

## CONTINUITY OF THE VOLUME OF SIMPLICES IN CLASSICAL GEOMETRY

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Received 12 February 2005

It is proved that the volume of spherical or hyperbolic simplices, when considered as a function of the dihedral angles, can be extended continuously to degenerated simplices. This verifies affirmatively a conjecture of John Minor.

*Keywords:* Hyperbolic space; spherical space; volume; simplex.

Mathematics Subject Classification 2000: 52A55, 52A41, 52A25

### 1. Introduction

#### 1.1. *Main results*

It is well known that the area of a spherical or a hyperbolic triangle can be expressed as an affine function of the inner angles by the Gauss–Bonnet formula. In particular, the area considered as a function of the inner angles can be extended continuously to degenerated spherical or hyperbolic triangles. It was conjectured by John Milnor [8] that the continuous extension property holds in any dimension. Namely, if a sequence of spherical (or hyperbolic)  $n$ -simplices has the property that their corresponding dihedral angles at codimension-2 faces converge, then the volumes of the simplices converge. The main result of the paper (Theorem 1.1) gives a proof of Milnor’s conjecture. The continuous extension property for volume of spherical simplexes was also known to Aomoto [1]. We remark that Igor Rivin has now a new proof of the continuous extension result using Schläefli formula [11]. His argument also works for a more general class of convex polytopes including simplexes.

Milnor [8] conjectured further that the extended volume function is non-zero except on the closure of the space of all Euclidean simplexes. In our recent work [4], Guo and the author have established the second part of Milnor volume conjecture using the volume formula (2.4) and (3.11) obtained in this paper.

If we consider the area as a function of the three edge lengths of a triangle, then there does not exist any continuous extension of the area to all degenerated

triangles. For instance, a degenerated spherical triangle of edge lengths  $0, \pi, \pi$  is represented geometrically as the intersection of two great circles at the north and south poles. However, its area depends on the intersection angle of these two geodesics and cannot be defined in terms of the lengths. This 2-dimensional simple phenomenon still holds in high dimension for both spherical and hyperbolic simplices.

To state our result, let us introduce some notations. Given an  $n$ -simplex with vertices  $v_1, \dots, v_{n+1}$ , the  $i$ th codimension-1 face is defined to be the  $(n-1)$ -simplex with vertices  $v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_{n+1}$ . The dihedral angle between the  $i$ th and  $j$ th codimension-1 faces is denoted by  $a_{ij}$ . As a convention, we define  $a_{ii} = \pi$  and call the symmetric matrix  $[a_{ij}]_{(n+1) \times (n+1)}$  the *angle matrix* of the simplex. It is well known that the angle matrix  $[a_{ij}]_{(n+1) \times (n+1)}$  determines the simplex up to isometry in spherical and hyperbolic geometry.

Let  $\mathbf{R}^{m \times m}$  be the space of all real  $m \times m$  matrices. Our main result is the following:

**Theorem 1.1.** *Let  $X_n(1)$  and  $X_n(-1) \subset \mathbf{R}^{(n+1) \times (n+1)}$  be the spaces of angle matrices of all  $n$ -dimensional spherical and hyperbolic simplices, respectively. The volume function  $V : X_n(k) \rightarrow \mathbf{R}$  can be extended continuously to the closure of  $X_n(k)$  in  $\mathbf{R}^{(n+1) \times (n+1)}$  for  $k = 1, -1$ .*

Note that both spaces  $X_n(1)$  and  $X_n(-1)$  are fairly explicitly known. Topologically, both of them are homeomorphic to the Euclidean space of dimension  $n(n+1)/2$ .

The proof of the theorem for spherical simplices is quite simple. It is an easy consequence of the continuity of the function which sends a semi-positive definite symmetric matrix to its square root. The case of the hyperbolic simplices is more subtle. It uses the continuity of the square roots of semi-positive definite symmetric matrices and the following property of hyperbolic simplices. We use  $B_R(x)$  to denote the ball of radius  $R$  centered at  $x$ .

**Theorem 1.2.** *For any  $\epsilon > 0$  and any  $r > 0$ , there is  $R = R(\epsilon, r, n)$  so that for any hyperbolic  $n$ -simplex  $\sigma$ , if  $x \in \sigma$  is a point whose distance to each totally geodesic codimension-1 hypersurface containing a codimension-1 face is at most  $r$ , then the volume of  $\sigma - B_R(x)$  is at most  $\epsilon$ .*

Recall that the *center* and the *radius* of a simplex are defined to be the center and the radius of its inscribed ball. The radius of a hyperbolic  $n$ -simplex is well known to be uniformly bounded from above. Applying Theorem 1.2 to the center of the  $n$ -simplex, we conclude that for any  $\epsilon > 0$ , there is  $R = R(\epsilon)$  so that the volume of  $\sigma - B_R(c)$  is less than  $\epsilon$  for any hyperbolic  $n$ -simplex  $\sigma$  with center  $c$ .

Recent work of [9] produces an explicit formula expressing volume of spherical and hyperbolic tetrahedra in terms of the dihedral angles using dilogarithmic

function. It is not clear if Theorem 1.1 in dimension 3 follows from their explicit formula.

## 1.2. A volume formula

Using the work of Aomoto [2], Kneser [6], and Vinberg [15], one may express the volume of a simplex in terms of an integral related to the Gaussian distribution (see (2.3) and (2.7)). To state Theorem 1.1 in terms of matrices, let us introduce some notations. For an  $n \times n$  matrix  $A$ , we use  $\text{ad}(A)$  to denote the adjacency matrix of  $A$ . The transpose of  $A$  is denoted by  $A^t$ . The  $ij$ th entry of  $A$  is denoted by  $A_{ij}$ . We use  $A > 0$  to denote the condition that all entries in  $A$  are positive. Evidently, if a matrix  $A$  is positive definite, or  $\text{ad}(A) > 0$ , then the following function  $F$  is well defined,

$$F(A) = \sqrt{|\det(\text{ad}(A))|} \int_{\mathbf{R}_{\geq 0}^n} e^{-x^t \text{ad}(A)x} dx, \quad (1.1)$$

where  $x \in \mathbf{R}^n$  is a column vector,  $\mathbf{R}_{\geq 0}$  is the set of all non-negative numbers and  $dx$  is the Euclidean volume form. Theorem 1.1 is equivalent to the following,

**Theorem 1.3.** *Let  $\mathcal{X}_n = \{A \in \mathbf{R}^{n \times n} \mid A^t = A, \text{ all } A_{ii} = 1, A \text{ is positive definite}\}$  and let  $\mathcal{Y}_n = \{A \in \mathbf{R}^{n \times n} \mid A^t = A, \text{ all } A_{ii} = 1, \text{ad}(A) > 0, \det(A) < 0, \text{ and all principal } (n-1) \times (n-1) \text{ submatrices of } A \text{ are positive definite}\}$ . Then, the function  $F: \mathcal{X}_n \cup \mathcal{Y}_n \rightarrow \mathbf{R}$  can be extended continuously to the closure of  $\mathcal{X}_n \cup \mathcal{Y}_n$  in  $\mathbf{R}^{n \times n}$ . Furthermore, the extended volume function on the closure of  $\mathcal{X}_n$  is*

$$F(A) = 1/\mu_{n-1} \int_{\mathbf{R}^{n+1}} e^{-x^t x} \chi(\sqrt{A}(x)) dx,$$

where  $\mu_{n-1} = \int_0^\infty x^{n-1} e^{-x^2} dx$  and  $\chi$  is the characteristic function of  $\mathbf{R}_{\geq 0}^{n+1}$  in  $\mathbf{R}^{n+1}$ . The extended volume function on the closure of  $\mathcal{Y}_n$  is in (3.11).

We do not know a proof of Theorem 1.3 without using hyperbolic geometry (i.e. Theorem 1.2).

## 1.3. Organization of the paper

The paper is organized as follows. In Sec. 2, we recall the basic set up and the Gram matrices of simplices. Also, we prove Theorem 1.1 for spherical simplices. In Sec. 3, we prove Theorem 1.1 for hyperbolic simplices assuming Theorem 1.2. We prove Theorem 1.2 in Sec. 4.

