G]^{\mathbb{Q}}_{\alpha+1}$ is a subset of $V[G]^{\mathbb{Q}}_\alpha \times \mathbb{Q}$, so it is equal to some π_G where π consists of certain pairs $(\mathrm{op}(\sigma, \dot{q}), p)$ where $\sigma \in X_\alpha$, $p \in \mathbb{P}$ and $\dot{q} \in \mathrm{Core}(\dot{\mathbb{Q}})$. Let $X_{\alpha+1}$ be the set of all such π. Lastly let $X = \bigcup_{\alpha\in\mathrm{Ord}} X_\alpha$. Now we define a map that inductively sends $\pi \in X_\alpha$ to $\bar{\pi} \in V^{\mathbb{P}*\dot{\mathbb{Q}}}_\alpha$, such that $\pi_G = \bar{\pi}_G$; this would prove the surjectivity of the map. Simply define $\bar{\pi}$ by replacing all $(\mathrm{op}(\sigma, \dot{q}), p) \in \pi$ with $(\bar{\sigma}, (p, \dot{q}))$. □

Theorem 8.8 (Factor lemma for general iteration)**.** *In $V[G_\beta]$, $\mathbb{R}_{\beta\alpha}$ densely embeds into an iteration $(\mathbb{P}^{(\beta)}_\xi : \xi \leq \alpha - \beta)$ built from some $(\dot{\mathbb{Q}}^{(\beta)}_\xi : \xi < \alpha - \beta)$.*

Proof. First fix some notations. Local to this proof we use $\mathbb{P} \Vdash \varphi$ to mean $\Vdash_{\mathbb{P}} \varphi$. For any $0 \le \xi \le \alpha - \beta$, there is a map that changes a $\mathbb{P}_{\beta+\xi}$-name to a $\mathbb{P}_\beta * \dot{\mathbb{R}}_{\beta,\beta+\xi}$-name, since the former densely embeds into the latter; then in $V[G_\beta]$ there is the above procedure that transforms a $\mathbb{P}_\beta * \dot{\mathbb{R}}_{\beta,\beta+\xi}$-name to $\mathbb{R}_{\beta,\beta+\xi}$-name. Denote the composition as t_ξ, which transforms a $\mathbb{P}_{\beta+\xi}$ name to a $\mathbb{R}_{\beta,\beta+\xi}$-name. So $t_\xi(\dot{\mathbb{Q}}_{\beta+\xi})$ is a $\mathbb{R}_{\beta,\beta+\xi}$-name for a poset. Working in $V[G_\beta]$, we inductively do the following for $\xi \le \alpha - \beta$: define $\mathbb{P}^{(\beta)}_\xi$, as well as a *surjective* dense embedding $i_\xi : \mathbb{R}_{\beta,\beta+\xi} \to \mathbb{P}^{(\beta)}_\xi$. Then the name translation map induced by i_ξ changes a $\mathbb{R}_{\beta,\beta+\xi}$-name to a $\mathbb{P}^{(\beta)}_\xi$-name; if $\xi < \alpha - \beta$, we let $\dot{\mathbb{Q}}^{(\beta)}_\xi = i_\xi(t_\xi(\dot{\mathbb{Q}}_{\beta+\xi}))$ and proceed to the next induction step.

Zero stage: $\mathbb{P}^{(\beta)}_0 = \{\varnothing\}$ must hold by our definition of iteration. $\mathbb{R}_{\beta,\beta}$ is by definition G_β, which when viewed as a poset is directed, so the constant map i_0 from $\mathbb{R}_{\beta,\beta}$ to $\mathbb{P}^{(\beta)}_0$ is a surjective dense embedding. As indicated above, we use i_0 to change the $\mathbb{R}_{\beta,\beta}$-name $i_0(\dot{\mathbb{Q}}_\beta)$ to a $\mathbb{P}^{(\beta)}_0$-name for a poset, denoted $\dot{\mathbb{Q}}^{(\beta)}_0$. It is essentially just $(\dot{\mathbb{Q}}_\beta)_{G_\beta}$.

