CONVERGENCE OF DISCRETE CONFORMAL GEOMETRY AND COMPUTATION OF UNIFORMIZATION MAPS

DAVID GU, FENG LUO, AND TIANQI WU

ABSTRACT. The classical uniformization theorem of Poincaré and Koebe states that any simply connected surface with a Riemannian metric is conformally diffeomorphic to the Riemann sphere, or the complex plane or the unit disk. Using the work by Gu-Luo-Sun-Wu [9] on discrete conformal geometry for polyhedral surfaces, we show that the uniformization maps for simply connected Riemann surfaces are computable.

1. INTRODUCTION

The Poincaré-Koebe uniformization theorem for Riemann surfaces is an important result in geometry and has a wide range of applications within and outside mathematics. The theorem states that given any connected surface with a Riemannian metric (S, g) (i.e., a Riemannian surface), there exists a complete constant curvature Riemannian metric g^* on S conformal to the given metric g. In particular, each simply connected Riemannian surface is conformally diffeomorphic to the 2-sphere $\mathbb{S}^2$, the plane $\mathbb{C}$, or the open unit disk $\mathbb{D}$. Computing the conformal diffeomorphism, to be called the *uniformization map*, has been a challenging problem. When the Riemannian surface is isometric to a domain in $\mathbb{C}$, the uniformization map is the Riemann mapping. There are many efficient algorithms and softwares which compute Riemann mapping. See the important works of [21], [20], [19], [2], [8], and others. These mathematical works also prove the convergence of the algorithms. See also the related work [13] and [14]. For non-flat Riemannian disks, efficient algorithms for computing the uniformization maps are mainly developed by computer scientists. These appeared in the important works of Gu et al. [11], Gu-Yau [12], Levy et al. [16], Desbrun et al. [7], Choi et al. [3], [4], [5], Yueh et al. [22] and others. While the convergence of these algorithms are quite evident in practice, mathematical proofs of the convergence of these algorithms were not addressed. Using the works [9] and [10], we prove in the paper,

Theorem 1.1. *There exist algorithms which compute uniformization maps for simply connected Riemannian surfaces.*

The key ingredient in our work is the notion of discrete conformality for polyhedral metrics on surfaces introduced in [9]. It is shown in [9] and [10] that a discrete counterpart of the uniformization theorem holds for compact polyhedral surfaces and the associated discrete uniformization maps are computable. The main result of this paper shows that discrete uniformization maps converge to the uniformization map when the approximation triangulation meshes are suitably chosen. As a consequence of the convergence and the computability of discrete uniformization maps, one sees that Theorem 1.1 follows.

The strategy of proving Theorem 1.1 is as follows. Suppose (S, g) is a simply connected Riemannian surface and f is a uniformization map sending (S, g) to $\mathbb{S}^2$, $\bar{\mathbb{C}}$ or $\mathbb{D}$. If S is the 2-sphere, by removing one point from S, we reduce the case to a non-compact simply connected surface. Since each non-compact simply connected surface is an increasing union of compact simply connected surfaces with non-empty boundary, using the Caratheodory kernel theorem, we can further reduce the computation of the uniformization map f to the case of compact simply connected surfaces with non-empty boundary. By a *Riemannian disk* we mean a Riemannian surface (Σ, g) such that Σ is diffeomorphic to the compact disk $\overline{\mathbb{D}}$. Given a Riemannian disk (Σ, g), by the uniformization theorem, there exists a conformal diffeomorphism h from the interior of Σ onto the open unit disk $\mathbb{D}$. The Caratheodory's extension theorem implies that h extends to a homeomorphism from Σ to the closed disk $\overline{\mathbb{D}}$. The extension map will be called a *conformal homeomorphism* from (Σ, g) to $\overline{\mathbb{D}}$. Theorem 1.1 is a consequence of the following,

Theorem 1.2. *There exists an algorithm which computes uniformization maps for Riemannian disks.*

The paper is organized as follows. In §2, preliminaries on polyhedral surfaces and discrete conformal geometry will be discussed. We prove a general existence theorem for discrete conformal factors in §3. In §4, we prove a result on the convergence of discrete conformal maps. Theorem 1.2 is proved in §5. In §6, we discuss the similar theorem for tori with Riemannian metrics.

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2. Polyhedral metrics and discrete conformal geometry

Suppose S is a compact topological surface which may have non-empty boundary. Let V be a non-empty finite subset of S. The pair (S, V) will be called a *marked surface*. A *triangulated surface* $(S, \mathcal{T})$ is the quotient of a disjoint union $\Delta_1 \sqcup ... \sqcup \Delta_m$ of Euclidean triangles by identifying pairs of edges by homeomorphisms. The quotient surface S admits a natural *triangulation* $\mathcal{T}$ whose triangles are the quotients of Δ_i's. We use $V = V(\mathcal{T})$ and $E = E(\mathcal{T})$ to denote the sets of vertices and oriented edges in $\mathcal{T}$ respectively and call $\mathcal{T}$ a *triangulation* of the marked surface (S, V). If the homeomorphisms used in identifying pairs of edges are isometries, then the quotient metric d on S is a *polyhedral metric*. We call (S, V, d) a *polyhedral surface* and d a PL metric on $(S, \mathcal{T})$ or simply a PL metric on $\mathcal{T}$. The *discrete curvature* K_d of a PL metric d is the function defined on V sending each interior (resp. boundary) vertex to 2π (resp. π) less the sum of all angles at the vertex. It is well known that the Gauss-Bonnet formula holds, i.e., $\sum_{v \in V} K_d(v) = 2\pi\chi(S)$ where $\chi(S)$ is the Euler characteristic of S. A topological triangulation $\mathcal{T}$ of a PL surface (S, V, d) is called *geometric* in d, if each triangle in $\mathcal{T}$ (in d metric) is isometric to a Euclidean triangle. A geometric triangulation $\mathcal{T}$ of (S, V, d) is called *Delaunay* if for each interior edge (i.e., an edge not in ∂S), the sum of the two inner angles facing the edge is at most π. It is well known that any compact polyhedral surface (S, V, d) admits some Delaunay triangulation. Two different Delaunay triangulations of the same polyhedral surface (S, V, d) are related by a sequence of diagonal switches such that each triangulation in the sequence is still Delaunay.

A PL metric d on a triangulated surface $(S, \mathcal{T})$ can represented by the *edge length function* $l_d : E(\mathcal{T}) \to \mathbb{R}_{>0}$ sending an edge e to its length. In this case, we also denote (S, V, d) by

$(S, \mathcal{T}, l)$ or $(\mathcal{T}, l)$ and call l a PL metric on $\mathcal{T}$. For any $u : V(\mathcal{T}) \to \mathbb{R}$, we define a new function $u * l_d : E(\mathcal{T}) \to \mathbb{R}_{>0}$ by

$$u * l_d(e) = e^{u(v)+u(v')} l_d(e) \tag{2.1}$$

where v, v' are the end points of the edge e. This is called a *vertex scaling* of l_d (or d) in the triangulation $\mathcal{T}$. Two PL metrics d, d' on $\mathcal{T}$ are related by a *vertex scaling* (with respect to triangulation $\mathcal{T}$) if $l_{d'} = u * l_d$ for some u. The function u is called a *discrete conformal factor*. We will prove in Proposition 5.2 that vertex scaling is a good approximation to the conformal change $e^{\mu} g$ of Riemannian metric g.