## 2. Preliminaries on Spherical and Hyperbolic Simplices

We recall some of the basic material related to the spherical and hyperbolic simplices in this section. In particular, we will recall the Gram matrices, the dual simplex and the volume formula. We also give a proof of Theorem 1.1 for spherical simplices. Here are the conventions and notations. Let  $\mathbf{R}^m$  denote the  $m$ -dimensional real vector space whose elements are column vectors. A diagonal matrix with diagonal

entries  $a_{11}, \dots, a_{nn}$  will be denoted by  $\text{diag}(a_{11}, \dots, a_{nn})$ . A diagonal matrix is *positive* if all diagonal entries are positive. The Kronecker delta is denoted by  $\delta_{ij}$ . The standard inner product in  $\mathbf{R}^m$  is denoted by  $(x, y) = x^t y$ . The length of a vector  $x \in \mathbf{R}^m$  is denoted by  $|x| = \sqrt{(x, x)}$ . We use  $dx = dx_1 dx_2 \cdots dx_m$  to denote the Euclidean volume element in  $\mathbf{R}^m$  and  $\mathbf{R}_{\geq 0}^m$  to denote the set  $\{(x_1, \dots, x_m) \in \mathbf{R}^m \mid x_i \geq 0 \text{ for all } i\}$ .

We will make a use of the continuity of the square root of symmetric semi-positive definite matrix. Recall that if  $A$  is a symmetric semi-positive definite matrix, then its square root  $\sqrt{A}$  is the symmetric semi-positive definite matrix so that it commutes with  $A$  and its square is  $A$ . It is well known that the square root matrix is unique. Furthermore, the square root operation, considered as a self map defined on the space of all symmetric semi-positive definite matrices, is continuous ([5, Theorem 6.2.37]).

2.1. Gram matrices of spherical simplices

Let  $\mathbf{R}^{n+1}$  be the Euclidean space with the standard inner product. The sphere  $S^n$  is  $\{x \in \mathbf{R}^{n+1} \mid (x, x) = 1\}$ . A spherical  $n$ -simplex  $\sigma^n$  has vertices  $v_1, \dots, v_{n+1}$  in  $S^n$  so that the vectors  $v_1, \dots, v_{n+1}$  are linearly independent. The codimension-1 face of  $\sigma^n$  opposite to  $v_i$  is denoted by  $\sigma_i^n$ . Let  $d_{ij}$  be the spherical distance between  $v_i$  and  $v_j$  and  $a_{ij}$  be the dihedral angle between the codimension-1 faces  $\sigma_i^n$  and  $\sigma_j^n$  for  $i \neq j$ . Define  $d_{ii} = 0$  and  $a_{ii} = \pi$ . Then, the *Gram matrix* of  $\sigma^n$  is defined to be the matrix  $G = [\cos(d_{ij})] = [(v_i, v_j)]$  and the *angle Gram matrix* of the simplex is the matrix  $G^* = [-\cos(a_{ij})]$ . Note that both of them are symmetric with diagonal entries being 1. The following is a well-known fact.

**Lemma 2.1.** *The Gram matrix  $G$  and the angle Gram matrix  $G^*$  of a simplex are related by the following formula*

$$G^* = DG^{-1}D, \tag{2.1}$$

where  $D$  is a positive diagonal matrix.

**Proof.** Let  $B = [v_1, \dots, v_{n+1}]$  be the  $(n + 1) \times (n + 1)$  matrix whose  $i$ th column is the  $i$ th vertex  $v_i$ . Then, the Gram matrix  $G$  of the simplex  $\sigma^n$  is  $B^t B$  due to the obvious formula  $v_i^t v_j = (v_i, v_j) = \cos(d_{ij})$ . To relate the matrix  $G^*$  with  $G$ , we consider the *dual simplex*. First, find  $n+1$  independent vectors  $w_1, \dots, w_{n+1} \in \mathbf{R}^{n+1}$  so that

$$(v_i, w_j) = \delta_{ij}. \tag{2.2}$$

Define  $v_i^* = w_i/|w_i|$ . Then, the dual simplex of  $\sigma^n$  is the spherical simplex with vertices  $\{v_1^*, \dots, v_{n+1}^*\}$ . If we use  $W = [w_1, \dots, w_{n+1}]$ , then Eq. (2.2) says  $B^t W = Id$ . In particular,  $W = (B^t)^{-1}$ . Thus,  $W^t W = (B^t B)^{-1} = G^{-1}$ . However, by the formula  $v_i^* = w_i/|w_i|$ , we see that the Gram matrix of the dual simplex is  $D(W^t W)D = DG^{-1}D$  where  $D$  is the diagonal matrix whose  $i$ th entry is  $|w_i|^{-1}$ .

On the other hand, by the definition of dual simplex, the Gram matrix of the dual is exactly the same as the angle Gram matrix of  $\sigma^n$ . Namely, the spherical distance between  $v_i^*$  and  $v_j^*$  is  $\pi - a_{ij}$ . Thus, Eq. (2.1) follows.  $\square$

The volume of the simplex  $\sigma^n$  can be calculated as follows (see [2, 6, 15]). For the simplex  $\sigma^n \subset S^n$ , let the cone in  $\mathbf{R}^{n+1}$  based at the origin over  $\sigma^n$  be  $K(\sigma^n) = \{rx \in \mathbf{R}^{n+1} \mid r \geq 0 \text{ and } x \in \sigma^n\}$ . Note that the linear transformation  $B : \mathbf{R}^{n+1} \rightarrow \mathbf{R}^{n+1}$  sending the vector  $x$  to  $Bx$  takes the standard basis element  $e_i$  to  $v_i$ . In particular  $B(\mathbf{R}_{\geq 0}^{n+1}) = K(\sigma^n)$ . Let  $\mu_k = \int_0^\infty x^k e^{-x^2} dx$ , i.e.  $\mu_{2k} = \sqrt{\pi}(1 \cdot 3 \cdots (2k-3)(2k-1))/2^{k+1}$  and  $\mu_{2k+1} = 2 \cdot 4 \cdots (2k-2)(2k)/2^{k+1}$ . Let the volume element on  $S^n$  be  $ds$ , then the volume  $V(\sigma^n)$  of the simplex  $\sigma^n$  is given by (see [2, 6, 15]),

$$\begin{aligned} V(\sigma^n) &= \int_{\sigma^n} ds \\ &= \mu_n^{-1} \int_{K(\sigma^n)} e^{-(x,x)} dx \\ &= \mu_n^{-1} \int_{B(\mathbf{R}_{\geq 0}^{n+1})} e^{-(x,x)} dx \\ &= \mu_n^{-1} \int_{\mathbf{R}_{\geq 0}^{n+1}} e^{-(By,By)} |\det B| dy \\ &= \mu_n^{-1} \sqrt{|\det G|} \int_{\mathbf{R}_{\geq 0}^{n+1}} e^{-y^t G y} dy. \end{aligned} \quad (2.3)$$

**Lemma 2.2.** *Let  $\chi$  be the characteristic function of the set  $\mathbf{R}_{\geq 0}^{n+1}$  in  $\mathbf{R}^{n+1}$ , then the volume  $V(\sigma^n)$  of a spherical simplex  $\sigma^n$  can be written as*

$$V(\sigma^n) = \mu_n^{-1} \int_{\mathbf{R}^{n+1}} e^{-(x,x)} \chi(\sqrt{G^*}(x)) dx. \quad (2.4)$$

**Proof.** Note that since  $G = B^t B$  is positive definite,  $G^{-1}$  is again symmetric and positive definite. Let  $A = \sqrt{G^{-1}}$  be the square root of  $G^{-1}$  so that  $A$  is symmetric positive definite and  $AGA = Id$ . Now, make a change of variable  $y = Az$  in Eq. (2.3) where  $z \in A^{-1}(\mathbf{R}_{\geq 0}^{n+1})$ . Then,  $V(\sigma^n) = \mu_n^{-1} \int_{A^{-1}(\mathbf{R}_{\geq 0}^{n+1})} e^{-(z,z)} dz$ . Note that the characteristic function of  $A^{-1}(\mathbf{R}_{\geq 0}^{n+1})$  is the same as the composition  $\chi \circ A$ . Thus, the volume is

$$V(\sigma^n) = \mu_n^{-1} \int_{\mathbf{R}^{n+1}} e^{-(x,x)} \chi(A(x)) dx.$$

Finally, note that if we make a change of variable of the form  $x = D(y)$  where  $D$  is a positive diagonal matrix, the integral (2.3) does not change. By Lemma 2.1, we have  $A = D\sqrt{G^*}D$  for a positive diagonal matrix  $D$ . Thus, Eq. (2.4) holds.  $\square$