Successor stage: At a successor $\xi+1$, let $\mathbb{P}^{(\beta)}_{\xi+1}$ be defined as in the definition of iteration forcing, namely it consists of all $\tilde{p}^\frown\dot{q}$ where $\tilde{p} \in \mathbb{P}^{(\beta)}_\xi$ and $\dot{q} \in \mathrm{Core}(\dot{\mathbb{Q}}^{(\beta)}_\xi)$. Define $i_{\xi+1} : \mathbb{R}_{\beta,\beta+\xi+1} \to \mathbb{P}^{(\beta)}_{\xi+1}$ by the procedure below. If $p \in \mathbb{R}_{\beta,\beta+\xi+1}$ then $p\restriction(\beta+\xi) \in \mathbb{R}_{\beta,\beta+\xi}$ and $p(\beta+\xi) \in \mathrm{Core}(\dot{\mathbb{Q}}_{\beta+\xi})$. Let $i_{\xi+1}(p) = i_\xi(p\restriction(\beta+\xi))^\frown\dot{q}$, where $\dot{q} \in \mathrm{Core}(\dot{\mathbb{Q}}^{(\beta)}_\xi)$ is determined as follows. Since $\mathbb{P}_{\beta+\xi} \Vdash p(\beta+\xi) \in \dot{\mathbb{Q}}_{\beta+\xi}$, we have

(a) $\mathbb{R}_{\beta,\beta+\xi} \Vdash t_\xi(p(\beta+\xi)) \in t_\xi(\dot{\mathbb{Q}}_{\beta+\xi})$

and therefore

(b) $\mathbb{P}^{(\beta)}_\xi \Vdash i_\xi(t_\xi(p(\beta+\xi))) \in \dot{\mathbb{Q}}^{(\beta)}_\xi$

We let $\dot{q} \in \mathrm{Core}(\dot{\mathbb{Q}}^{(\beta)}_\xi)$ be the unique element such that

(c) $\mathbb{P}^{(\beta)}_\xi \Vdash i_\xi(t_\xi(p(\beta+\xi))) = \dot{q}$

and then $\dot{\mathbb{Q}}^{(\beta)}_{\xi+1}$ is defined using $i_{\xi+1}$ as before. To maintain the induction hypothesis we need to check $i_{\xi+1}$ is a surjective dense embedding. Surjectivity of $i_{\xi+1}$ boils down to the surjectivity of $p(\beta+\xi) \mapsto \dot{q}$. Suppose $\dot{q} \in \mathrm{Core}(\dot{\mathbb{Q}}^{(\beta)}_\xi)$, so in particular

(1) $\mathbb{P}^{(\beta)}_\xi \Vdash \dot{q} \in \dot{\mathbb{Q}}^{(\beta)}_\xi$

Using the induction hypothesis that i_ξ is a dense embedding, there is a $\mathbb{R}_{\beta,\beta+\xi}$-name $\dot{x}$ such that

(2) $\mathbb{P}^{(\beta)}_\xi \Vdash i_\xi(\dot{x}) = \dot{q}$

which together with (1) implies

(3) $\mathbb{R}_{\beta,\beta+\xi} \Vdash \dot{x} \in t_\xi(\dot{\mathbb{Q}}_{\beta+\xi})$

By "essential surjectivity" of t_ξ there exists a $\mathbb{P}_{\beta+\xi}$-name σ such that

(4) $\mathbb{R}_{\beta,\beta+\xi} \Vdash t_\xi(\sigma) = \dot{x}$

To show surjectivity, we need to find a suitable $p \in \mathbb{R}_{\beta,\beta+\xi+1}$ s.t. $p(\beta+\xi) = \sigma$. But that requires $\sigma \in \mathrm{Core}(\dot{\mathbb{Q}}_{\beta+\xi})$, which may not be true. To fix this, we use Lemma 6.12 to get σ such that

(5) $\mathbb{P}_{\beta+\xi} \Vdash \sigma \in \dot{\mathbb{Q}}_{\beta+\xi}$ and $\mathbb{P}_{\beta+\xi} \Vdash \sigma = \sigma \leftrightarrow \sigma \in \dot{\mathbb{Q}}_{\beta+\xi}$