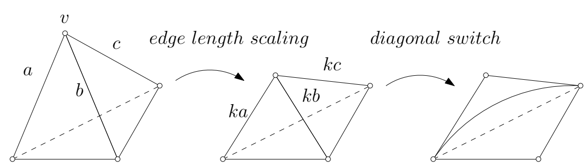

FIGURE 1. Discrete conformal change of PL metrics. All triangulations are Delaunay

Definition 2.1 (Discrete conformal equivalence of PL metrics, [9]). *Two PL metrics d and d' on a closed marked surface (S, V) are discrete conformal if there is a sequence of PL metrics $d_1 = d, d_2, ..., d_n = d'$ and a sequence of triangulations $\mathcal{T}_1$, $\mathcal{T}_2$, ..., $\mathcal{T}_n$ of (S, V) such that*

(a) each $\mathcal{T}_i$ is Delaunay in d_i,

(b) if $\mathcal{T}_i \neq \mathcal{T}_{i+1}$, then there is an isometry h_i from (S, V, d_i) to (S, V, d_{i+1}) so that h_i is homotopic to the identity map on (S, V), and

(c) if $\mathcal{T}_i = \mathcal{T}_{i+1}$, then the edge length functions $l_{d_{i+1}}$ and l_{d_i} are related by a vertex scaling, i.e., $l_{d_{i+1}} = u_i * l_{d_i}$ on $\mathcal{T}_i$ for some $u_i \in \mathbb{R}^V$.

Two PL metrics d, d' on a compact marked surface (Σ, V) with non-empty boundary are discrete conformal if their metric doubles are discrete conformal.

Theorem 2.2. *([9]) Given any PL metric d on a connected closed marked surface (S, V) and any $\tilde{K} : V \to (-\infty, 2\pi)$ so that $\sum_{v \in V} \tilde{K}(v) = 2\pi\chi(S)$, there exists a PL metric $\tilde{d}$, unique up to scaling and isometries homotopic to the identity on (S, V), such that*

(a) $\tilde{d}$ is discrete conformal to d, and

(b) the discrete curvature $K_{\tilde{d}} = \tilde{K}$.

Furthermore, the PL metric $\tilde{d}$ can be found algorithmically using a finite dimensional variational principle.

We will be working mainly on compact polyhedral surfaces (S, V, d) with non-empty boundary which admit acute geometric triangulations $\mathcal{T}$, i.e, all inner angles are less than $\frac{\pi}{2}$. The metric double of them are Delaunay triangulations (of a closed surface). Furthermore, we will produce a discrete conformal factor $w : V \to \mathbb{R}$ such that $(\mathcal{T}, w * l_d)$ is still acute. This implies that $(S, \mathcal{T}, l)$ and $(S, \mathcal{T}, w * l)$ are discrete conformal.

Definition 2.3. *Suppose $\delta > 0$.*

(a) A δ-triangulation $\mathcal{T}$ of a compact polyhedral surface (S, V, d) is a geometric triangulation such that all inner angles of triangles in $\mathcal{T}$ are in the interval $(\delta, \frac{\pi}{2} - \delta)$.

(b) Suppose $(S, \mathcal{T}, l^*)$ is a triangulated compact polyhedral surface. A sequence of geometric subdivisions $(\mathcal{T}_n, l_n^*)$ of $(\mathcal{T}, l^*)$ is (δ, c)-regular if there exist $\delta > 0$, $c > 0$ and a sequence of positive numbers $q_n \to \infty$ such that $(\mathcal{T}_n, l_n^*)$ are δ-triangulations and

$$l_n^*(e) \in (\frac{1}{cq_n}, \frac{c}{q_n}) \tag{2.2}$$

for all $e \in E(\mathcal{T}_n)$.

The main technical theorem which implies the convergence is the following approximation result.

Theorem 2.4. *Suppose $(S, \mathcal{T}_n, l_n^*)$ is a (δ, c)-regular sequence of geometric subdivisions of a compact triangulated polyhedral surface $(S, \mathcal{T}, l^*)$ satisfying (2.2) and $(S, \mathcal{T}_n, l_n)$ is another sequence of polyhedral metrics such that there exists $c_0 > 0$ for which*

$$|l_n(e) - l_n^*(e)| \leq \frac{c_0}{q_n^3}, \quad \text{for all } e \in E(\mathcal{T}_n). \tag{2.3}$$

Then there exist a constant $c_1 = c_1(l^*, \delta, c_0, c)$ and $w_n \in \mathbb{R}^{V(\mathcal{T}_n)}$ for sufficiently large n such that,

(a). $(\mathcal{T}_n, w_n * l_n)$ are $\frac{\delta}{2}$-triangulations,

(b). $K_{w_n * l_n} = K_{l_n^*}$,

(c). $||w_n||_\infty \leq \frac{c_1(l^*, \delta, c_0, c)}{\sqrt{q_n}}$. In particular,

$$|l_n^*(e) - w_n * l_n(e)| \leq \frac{c_2(l^*, \delta, c_0, c)}{q_n \sqrt{q_n}}, \quad \text{for all } e \in E(\mathcal{T}_n). \tag{2.4}$$

3. A general existence theorem on discrete conformal factors

Suppose $(S, \mathcal{T})$ is a compact connected triangulated surface and $l, l^* : E(\mathcal{T}) \to \mathbb{R}_{>0}$ are two PL metrics on $\mathcal{T}$. The problem we intend to solve is to find a discrete conformal factor $w \in \mathbb{R}^{V(\mathcal{T})}$ such that $w * l$ is a PL metric on $\mathcal{T}$ and

$$K_{w*l} = K_{l^*}. \tag{3.1}$$

The work of Bobenko-Pinkall-Springborn [1] shows that if the solution w exists, then it is unique up to scalar addition (i.e., replacing w by $w + k(1, 1, ..., 1)$). We will derive a system of ordinary differential equations on $w(t) \in \mathbb{R}^{V(\mathcal{T})}$ with $w(0) = 0$ such that $w(1)$ solves (3.1).