### 2.2. A proof of Theorem 1.1 for spherical simplices

We give a proof of Theorem 1.1 for spherical simplices in this section. Let  $X_{n-1}(1)$  be the space of all angle matrices  $[a_{ij}]_{n \times n}$  of spherical  $(n-1)$ -simplices where  $a_{ij} = a_{ji}$  and  $a_{ii} = \pi$ . The map sending  $[a_{ij}]$  to the angle Gram matrix  $G^* = [-\cos(a_{ij})]$  is an embedding of the closure of  $X_{n-1}(1)$  into the space of all semi-positive definite, symmetric matrices whose diagonal entries are 1. Thus, to prove the continuity of the volume function on  $X_{n-1}(1)$ , by (2.4) it suffices to show the continuity of the function  $W : \mathcal{X}_n \rightarrow \mathbf{R}$  sending a matrix  $A$  to

$$W(A) = \int_{\mathbf{R}^{n+1}} e^{-(x,x)} \chi \circ \sqrt{A}(x) dx. \quad (2.5)$$

To this end, take a sequence  $\{A_m\}$  in  $\mathcal{X}_n$  so that  $\lim_{m \rightarrow \infty} A_m = A$  in  $\mathbf{R}^{n \times n}$ . To establish the existence of  $\lim_{m \rightarrow \infty} W(A_m)$ , we first use the fact that the function sending a semi-positive definite matrix to its square root is continuous ([5, Theorem 6.2.37]). In particular,  $\sqrt{A_m}$  converges to  $\sqrt{A}$ .

**Lemma 2.3.** *Suppose  $B_m$  is a convergent sequence of  $n \times n$  matrices so that  $\lim_{m \rightarrow \infty} B_m = B$ . If each row vector of  $B$  is non-zero, then the function  $\chi \circ B_m$  converges almost everywhere to  $\chi \circ B$  in  $\mathbf{R}^n$ .*

Assuming this lemma, we finish the proof as follows. Since all diagonal entries of  $A$  are 1, we conclude that no row vector in  $\sqrt{A}$  is zero. Thus by the lemma,  $\chi \circ \sqrt{A_m}$  converges almost everywhere to  $\chi \circ \sqrt{A}$  in  $\mathbf{R}^n$ . Since the integrand in  $W(A)$  is bounded by the integrable function  $e^{-(y,y)}$ , the dominant convergent theorem implies that  $\lim_{m \rightarrow \infty} W(A_m)$  exists and is equal to  $W(A)$ .

To prove Lemma 2.3, let  $R_i = \{x \in \mathbf{R}^n \mid x_i = 0\}$  be the coordinate planes. Then,  $B^{-1}(R_i)$  is a proper subspace of  $\mathbf{R}^n$ . Indeed, if otherwise, say for some index  $i$ ,  $B(\mathbf{R}^n) \subset R_i$ , then the  $i$ th row of  $B$  must be zero. This contradicts the assumption. Therefore, the Lebesgue measure of  $B^{-1}(R_i)$  is zero for all indices  $i$ . Now, we claim for every point  $x \in \mathbf{R}^n - \bigcup_{i=1}^n B^{-1}(R_i)$ , the sequence  $\chi \circ B_m(x)$  converges to  $\chi \circ B(x)$ . Indeed, by the assumption,  $B_m(x)$  converges to  $B(x) \in \mathbf{R}^n - \bigcup_{i=1}^n R_i$ . Thus, we have  $\chi(B_m(x))$  converges to  $\chi(B(x))$ .

The above also produces a proof of Theorem 1.3 for the case of continuous extension of  $F$  to the closure of  $\mathcal{X}_n$ .

### 2.3. Volume and Gram matrices of hyperbolic simplices

The  $(n+1)$ -dimensional Minkowski space  $\mathbf{R}^{n,1}$  is  $\mathbf{R}^{n+1}$  together with the symmetric non-singular bilinear form  $\langle x, y \rangle = \sum_{i=1}^n x_i y_i - x_{n+1} y_{n+1} = x^t S y$  where  $S = \text{diag}(1, 1, \dots, 1, -1)$  is an  $(n+1) \times (n+1)$  diagonal matrix. We define the hyperboloid of two sheets to be  $S(-1) = \{x \in \mathbf{R}^{n,1} \mid \langle x, x \rangle = -1\}$  and the unit sphere  $S(1) = \{x \in \mathbf{R}^{n,1} \mid \langle x, x \rangle = 1\}$ . The space  $S(-1)$  has two connected components, denoted by  $H^n$  and  $-H^n$ . It is well known that each of them can be taken as a model for the  $n$ -dimensional hyperbolic space  $H^n$ . For simplicity, we take  $H^n$

to be the component with positive last coordinates, i.e.  $H^n = S(-1) \cap \{x_{n+1} > 0\}$ . Given a vector  $u \in S(1)$ , let  $u^\perp$  be the totally geodesic codimension-1 space  $\{x \in H^n \cup (-H^n) \mid \langle x, u \rangle = 0\}$ . The following lemma is well known (see for instance [15]).

**Lemma 2.4.** *Suppose  $u, v \in S(1) \cup S(-1)$ . The following holds:*

- (1) *If  $u, v \in H^n$ , then  $\langle u, v \rangle \leq -1$  and the distance between  $u, v$  in  $H^n$  is*

$$\cosh^{-1}(-\langle u, v \rangle).$$

- (2) *If  $u, v \in S(1)$ , then  $u^\perp$  intersects  $v^\perp$  if and only if  $|\langle u, v \rangle| < 1$ . In this case, the dihedral angle of the intersection  $u^\perp, v^\perp$  in the region  $\{x \in H^n \mid \langle x, u \rangle \langle x, v \rangle \geq 0\}$  is  $\arccos(-\langle u, v \rangle)$ .*

- (3) *If  $u \in S(-1)$  and  $v \in S(1)$ , then the distance from  $u$  to  $v^\perp$  is  $\cosh^{-1}(\sqrt{1 + \langle u, v \rangle^2})$ .*

A hyperbolic  $n$ -simplex  $\sigma^n$  has vertices  $v_1, \dots, v_{n+1}$  in  $H^n$  (or in  $-H^n$ ) so that these vectors are linearly independent in  $\mathbf{R}^{n,1}$ . We denote the codimension-1 face of  $\sigma^n$  opposite to  $v_i$  by  $\sigma_i^n$ . The hyperbolic distance between  $v_i$  and  $v_j$  is denoted by  $d_{ij}$  and the dihedral angle between  $\sigma_i^n$  and  $\sigma_j^n$  is denoted by  $a_{ij}$  for  $i \neq j$ . As a convention,  $d_{ii} = 0$  and  $a_{ii} = \pi$ . As in the case of spherical simplices, we define the *Gram matrix*  $G$  of  $\sigma^n$  to be  $G = [\cosh d_{ij}] = [-\langle v_i, v_j \rangle]$  and the *angle Gram matrix* of  $\sigma^n$  to be  $G^* = [-\cos(a_{ij})]$ . Note that both of these matrices are symmetric with diagonal entries  $\pm 1$ .

The counterpart of Lemma 2.1 holds, it is the following,

**Lemma 2.5.** *Suppose  $G$  and  $G^*$  are the Gram matrix and the angle Gram matrix of a hyperbolic  $n$ -simplex, then there is a positive diagonal matrix  $D$  so that*

$$G^* = -DG^{-1}D.$$

**Proof.** By Lemma 2.4,  $\cosh d_{ij} = -\langle v_i, v_j \rangle = -v_i^t S v_j$ . Let  $B = [v_1, \dots, v_{n+1}]$  be the square matrix whose  $i$ th column is the  $i$ th vertex  $v_i$ , then by definition the Gram matrix  $G$  is  $-B^t S B$  where  $S = \text{diag}(1, 1, \dots, 1, -1)$ . To relate  $G^*$  with  $G$ , we find vectors  $w_1, \dots, w_{n+1}$  in  $\mathbf{R}^{n,1}$  so that  $\langle v_i, w_j \rangle = \delta_{ij}$ . Indeed, these vectors can be found by taking the matrix  $W = [w_1, \dots, w_{n+1}]$ . The condition  $\langle v_i, w_j \rangle = \delta_{ij}$  translates to the equation,  $B^t S W = Id$ , i.e.  $W = S(B^t)^{-1}$ . By the construction of vertices  $\{v_1, \dots, v_{n+1}\}$ , the bilinear form  $\langle, \rangle$  restricted to the codimension-1 linear space spanned by  $\{v_1, \dots, v_{n+1}\} - \{v_i\}$  has signature  $(n-1, 1)$ . This implies that  $\langle w_i, w_i \rangle$  is positive. Define  $v_i^* = w_i / \sqrt{\langle w_i, w_i \rangle}$ . Then,  $v_i^* \in S(1)$  and  $\langle v_i^*, v_j \rangle = (\sqrt{\langle w_i, w_i \rangle})^{-1} \delta_{ij}$ . The last equation shows that  $v_i^*$  is the unit vector in  $S(1)$  orthogonal to the  $i$ th codimension-1 face  $\sigma_i^n$  so that  $\langle v_i, v_i^* \rangle > 0$ . By Lemma 2.4(2), the intersection angle  $a_{ij}$  between  $\sigma_i^n$  and  $\sigma_j^n$  is given by the equation  $-\cos a_{ij} = \langle v_i^*, v_j^* \rangle$ . This shows that the Gram matrix  $A = [\langle v_i^*, v_j^* \rangle]$  of the vectors  $\{v_1^*, \dots, v_{n+1}^*\}$  is equal to the angle Gram matrix  $G^*$ . On the other