By the first clause we may assume $\sigma \in \mathrm{Core}(\dot{\mathbb{Q}}_{\beta+\xi})$. The second one tells us

(6) $\mathbb{R}_{\beta,\beta+\xi} \Vdash t_\xi(\sigma) = t_\xi(\sigma) \leftrightarrow t_\xi(\sigma) \in t_\xi(\dot{\mathbb{Q}}_{\beta+\xi})$

This together with (3) and (4) implies

(7) $\mathbb{R}_{\beta,\beta+\xi} \Vdash t_\xi(\sigma) = \dot{x}$

So $\mathbb{P}^{(\beta)}_\xi \Vdash i_\xi(t_\xi(\sigma)) = \dot{q}$, which by definition of $i_{\xi+1}$ means $i_{\xi+1}$ is surjective "at ξ". Combining this with the surjectivity of i_ξ we are done.

It is relatively easy to see $i_{\xi+1}$ is order-preserving. Preservation of incompatibility takes a bit more work. Suppose $p_k \in \mathbb{R}_{\beta,\beta+\xi+1},\ k = 1, 2$ and they have some common extension in $\mathbb{P}^{(\beta)}_\xi$. By surjectivity there exists $p \in \mathbb{R}_{\beta,\beta+\xi+1}$ whose image under $i_{\xi+1}$ is a common extension. This means $i_\xi(p\restriction(\beta+\xi)) \le i_\xi(p_k\restriction(\beta+\xi))$, and

$$i_\xi(p\restriction(\beta+\xi)) \Vdash i_\xi(t_\xi(p(\beta+\xi))) \le i_\xi(t_\xi(p_k(\beta+\xi)))$$

for $k = 1, 2$. A dense embedding i has the property that $p_1, \ldots, p_n$ have a common extension iff $i(p_1), \ldots, i(p_n)$ do. Since by induction hypothesis i_ξ is a dense embedding, and $i_\xi(p\restriction(\beta+\xi)), i_\xi(p_1\restriction(\beta+\xi)), i_\xi(p_2\restriction(\beta+\xi))$ have a common extension, we may assume $p\restriction(\beta+\xi) \le p_k\restriction(\beta+\xi)$ by possibly strengthening $p\restriction(\beta+\xi)$. We also have

$$p\restriction(\beta+\xi) \Vdash t_\xi(p(\beta+\xi)) \le t_\xi(p_k(\beta+\xi)) \text{ w.r.t. } \mathbb{R}_{\beta,\beta+\xi} \text{ in } V[G_\beta]$$

By Lemma 8.7, and viewing $\mathbb{P}_{\beta+\xi}$ as $\mathbb{P}_\beta * \mathbb{R}_{\beta,\beta+\xi}$,

$$p'\restriction(\beta+\xi) \Vdash p(\beta+\xi) \le p_k(\beta+\xi) \text{ w.r.t. } \mathbb{P}_{\beta+\xi} \text{ in } V.$$

where p' is obtained from p by strengthening $p\restriction\beta$. Then $p' \in \mathbb{R}_{\beta,\beta+\xi}$ and is a common extension of p_1, p_2, as desired.

Limit stage: Suppose we are at a limit stage γ. It follows from our construction that if $p \in \mathbb{R}_{\beta,\beta+\gamma}$ and $\xi_1 < \xi_2 < \gamma$, then $i_{\xi_1}(p\restriction(\beta+\xi_1))$ is an initial segment of $i_{\xi_2}(p\restriction(\beta+\xi_2))$. We define $i_\gamma(p)$ to be the limit of the sequences $i_\xi(p\restriction(\beta+\xi))$ as $\xi \to \gamma$; in other words $[i_\gamma(p)](\xi) = [i_{\xi+1}(p\restriction(\beta+\xi+1))](\xi)$. Then we let $\mathbb{P}^{(\beta)}_\gamma$ be the image of i_γ, so that i_γ is trivially surjective. As before order-preservation is immediate. Suppose $p_1, p_2 \in \mathbb{P}_{\beta+\gamma}$ and $i_\gamma(p_i),\ i = 1, 2$ have a common extension. As before there exists p s.t. $i_\gamma(p)$ is a common extension, which means