3.1. **Preliminaries, notations and comparing discrete curvatures.** We begin with some notations. Given two vertices $i, j \in V(\mathcal{T})$, we use $i \sim j$ to denote that i, j are adjacent in $\mathcal{T}$ and use $[ij] \in E(\mathcal{T})$ to denote the oriented edge from i to j. Here $E = E(\mathcal{T}) = \{[ij] | i \sim j\}$ is the set of all oriented edges in $\mathcal{T}$. Given a PL metric l on $\mathcal{T}$, let $\alpha_{jk}^i = \alpha_{jk}^i(l)$ be the angle at vertex i in the triangle $\Delta ijk \in \mathcal{T}$ in the metric l. If $x : V \to \mathbb{R}$ or $y : E(\mathcal{T}) \to \mathbb{R}$, we use x_i and y_{ij} to denote

$x(i)$ and $y([ij])$ respectively. Define $\mathbb{R}^E_A = \{x \in \mathbb{R}^E | x_{ij} = -x_{ji}\}$ and the divergence operator $div : \mathbb{R}^E_A \to \mathbb{R}^V$ by

$$div(x)_i = \sum_{[ij] \in E} x_{ij}.$$

Lemma 3.1. *Suppose l and l^* are two PL metrics on a triangulated surface $(S, \mathcal{T})$ with discrete curvatures K and K^*. Then there exists $u \in \mathbb{R}^E_A$ such that $div(u) = K - K^*$ and*

$$||u||_\infty \leq 2 \max\{|\alpha^i_{jk}(l) - \alpha^i_{jk}(l^*)| \Delta v_i v_j v_k \in \mathcal{T}\}.$$

Proof. By definition, $K_i - K^*_i = \sum_{\Delta ijk \in \mathcal{T}} (\alpha^i_{jk}(l^*) - \alpha^i_{jk}(l))$. Fix an edge e_τ in each triangle τ. For each triangle $\tau = \Delta ijk$, we associate $u(\tau) \in \mathbb{R}^E_A$ such that $u(\tau)_{ab} = 0$ for all edges $[ab]$ not in the triangle τ and in the triangle Δijk

$$u_{ki}(\tau) + u_{kj}(\tau) = \alpha^k_{ij}(l^*) - \alpha^k_{ij}(l), \qquad u_{ij}(\tau) + u_{ji}(\tau) = 0. \tag{3.2}$$

This is a system of six linear equations in six variables. Since the sum of the constant terms in (3.2) is zero, the equation is solvable and the solution space is 1-dimensional. Set $u(\tau)_{e_\tau} = 0$. Then there is exactly one solution $u(\tau)$ such that $u_{ij}(\tau) \in \{0, \pm(\alpha^a_{bc}(l^*) - \alpha^a_{bc}(l))\}$. Indeed, (3.2) can be written as: $x_{ij} = -x_{ji}, x_{12} + x_{13} = c_1, x_{21} + x_{23} = c_2, x_{31} + x_{32} = c_3$ with $c_1 + c_2 + c_3 = 0$. When $x_{12} = 0$, then the solution is $x_{13} = c_1 = -x_{31}, x_{23} = c_2 = -x_{32}$, and $x_{21} = 0$. For an interior edge $[ij]$, define $u_{ij} = u_{ij}(\tau) + u_{ij}(\tau')$ where τ and τ' are the triangles adjacent $[ij]$ and $u_{ij} = u_{ij}(\tau)$ otherwise. Then by definition we have $div(u) = K - K^*$. Clearly $||u||_\infty \leq 2 \max\{|\alpha^i_{jk}(l) - \alpha^i_{jk}(l^*)| : \Delta v_i v_j v_k \in \mathcal{T}\}$. □

3.2. **Discrete gradient and Hodge decomposition.** The discrete Laplace operator is defined on a graph (V, E) with an edge weight $\eta : E \to \mathbb{R}_{>0}$ such that $\eta_{ij} = \eta_{ji}$. For a triangulated connected PL surface $(S, \mathcal{T}, l)$ such that all angles $\alpha^i_{jk} < \pi/2$, define the associated edge weight $\eta \in \mathbb{R}^E$ as follows. If $[ij] \in E$ is an interior edge, $\eta_{ij} = \cot(\alpha^k_{ij}) + \cot(\alpha^h_{ij})$ and if $[ij] \subset \partial S$ is a boundary edge $\eta_{ij} = \cot(\alpha^k_{ij})$. The gradient operator $\nabla_l : \mathbb{R}^V \to \mathbb{R}^E_A$ is given by $\nabla_l(f)_{ij} = \eta_{ij}(f_i - f_j)$ and the Laplace operator associated to l is the composition $div \circ \nabla_l$. Let inner products (x, y) on $\mathbb{R}^V$ and $< x, y >_\eta$ on $\mathbb{R}^E_A$ be $(x, y) = \sum_{i \in V} x_i y_i$ and $< x, y >_\eta = \sum_{[ij] \in E} \frac{1}{\eta_{ij}} x_{ij} y_{ij}$ respectively.

Lemma 3.2. *If $x \in \mathbb{R}^V, y \in \mathbb{R}^E_A$, then*

$$< \nabla_l x, y >_\eta = 2(x, div(y)).$$

Indeed, the left-hand-side is

$$\sum_{[ij] \in E} \frac{1}{\eta_{ij}} (\nabla_l x)_{ij} y_{ij} = \sum_{[ij] \in E} (x_i - x_j) y_{ij} = 2 \sum_{[ij] \in E} x_i y_{ij}.$$

The Right-hand-side is $2 \sum_{i \in V} x_i div(y)_i = 2 \sum_{i \in V} \sum_{j \sim i} x_i y_{ij} = 2 \sum_{[ij]} x_i y_{ij}$.

As a consequence of Lemma 3.2, we have the orthogonal Hodge decomposition that $\mathbb{R}^E_A = Im(\nabla_l) + kernel(div)$, i.e., each $u \in \mathbb{R}^E_A$ can be written uniquely as

$$u = \nabla_l(\phi) + \psi, \tag{3.3}$$

where $\phi \in \mathbb{R}^V_0 = \{x \in \mathbb{R}^V | \sum_{i \in V} x_i = 0\}$ and $div(\psi) = 0$. Note that ϕ depends linearly on u.