hand,  $v_i^* = w_i/\sqrt{\langle w_i, w_i \rangle}$ . Thus, the Gram matrix  $A$  can be expressed as  $DFD$  where  $D$  is a diagonal matrix with positive diagonal entries and  $F$  is the Gram matrix  $[\langle w_i, w_j \rangle]$ . By definition,  $F = W^tSW$ . Since  $W = S(B^t)^{-1}$ , we have  $F = W^tSW = (B^tSB)^{-1} = -G^{-1}$ . This establishes  $G^* = -DG^{-1}D$ .  $\square$

Let the volume element on  $H^n$  be  $ds$ , let  $K(\sigma^n) = \{rx \in \mathbf{R}^{n+1} \mid r \geq 0, x \in \sigma^n\}$  be the cone based at the vertex 0 spanned by the simplex  $\sigma^n$  in the vector space  $\mathbf{R}^{n+1}$  and  $dx = dx_1 \cdots dx_{n+1}$  be the Euclidean volume form in the Euclidean metric in  $\mathbf{R}^{n+1}$ . Then, the hyperbolic volume  $V(\sigma^n)$  is given by (see [15, p. 28], note the Gram matrix used in [15] is the angle Gram matrix in our case),

$$\begin{aligned} V(\sigma^n) &= \int_{\sigma^n} ds \\ &= \mu_n^{-1} \int_{K(\sigma^n)} e^{\langle x, x \rangle} dx \\ &= \mu_n^{-1} \int_{B(\mathbf{R}_{\geq 0}^{n+1})} e^{\langle x, x \rangle} dx \\ &= \mu_n^{-1} \int_{\mathbf{R}_{\geq 0}^{n+1}} e^{\langle By, By \rangle} |\det B| dy \\ &= \mu_n^{-1} \sqrt{|\det G|} \int_{\mathbf{R}_{\geq 0}^{n+1}} e^{y^t B^t S B y} dy \\ &= \mu_n^{-1} \sqrt{|\det G|} \int_{\mathbf{R}_{\geq 0}^{n+1}} e^{-y^t G y} dy. \end{aligned} \tag{2.6}$$

Since the integration in (2.6) remains unchanged if we replace  $G$  by  $DGD$  for a positive diagonal matrix, by Lemma 2.5, (2.6) is the same as

$$\begin{aligned} V(\sigma^n) &= \mu_n^{-1} (\sqrt{|\det G^*|})^{-1} \int_{\mathbf{R}_{\geq 0}^{n+1}} e^{y^t (G^*)^{-1} y} dy \\ &= \mu_n^{-1} \sqrt{|\det(\text{ad}(G^*))|} \int_{\mathbf{R}_{\geq 0}^{n+1}} e^{-y^t \text{ad}(G^*) y} dy. \end{aligned} \tag{2.7}$$

To summarize, we have:

**Lemma 2.6 ([15]).** *Suppose a hyperbolic  $n$ -simplex has angle Gram matrix  $G^*$ . Then, the volume of the simplex is a function of  $G^*$  given by Eq. (2.7).*

2.4. Some results from Matrix Perturbation Theory

The following two results will be used frequently in the paper. See [13,16] for proofs. The first theorem states the continuous dependence of eigenvalues on the matrices.

**Theorem 2.7 (Ostrowski).** *Let  $\lambda$  be an eigenvalue of  $A$  of algebraic multiplicity  $m$ . Then, for any matrix norm  $\|\cdot\|$  and all sufficiently small  $\epsilon > 0$ , there is  $\delta > 0$*



so that if  $\|B - A\| \leq \delta$ , the disk  $\{z \in \mathbf{C} \mid \|z - \lambda\| \leq \epsilon\}$  contains exactly  $m$  eigenvalues of  $B$  counted with multiplicity.

The next theorem concerns the continuous dependence of eigenvectors on the matrices. We state the result in the form applicable to our situation. Recall that an eigenvalue of a matrix is called *simple* if it is the simple root of the characteristic polynomial.

**Theorem 2.8** (see [16, p. 67]). *Suppose  $A_m$  is a sequence of  $n \times n$  matrices converging to  $B$ . Suppose  $\lambda$  is a simple eigenvalue of  $B$  and  $\lambda_m$  is a simple eigenvalue of  $A_m$  so that  $\lim_{m \rightarrow \infty} \lambda_m = \lambda$ . Then, there exists a sequence of eigenvectors  $v_m$  of  $A_m$  associated to  $\lambda_m$  so that these eigenvectors converge to an eigenvector of  $B$  associated to  $\lambda$ .*

This theorem follows from the fact that if  $\lambda$  is simple eigenvalue, then the adjacency matrix  $\text{ad}(B - \lambda Id)$  has rank 1 and its non-zero column vectors are the eigenvectors of  $B$  associated to  $\lambda$ .

### 3. A Proof of Theorem 1.1 for Hyperbolic Simplices Assuming

#### Theorem 1.2

Recall that  $X_n(-1)$  denotes the space of all angle matrices  $[a_{ij}]$  of hyperbolic  $n$ -simplices. The map  $\cos(x)$  is a homeomorphism from  $[0, \pi]$  to  $[-1, 1]$ . Thus, the angle Gram matrix  $G^* = [-\cos(a_{ij})]$  is a map which embeds the closure of  $X_n(-1)$  in  $\mathbf{R}^{(n+1) \times (n+1)}$  into the space of all symmetric matrices. The characterization of angle Gram matrix  $[-\cos(a_{ij})]$  was known.

**Lemma 3.1** ([7, 8]). *An  $(n + 1) \times (n + 1)$  symmetric matrix  $A$  with diagonal entries being one is the angle Gram matrix of a hyperbolic  $n$ -simplex if and only if*

(3.1) *all principal  $n \times n$  submatrices of  $A$  are positive definite,*

(3.2)  *$\det(A) < 0$  and*

(3.2) *all entries of the adjacency matrix  $\text{ad}(A)$  are positive.*

Recall that  $\mathcal{Y}_{n+1}$  is the space of all real matrices satisfying conditions in Lemma 3.1 and the scaled volume function  $F : \mathcal{Y}_{n+1} \rightarrow \mathbf{R}$  (i.e.  $\mu_n \text{vol} : \mathcal{Y}_{n+1} \rightarrow \mathbf{R}$ ) is

$$F(A) = \sqrt{|\det(A^{-1})|} \int_{\mathbf{R}_{\geq 0}^{n+1}} e^{x^t A^{-1} x} dx. \quad (3.4)$$

By changing the variable  $x$  to  $D^{-1}(x)$  for a positive diagonal matrix  $D$ , we see that  $F(A) = F(DAD)$ . Thus, to establish Theorem 1.1 for hyperbolic  $n$ -simplices, it suffices to show for any sequence of matrices  $A_m \in \mathcal{Y}_{n+1}$  so that  $\lim_{m \rightarrow \infty} A_m = A_\infty$  where  $A_\infty \in \mathbf{R}^{(n+1) \times (n+1)}$ ,  $\lim_{m \rightarrow \infty} F(DA_m D)$  exists for some positive diagonal matrix  $D$ . We will also identify the limit value.

### 3.1. Perturbation of eigenvalues of $A_\infty$

By definition, all diagonal entries of  $A_\infty$  are 1 and all principal proper submatrices of  $A_\infty$  are semi-positive definite. This shows that  $A_\infty$  is semi-positive definite if  $\det(A_\infty) = 0$ , and  $A_\infty$  has signature  $(n, 1)$  if  $\det(A_\infty) \neq 0$ .

Recall that an eigenvalue of a matrix is called *simple* if it is a simple root of the characteristic polynomial.

