Summary up to 3.4.

- Principle of Superposition:

For any second order linear homogeneous ODE

$$y'' + p(t)\, y' + q(t)\, y = 0,$$

if $y_1(t)$, $y_2(t)$ are solutions, then for ~~any~~ any constants $C_1, C_2$, $C_1 y_1(t) + C_2 y_2(t)$ is a solution.

Moreover, if $y_1, y_2$ are linearly independent, i.e., the Wronskian $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ is not a constantly zero function, then ALL solutions are of the form

$$y(t) = C_1 y_1(t) + C_2 y_2(t).$$

So in order to solve second order linear homog. ODE, it suffices to find a couple of independent solution.

Example: Second order linear homog. ODE with constant coefficients

$$ay'' + by' + cy = 0.$$

We guess that $y = e^{rt}$ is a solution. Put it into the ODE: $y' = re^{rt}$, $y'' = r^2e^{rt}$,

$$ay'' + by' + cy = ar^2e^{rt} + bre^{rt} + ce^{rt}.$$
$$= (ar^2 + br + c)e^{rt} = 0.$$

In case that $ar^2 + br + c = 0$, then $e^{rt}$ is a solution.
However, we have the following 3 cases:

(i) The roots $r_1, r_2$ are real, $r_1 \neq r_2$.

In this case, $e^{r_1t}$, $e^{r_2t}$ are two independent functions that solves the ODE, therefore by the principle, the general sol'n is

$$y(t) = C_1e^{r_1t} + C_2e^{r_2t}.$$

(ii) The roots $r_1, r_2$ are complex. ($r_1 \neq r_2$ automatically).

Rigorously speaking, $e^{r_1t}$ and $e^{r_2t}$ are ALSO solution to the ODE. However they are complex functions. For calc 4, we want real functions as solutions.

Important Fact: If a complex function ~~$y=f(x)$~~

$$y = u(x) + i\,v(x)$$

is a solution to the second order linear homogeneous ODE

$$y'' + p(x)\,y' + q(x)\,y = 0.$$

Then the real part $u(x)$ and the imaginary part $v(x)$ are also solutions.

Proof: $u(x)+iv(x)$ satisfies the ODE.

$$\Rightarrow (u(x)+iv(x))'' + p(x)(u(x)+iv(x))' + q(x)(u(x)+iv(x)) = 0.$$

$$\Rightarrow u''(x) + iv''(x) + p(x)u'(x) + ip(x)v'(x) + q(x)u(x) + iq(x)v(x) = 0$$

$$\Rightarrow u''(x) + p(x)u'(x) + q(x)u(x) + i\left(v''(x) + p(x)v'(x) + q(x)v(x)\right) = 0$$

Note that $a+ib=0 \Leftrightarrow a=0,\ b=0$, so we have

$$\begin{cases} u''(x) + p(x)u'(x) + q(x)u(x) = 0. \\ v''(x) + p(x)v'(x) + q(x)v(x) = 0. \end{cases}$$

$\boxtimes$.

Since $a, b, c$ in $ay''+by'+cy=0$ are all real,
we know that the complex roots of $ar^2+br+c=0$
can be written as $r_1 = \alpha+i\beta$, $r_2 = \alpha-i\beta$, $\alpha, \beta$ real numbers.

It suffices to look at only one of the roots.

$$y = e^{r_1 t} = e^{(\alpha+i\beta)t} = e^{\alpha t}\cdot e^{i\beta t}$$

$$= e^{\alpha t}(\cos\beta t + i\sin\beta t)$$

Recall $e^{iu} = \cos u + i\sin u$. Famous Euler formula.

$$= \underbrace{e^{\alpha t}\cos\beta t}_{\text{Real part}} + i\,\underbrace{e^{\alpha t}\sin\beta t}_{\text{Imaginary part}}.$$

By the important fact above,

$$y_1(t) = e^{\alpha t}\cos\beta t, \quad y_2(t) = e^{\alpha t}\sin\beta t$$

are solutions to the ODE. Also the Wronskian.

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1' = \cancel{e^{\alpha t}\cos\beta t\,(\alpha e}$$

$$= e^{\alpha t}\cos\beta t\,(\alpha e^{\alpha t}\sin\beta t + \beta e^{\alpha t}\cos\beta t) - e^{\alpha t}\sin\beta t\,(\alpha e^{\alpha t}\cos\beta t - \beta e^{\alpha t}\sin\beta t)$$

$$= \cancel{\alpha\cdot e^{2\alpha t}\cos\beta t\sin\beta t} + \beta e^{2\alpha t}\cos^2\beta t - \cancel{\alpha e^{2\alpha t}\sin\beta t\cos\beta t} + \beta e^{2\alpha t}\sin^2\beta t$$

$$= \beta e^{2\alpha t}(\cos^2\beta t + \sin^2\beta t) = \beta e^{2\alpha t} \neq 0.$$ ($\beta \neq 0$ by assumption)

Therefore by the principle of superposition,

$$y(t) = C_1 e^{\alpha t}\cos\beta t + C_2 e^{\alpha t}\sin\beta t$$

is the general solution.

