MATH 244 Spring 2015
Exam 1 Study Guide

(for Dr. Goonetilleke's sections)

February 18, 2015

Contents

1 Exam 1 Format

The exam will be during the regularly scheduled class on February 26^{th} or 27^{th} (depending on your section).

Format of the exam:

- Eight questions: 4 questions worth 15 points each and another 4 worth 10 points each.
- No calculators
- No formulas will be given except those used in numerical approximations (The three Euler approximation formulas and the Runge-Kutta method).

It is crucial to know integration of basic functions and integral methods from your calculus course. For Example, various substitutions, integration by parts, and integrals of standard functions including those of $\tan, \sec, \csc$ will all be utilized.

2 Topics from 1^{st} Order Differential Equations

1. Know how to solve 1st-order differential equations and initial value problems via:
   - Separation of variable
   - Integrating factor method
   - Exact equations
2. Know how to sketch slope fields by hand
3. Know how to find equilibria and the phase line for autonomous equations.
4. Be able to find the interval of existence of a solution
5. Know how to set up models such as:
   - Population model
   - Interest rate models
   - Newton's law of cooling (do not have to memorize the formula)
6. Linear approximations
   - Be able to do a simple calculation involving Euler's or Runge-Kutta methods. (Formulas will be given).
   - Know how to apply local truncation error formula for the Euler method
   - Know that the local errors in the Euler, Improved Euler, and Runge-Kutta methods are proportional to $h^2, h^3, and h^5$ respectively.

This material comes from chapters 1 and 2. In chapter 1 pay attention to following topics/terms:

- Direction (Slope) Fields, Equilibrium Solutions
- General Solution, Integral curves, Initial Condition, Initial Value Problem (IVP)
- Clasifications: Linear and Nonlinear differential equations

In chapter 2 pay attention to following topics, terms, and theorems:

- First Order Ordinary Differential Equation
- Integrating Factor method
- Separable Equations
- Exact equations, Test for Exactness, Implicit solutions
- Logistic equations, intrinsic growth rate
- Existence and Uniqueness of Solutions
- Autonomous equations, Logistic Growth, Equilibrium Solutions, Stable, asymptotically stable, unstable, and semi-stable equilibrium solutions

3 Topics from 2^{nd} order Differential Equations

1. Know how to solve second-order linear non homogeneous equations with Characteristic equation having:
   - two different real roots;
   - repeated root;
   - complex roots;
2. Know how to use the initial conditions to solve an initial value problem
3. Be able to find the long term behavior of solutions
4. Be able to calculate the Wronskian (including the use of Abel's theorem)

This material comes from chapter 3. Pay attention to following topics/terms:

- Linear, nonlinear, homogeneous, nonhomogeneous second order equations
- Characteristic Equation
- General Solution
- Fundamental Set of Solutions
- Principle of superposition
- Wronskian

- Theorem 3.2.1: Existence and uniqueness of solutions
- Theorem 3.2.2: Principle of Superposition
- Theorem 3.2.3: Wronskian at the initial conditions
- Theorem 3.2.4: Representing general solutions to second order linear homogeneous ODE's
- Theorem 3.2.5: Existence of a fundamental set of solutions
- Theorem 3.2.6: Abel's Theorem
- Complex roots of characteristic equation
- Repeated roots of the characteristic equation

4 Practice Problems

1. Solve $\frac{dy}{dx} + y\cos t = \cos t$
2. Solve the initial value problem (IVP) $\frac{y'}{\cos(x^2)} - 2xe^{-y} = 0$, $y(0) = 1$
3. Solve the IVP $\frac{2ty}{t^2+1} - 2t - (2 - \ln(t^2+1))y' = 0, \quad y(5) = 0$
4. Solve $\frac{dy}{dx} = -\frac{5x}{3y}$, $y \neq 0$, $y(1) = -3$. Obtain your solution as an explicit function of y and find the interval of existence of the solution.
5. Solve $y'' + y = 0, \quad y(\frac{\pi}{3}) = 2, y'(\frac{\pi}{3}) = -4$
6. Solve $16y'' - 40y' + 25y = 0 \quad y(0) = 3 \quad y'(0) = -\frac{9}{4}$
7. A population of insects will grow at a rate that is proportional to their current population. In the absence of any outside factors the population will triple in 10 days.
   (a). Find an expression for the rate of proportionality
   (b). Suppose initially there are 1000 insects, and that on any given day 20 insects are lost due to prey and other causes. Find an expression for the time T_e for the time it takes the population to be extinct.
8. Consider the differential equation $\frac{dy}{dx} = \frac{-xy}{\ln y}$, $y > 0$.
   (a). Find the general solution of the differential equation.
   (b). Find the solution that satises the initial condition $y(0) = e^2$. Express your answer in the form $y = f(x)$.
   (c). Explain why $x = 2$ is not in the domain of the solution found in part (b).
9. Consider the differential equation $\frac{dy}{dx} = k(1-y)^2$, $k > 0$.
   (a). Determine the critical point(s) and sketch a few solution curves including equilibrium solution(s).
   (b). Classify the equilibrium solution(s) as stable, unstable, or semi-stable.
   (c). Solve the differential equation subject to the initial condition $y(0) = y_0$ and confirm your conclusions above.

10. Show that the following differential equation is exact and solve the initial value problem. $(2xy - 9x^2) + (2y + x^2 + 1)\frac{dy}{dx} = 0$, $y(0) = -3$

11. Consider the differential equation $y'' + 5y' + 6y = 0$
(a). Find the general solution. (b). Find the solution that corresponds to the initial values $y(0) = 1$, $y'(0) = 1$.
(c). Describe the behavior of the solution as $t \to \infty$.

12. Without solving the differential equation $t^2y'' - t(t+2)y' + (t+2)y = 0$;
a) Find the Wronskian of two solutions.
b) If $y(1) = 1$ and $y'(1) = 2$, determine the longest interval where a solution would exist.

13. Consider the initial value problem $4y'' + 12y' + 9y = 0$, $y(0) = 1, y'(0) = -4$.
(a). Find the solution.
(b). Change the second initial condition to $y'(0) = \beta$ and find the solution as a function of β. Then find the critical value of β that separates solutions that always remain positive from those that eventually become negative.

14. According to Newton's law of cooling the rate of change of the temperature T of an object with respect to time t is given by

$$\frac{dT}{dt} = -k(T - T_a)$$

where T_a is the ambient (or room) temperature and k is a positive constant.
A pot of liquid is put on the stove to boil. The temperature of the liquid reaches 170 °F and then the pot is taken off the burner and placed on a counter in the kitchen. The temperature of the air in the kitchen is 76 °F.

(a). Obtain an expression for $T(t)$, the temperature of the liquid at time t. *(Your expression will include k)*

(b). After two minutes the temperature of the liquid in the pot is 123 °F. Find an expression for k

15. Suppose you are trying to solve $y' = \sqrt{t+y}$, $y(0) = 1$ numerically.
a). If you use the Euler method what will be the local truncation error? Give your answer in terms of t, step size h and the solution Φ.

b). Suppose you tried the Runge-Kutta method with step size $h = 0.1$ and got an error of 2×10^{-1}. What do you expect the error to be if you change the step size to 0.01?

16. Consider the initial value problem $y' = 2(t-1)$, $y(1) = 0$. Let $\Phi(t)$ be the solution. Use Euler's approximation method (Euler formula) to estimate $\Phi(3)$ using a step size of $h = 0.5$.