**Lemma 3.2.** *Given a symmetric  $n \times n$  matrix  $A$  of signature  $(k, 0)$  or  $(k, 1)$ , and  $\epsilon > 0$ , there exists a positive diagonal matrix  $D$  so that  $|D - Id| \leq \epsilon$  and all non-zero eigenvalues of  $DAD$  are simple.*

This is an easy consequence of the work on multiplicative inverse eigenvalue problem (see, for instance, [3]). For completeness, we provide a simple proof of it in the appendix.

Applying this lemma, we find a positive diagonal matrix  $D$  so that all non-zero eigenvalues of  $DA_\infty D$  are simple. Let  $k$  be the rank of  $A_\infty$ . Let  $\lambda_i(\infty)$ ,  $i = 1, \dots, n+1$ , be the eigenvalues of  $A_\infty$  so that either  $k = n+1$  and

$$\lambda_1(\infty) > \lambda_2(\infty) > \dots > \lambda_n(\infty) > 0 > -\lambda_{n+1}(\infty),$$

or  $k \leq n$  and

$$\lambda_1(\infty) > \lambda_2(\infty) > \dots > \lambda_k(\infty) > \lambda_{k+1}(\infty) = \dots = \lambda_{n+1}(\infty) = 0.$$

Let the eigenvalues of  $DA_m D$  be  $\{\lambda_i(m) \mid i = 1, 2, \dots, n+1\}$  so that

$$\lambda_1(m) \geq \lambda_2(m) \geq \dots \geq \lambda_n(m) > 0 > -\lambda_{n+1}(m).$$

Then by Theorem 2.7, we have  $\lim_m \lambda_i(m) = \lambda_i(\infty)$  for all  $i = 1, \dots, n+1$ . In particular, we may assume, by taking  $m$  large, that

$$\lambda_1(m) > \lambda_2(m) > \dots > \lambda_{\min\{k, n\}}(m) > 0$$

and

$$\lim_m \lambda_i(m) = 0 \tag{3.5}$$

for  $i > k$ .

### 3.2. Decomposition of the matrix $DAD$

We need the following canonical decomposition of matrices  $DAD$  where  $A \in \mathcal{Y}_{n+1}$  and  $D$  is any positive diagonal matrix. Note that  $(DAD)^2$  is symmetric and positive definite. In particular, the symmetric positive definite matrix  $B = \sqrt{\sqrt{(DAD)^2}}$  exists and depends continuously on  $DAD$ . Suppose the eigenvalues of  $DAD$  are  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n > 0 > -\lambda_{n+1}$ . Then, there exists an orthogonal matrix

$U = [v_1, \dots, v_{n+1}]$  whose column vectors  $v_i$  are eigenvectors of  $DAD$  of length one so that

$$DAD = U \operatorname{diag}(\lambda_1, \dots, \lambda_n, -\lambda_{n+1}) U^t. \quad (3.6)$$

By replacing the last column  $v_{n+1}$  by  $-v_{n+1}$  if necessary, we may assume that one of the entry in  $v_{n+1}$  is positive.

We can recover  $B$  from (3.6) by the formula  $B = U \operatorname{diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n}, \sqrt{\lambda_{n+1}}) U^t$ . Furthermore,  $Bv_i = \sqrt{\lambda_i} v_i$ , and

$$BU = [\sqrt{\lambda_1} v_1, \dots, \sqrt{\lambda_n} v_n, \sqrt{\lambda_{n+1}} v_{n+1}].$$

In particular, we have

$$DAD = BUSU^t B \quad \text{and} \quad A = D^{-1} BUSU^t BD^{-1}. \quad (3.7)$$

Write  $W = SU^t BD^{-1}$  and  $W^t = D^{-1} BUS$ . Then, (3.7) says  $A = W^t S W$ . If  $w_1, \dots, w_{n+1}$  are the column vectors of  $W$ , i.e.  $W = [w_1, \dots, w_{n+1}]$ , then  $A = [\langle w_i, w_j \rangle]$ . Since  $A$  is assumed to be the angle Gram matrix of a hyperbolic simplex, we may take the simplex to be  $\sigma = \{v \in H^n \cup (-H^n) \mid \langle w_i, v \rangle \geq 0 \text{ for all } i\}$  and take  $\{w_1, \dots, w_{n+1}\}$  to be the unit outward normal vectors to the codimension-1 faces of  $\sigma$ . (The condition  $A \in \mathcal{Y}_{n+1}$  implies either  $\sigma \subset H^n$  or  $\sigma \subset -H^n$ .) We claim that vertices of  $\sigma$  are given by

$$v_i = u_i / |\langle u_i, u_i \rangle|^{1/2},$$

where  $u_i = U^t B^{-1} D^{-1} (e_i)$  and  $e_i = [0, \dots, 0, 1, 0, \dots, 0]^t$  is the  $i$ th standard basis vector of  $\mathbf{R}^{n+1}$ . Indeed,  $W^t S (U^t B^{-1} D^{-1}) = Id$  says  $\langle w_i, u_j \rangle = \delta_{ij}$ , the Kronecker delta. By the definition of  $A \in \mathcal{Y}_{n+1}$ , the claim follows. This shows that the simplex  $\sigma = U^t B^{-1} D^{-1} (\mathbf{R}_{\geq 0}^{n+1}) \cap (H^n \cup (-H^n)) = U^t B^{-1} (\mathbf{R}_{\geq 0}^{n+1}) \cap (H^n \cup (-H^n))$ .

**Proposition 3.3.** *The point  $e_{n+1} = [0, \dots, 0, 1]^t$  is in the simplex  $\sigma = U^t B^{-1} (\mathbf{R}_{\geq 0}^{n+1}) \cap (H^n \cup -H^n)$  and the distance from  $e_{n+1}$  to the totally geodesic hypersurface containing the  $i$ th codimension-1 face of  $\sigma$  is at most  $\cosh^{-1}(\sqrt{1 + \lambda_{n+1}/d_i^2})$  for all  $i$  where  $D = \operatorname{diag}(d_1, \dots, d_{n+1})$ .*

**Proof.** To find the distance from  $e_{n+1}$  to the codimension-1 totally geodesic hypersurface containing the  $i$ th codimension-1 face, we compute  $\langle w_i, e_{n+1} \rangle$ . Since

$$\begin{aligned} W^t S e_{n+1} &= D^{-1} B U S e_{n+1} = D^{-1} B U e_{n+1} \\ &= D^{-1} B v_{n+1} = \sqrt{\lambda_{n+1}} D^{-1} v_{n+1} \end{aligned} \quad (3.8)$$

and the eigenvector  $v_{n+1}$  has Euclidean norm 1, we obtain  $|\langle w_i, e_{n+1} \rangle| \leq \sqrt{\lambda_{n+1}}/d_i$ . By Lemma 2.4(3), we conclude that the distance from  $e_{n+1}$  to the hypersurface containing  $i$ th codimension-1 face is at most  $\cosh^{-1}(\sqrt{1 + \lambda_{n+1}/d_i^2})$ .

Finally, we need to show that  $e_{n+1}$  is in the simplex  $\sigma$ . Due to Eq. (3.8), this is the same as showing that all entries of the eigenvector  $v_{n+1}$  are positive. By the

assumption on  $v_{n+1}$  that one of its entry is positive, it suffices to show that all entries of the eigenvector  $v_{n+1}$  have the same sign. To this end, we need:

**Lemma 3.4.** *Suppose  $B$  is a symmetric  $(n+1) \times (n+1)$  matrix so that all  $n \times n$  principal submatrices in  $B$  are positive definite and  $\det(B) \leq 0$ . Then, no entry in the adjacent matrix  $\text{ad}(B)$  is zero.*

To prove Lemma 3.4, we first note that  $B \text{ad}(B) = \det(B) \text{Id}$ . Also, the positive definiteness of the principal submatrices shows that  $\text{ad}(B)_{ii} > 0$  for all  $i$ . If there is an entry  $\text{ad}(B)_{ij} = 0$ , then  $i \neq j$ . Without loss of generality, let us assume that  $\text{ad}(B)_{1(n+1)} = 0$ . Let  $w$  be the first column of  $\text{ad}(B)$ . The vector  $w$  is not the zero vector due to  $\text{ad}(B)_{11} > 0$ . By the assumption that the principal submatrix  $P$  obtained by removing the last row and column is positive definite, we have  $w^t B w = w^t P w > 0$ . On the other hand,  $B w = \det(B)[1, 0, \dots, 0]^t$  by definition and  $w^t B w = \det(B) \text{ad}(B)_{11} \leq 0$  due to  $\det(B) \leq 0$  and  $\text{ad}(B)_{11} > 0$ . This is a contradiction.  $\square$