Recall that the *combinatorial diameter* of a triangulation, denoted by $|\mathcal{T}|$, is the minimum integer k such that any two vertices $i, j \in V(\mathcal{T})$ can be joint by an edge path of combinatorial length at most k. If $y \in \mathbb{R}^X$, then its L_∞-norm is $||y||_\infty = \max\{|y_i| i \in X\}$. We use $|E|$ to denote the number of edges in $\mathcal{T}$. The key estimate we need is the following,

Lemma 3.3. *Let $C_0 = \max\{\frac{1}{\eta_{ij}} : [ij] \in E\}$. Then in the Hodge decomposition (3.3), we have*

$$||\phi||_\infty \leq C_0 \sqrt{|\mathcal{T}||E|}||u||_\infty. \tag{3.4}$$

Proof. Indeed, by the orthogonality of the Hodge decomposition $u = \nabla_l(\phi) + \psi$, we have $||\nabla_l(\phi)||_2^2 \leq ||u||_2^2 \leq C_0|E|||u||^2_\infty$. Here the L^2-norm is with respect to the inner product $<,>_\eta$. We estimate $||\phi||_\infty$ as follows. For any two vertices $i, j \in V$, there exists an edge path $i = k_1 \sim k_2 \sim k_3 \sim ... \sim k_r = j$ of length $r \leq |\mathcal{T}|$. Therefore, by the Cauchy-Schwarz inequality,

$$\begin{aligned}
|\phi_i - \phi_j|^2 &= (\sum_{s=1}^{r-1} (\phi_{k_s} - \phi_{k_{s+1}}))^2 \\
&\leq |\mathcal{T}| \sum_{s=1}^{r-1} (\phi_{k_s} - \phi_{k_{s+1}})^2 \\
&\leq C_0 |\mathcal{T}| [\sum_{s=1}^{r-1} \eta_{k_s k_{s+1}} (\phi_{k_s} - \phi_{k_{s+1}})^2] \\
&\leq C_0 |\mathcal{T}| ||\nabla_l(\phi)||_2^2 \\
&\leq C_0^2 |\mathcal{T}||E|||u||^2_\infty.
\end{aligned}$$

Let $||\phi||_\infty = |\phi_j|$ for vertex j. Since $\sum_{i \in V} \phi_i = 0$, there exists i such that $\phi_i \phi_j \leq 0$. Therefore $||\phi||_\infty \leq |\phi_i - \phi_j| \leq C_0 \sqrt{|\mathcal{T}||E|}||u||_\infty$. □

Remark. If all angles $\alpha^i_{jk} \in (\delta, \pi/2 - \delta)$, then we can take C_0 to be $\cot(\delta)$ in Lemma 3.3.

3.3. **A system of ordinary differential equations.** We now construct a system of ordinary differential equation for solving the problem (3.1) that $K_{w*l} = K_{l^*}$. For the given PL metric l on $\mathcal{T}$, assume that there exists $\delta > 0$ such that $\alpha^i_{jk}(l) \in (\delta, \pi/2 - \delta)$ for all angles.

For simplicity, denote the discrete curvatures of l and l^* by $K(0) = K_l$ and $K^* = K_{l^*}$ respectively. Using Lemma 3.1, we can write $K^* - K(0) = div(u)$ for some $u \in \mathbb{R}^E_A$ satisfying the inequality in Lemma 3.1. Consider a smooth family of vectors $w(t) \in \mathbb{R}^V$ such that $w(0) = 0$ and $w(t) * l$ are PL metrics on $\mathcal{T}$ with all angles less than $\pi/2$. For the edge weight function η associate to $w(t) * l$, we obtain a Hodge decomposition

$$u = \nabla_{w(t)*l} \phi(t) + \psi(t),$$

where $div(\psi(t)) = 0$ and $\sum_{i \in V} \phi(t)_i = 0$.

The system of ordinary differential equation is defined to be

$$\frac{dw(t)}{dt} = \phi(t), \quad w(0) = 0. \tag{3.5}$$

Lemma 3.4. *If $w(t)$ solves (3.5), then the associated discrete curvature $K(t) = K_{w(t)*l}$ satisfies*

$$\frac{dK(t)}{dt} = -K(0) + K^*, \quad \textit{and} \quad K(t) = (1-t)K(0) + tK^*. \tag{3.6}$$

Proof. The basic formula (Lemma 4.2 in [10], Theorem 2.1 [17]) for curvature involution under any smooth family $w(t)$ of vertex scaling is the following,

$$\frac{dK(t)}{dt} = div(\nabla_{w(t)*l}\frac{dw(t)}{dt}).$$

Therefore in our case,

$$\begin{aligned}\frac{dK(t)}{dt} &= div(\nabla_{w(t)*l}\phi(t))\\ &= div(u)\\ &= -K(0) + K^*.\end{aligned}$$

As a consequence, $K(t) = (1-t)K(0) + tK^*$. □

Our goal is to show that under some conditions on $\mathcal{T}$ and l, the equation (3.5) has a solution in the interval $[0, 1+\epsilon_0)$ for some $\epsilon_0 > 0$. Then by Lemma 3.4, the required solution $w \in \mathbb{R}^V$ to the equation $K_{w*l} = K_{l^*}$ is given by $w(1)$.

3.4. **Euclidean triangles.** Suppose $x = (x_1, x_2, x_3)$ is the edge length vector of a triangle whose inner angles are a_1, a_2, a_3. For $\delta > 0$, let

$$W_\delta = \{x \in \mathbb{R}^3_{>0} | x_i + x_j > x_k, a_i \in (\delta, \pi/2 - \delta)\}$$

be the space of all Euclidean triangles whose inner angles are in $(\delta, \pi/2-\delta)$. Note that the Law of sines implies $x_i \leq \frac{x_j}{\sin(\delta)}$ for $x \in W_\delta$. In particular, there exists an explicit constant $\lambda = \lambda(\delta) > 1$ such that $x_i + x_j \geq \lambda x_k$ for all $x \in W_\delta$. As a consequence, there is an explicit constant $\epsilon = \epsilon(\delta) > 0$ such that if $y \in \mathbb{R}^3_{>0}$, $x \in W_\delta$ and $|\ln(x_i) - \ln(y_i)| < \epsilon$ for all i, then $y \in W_{\frac{\delta}{2}}$. The scalar multiplication of $\mathbb{R}_{>0}$ on W_δ preserves the inner angles. The quotient space $W_\delta/\mathbb{R}_{>0}$ has a compact closure in $W_{\frac{\delta}{2}}/\mathbb{R}_{>0}$. The compactness of the closure implies immediately the following lemma whose proof will be omitted.

Lemma 3.5. *For any $\delta > 0$, there exist explicit constants $\epsilon = \epsilon(\delta) > 0$ and $c_1 = c_1(\delta)$ such that*
(a) if $x \in W_\delta$ and $y \in \mathbb{R}^3_{>0}$ such that $|\ln(x_i) - \ln(y_i)| \leq \epsilon$ for $i = 1, 2, 3$, then $y \in W_{\delta/2}$;
(b) if $x, y \in W_\delta$, then for all i,

$$|a_i(x) - a_i(y)| \leq c_1(\delta)(\sum_{j=1}^{3} |\ln(x_j) - \ln(y_j)|).$$

As a consequence,

Corollary 3.6. *Suppose $(\mathcal{T}, l)$ is a δ-triangulation and $w \in \mathbb{R}^V$ such that $||w||_\infty < \frac{\epsilon}{2}$, then $w * l$ is a PL metric on $\mathcal{T}$ and $(\mathcal{T}, w * l)$ is a $\frac{\delta}{2}$-triangulation.*

3.5. **An existence theorem for discrete conformal factors.**

Theorem 3.7. *Suppose $(\mathcal{T}, l)$ is a δ-triangulated connected surface, $\epsilon = \epsilon(\delta)$ is the constant in Lemma 3.5 and $C_0 = \cot(\frac{\delta}{2})$. Then the solution to the system of ordinary differential equation (3.5) exists in the time interval $[0, T_0)$ where*

$$T_0 = \frac{\epsilon}{5C_0\sqrt{|\mathcal{T}||E|}\max\{|\alpha^i_{jk}(l) - \alpha^i_{jk}(l^*)| : \Delta ijk \in \mathcal{T}\}}. \tag{3.7}$$

In particular, if $T_0 > 1$, then there exists $w \in \mathbb{R}^V$ such that $K_{w*l} = K_{l^*}$.