(iii). $r_1 = r_2 = r$ ( $r$ is automatically real).

In this case we only managed to find one solution

$$y_1(t) = e^{rt}.$$

By cheating, one can guess that

$$y_2(t) = te^{rt}$$

$y_2' = (1+rt)e^{rt}$
$y_2'' = (2r+r^2t)e^{rt}$.

is also a solution. In fact it's easy to verify that

$$ay_2'' + by_2' + cy_2 = a(2r+r^2t)e^{rt} + b(1+rt)e^{rt} + ce^{rt}.$$

$$= te^{rt}(ar^2+br+c) + e^{rt}(2ar+b)$$

<u>Fact</u>: If $r$ is the repeated root of $ar^2+br+c=0$, then $r = -\frac{b}{2a}$. (Very easily seen from quadratic formula if noticing $b^2-4ac=0$.)

So we got zero and thus $y_2$ is a solution.

What if we don't want to cheat and want to find that $y_2(t) = te^{rt}$ is a solution?

· Reduction of order.

Suppose $y_1(t)$ is a solution for the ODE

$$y'' + p(t)y' + q(t)y = 0.$$

We guess that $y_2(t) = v(t)\,y_1(t)$ is also a solution.

Plug it in: $y_2'' + p(t)y_2' + q(t)y_2 = (v(t)y_1)'' + p(t)(v(t)y_1)' + q(t)vy_1$

$$= v''(t)y_1(t) + 2v'(t)y_1'(t) + v(t)y_1''(t) + p(t)\left(v'(t)y_1(t) + v(t)y_1'(t)\right) + q(t)v(t)y_1(t).$$

$$= v''(t)y_1(t) + \left(2y_1'(t) + p(t)y_1(t)\right)v'(t) + v(t)\underbrace{\left(y_1''(t) + p(t)y_1'(t) + q(t)y_1(t)\right)}_{=0} = 0.$$

$$\Rightarrow v''(t)y_1(t) + v'(t)\left(2y_1'(t) + p(t)y_1(t)\right) = 0.$$

This is indeed a separable ODE concerning $v'(t)$:

$$\frac{v''(t)}{v'(t)} = -\frac{2y_1'(t) + p(t)y_1(t)}{y_1(t)}.$$

RHS is assumed to be known.

Applied to our scenario:

$r_1 = r_2 = r$ root of $ar^2 + br + c = 0$.

$\Rightarrow ax^2 + bx + c = a(x-r)^2 = ax^2 - 2arx + ar^2$.

i.e., $b = -2ar$, $c = ar^2$. So the ODE is

$$ay'' - 2ary' + ar^2 y = 0.$$

Reduction of order works on standard form $\Rightarrow$ Standard form. $y'' - 2ry' + r^2 y = 0$.

So $y_1(t) = e^{rt}$, $p(t) = -2r$. Form the ODE:

$$v''(t) \cdot e^{rt} + v'(t) \cdot (2re^{rt} - 2r \cdot e^{rt}) = 0.$$

$\Rightarrow v''(t) = 0 \Rightarrow v'(t) = 1$ (we don't care about constants: as long as it's nonzero).

$\Rightarrow v(t) = t$

So $y_2(t) = v(t)\, y_1(t) = te^{rt}$ is another solution.

If you care about the constants: $v''(t) = 0 \Rightarrow v'(t) = C_1'$

$\Rightarrow v(t) = C_1' t + C_2'$.

So $y_2(t) = (C_1' t + C_2')e^{rt}$ is another solution.

Gen. sol'n: $y(t) = C_1 e^{rt} + C_2 (C_1' t + C_2') e^{rt}$.

$= \underbrace{(C_1 + C_2 C_2')}_{D_1} e^{rt} + \underbrace{C_2 C_1'}_{D_2} te^{rt}$.

absorbed by

Example: Euler's equation:

$$at^2y'' + bty' + cy = 0 . \quad t > 0 .$$

for generic $t$ just replace $t$ with absolute value of $t$, i.e., $t \to |t|$.

Guess: $y(t) = t^r$ is a sol'n.