Using Lemma 3.4, we prove that all entries of the eigenvector  $v_{n+1}$  have the same sign as follows. For the variable  $t \in [0, \lambda_{n+1}]$ , consider the matrix  $C(t) = DAD + t \text{Id}$ . By definition, all  $n \times n$  principal submatrix of  $C(t)$  are positive definite. Furthermore,  $\det(C(t)) \leq 0$  since the smallest eigenvalue of  $C(t)$  is  $t - \lambda_{n+1} \leq 0$  and all other eigenvalues are positive. By Lemma 3.4, all entries  $\text{ad}(C(t))_{ij}$  are non-zero. On the other hand,  $\text{ad}(C(t))_{ij}$  is continuous in  $t$  and is positive when  $t = 0$  by (3.3). Thus, all  $\text{ad}(C(t))_{ij} > 0$ . Now, for  $t = \lambda_{n+1}$ , a non-zero column of  $\text{ad}(C(\lambda_{n+1}))$  is an eigenvector of  $DAD$  associated to  $-\lambda_{n+1}$ . Since this negative eigenvalue is simple for  $DAD$ , any two associated eigenvectors are multiple of each other. This  $v_{n+1}$  is proportional to the non-zero column. Therefore, all entries in  $v_{n+1}$  are positive.

3.3. The convergence of volume

We now prove Theorem 1.3. Given a sequence  $A_m \in \mathcal{Y}_{n+1}$  converging to a matrix  $A_\infty$  in  $\mathbf{R}^{(n+1) \times (n+1)}$ , we choose  $D$  as in (3.1) and will prove that  $\lim_m F(DA_m D)$  exists and identify the limit. Let  $k$  be the rank of  $A_\infty$ . By Theorem 2.8 and the fact that  $\lambda_i(m)$  and  $\lambda_i(\infty)$  are simple eigenvalues for  $i \leq k$ , we choose eigenvectors  $v_1(m), \dots, v_k(m)$  of unit length for  $DA_m D$  associated to the simple eigenvalues  $\lambda_i(m)$  so that

$$\lim_{m \rightarrow \infty} v_i(m) = v_i(\infty)$$

for  $i = 1, 2, \dots, k$  and  $v_i(\infty)$  is an eigenvector of norm 1 for  $DA_\infty D$ . Produce the decomposition

$$A_m = D^{-1} B_m U_m S U_m^t B_m D^{-1} \tag{3.9}$$

as in Eq. (3.7) where  $B_m = \sqrt{\sqrt{(DA_m D)^2}}$  and the first  $k$  column vectors of the orthogonal matrix  $U_m$  are  $v_1(m), \dots, v_k(m)$ . By (3.5), it follows that the matrices

$$B_m U_m = [\sqrt{\lambda_1(m)} v_1(m), \dots, \sqrt{\lambda_k(m)} v_k(m), \\ \sqrt{\lambda_{k+1}(m)} v_{k+1}(m), \dots, \sqrt{\lambda_{n+1}(m)} v_{n+1}(m)]$$

converge to the matrix  $M = [\sqrt{\lambda_1(\infty)} v_1(\infty), \dots, \sqrt{\lambda_k(\infty)} v_k(\infty), 0, \dots, 0]$  if  $k \leq n$  or to  $M = [\sqrt{\lambda_1(\infty)} v_1(\infty), \dots, \sqrt{\lambda_n(\infty)} v_n(\infty), \sqrt{\lambda_{n+1}(\infty)} v_{n+1}(\infty)]$  if  $k = n+1$ .

By (3.4), the function  $F(DA_m D)$  is

$$F(DA_m D) = \sqrt{|\det(DA_m D)^{-1}|} \int_{\mathbf{R}_{\geq 0}^{n+1}} e^{x^t D^{-1} A_m^{-1} D^{-1} x} dx.$$

Making a change of variable  $x = B_m U_m(y)$  in the above integral and using (3.9), we obtain

$$\begin{aligned} F(DA_m D) &= \int_{(B_m U_m)^{-1}(\mathbf{R}_{\geq 0}^{n+1})} e^{y^t S y} dy \\ &= \int_{\mathbf{R}^{n+1}} e^{\langle y, y \rangle} \chi \circ (B_m U_m)(y) dy, \end{aligned} \quad (3.10)$$

where  $\chi$  is the characteristic function of  $\mathbf{R}_{\geq 0}^{n+1}$  in  $\mathbf{R}^{n+1}$ .

By the construction,  $B_m U_m$  converges to the matrix  $M$  in  $\mathbf{R}^{(n+1) \times (n+1)}$ . We claim that the sequence of functions  $\chi \circ (B_m U_m)$  converges almost everywhere to  $\chi \circ M$  in  $\mathbf{R}^{n+1}$ . In fact, by Lemma 2.3, it suffices to verify that no row vector in  $\lim_{m \rightarrow \infty} B_m U_m$  is zero. Suppose otherwise, say the  $i$ th row is zero. Then, the  $i$ th entry in  $\lim_{m \rightarrow \infty} B_m U_m S U_m^t B_m$  is zero. But by assumption, the  $i$ th entry in  $B_m U_m S U_m^t B_m$  is  $d_i^2 \neq 0$  where  $D = \text{diag}(d_1, \dots, d_{n+1})$ .

To summarize, we see that the integrand in (3.10) converges almost everywhere in  $\mathbf{R}^{n+1}$  to  $e^{\langle y, y \rangle} \chi \circ M(y)$ . To prove that the limit  $\lim_{m \rightarrow \infty} F(A_m)$  exists, we will use the following well known lemma from real analysis. See, for instance, [12] for a proof.

**Lemma 3.5.** *Suppose  $\{f_m\}$  is a sequence of integrable non-negative functions converging almost everywhere to  $f$  in  $\mathbf{R}^n$ . If for any  $\epsilon > 0$ , there exists a measurable set  $E \subset \mathbf{R}^n$  so that*

- (a) *the restriction  $f_m|_E$  converges a.e. to  $f|_E$  and is dominated by an integrable function  $g$  on  $E$ , and*
- (b)  *$\int_{\mathbf{R}^n - E} f_m dx \leq \epsilon$  for all integer  $m \geq 1$ , then the function  $f$  is integrable over  $\mathbf{R}^n$ ,  $\lim_{m \rightarrow \infty} \int_{\mathbf{R}^n} f_m dx$  exists and is equal to  $\int_{\mathbf{R}^n} f dx$ .*

To apply this lemma, we will produce a decomposition of integral (3.10) as follows. For any positive number  $p \in (0, 1)$ , consider the set  $\Omega_p = \{x \in \mathbf{R}^{n+1} \mid \langle x, x \rangle \leq -p(x, x)\}$  where  $(x, x) = x^t x$  is the Euclidean inner product. The intersection  $\Omega_p \cap \pm H^n$  is equal to the hyperbolic ball of radius  $r = \cosh^{-1}(\sqrt{(1+p)/2p})$  centered at  $\pm e_{n+1}$ . Indeed, we have  $(x, x) = \langle x, x \rangle + 2(x, e_{n+1})^2$ . Thus,  $\langle x, x \rangle \leq -p(x, x)$

inside  $\pm H^n$  is the same as  $|\langle x, e_{n+1} \rangle| \leq \sqrt{(1+p)/2p}$ . By Lemma 2.4(1), it follows  $\Omega_p \cap \pm H^n = B_r(\pm e_{n+1})$ . Now, in the region  $\Omega_p$ , the integral  $\int_{\Omega_p} e^{\langle y, y \rangle} \chi \circ (B_m U_m)(y) dy$  converges to  $\int_{\Omega_p} e^{\langle y, y \rangle} \chi \circ (M)(y) dy$  since the integrant is dominated by the integrable function  $e^{-p\langle y, y \rangle}$ . On the other hand, the integral  $\int_{\mathbf{R}^{n+1} - \Omega_p} e^{\langle y, y \rangle} \chi \circ (B_m U_m)(y) dy$  is the same as  $\mu_n \text{vol}(\sigma_m - B_r(\pm e_{n+1}))$  where  $\sigma_m = (B_m U_m)^{-1}(\mathbf{R}_{\geq 0}^{n+1}) \cap (H^n \cup -H^n)$  is the hyperbolic  $n$ -simplex whose angle Gram matrix is  $A_m$ . By Proposition 3.3 and the existence of  $\lim_{m \rightarrow \infty} \lambda_{n+1}(m)$ , there is a constant  $C$  independent of  $m$  so that  $\pm e_{n+1}$  is within distance  $C$  to each codimension-1 totally geodesic surface containing a codimension-1 face of the simplex  $\sigma_m$ . By Theorem 1.2 and Proposition 3.3, the volume  $\text{vol}(\sigma_m - B_r(\pm e_{n+1}))$  can be made arbitrary small for all  $n$ -simplices  $\sigma_m$  if the radius  $r$  is large. Thus, by Lemma 3.5, the function  $e^{\langle y, y \rangle} \chi \circ M(y)$  is integrable over  $\mathbf{R}^{n+1}$ , the limit  $\lim_{m \rightarrow \infty} F(A_m)$  exists and is equal to  $\int_{\mathbf{R}^{n+1}} e^{\langle y, y \rangle} \chi \circ M(y) dy$ .