Proof. Let t_0 be the maximum time for which the solution $w(t)$ of (3.5) exists on $[0, t_0)$ such that $w(t) * l$ is a PL metric and $||w(t)||_\infty < \epsilon/2$ for all $t \in [0, t_0)$. By Corollary 3.6, $(\mathcal{T}, w(t) * l)$ is a $\frac{\delta}{2}$-triangulation for all $t \in [0, t_0)$. We estimate t_0 as follows. If $t \in [0, t_0)$ and vertex $v \in V$, using Lemma 3.3 and Lemma 3.1, we have

$$\begin{aligned} |w(t)(v)| &\leq \int_0^{t_0} |\phi(s)(v)| ds \\ &\leq t_0 C_0 \sqrt{|\mathcal{T}||E|}||u(0)||_\infty \\ &\leq 2t_0 C_0 \sqrt{|\mathcal{T}||E|}\max\{|\alpha^i_{jk}(l) - \alpha^i_{jk}(l^*)| : \Delta ijk \in \mathcal{T}\}. \end{aligned} \tag{3.8}$$

Now if

$$t_0 \leq \frac{\epsilon}{5C_0\sqrt{|\mathcal{T}||E|}\max\{|\alpha^i_{jk}(l) - \alpha^i_{jk}(l^*)| : \Delta ijk \in \mathcal{T}\}},$$

then $\limsup_{t\to t_0^+} |w(t)(v)| \leq \frac{2\epsilon}{5} < \frac{\epsilon}{2}$. By corollary 3.6, we can further extend the solution $w(t)$ to the interval $[0, t_0 + \epsilon_0)$ for some $\epsilon_0 > 0$. This contradicts the maximality of t_0.

□

4. A PROOF OF THEOREM 2.4

To prove Theorem 2.4, we will verify that the condition $T_0 > 1$ in Theorem 3.7 holds for $(\mathcal{T}_n, l_n)$ for large n. To this end, we will establish several estimates.

4.1. **An estimate of inner angles.** One consequence of Lemma 3.5 is,

Corollary 4.1. *Suppose $x, y \in W_\delta$ such that $|x_i - y_i| \leq \frac{c}{q_n^3}$ and $x_i, y_i \in (\frac{1}{cq_n}, \frac{c}{q_n})$ for $i = 1, 2, 3$. Then there exists an explicit constant $c' = c'(c, \delta)$ such that*

$$|a_i(x) - a_i(y)| \leq \frac{c'}{q_n^2}. \tag{4.1}$$

Indeed, this follows from Lemma 3.5 and the inequality that $|\ln(t) - \ln(s)| \leq \frac{1}{\min\{s,t\}}|s - t|$ where $s, t > 0$.

4.2. **An estimate of diameters of (δ, c)-regular sequence of triangulations.**

Lemma 4.2. *Suppose $(\mathcal{T}_n, l_n^*)$ is a (δ, c)-regular sequence of subdivisions of a triangulated polyhedral surface $(S, \mathcal{T}, l^*)$ satisfying (2.2). Then there exists an explicit constant $c_1(\delta, c, l^*, \mathcal{T}) > 0$ such that the combinatorial diameter $|\mathcal{T}_n| \leq c_1 q_n$ and $|E(\mathcal{T}_n)| \leq c_1 q_n^2$.*

Proof. Since the number of triangles in $\mathcal{T}$ is fixed and $(\mathcal{T}_n, l_n^*)$ are geometric subdivisions, we may assume that $(S, \mathcal{T}, l^*)$ is a single Euclidean triangle. By definition, all inner angles in $(\mathcal{T}_n, l_n^*)$ are at least δ and all edge lengths are in $(\frac{1}{cq_n}, \frac{c}{q_n})$. Therefore, there exists an explicit constant $c_2 (= \frac{\sin(\delta)}{2c^2})$ such that the area $A(t)$ of any triangle t in $(\mathcal{T}_n, l_n^*)$ is at least $\frac{c_2}{q_n^2}$. Since the sum of area of all triangles t in $\mathcal{T}_n$ is the area $A(S, l^*)$ of $(S, \mathcal{T}, l^*)$, it follows that the total number of triangles in $\mathcal{T}_n$ is at most $\frac{A(S,l^*)}{\min_t\{A(t)\}} \leq \frac{A(S,l^*)}{c_2} q_n^2$. But $\frac{1}{6}|E(\mathcal{T}_n)|$ is at most the number of triangles in $\mathcal{T}_n$, therefore, $|E(\mathcal{T}_n)| \leq \frac{6A(S,l^*)}{c_2} q_n^2$.

To bound the combinatorial diameter of $\mathcal{T}_n$, take two vertices u, v in $(\mathcal{T}_n, l_n)$. Let $[u, v]$ be the Euclidean line segment in the Euclidean triangle $(S, \mathcal{T}, l^*)$ connecting u, v. Then for any triangle t in $(\mathcal{T}_n, l_n)$ which intersects $[u, v]$, we have that $t \subset B([u, v], \frac{c}{q_n}) =: \{x \in S | d(x, [u, v]) \leq \frac{c}{q_n}\}$. Here d is the polyhedral metric on S associated to the edge length function l^*. The area of $B([u, v], \frac{c}{q_n})$ is at most $\frac{2cc_3}{q_n}$ where c_3 is the diameter of $(S, \mathcal{T}, l^*)$. Hence, by the area estimate that $A(t) \geq \frac{c_2}{q_n^2}$, we obtain,

$$\frac{2cc_3}{q_n} \geq Area(B([u, v], \frac{c}{q_n})) \geq \min_t A(t) |\{t : t \cap [u, v] \neq \emptyset\}| \geq \frac{c_2}{q_n^2} \cdot |\{t : t \cap [u, v] \neq \emptyset\}|.$$

This implies $|\{t : t \cap [u, v] \neq \emptyset\}| \leq \frac{2cc_3}{c_2} q_n$.

