Math 244
Summer 2015
Hour Exam 3
7/8/14
Time Limit: 80 Minutes

Name (Print): ____________________

This exam contains 10 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	35	
2	10	
3	20	
4	15	
5	20	
Total:	100	

1. For the ODE

$$(1-x)y'' + y = 0$$

(a) (5 points) Set up the series solution about the point $x_0 = 0$ and find the derivatives.

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

(b) (10 points) Put the series into the ODE and write the left hand side as a single power series.

$$(1-x)y'' + y$$

$$= (1-x)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} n(n-1)a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}x^m - \sum_{m=0}^{\infty}(m+1)m\, a_{m+1}x^m + \sum_{m=0}^{\infty} a_m x^m$$

$$= \sum_{m=0}^{\infty}\left[(m+2)(m+1)a_{m+2} - m(m+1)a_{m+1} + a_m\right]x^m$$

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(c) (5 points) Find the recurrence relation.

$$(m+2)(m+1)a_{m+2} - (m+1)m a_{m+1} + a_m = 0$$

$$a_{m+2} = \frac{m}{m+2} a_{m+1} - \frac{1}{(m+2)(m+1)} a_m$$

(d) (5 points) Solve a_n in terms of a_0 and a_1 for $n = 2, 3, 4, 5$.

$m=0.\quad a_2 = 0 - \frac{1}{2}a_0 = -\frac{1}{2}a_0$

$m=1,\quad a_3 = \frac{1}{3}a_2 - \frac{1}{6}a_1 = -\frac{1}{6}a_1 - \frac{1}{6}a_0$

$m=2,\quad a_4 = \frac{2}{4}a_3 - \frac{1}{12}a_2 = -\frac{1}{12}a_1 - \frac{1}{24}a_0$

$m=3,\quad a_5 = \frac{3}{5}a_4 - \frac{1}{20}a_3 = -\frac{1}{20}a_1 - \frac{1}{40}a_0 + \frac{1}{120}a_1 + \frac{1}{120}a_0$

$= -\frac{1}{24}a_1 - \frac{1}{60}a_0$

(e) (5 points) Find the first four terms of the series solution.

$$y = a_0\left(1 - \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{24}x^4 + \cdots\right) + a_1\left(x - \frac{1}{6}x^3 - \frac{1}{12}x^4 - \frac{1}{24}x^5 + \cdots\right)$$

(f) (5 points) Estimate the lower bound of radius of convergence for the series solution you obtained.

Singularity: 1 distance between $x_0 = 0$ and 1 is 1

Radius of Convergence ≥ 1

2. (10 points) Determine if the following vectors are linearly dependent. If they are linearly dependent, find a linear relation among them.

$$\vec{v_1} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \vec{v_2} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \vec{v_3} = \begin{bmatrix} 4 \\ 5 \\ 3 \end{bmatrix}.$$

$$\left[\begin{array}{ccc|c} 2 & 0 & 4 & 0 \\ 1 & 1 & 5 & 0 \\ 1 & 1 & 3 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right]$$

linearly independent

3. For the linear system

$$\vec{x}' = \begin{bmatrix} 3 & 5 \\ 1 & 7 \end{bmatrix} \vec{x}$$

(a) (10 points) Find the general solution.

$$\begin{vmatrix} 3-\lambda & 5 \\ 1 & 7-\lambda \end{vmatrix} = \lambda^2 - 10\lambda + 16 = 0 \Rightarrow \lambda_1 = 2, \quad \lambda_2 = 8$$

$\lambda_1 = 2$

$$\left[\begin{array}{cc|c} 3-2 & 5 & 0 \\ 1 & 7-2 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 5 & 0 \\ 1 & 5 & 0 \end{array}\right] \Rightarrow \vec{v^{(1)}} = \begin{bmatrix} -5 \\ 1 \end{bmatrix}$$

$\lambda_2 = 8$

$$\left[\begin{array}{cc|c} 3-8 & 5 & 0 \\ 1 & 7-8 & 0 \end{array}\right] = \left[\begin{array}{cc|c} -5 & 5 & 0 \\ 1 & -1 & 0 \end{array}\right] \Rightarrow \vec{v^{(2)}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Gen. sol'n: $\vec{x}(t) = C_1 e^{2t} \begin{bmatrix} -5 \\ 1 \end{bmatrix} + C_2 e^{8t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

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(b) (5 points) With the initial value specified at

$$\vec{x}(0) = \begin{bmatrix} -5 \\ 1 \end{bmatrix}$$

Find the solution.

$$C_1 \begin{bmatrix} -5 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 1 \end{bmatrix}$$

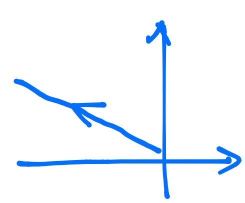
$$\Rightarrow \begin{cases} -5C_1 + C_2 = -5 \\ C_1 + C_2 = 1 \end{cases} \Rightarrow \begin{cases} C_1 = 1 \\ C_2 = 0 \end{cases}$$

Sol'n: $\vec{x}(t) = e^{2t} \begin{bmatrix} -5 \\ 1 \end{bmatrix}$

(c) (5 points) Briefly describe the trajectory of the solution over the phase plane.