The above proof also produces a formula for the extended volume of matrices in the closure of the hyperbolic angle Gram matrices.

**Corollary 3.6.** *Let  $A$  be a matrix in the closure of  $\mathcal{Y}_{n+1}$  in  $\mathbf{R}^{(n+1) \times (n+1)}$ ,  $D$  be a positive diagonal matrix so that all non-zero eigenvalues of  $DAD$  are simple, and  $U$  be an orthogonal matrix whose column vectors are the eigenvectors of  $DAD$ . Then, the extended volume  $\text{vol}(A)$  is given by the convergent integral*

$$\text{vol}(A) = 1/\mu_n \int_{\mathbf{R}^{n+1}} e^{\langle y, y \rangle} \chi \circ (\sqrt{\sqrt{(DAD)^2} U})(y) dy, \tag{3.11}$$

where  $\mu_n = \int_0^\infty x^n e^{-x^2} dx$ . In particular, if  $F(A) = 0$ , then  $\det(A) = 0$ .

Note that if  $A \in \mathcal{Y}_{n+1}$ , then we can take  $D = Id$  in (3.11). It seems that the extension formula (3.11) should hold for all  $A$  in the closure of  $\mathcal{Y}_{n+1}$  with  $D = id$ .

4. A Proof of Theorem 1.2

We prove Theorem 1.2 in this section. Recall that  $B_R(x)$  denotes the ball of radius  $R$  centered at  $x$ .

**Theorem 1.2.** *For any  $\epsilon > 0$  and  $r > 0$ , there exists a positive number  $R = R(\epsilon, r, n)$  so that for any hyperbolic  $n$ -simplex  $\sigma$ , if  $x \in \sigma$  is a point whose distance to each totally geodesic hyperplane containing a codimension-1 face is at most  $r$ , then the volume of  $\sigma - B_R(x)$  is at most  $\epsilon$ .*

The theorem will follow from a sequence of propositions and lemmas on hyperbolic simplices. To begin with, we fix the notations and conventions as follow. The projective disk model of  $H^n$  is denoted by  $D^n = \{(x_1, \dots, x_n) \in \mathbf{R}^n \mid \sum_{i=1}^n x_i^2 < 1\}$ . The compact closure of  $D^n$  is denoted by  $\overline{D^n}$  which is the compactification of the hyperbolic space by adding the ideal points. The hyperbolic distance in  $H^n$  or  $D^n$  will be denoted by  $d$ . If  $\{v_1, \dots, v_k\}$  is a set of points in  $\overline{D^n}$ , the convex hull of it will be denoted by  $C(v_1, \dots, v_k)$ . The volume of  $C(v_1, \dots, v_{n+1})$  in  $D^n$ , denoted by

$\text{vol}(C(v_1, \dots, v_k))$ , is the hyperbolic volume of  $C(v_1, \dots, v_{n+1}) \cap D^n$ . If  $v_1, \dots, v_{n+1}$  are pairwise distinct, we call  $\sigma = C(v_1, \dots, v_{n+1})$  a *generalized  $n$ -simplex* in  $\overline{D^n}$ . Its  $i$ th codimension-1 face, denoted by  $\sigma^i$  is  $C(v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_{n+1})$ . A generalized  $n$ -simplex is said to be *non-degenerated* if it has positive volume. Evidently, a generalized  $n$ -simplex  $C(v_1, \dots, v_{n+1})$  in  $\overline{D^n}$  is non-degenerated if and only if the vectors  $\{v_1, \dots, v_{n+1}\}$  are linearly independent in  $\mathbf{R}^{n+1}$ . The *center* and the *radius* of a non-degenerated generalized  $n$ -simplex are defined to be the center and the radius of its inscribed ball. Given a finite set  $X \in \overline{D^n}$  so that  $X$  contains at least two points, the smallest complete totally geodesic submanifold containing  $X$  in its closure is denoted by  $\text{sp}(X)$ . For a measurable subset  $X$  of  $H^n$ , or  $D^n$ , we use  $\text{vol}(X)$  to denote the volume of the set. If  $X$  lies in a totally geodesic submanifold of dimension- $k$   $H^k$  in  $H^n$ , we use  $\text{vol}_k(X)$  to denote the volume of  $X$  in the subspace  $H^k$ .

#### 4.1. Continuous dependence of volume on vertices

We will establish the following propositions and lemmas in order to prove Theorem 1.2.

The first proposition generalizes a result of Ratcliffe.

**Proposition 4.1** (see [10, Theorem 11.3.2]). *Suppose  $\sigma_m = C(v_1(m), \dots, v_{n+1}(m))$  is a sequence of generalized  $n$ -simplices in  $D^n$  so that  $\lim_{m \rightarrow \infty} v_i(m) = u_i$  exists in  $\overline{D^n}$  for all  $i = 1, \dots, n+1$  and either  $\{u_1, \dots, u_{n+1}\}$  contains at least three points or  $\{u_1, \dots, u_{n+1}\}$  consists of two distinct points  $\{p, q\}$  so that both sets  $\{i \mid u_i = p\}$  and  $\{i \mid u_i = q\}$  contain more than one point. Then,  $\lim_{m \rightarrow \infty} \text{vol}(\sigma_m) = \text{vol}(C(u_1, \dots, u_{n+1}))$ .*

Note that Ratcliffe proved the proposition when  $C(u_1, \dots, u_{n+1})$  is a non-degenerated generalized  $n$ -simplex. (In [10], a non-degenerated generalized simplex in our sense is called a generalized  $n$ -simplex.) However, if one exams his proof carefully in ([10, pp. 527–529]), the non-degeneracy condition is never used. Ratcliffe in fact already proved the proposition under the assumption that  $\{u_1, \dots, u_{n+1}\}$  are pairwise distinct. Thus, it suffices to prove the proposition in the case that the number of elements in  $\{u_1, \dots, u_{n+1}\}$  is at most  $n$  and is at least 2 as specified in the proposition. This will be proved in Sec. 4.3.

**Proposition 4.2.** *For any  $\epsilon > 0$ , there is a number  $\delta > 0$  so that if the radius of the inscribed ball of a hyperbolic  $n$ -simplex is less than  $\delta$ , the volume of the simplex is less than  $\epsilon$ .*

**Lemma 4.3.** *For any  $\delta > 0$  and  $r > 0$ , there exists  $R = R(\delta, r, n)$  so that for any hyperbolic  $n$ -simplex  $\sigma$  of radius at least  $\delta$ , if  $x \in \sigma$  is a point whose distance to each codimension-1 totally geodesic surface containing a codimension-1 face is at most  $r$ , then  $d(x, c) \leq R$  where  $c$  is the center of  $\sigma$ .*

Finally, we recall the following useful lemma of Thurston,

**Lemma 4.4.** *Given a generalized hyperbolic  $n$ -simplex  $\sigma = C(v_1, \dots, v_{n+1})$  where  $n \geq 2$ , let  $\tau = C(v_1, \dots, v_n)$  be a codimension-1 face of  $\sigma$ , then*

$$\text{vol}_n(\sigma) \leq 1/(n-1)\text{vol}_{n-1}(\tau).$$

See [14, Chap. 6], or [10, pp. 518–528], especially p. 528 for a proof.

#### 4.2. A proof of Theorem 1.2

Assuming the results above, we finish the proof of Theorem 1.2 as follows. Suppose otherwise that Theorem 1.2 is not true. Then, there are  $\epsilon_0 > 0$ ,  $r_0 > 0$ , a sequence of hyperbolic  $n$ -simplices  $\sigma_m$ , and a point  $x_m \in \sigma_m$  so that,

(4.1) The distance of  $x_m$  to the totally geodesic codimension-1 surface containing each codimension-1 face of  $\sigma_m$  is at most  $r_0$  and

(4.2)  $\text{vol}(\sigma_m - B_m(x_m)) \geq \epsilon_0$ .