$$\forall\xi < \gamma\ i_\xi(p\restriction(\beta+\xi)) \Vdash i_\xi(t_\xi(p(\beta+\xi))) \le i_\xi(t_\xi(p_i(\beta+\xi))), \text{ so}$$
$$\forall\xi < \gamma\ p\restriction(\beta+\xi) \Vdash t_\xi(p(\beta+\xi)) \le t_\xi(p_i(\beta+\xi))$$

□

Of course, knowing the remainder $\mathbb{R}_{\beta\alpha}$ is an iteration built from some random $\dot{\mathbb{Q}}^{(\beta)}_\xi$ is not enough. From Lemma 8.7 we see that if $\mathbb{P}_{\beta+\xi}$ forces that $\dot{\mathbb{Q}}_{\beta+\xi}$ is the unique poset with some property φ, e.g. the property of being the Cohen poset $\mathrm{Fn}_\kappa(\lambda, 2)$, then $\mathbb{P}^{(\beta)}_\xi$ forces $\dot{\mathbb{Q}}^{(\beta)}_\xi$ to have the same property. This is implicitly used in virtually all factoring arguments.

We would also like $\mathbb{P}^{(\beta)}_{\alpha-\beta}$ to be the same type of iteration (finite support, countable support, Easton support, etc.) as $\mathbb{P}_\alpha$, which unfortunately is not always the case. If $\gamma \le \alpha - \beta$ is a limit and $\mathbb{P}_\alpha$ is an iteration that takes direct limit at $\beta + \gamma$, then (in $V[G_\beta]$) the iteration $\mathbb{P}^{(\beta)}_{\alpha-\beta}$ also takes direct limit at γ. This is not true of inverse limit in general, i.e., it is not true that $\mathbb{P}_\alpha$ taking inverse limit at stage $\beta+\gamma$ implies $\mathbb{P}^{(\beta)}_{\alpha-\beta}$ doing so at stage γ, but it is true in some important cases. The following warm-up example is, similar to the factor lemma, essentially trivial modulo all the name translations.

Lemma 8.9. *Suppose the iteration $\mathbb{P}_\alpha$ takes inverse limit at every limit stage, then so does $\mathbb{P}^{(\beta)}_{\alpha-\beta}$.*

Proof. If $G_{\beta+\xi}$ is $(V,\mathbb{P}_{\beta+\xi})$-generic, let G_β be the induced $(V,\mathbb{P}_\beta)$-generic filter, and $G^{(\beta)}_\xi$ be the $(V[G_\beta],\mathbb{P}^{(\beta)}_\xi)$-generic filter that corresponds to the $(V[G_\beta],\mathbb{R}_{\beta,\beta+\xi})$-generic filter via the dense embedding i_ξ; it can then be checked that $(\dot{\mathbb{Q}}_{\beta+\xi})_{G_{\beta+\xi}}=(\dot{\mathbb{Q}}^{(\beta)}_\xi)_{G^{(\beta)}_\xi}$.

Back in V, let $\dot{\mathbb{P}}^{(\beta)}_{\alpha-\beta}\in V^{\mathbb{P}_\beta}$ be the name for the iteration $\mathbb{P}^{(\beta)}_{\alpha-\beta}$. Suppose $\dot f\in V^{\mathbb{P}_\beta}$ is such that $\Vdash_{\mathbb{P}_\beta}$ "$\dot f$ is a sequence of length $\alpha-\beta$ such that for any $\xi<\alpha-\beta$, $\dot f\restriction\xi\in\dot{\mathbb{P}}^{(\beta)}_\xi$"; in particular, it is forced that the ξ-th element of $\dot f$ is in the core (as constructed in the extension by $\mathbb{P}_\beta$) of $\dot{\mathbb{Q}}^{(\beta)}_\xi$. Our goal is to show that $\Vdash_{\mathbb{P}_\beta}\dot f\in\dot{\mathbb{P}}^{(\beta)}_{\alpha-\beta}$.