By definition the set $\cup\{t | t \cap [u, v] \neq \emptyset\}$ is path connected. Therefore, there exists a sequence of triangles $t_1, ..., t_k$ in $\mathcal{T}_n$ such that $u \in t_1$, $v \in t_k$, each t_i intersects $[u, v]$, t_i is adjacent to t_{i+1}, and $k \leq \frac{2cc_3}{c_2} q_n$. For each i, pick a vertex v_i in $t_i \cap t_{i+1}$. Then the edge path $u \sim v_1 \sim, ..., \sim v_k \sim v$ joins u to v and has length at most $\frac{2cc_3}{c_2} q_n$. We finish the proof by taking $c_1 = \max\{\frac{6A(S,l^*)}{c_2}, \frac{2cc_3}{c_2}\}$. $\square$

4.3. **Proof of Theorem 2.4.** Recall that $(\mathcal{T}_n, l_n^*)$ is a (δ, c)-regular sequence of geometric subdivisions of the triangulated PL surface $(S, \mathcal{T}, l^*)$. By lemma 4.2, we have that $|\mathcal{T}_n| \leq C_1 q_n$, $|E(\mathcal{T}_n)| \leq C_1 q_n^2$ for some explicit constant C_1. Furthermore, by the given condition (2.3) on l_n and Corollary 4.1, we have $|\alpha_{jk}^i(l_n) - \alpha_{jk}^i(l_n^*)| \leq C_2/q_n^2$. This shows that

$$\sqrt{|\mathcal{T}_n||E(\mathcal{T}_n)|} \max_{\Delta ijk \in \mathcal{T}_n} \{|\alpha_{jk}^i(l_n) - \alpha_{jk}^i(l_n^*)|\} \leq \frac{C_1 C_2 \sqrt{q_n^3}}{q_n^2} = \frac{C_1 C_2}{\sqrt{q_n}}.$$

Therefore, the time T_0 in Theorem 3.7 satisfies $T_0 \geq \frac{\epsilon(\delta)\sqrt{q_n}}{5C_0 C_1 C_2} > 1$ for large n. This shows that the required vector $w_n \in \mathbb{R}^{V(\mathcal{T}_n)}$ exists for n large. Parts (a) and (b) of Theorem 2.4 hold by the construction. The first part of (c) follows from (3.8) that $||w_n||_\infty \leq 2C_0\sqrt{|\mathcal{T}_n||E|} \max\{|\alpha_{jk}^i(l) - \alpha_{jk}^i(l^*)| : \Delta ijk \in \mathcal{T}_n\} \leq C(l^*, \delta, c_0, c)/\sqrt{q_n}$. The second part of (c) follows from the first part of (c) and the fact that $l_n^*(e) \in (\frac{1}{cq_n}, \frac{c}{q_n})$.

5. A PROOF OF THEOREM 1.2 USING THEOREM 2.4

We reduce the proof of Theorem 1.2 as follows. Suppose (Σ, g) is a given Riemannian disk with $p, q, r \in \partial\Sigma$ and $h : (\Sigma, g, (p, q, r)) \to (\bar{\mathbb{D}}, (1, i, -1))$ is the conformal map sending (p, q, r) to $(1, i, -1)$. Let ΔABC be the equilateral Euclidean triangle of edge lengths one and $h^* : \bar{\mathbb{D}} \to \Delta ABC$ be the conformal homeomorphism such that $(h^*(1), h^*(i), h^*(-1)) = (A, B, C)$. Due to the uniqueness of h^*, it suffices to show that the conformal map $\phi = h^* \circ h$ is computable. To this end, we begin by modifying algorithmically the Riemannian metric g as follows. Construct algorithmically a smooth function $\lambda_1 : \Sigma - \{p, q, r\} \to \mathbb{R}_{>0}$ such that the surface $(\Sigma, \lambda_1 g)$ has three corners at p, q, r of angle $\frac{\pi}{3}$. This follows from the growth condition that $\lim_{z\to z_0} \lambda_1(z) d_g(z_0, z)^{\frac{2}{3}} = 1$ where d_g is the Riemannian distance associated to g. Let the standard Riemannian metric on $\mathbb{C}$ be $|dz|^2$. By the construction, the two Riemannian metrics $\phi^*(|dz|^2)$ and $g_1 = \lambda_1 g$ are conformal on Σ, i.e., there exists a smooth function $\lambda : \Sigma \to \mathbb{R}_{>0}$ such that $\phi^*(|dz|^2) = \lambda g_1$. The goal is to show that λ is computable. As a consequence, the uniformization map ϕ which is the isometry map from $(\Sigma, \lambda g_1)$ to ΔABC is computable.

The above discussion can be phrased in terms of Riemannian triangles. By definition, a Riemannian triangle $(S, g, (p, q, r))$ is a compact C^∞-smooth surface S which has three corners $p, q, r \in \partial S$ and is C^∞-diffeomorphic to ΔABC where g is a smooth Riemannian metric S. We call p, q, r the *vertices* of S. Here by a smooth Riemannian metric g on S we mean that there exists a smooth embedding f of S into a smooth surface Y without boundary such that $g = f^*(\hat{g})$ for some smooth Riemannian metric $\hat{g}$ on Y. The uniformization map ϕ for a Riemannian triangle $(S, g, (p, q, r))$ is the conformal map from S to ΔABC sending (p, q, r) to (A, B, C).

Given a Riemannian triangle (S, g), by a *geodesic triangulation* $\mathcal{T}$ of $(S, g, (p, q, r))$, we mean a triangulation $\mathcal{T}$ such that each interior edge e of $\mathcal{T}$ is the shortest g-geodesic segment between end points of e, each triangle in $\mathcal{T}$ has angles less than π and $\{p, q, r\} \subset V(\mathcal{T})$. Suppose $(S, \mathcal{T}, l)$ and $(S, \mathcal{T}, w * l)$ are two PL metrics on $\mathcal{T}$. We define the *discrete conformal map* $\psi : (S, \mathcal{T}, l) \to (S, \mathcal{T}, w * l)$ to be the piecewise linear homeomorphism sending each triangle t back to itself and is linear with respect to the metrics l and $w*l$ on t. If $(S, \mathcal{T}, w*l)$ is isometric to the equilateral triangle ΔABC by an isometry ρ, then we call the composition $\rho \circ \psi : (S, \mathcal{T}, l) \to \Delta ABC$ the *discrete uniformization map*. We remark that a better definition of discrete conformal map introduced in [1] uses piecewise projective instead of piecewise linear. However, in the case that all triangulations are (δ, c)-regular, these two definitions give the same convergence results. Furthermore, piecewise linear maps are easier to be implemented in practice.

The main theorem of this section which implies Theorem 1.2 is the following.