A ray passing through $(-5, 1)$ with direction $\begin{bmatrix} -5 \\ 1 \end{bmatrix}$.

in the second quadrant.



4. For the linear system

$$\vec{x}' = \begin{bmatrix} 3 & 2 \\ -4 & 7 \end{bmatrix} \vec{x}$$

(a) (10 points) Find the general solution.

Eigenvalues: $\lambda = 5 \pm 2i$

$\lambda = 5+2i$,

$$\left[\begin{array}{cc|c} 3-5-2i & 2 & 0 \\ -4 & 7-5-2i & 0 \end{array}\right] = \left[\begin{array}{cc|c} -2-2i & 2 & 0 \\ -4 & 2-2i & 0 \end{array}\right]$$

first row $\Rightarrow (-2-2i)k_1 + 2k_2 = 0 \Rightarrow k_2 = (1+i)k_1$

Set $k_1 = 1$, $k_2 = 1+i$

Eigenvector $\begin{bmatrix} 1 \\ 1+i \end{bmatrix}$

Cplx sol'n $e^{(5+2i)t}\begin{bmatrix} 1 \\ 1+i \end{bmatrix} = e^{5t}(\cos 2t + i\sin 2t)\begin{bmatrix} 1 \\ 1+i \end{bmatrix}$

$$= e^{5t}\begin{bmatrix} \cos 2t + i\sin 2t \\ \cos 2t - \sin 2t + i(\cos 2t + \sin 2t) \end{bmatrix}$$

Gen. sol'n: $\vec{x}(t) = C_1 e^{5t}\begin{bmatrix} \cos 2t \\ \cos 2t - \sin 2t \end{bmatrix} + C_2\begin{bmatrix} \sin 2t \\ \cos 2t + \sin 2t \end{bmatrix}$

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(b) (5 points) With the initial value specified at

$$\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Find the solution.

$$C_1 \begin{bmatrix} 1 \\ 1-0 \end{bmatrix} + C_2 \begin{bmatrix} 0 \\ 1+0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} C_1 = 1 \\ C_1 + C_2 = 0 \end{cases} \Rightarrow \begin{cases} C_1 = 1 \\ C_2 = -1 \end{cases}$$

Soln: $\vec{x}(t) = e^{5t} \begin{bmatrix} \cos 2t \\ \cos 2t - \sin 2t \end{bmatrix} - e^{5t} \begin{bmatrix} \sin 2t \\ \cos 2t + \sin 2t \end{bmatrix}$

$$= e^{5t} \begin{bmatrix} \cos 2t - \sin 2t \\ -2 \sin 2t \end{bmatrix}$$

5. For the linear system

$$\vec{x}' = \begin{bmatrix} 3 & -2 \\ 2 & 7 \end{bmatrix} \vec{x}$$

(a) (10 points) Find the general solution.

Eigenvalue: $\lambda_1 = \lambda_2 = 5$

Eigenvector: $\left[\begin{array}{cc|c} 3-5 & -2 & 0 \\ 2 & 7-5 & 0 \end{array}\right] = \left[\begin{array}{cc|c} -2 & -2 & 0 \\ 2 & 2 & 0 \end{array}\right] \Rightarrow \vec{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Gen. e-vector: $\left[\begin{array}{cc|c} 3-5 & -2 & -1 \\ 2 & 7-5 & 1 \end{array}\right] = \left[\begin{array}{cc|c} -2 & -2 & 1 \\ 2 & 2 & 1 \end{array}\right] \Rightarrow \vec{w} = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$

Gen. sol'n: $\vec{x}(t) = C_1 e^{5t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + C_2 e^{5t} \left(t \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} \right)$

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(b) (5 points) With the initial value specified at

$$\vec{x}(0) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Find the solution.

$$C_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + C_2 \left(0 + \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{cases} -C_1 + \frac{1}{2} C_2 = -1 \\ C_1 = 1 \end{cases} \Rightarrow \begin{cases} C_1 = 1 \\ C_2 = 0 \end{cases}$$

Sol'n: $\vec{x}(t) = e^{5t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

(c) (5 points) Briefly describe the trajectory of the solution over the phase plane.

A ray passing through $(-1, 1)$ with direction $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ in the second quadrant