If so, then $at^2y'' + bty' + cy$

$$= at^2 \cdot r(r-1)t^{r-2} + bt \cdot rt^{r-1} + ct^r .$$

$$= t^r(ar(r-1) + br + c) = 0 .$$

So if $r$ satisfies the characteristic equation

$$ar(r-1) + br + c = 0$$

Then $y = t^r$ is a solution. As before we have 3 cases:

(1) $r_1 \neq r_2$ real. Then $t^{r_1}, t^{r_2}$ are solutions to the ODE with $W(t^{r_1}, t^{r_2}) = \begin{vmatrix} t^{r_1} & t^{r_2} \\ r_1 t^{r_1-1} & r_2 t^{r_2-1} \end{vmatrix} = (r_2 - r_1)t^{r_1+r_2-1} \neq 0$

So the general solution

$$y(t) = C_1 t^{r_1} + C_2 t^{r_2} .$$

(ii) $r_1 \neq r_2$ complex. Write $r_1 = \alpha + i\beta$, $r_2 = \alpha - i\beta$.

Like before, it suffices to find the real part and imaginary part of $t^{r_1} = t^{\alpha+i\beta}$.

$$t^{\alpha+i\beta} = t^{\alpha} \cdot t^{i\beta} = t^{\alpha} \cdot e^{i\beta \ln t}$$

Change of base: $a^b = e^{b \ln a}$.

$$= t^{\alpha}\left(\cos(\beta \ln t) + i \sin(\beta \ln t)\right).$$

$$= \underbrace{t^{\alpha}\cos(\beta \ln t)}_{\text{real part}} + i\, \underbrace{t^{\alpha}\sin(\beta \ln t)}_{\text{imagine part}}.$$

So $y_1 = t^{\alpha}\cos(\beta \ln t)$, $y_2 = t^{\alpha}\sin(\beta \ln t)$ are solutions.

$$W(y_1, y_2) = y_1 y_2' - y_1' y_2$$

$$= t^{\alpha}\cos(\beta \ln t) \cdot \left[\cancel{\alpha t^{\alpha-1}\sin(\beta \ln t)} + t^{\alpha} \cdot \cos(\beta \ln t) \cdot \frac{\beta}{t}\right]$$

$$- t^{\alpha}\sin(\beta \ln t)\left[\cancel{\alpha t^{\alpha-1}\cos(\beta \ln t)} + t^{\alpha} \cdot (-\sin(\beta \ln t)) \cdot \frac{\beta}{t}\right]$$

$$= t^{2\alpha-1} \cdot \beta \cos^2(\beta \ln t) + t^{2\alpha-1} \cdot \beta \sin^2(\beta \ln t)$$

$$= t^{2\alpha-1} \cdot \beta \not\equiv 0.$$

General solution:

$$y(t) = C_1 t^{\alpha}\cos(\beta \ln t) + C_2 t^{\alpha}\sin(\beta \ln t)$$

(iii) $r_1 = r_2 = r$. In this case we only got one solution $y_1(t) = t^r$. To get another, we use reduction of order. First, since the roots of

$$a(x-1)x + bx + c = 0 \quad \text{are} \quad r_1 = r_2 = r.$$

i.e. $ax^2 - ax + bx + c = a(x-r)^2 = ax^2 - 2arx + ar^2$.

$\Rightarrow \quad -a+b = -2ar, \quad c = ar^2.$

i.e. $b = a - 2ar$.

Standard form
$$y'' + \frac{b}{at}y' + \frac{c}{at^2}y$$
$$= y'' + \frac{1-2r}{t}y' + \frac{r^2}{t^2}y = 0.$$

Form the ODE: $v''(t)\cdot \cancel{y_1}(t) + v'(t)\left(2y_1'(t) + p(t)y_1(t)\right) = 0$.

$$v''(t)\cdot t^r + v'(t)\left(2rt^{r-1} + \frac{1-2r}{t}\cdot t^r\right) = 0.$$

$$\frac{v''(t)}{v'(t)} = -\frac{t^{r-1}}{t^r} = -\frac{1}{t}.$$

Integrate $\Rightarrow \ln v'(t) = -\ln t \Rightarrow v'(t) = \frac{1}{t} \Rightarrow v(t) = \ln t.$

So another solution $y_2(t) = \ln t \cdot y_1(t) = t^r \ln t$.

General solution: $y(t) = C_1 t^r + C_2 t^r \ln t$.

Remarks:

① In Chap. 7, you will use the same idea to get the solutions to SYSTEM of LINEAR ODEs. In case the eigenvalues (corresp. analogue to roots) are complex, it suffices to look into ONE of them just as what we did here.

~~② The idea of reduction of order will be used also in 3.6, namely, Also it can be used in 2.1.~~

② The idea of setting $y_2(t) = v(t)\, y_1(t)$ is called variation of parameter. This is ~~so~~ a VERY IMPORTANT technique in finding solutions. You will use it in 3.6, as well as 2.1 where the formula $\int \mu(t) g(t)\, dt$ is indeed $v(t)$. Try to apply the method in 3.6 to first order linear ODE and you'll see the point.