Theorem 5.1. *Given a Riemannian triangle (S, g) of angles $\pi/3$ at three vertices p, q, r and a (δ, c)-regular sequence of geodesic triangulations $\mathcal{T}_n$ of (S, g), let $L_n(e)$ be the length of edge $e \in E(\mathcal{T}_n)$ in g-metric. Then there exists $w_n \in \mathbb{R}^{V(\mathcal{T}_n)}$ such that for sufficiently large n,*

(a) $(S, \mathcal{T}_n, w_n * L_n)$ is isometric to the equilateral triangle ΔABC and $(\mathcal{T}_n, w_n * L_n)$ is a $\frac{\delta}{2}$-triangulation,

(b) The discrete uniformization map $\phi_n : (\mathcal{T}_n, w_n * L_n) \to \Delta ABC$ converges uniformly to the uniformization map ϕ in the sense that $\lim_{n\to\infty} ||\phi_n|_{V(\mathcal{T}_n)} - \phi|_{V(\mathcal{T}_n)}||_\infty = 0$.

In particular, the uniformization map ϕ is computable.

We remark that any Riemannian triangle admits a (δ, c)-regular sequence of geodesic triangulations for some $\delta > 0$ and $c > 0$. See for instance the work of Colin de Verdiere [6].

We begin with a simple fact on conformal change of lengths.

5.1. **A cubic estimate.** Given a smooth path γ in a Riemannian manifold (S, g), we use $l_g(\gamma)$ to denote the length of γ in the metric g.

Proposition 5.2. *Suppose (S, g_1) is a C^2-smooth compact Riemannian surface with possibly non-empty boundary and corners and $g_2 = e^{2u} g_1$ is a conformal metric where $u \in C^2(S)$. Then there exists a constant $c = c(S, g_1, u)$ such that for any g_1-geodesic γ from p to q, or a smooth arc γ in ∂S, we have*

$$|l_{g_2}(\gamma) - e^{\frac{u(p)+u(q)}{2}} l_{g_1}(\gamma)| \leq c(S, g_1, u) l_{g_1}^3(\gamma).$$

Proof. We begin with the mid-point rule approximation of definite integrals. It states that if $f \in C^2([0, l])$, then

$$|\int_0^l f(x)dx - \int_0^l f(\frac{l}{2})dx| \leq \frac{l^3}{24} \max_{x \in [0,l]} |f''(x)|. \tag{5.1}$$

On the other hand, by the Taylor expansion for $w \in C^2([0, l])$, there exists $M_1 = M_1(\max |w|, \max |w'|, \max |w''|)$ such that

$$|e^{w(l/2)} - e^{\frac{w(0)+w(l)}{2}}| \leq M_1 l^2. \tag{5.2}$$

This follows from the Taylor expansion of $h \in C^2([0, l])$ that $h(0) = h(\frac{l}{2}) - \frac{1}{2} h'(\frac{l}{2}) l + \frac{1}{8} h''(\alpha) l^2$ and $h(l) = h(\frac{l}{2}) + \frac{1}{2} h'(\frac{l}{2}) l + \frac{1}{8} h''(\beta) l^2$. Therefore $|h(0)h(l) - h^2(\frac{l}{2})| \leq M_1 l^2$. Now take $h(t) = e^{\frac{1}{2} w(t/2)}$. We obtain (5.2).

Combining (5.1) and (5.2), we obtain for $w \in C^2([0, l])$

$$|\int_0^l e^{w(x)} dx - \int_0^l e^{\frac{w(0)+w(l)}{2}} dx| \leq M_2 l^3. \tag{5.3}$$

Now suppose $\gamma : [0, l] \to S$ is a g_1-geodesic from p to q such that γ is parameterized by the arc length parameter $t \in [0, l]$. Here $l = l_{g_1}(\gamma)$. The geodesic equation in local coordinate states $\frac{d^2}{dt^2} \gamma^i + \sum_{j,k} \Gamma^i_{jk} (\frac{d}{dt} \gamma^j)(\frac{d}{dt} \gamma^k) = 0$. Using $|\gamma'(t)|_{g_1} = 1$ and the geodesic equation, we see that for $w(t) = u(\gamma(t))$, the expression $M_1(\max |w|, \max |w'|, \max |w''|)$ is bounded by a constant independent of p, q, and γ (depending only on (S, g_1, u)). By definition $l_{g_2}(\gamma) = \int_0^l e^{u(\gamma(t))} dt$ and $l_{g_1}(\gamma) = \int_0^l dt$. Therefore the result follows from (5.3). The same argument works if γ is a smooth arc in ∂S. $\square$

5.2. **Proof of Theorem 5.1.** Let $\mathcal{T}$ be the triangulation of S with only one triangle such that $V(\mathcal{T}) = \{p, q, r\}$. The triangulations $\mathcal{T}_n$ are geodesic subdivisions of $\mathcal{T}$ in the metric g. Let the pull back metric $\phi^*(|dz|^2)$ on S be g^*. We can write $g^* = \lambda^{-4} g$ for some smooth function $\lambda : S \to \mathbb{R}_{>0}$. By definition, the uniformization map ϕ is the isometry from (S, g^*) to the equilateral triangle ΔABC. In particular, the surface (S, g^*) is convex. If e is an edge in $\mathcal{T}_n$, let $L_n(e)$ be the length of e in the g-metric. Thus if e is an interior edge, $L_n(e)$ is the length of the shortest path between end points of e in the g-metric. For each interior edge e, let e^* be the shortest g^*-geodesic segment between the end points of e such that e^* is homotopic to e fixing the end points. Due to convexity of (S, g^*), the interior of e^* is in the interior of S. If $e \subset \partial S$, define $e^* = e$. Let $l_n^*(e)$ be the length of e^* in the g^*-metric. Note that $l_n^*(e)$ is the length of the shortest path between the end points of e in the g^*-metric. Then $(S, \mathcal{T}_n, l_n^*)$ is a geodesic triangulation of an equilateral Euclidean triangle. We claim that there exists a constant $c_1 = c_1(g, \delta, c, g^*)$ such that

$$|l_n^*(e) - \mu_n * L_n(e)| \leq \frac{c_1}{q_n^3} \quad \text{for all edges } e \in \mathcal{T}_n, \tag{5.4}$$

where $\mu_n(v) = \ln(\lambda(v))$ and $g^* = \lambda^{-4} g$. Indeed, by Proposition 5.2, there exists a constant $c_2 = c_2(g, \delta, c, g^*)$ such that for any g-geodesic path γ from z_1 to z_2,

$$|l_{g^*}(\gamma) - \frac{1}{\lambda(z_1)\lambda(z_2)} l_g(\gamma)| \leq c_2 l_g^3(\gamma)$$

and any g^*-geodesic path γ^* from z_1 to z_2,

$$|l_{g^*}(\gamma^*) - \frac{1}{\lambda(z_1)\lambda(z_2)} l_g(\gamma^*)| \leq c_2 l_{g^*}^3(\gamma^*).$$