Still in V, let $\dot x_\xi$ be the $\mathbb{P}_{\beta+\xi}$-name defined as follows; imagine there is a $(V,\mathbb{P}_{\beta+\xi})$-generic filter $G_{\beta+\xi}$, then $\dot f$ is interpreted as a sequence $f=\dot f_{G_\beta}$ in $V[G_\beta]$, and $f(\xi)\in\mathrm{Core}(\dot{\mathbb{Q}}^{(\beta)}_\xi)$; thus $f(\xi)_{G^{(\beta)}_\xi}\in(\dot{\mathbb{Q}}^{(\beta)}_\xi)_{G^{(\beta)}_\xi}=(\dot{\mathbb{Q}}_{\beta+\xi})_{G_{\beta+\xi}}$; if we consider some $\mathbb{P}_{\beta+\xi}$-name σ for this element, then $\Vdash_{\mathbb{P}_{\beta+\xi}}\sigma\in\dot{\mathbb{Q}}_{\beta+\xi}$, so σ is equivalent to a unique member of $\mathrm{Core}(\dot{\mathbb{Q}}_{\beta+\xi})$, which we define as $\dot x_\xi$. Define a sequence g of length α by $g(\beta+\xi)=\dot x_\xi$ and $g(\eta)=\dot{\mathbb{1}}_{\mathbb{Q}_\eta}$ for $\eta<\beta$. Clearly $g\in\mathbb{P}_\alpha$ since inverse limit is taken everywhere. Then $g\in\mathbb{R}_{\beta\alpha}$, and by construction, for any G_β we have in $V[G_\beta]$ that $h:=i_{\alpha-\beta}(g)$ and f are pointwise equivalent. Pointwise equivalence means $\Vdash_{\mathbb{P}^{(\beta)}_\xi}h(\xi)=f(\xi)$ for every $\xi<\beta-\alpha$; this is because $h(\xi)_{G^{(\beta)}_\xi}=i_\xi(t_\xi(\dot x_\xi))_{G^{(\beta)}_\xi}=(\dot x_\xi)_{G_{\beta+\xi}}=f(\xi)_{G^{(\beta)}_\xi}$. Since we are using cores, in fact $h(\xi)=f(\xi)$, and thus $f=h=i_{\alpha-\beta}(g)\in\mathbb{P}^{(\beta)}_{\alpha-\beta}$. $\square$

The next lemma is more useful. In particular it implies that the remainder of Easton support iteration also has Easton support, which is used in Silver's forcing construction of a measurable κ s.t. $2^\kappa=\kappa^{++}$; following that by a Prikry forcing, we get the failure of Singular Cardinal Hypothesis.

Again, let $\mathbb{P}_\alpha$ be an iteration and $\beta<\alpha$. Let $K_\beta=\{\xi<\alpha-\beta:\mathbb{P}_\alpha$ takes direct limit at $\beta+\xi\}$. Call a set $X\subseteq\alpha-\beta$ *K_β-thin* if $\sup(X\cap\xi)<\xi$ for any $\xi\in K_\beta$; being K_β-thin is absolute.

Lemma 8.10. *(i) Suppose α is a limit, and the iteration $\mathbb{P}_\alpha$ takes either direct or inverse limit at every limit stage; furthermore, every K_β-thin set X in $V[G_\beta]$ is covered by some K_β-thin set Y in V. Then $\mathbb{P}^{(\beta)}_{\alpha-\beta}$ takes inverse limit at $\alpha-\beta$ iff $\mathbb{P}_\alpha$ takes inverse limit at α.*

(ii) Suppose α is a limit, $\mathbb{P}_\alpha$ takes only direct and inverse limits, and also it takes inverse limit at every limit $\gamma>\beta$ such that $\mathrm{cf}(\gamma)\le|\mathbb{P}_\beta|$, then for every $\xi<\alpha-\beta$, $\mathbb{P}^{(\beta)}_{\alpha-\beta}$ takes inverse limit at ξ iff $\mathbb{P}_\alpha$ takes inverse limit at $\beta+\xi$.

