Math 244
Summer 2015
Hour Exam 1
6/8/15
Time Limit: 60 Minutes

Name (Print): ____________________

This exam contains 7 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	10	
2	15	
3	10	
4	10	
5	15	
6	10	
Total:	70	

1. For the following initial value problem

$$y' = \frac{e^x}{y}, y(0) = 1$$

(a) (5 points) Find the explicit solution.

$y\,dy = e^x dx \Rightarrow \frac{1}{2}y^2 = e^x + C \Rightarrow y^2 = 2e^x + C$

$y(0) = 1 \Rightarrow 1^2 = 2e^0 + C \Rightarrow C = -1$

$y^2 = 2e^x - 1 \Rightarrow y = \sqrt{2e^x - 1}$ (cannot take negative b/c $y(0) > 0$)

(b) (5 points) Determine the interval where the solution makes sense.

$2e^x - 1 \geq 0 \Rightarrow e^x \geq \frac{1}{2} \Rightarrow x \geq \ln\frac{1}{2}$

2. For the following ordinary differential equation

$$2y - 9x + \left(\frac{2y+1}{x} + x\right)\frac{dy}{dx} = 0$$

(a) (5 points) Find an integrating factor to make the ODE exact.

$M_y = 2, \quad N_x = -\frac{2y+1}{x^2} + 1 \qquad M_y - N_x = 1 + \frac{2y+1}{x^2} = \frac{x^2+2y+1}{x^2}$

$$\frac{M_y - N_x}{N} = \frac{\frac{x^2+2y+1}{x^2}}{\frac{2y+1+x^2}{x}} = \frac{1}{x} \quad \text{indep. of } y.$$

$$\frac{d\mu}{\mu} = \frac{1}{x}dx \Rightarrow \ln\mu = \ln x \Rightarrow \mu = x$$

(b) (5 points) Find the general solution.

$$2xy - 9x^2 + (2y+1+x^2)y' = 0$$

$M_y = 2x, \ N_x = 2x,$ exact.

$$\psi(x,y) = \int M\,dx = \int (2xy - 9x^2)dx = x^2y - 3x^3 + h(y)$$

$$\frac{\partial\psi}{\partial y} = x^2 + h'(y) = 2y + 1 + x^2 \Rightarrow h(y) = y^2 + y$$

Gen. sol'n: $x^2y - 3x^3 + y^2 + y = C$

(c) (5 points) With the initial value $y(0) = -3$, find the explicit solution. (Hint: Use the quadratic formula)

$x = 0, \ y = -3, \ C = 9 - 3 = 6$

$$y^2 + (1+x^2)y - 3x^3 - 6 = 0$$

$$y = \frac{1}{2}\left(-(1+x^2) \pm \sqrt{(1+x^2)^2 + 12x^3 + 24}\right)$$

$$= \frac{1}{2}\left(-(1+x^2) - \sqrt{x^4 + 12x^3 + 2x^2 + 25}\right)$$

(pick negative to get $y(0) = -3$)

3. For the initial value problem

$$(t+1)y' + ty = -e^{-t}, y(0) = 1$$

(a) (5 points) Find the integrating factor (Hint: Try either u-substitution or partial fraction)

Std. form: $y' + \frac{t}{t+1} y = -\frac{e^{-t}}{t+1}$

Int. factor: $\mu(t) = e^{\int \frac{t}{t+1} dt}$

$$\int \frac{t}{t+1} dt \overset{u=t+1}{=} \int \frac{u-1}{u} du = u - \ln u = (t+1) - \ln(t+1)$$

$$\mu = e^{\int \frac{t}{t+1} dt} = \frac{e^{t+1}}{t+1}$$

(b) (5 points) Solve the IVP. (Hint: the solution is simple)

Gen. sol'n: $y = \dfrac{\int \frac{e^{t+1}}{t+1} \cdot \frac{-e^{-t}}{t+1} dt}{\frac{e^{t+1}}{t+1}}$

Numerator $= e \int \frac{-1}{(t+1)^2} dt = e\left(\frac{1}{1+t} + C\right)$

$$y = \frac{(t+1) \cdot e\left(\frac{1}{1+t} + C\right)}{e^{t+1}} = \frac{1 + C(t+1)}{e^t}$$