Take an edge e in $(\mathcal{T}_n, l_n)$ whose end points are z_1, z_2. Apply the above inequality to the g-geodesic e or $e \subset \partial S$, one obtains

$$l_n^*(e) \leq l_{g^*}(e) \leq \frac{1}{\lambda(z_1)\lambda(z_2)} l_g(e) + c_2 l_g^3(e) \leq \mu_n * L_n(e) + \frac{c_2'}{q_n^3}.$$

Since $L_n(e) \leq \frac{c}{q_n}$, this implies that $l_n^*(e) \leq \frac{c'}{q_n}$. On the other hand, applying Proposition 5.2 to the g^*-geodesic edge e^* or $e^* \subset \partial S$ isotopic to e with the same end points, one obtains

$$\mu_n * L_n(e) = \frac{1}{\lambda(z_1)\lambda(z_2)} l_g(e) \leq \frac{1}{\lambda(z_1)\lambda(z_2)} l_g(e^*) \leq l_{g^*}(e^*) + c_2 l_{g^*}^3(e^*) \leq l_n^*(e) + \frac{c_2''}{q_n^3}$$

due to $l_{g^*}(e^*) = l_n^*(e) \leq \frac{c'}{q_n}$. Combining these two together and take $c_1 = \max(c_2', c_2'')$, we see (5.4) holds.

Consider the new PL metric $l_n = \mu_n * L_n$ on $\mathcal{T}_n$. Since the function λ is uniformly continuous, for n large μ_n is locally near a constant. Therefore, for large n, $(\mathcal{T}_n, l_n)$ is a $(\frac{\delta}{2}, c_1')$-regular sequence of triangulations. By (5.4) and Theorem 2.4, there exists a discrete conformal factor $\nu_n \in \mathbb{R}^{V(\mathcal{T}_n)}$ such that $(\mathcal{T}_n, \nu_n * l_n) = (\mathcal{T}_n, (\nu_n + \mu_n) * L_n))$ is a $\frac{\delta}{4}$-triangulation whose discrete curvature is the same as the discrete curvature of $(S, \mathcal{T}_n, l_n^*)$. Furthermore, for $w_n = \nu_n + \mu_n \in \mathbb{R}^{V(\mathcal{T}_n)}$

$$|l_n^*(e) - w_n * L_n(e)| \leq \frac{c_3(g, \delta, c, \lambda)}{q_n \sqrt{q_n}}. \tag{5.5}$$

Following the argument used in [19], we derive part (b) of Theorem 5.1 from (5.5) as follows. Let $\mathcal{T}_n'$ and $\mathcal{T}_n''$ be the geometric triangulations of ΔABC which are the images of $\mathcal{T}_n$ under the discrete uniformization maps ϕ_n and ϕ respectively. In particular, for an edge $e \in \mathcal{T}_n$, the length of $\phi_n(e) \in \mathcal{T}_n'$ and $\phi(e) \in \mathcal{T}_n''$ in ΔABC are $w_n * L_n(e)$ and $l_n^*(e)$ respectively. To prove part (b), it suffices to show that the piecewise linear homeomorphism $F_n = \phi \circ \phi_n^{-1} : (\Delta ABC, \mathcal{T}_n', (A, B, C)) \to (\Delta ABC, \mathcal{T}_n'', (A, B, C))$ converges uniformly to the identity map. Since $(\mathcal{T}_n, l_n^*)$ and $(\mathcal{T}_n, w_n * L_n)$ are $\frac{\delta}{5}$-triangulations, there exists a lower bound for all angles in triangles in $\mathcal{T}_n'$ and $\mathcal{T}_n''$. Therefore, there exists a constant $K > 0$ such that F_n are K-quasiconformal. By the reflection principle for quasiconformal maps ([15], page 47), we can extend F_n to be a K-quasiconformal map, still denoted by F_n, from a domain Ω to itself where Ω contains ΔABC and Ω is independent of n (in fact we can take $\Omega = \mathbb{C}$). It is well known ([15]) that the space of all K-quasiconformal maps of Ω fixing A, B, C is compact. Hence there exists a subsequence F_{n_i} converging uniformly on compact sets in Ω to a K-quasiconformal map $f : (\Omega, (A, B, C)) \to (\Omega, (A, B, C))$. We claim that f is a 1-quasiconformal map. Indeed, take a triangle t in $\mathcal{T}_n'$ of edge lengths a_n, b_n, c_n. Let its image triangle $F_n(t)$ in $\mathcal{T}_n''$ have corresponding edge lengths a_n', b_n', c_n'. By (5.5), we have $\lim_n \frac{a_n}{a_n'} = \lim_n \frac{b_n}{b_n'} = \lim_n \frac{c_n}{c_n'} = 1$ since all these lengths $a_n, ..., c_n' \in (\frac{1}{c'' q_n}, \frac{c''}{q_n})$ for some constant c'' independent of n. Therefore, the two triangles t and $F_n(t)$ are nearly similar. Since $F_n|_t : t \to F_n(t)$ is linear, this shows that the quasi-conformal constant of $F_n|_t$ tends to 1 as n approaches infinity. Therefore f is 1-quasiconformal on ΔABC. On the other hand, f fixes A, B, C. It follows that f is the identity map. The above argument shows all limits of convergent subsequences of F_n are the identity map. Therefore, F_n converges uniformly on ΔABC to the identity map.

6. The torus case

The same proof as in §5 shows the following theorem that discrete uniformization metrics converge to the uniformization metric on tori.

Theorem 6.1. *Suppose $(\mathbb{S}^1 \times \mathbb{S}^1, g)$ is a torus with a Riemannian metric g whose associated uniformization metric is g^* (the flat Riemannian metric of area one conformal to g). Let $\mathcal{T}_n$ be a sequence of (δ, c)-regular geodesic subdivision of $(\mathbb{S}^1 \times \mathbb{S}^1, g)$ and $l_n : E(\mathcal{T}_n) \to \mathbb{R}_{>0}$ be the length of an edge in g-metric. Then there exists a sequence of $w_n \in \mathbb{R}^{V(\mathcal{T}_n)}$ such that $(\mathbb{S}^1 \times \mathbb{S}^1, \mathcal{T}_n, w_n * l_n)$ is a $\delta/2$-triangulation of a flat torus $(\mathbb{S}^1 \times \mathbb{S}^1, g_n)$ and g_n converges to g uniformly on $\mathbb{S}^1 \times \mathbb{S}^1$.*

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DEPARTMENT OF COMPUTER SCIENCE, STONY BROOK UNIVERSITY, STONY BROOK, NY, 11794

E-mail address: gu@cs.stonybrook.edu

DEPARTMENT OF MATHEMATICS, RUTGERS UNIVERSITY, PISCATAWAY, NJ, 08854

E-mail address: fluo@math.rutgers.edu

COURANT INSTITUTE OF MATHEMATICS, NEW YORK UNIVERSITY, NEW YORK, NY 251 MERCER STREET NEW YORK, N.Y. 10012

E-mail address: tianqi@cims.nyu.edu