Proof. (i) Since $\mathbb{P}_\alpha$ only takes direct and inverse limits, it can be shown by induction that if p is a sequence of length α, $p\restriction\beta\in\mathbb{P}_\beta$ and $p(\beta+\xi)\in\dot{\mathbb{Q}}_{\beta+\xi}$ for every $\xi<\alpha-\beta$, then $p\in\mathbb{P}_\alpha$ iff $\mathrm{supp}(p):=\{\xi<\alpha-\beta:p(\beta+\xi)\ne\dot{\mathbb{1}}_{\mathbb{Q}_{\beta+\xi}}\}$ is K_β-thin.

Suppose $f\in V[G_\beta]$ is a sequence of length $\alpha-\beta$ such that $f\restriction\xi\in\mathbb{P}^{(\beta)}_\xi$ for every $\xi<\alpha-\beta$. We may assume $\dot f\in V^{\mathbb{P}_\beta}$ is forced by $\mathbb{1}_{\mathbb{P}_\beta}$ to have these properties. Then $\mathrm{supp}(f)$ is K_β-thin, because supp$(f) \cap \xi$ is K_β-thin for every $\xi < \alpha - \beta$, since $f \restriction \xi = \iota_\xi(p)$ for some $p \in \mathbb{R}_{\beta,\beta+\xi} \subseteq \mathbb{P}_{\beta+\xi}$. By assumption there exists a K_β-thin set $Y \in V$ s.t. supp$(f) \subseteq Y$; by truth lemma there exists $p_0 \in G_\beta$ s.t. $p_0 \Vdash \text{supp}(\dot{f}) \subseteq Y$. Define g as in the proof of previous lemma, except that $g(\beta + \xi) = \dot{x}_\xi$ only for $\xi \in Y$, and g is 1 everywhere else. By the first paragraph we have $g \in \mathbb{P}_\alpha$; then it can be shown as before that $i_{\alpha-\beta}(g) = f$ in $V[G_\beta]$, so $f \in \mathbb{P}^{(\beta)}_{\alpha-\beta}$.

(ii) It suffices to show that $\mathbb{P}_{\beta+\gamma}$ satisfies the covering assumption in (i) for every $\gamma \leq \alpha - \beta$. Suppose $\Vdash_{\mathbb{P}_\beta} \dot{X}$ is a K_β-thin subset of γ. We claim that $Y := \{\xi < \gamma : \exists p \in \mathbb{P}_\beta \ p \Vdash \xi \in \dot{X}\}$ is K_β-thin, which clearly contains X. This is because $\mathbb{P}_\beta$ is $|\mathbb{P}_\beta|^+$-cc, so if $\zeta < \gamma$ has cofinality greater than $|\mathbb{P}_\beta|$, then the set $\{\xi < \zeta : \exists p \in \mathbb{P}_\beta \ p \Vdash \sup(\dot{X} \cap \zeta) = \xi\}$ has size at most $|\mathbb{P}_\beta|$, so it is bounded below ζ. It follows that Y is bounded below ζ too. □

Factor lemma is not needed in the consistency proof of Martin's Axiom MA, which is one of the first applications of iterated forcing. However, it does seem needed for the consistency proof of Martin's Axiom for Knaster forcings (cf. Kunen [12, Theorem V.4.12]), because at some point one would like to argue that if $\mathbb{P}_\kappa$ is an iteration of Knaster forcings, $\mathbb{Q} \in V[G_\beta]$ and $\mathbb{Q}$ is Knaster in $V[G]$, then it is also Knaster in $V[G_\beta]$. In the case of MA one could simply use the fact that being ccc is downward absolute (if cardinals are preserved), but that is not true of Knaster forcing. Instead we argue that the quotient forcing $\mathbb{R}_{\beta,\alpha}$ is an iteration of Knaster forcing, so $\mathbb{R}_{\beta,\alpha} * \dot{\mathbb{Q}}$ is Knaster, but that's the same as $\mathbb{R}_{\beta,\alpha} \times \mathbb{Q}$, which means $\mathbb{Q}$ is Knaster.

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