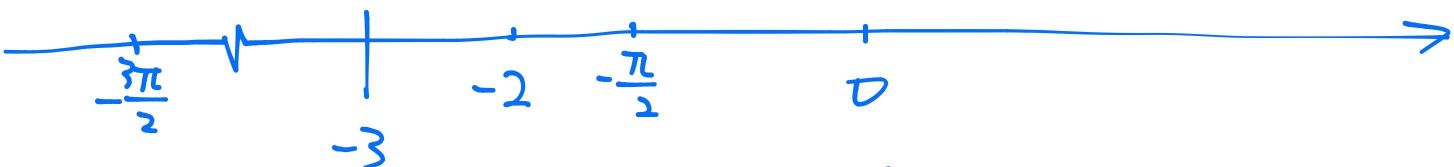
$$y(0) = 1 \Rightarrow 1 + C = 1 \Rightarrow C = 0 \Rightarrow y = \frac{1}{e^t} = e^{-t}$$

4. (a) (5 points) Determine the interval of existence for the solution to the IVP

$$y' - \frac{\ln|t|}{t^3 - 4t} y = \tan t, y(-3) = 2,$$

$\frac{\ln|t|}{t^3-4t}$ continuous $\Rightarrow t \neq 0, \pm 2$

$\tan t$ continuous $\Rightarrow \cos t \neq 0 \Rightarrow t \neq k\pi + \frac{\pi}{2}$, $k = 0, \pm 1, \pm 2, \ldots$



Interval of existence $= \left(-\frac{3\pi}{2}, -2\right)$

(b) (5 points) Determine if the nonlinear IVP is reasonably formulated.

$$y' = \sqrt[3]{y + x}, y(1) = -1$$

$\sqrt[3]{y+x}$ is continuous everywhere

$\frac{\partial}{\partial y}\sqrt[3]{y+x} = \frac{1}{3}(y+x)^{-\frac{2}{3}}$ is continuous when $y + x \neq 0$

Initial value $(x_0, y_0) = (1, -1)$ lies out of the continuous region $\Rightarrow$ not reasonably formulated.

5. For the following initial value problem

$$y' = 2(t-1), y(1) = 0$$

(a) (5 points) Use Euler's method to find the numerical solution $y(3)$ with step-size $h = 0.5$.

$t_0 = 1$, $y_0 = 0$, $t_n = 1 + 0.5n$

$y_{n+1} = y_n + hf(t_n, y_n) = y_n + 0.5 \times 2(t_n - 1) = y_n + 1 + 0.5n - 1$

$= y_n + 0.5n$

$n=0$. $y_1 = y_0 + 0.5 \times 0 = 0$ $t_1 = 1 + 0.5 = 1.5$

$n=1$, $y_2 = y_1 + 0.5 \times 1 = 0.5$, $t_2 = 2$

$n=2$ $y_3 = y_2 + 0.5 \times 2 = 1.5$, $t_3 = 2.5$

$n=3$, $y_4 = y_3 + 0.5 \times 3 = 3$, $t_4 = 3$

$y(3) \approx 3$

(b) (5 points) Find the actual solution to the ODE and get the actual error.

$y = (t-1)^2 + C$, $C = 0$

Actual sol'n $y = (t-1)^2$, $y(3) = 2^2 = 4$

Error $= 1$

(c) (5 points) What will the error roughly be if you use step-size $h = 0.005$? (Hint: Use the proportionality of the error corresponding to the step-size)

$E = kh$

$1 = k \times 0.5 \Rightarrow k = 2$

The error when $h = 0.005$ is $k \times 0.005 = 0.01$

6. For the autonomous differential equation

$$y' = \cos y$$

(a) (5 points) Find all the equilibrium solutions. (Hint: you should get infinitely many)

$\cos y = 0 \Rightarrow y = k\pi + \frac{\pi}{2}, \; k = 0, \pm 1, \pm 2, \cdots$

(b) (5 points) Draw the phase line with y ranges from $-\pi$ to 2π and determine the stability of each equilibrium involved.