By Proposition 4.2 and condition (4.2), we may assume that the radius  $r_m$  of  $\sigma_m$  is at least  $\delta_0 > 0$  for all  $m$ . By Lemma 4.3 for  $\delta_0$  and  $r_0$ , we find a constant  $R_0$  so that  $d(x_m, c_m) \leq R_0$  for all  $m$  where  $c_m$  is the center of the simplex  $\sigma_m$ . In particular,  $B_{m-R_0}(c_m) \subset B_m(x_m)$ . This implies  $\sigma_m - B_m(x_m) \subset \sigma_m - B_{m-R_0}(c_m)$  and

$$\text{vol}(\sigma_m - B_{m-R_0}(c_m)) \geq \epsilon_0, \tag{4.3}$$

for all  $m$ .

In the projective disk model  $D^n$ , we put the center  $c_m$  to the Euclidean center 0 of  $D^n$ . By taking a subsequence if necessary, we may assume that  $\sigma_m = C(v_1(m), \dots, v_{n+1}(m))$  where the limit  $\lim_{m \rightarrow \infty} v_i(m) = u_i$  exists in  $\overline{D^n}$ .

**Lemma 4.5.** *Suppose  $\sigma_m = C(v_1(m), \dots, v_{n+1}(m))$  is a sequence of hyperbolic  $n$ -simplices with center 0 in the projective model  $D^n$  so that the limit  $\lim_{m \rightarrow \infty} v_i(m) = u_i$  exists in  $\overline{D^n}$  for all  $i$ . If  $\liminf_{m \rightarrow \infty} \text{vol}(\sigma_m) > 0$ , then either  $\{u_1, \dots, u_{n+1}\}$  consists of at least three points, or  $\{u_1, \dots, u_{n+1}\} = \{p, q\}$ ,  $p \neq q$ , so that both sets  $\{i \mid u_i = p\}$  and  $\{j \mid u_j = q\}$  contain at least two points.*

To prove this lemma, suppose otherwise, there are two possibilities. In the first possibility,  $\{u_1, \dots, u_{n+1}\}$  consists of one point  $\{p\}$ . Then, for all  $m$  large, the points  $v_i(m)$  are close to  $p$  in the Euclidean metric in  $\overline{D^n}$ . If  $p$  is in  $D^n$ , then the volume of  $\sigma_m$  tends to zero which contradicts the assumption. If  $p$  is in  $S^{n-1}$ , then  $\sigma_m$  cannot have the center to be 0 for  $m$  large. In the second possibility, we may assume that  $u_2 = \dots = u_{n+1} \neq u_1$ . In this case, consider the codimension-1 face  $\sigma_m^1 = C(v_2(m), \dots, v_{n+1}(m))$ . This  $(n-1)$ -simplex is close to  $u_2$  for  $m$  large in the Euclidean metric. Since the face is tangent to 0, it follows that  $u_2 = \dots = u_{n+1} = 0$ . This implies that the  $(n-1)$ -dimensional volume  $\text{vol}_{n-1}(\sigma_m^1)$  tends to zero. By Thurston's inequality, i.e. Lemma 4.4, this implies that the volume of  $\sigma_m$  tends to zero. This is again a contradiction.  $\square$



Thus, by Proposition 4.1 and (4.2), the simplex  $\sigma = \sigma(u_1, \dots, u_{n+1})$  has positive volume. This implies that  $\sigma$  is a non-degenerated  $n$ -simplex in  $D^n$  whose center is 0. Let  $\chi_m$  and  $\chi$  be the characteristic functions of  $\sigma_m$  and  $\sigma$  in  $\overline{D^n}$ . Then by definition, the function  $\chi_m$  converges almost everywhere to  $\chi$  in  $D^n$ . Furthermore, by Proposition 4.1, the integral  $\int_{D^n} \chi_m dv$  converges to  $\int_{D^n} \chi dv$  where  $dv$  is the hyperbolic volume element in  $D^n$ . By Fatou's lemma (see for instance [12, p. 86, Problem 9]), this implies that for any ball of radius  $R$  centered at 0,  $\text{vol}(\sigma_m - B_R(0))$  converges to  $\text{vol}(\sigma - B_R(0))$ . Choose  $R$  so large that  $\text{vol}(\sigma - B_R(0)) \leq \epsilon_0/2$ . Then, for  $m$  large, we have  $\text{vol}(\sigma_m - B_R(0)) < \epsilon_0$ . However, this contradicts (4.3) for  $m$  large.  $\square$

### 4.3. A proof of Proposition 4.1

By the work of Ratcliffe [10], it suffices to show the proposition in two cases. In the first case, the number of elements in the set  $\{u_1, \dots, u_{n+1}\}$  is between 3 and  $n$ . In the second case,  $\{u_1, u_2, \dots, u_{n+1}\}$  consists of two elements  $\{p, q\}$ ,  $p \neq q$ , so that both sets  $\{i \mid u_i = p\}$  and  $\{j \mid u_j = q\}$  contain at least two points. The goal is to show that  $\lim_{m \rightarrow \infty} \text{vol}(\sigma_m) = 0$  in both cases.

The proposition holds for  $n = 2$ . Indeed, in this case,  $u_1, u_2, u_3$  are pairwise distinct. Thus, the result was proved by Ratcliffe. Assume from now on that  $n \geq 3$ .

First of all, we claim

**Claim.** *If  $u_i = u_j$  for  $i \neq j$  so that  $u_i$  is in  $D^n$ , then  $\lim_{m \rightarrow \infty} \text{vol}(\sigma_m) = 0$ .*

Indeed, by Lemma 4.4, we can estimate  $\text{vol}(\sigma_m) \leq 1/(n-1)! \text{vol}_1(v_i(m), v_j(m))$ . Now,  $\text{vol}_1(v_i(m), v_j(m)) = d(v_i(m), v_j(m))$  tends to  $d(u_i, u_j) = 0$ .

By this claim, we may assume from now on that if  $u_i = u_j$ ,  $i \neq j$ , then  $u_i \in S^{n-1}$ .

By the assumption on  $\{u_1, \dots, u_{n+1}\}$ , we may choose four points, say  $u_1, u_2, u_3, u_4$  so that  $u_1 = u_2$  and either  $u_3 = u_4 \neq u_1$ , or  $\{u_1, u_2, u_3, u_4\}$  consists of three points. By Lemma 4.4, we have  $\text{vol}(\sigma_m) \leq 1/((n-1) \cdots 4 \cdot 3) \text{vol}_3(C(v_1(m), v_2(m), v_3(m), v_4(m)))$ . This implies that it suffices to prove the proposition for  $n = 3$  which we will assume.

To prove the proposition, there are two cases to be considered: case 1, all  $u_i$ 's are in  $S^2$ , and case 2, some  $u_i$  are in  $D^3$ .

In the first case that all  $u_i$ 's are in  $S^2$ , let  $w_1(m), \dots, w_4(m)$  be four points in  $S^2$  so that  $v_1(m), v_3(m)$  lie in the geodesic from  $w_1(m)$  to  $w_3(m)$  and  $v_2(m), v_4(m)$  lie in the geodesic from  $w_2(m)$  to  $w_4(m)$ . We choose  $w_1(m)$  to be the endpoint in the ray from  $v_3(m)$  to  $v_1(m)$  and  $w_2(m)$  similarly. By the construction, we still have  $\lim_{m \rightarrow \infty} w_i(m) = u_i$  for  $i = 1, 2, 3, 4$ . Furthermore, by the construction  $C(w_1(m), \dots, w_4(m))$  contains the tetrahedron  $C(v_1(m), v_2(m), v_3(m), v_4(m))$ . In particular,

$$\text{vol}(C(v_1(m), \dots, v_4(m))) \leq \text{vol}(C(w_1(m), \dots, w_4(m))).$$

Now, the volume of the ideal tetrahedra  $C(w_1(m), \dots, w_4(m))$  can be calculated from the cross ratio of the four vertices  $w_1(m), \dots, w_4(m)$ . To be more precise,

by [14], the volume of an ideal hyperbolic tetrahedron with vertices  $z_1, z_2, z_3, z_4 \in \mathbb{C}$  depends continuously on the cross ratio  $[z_1, z_2, z_3, z_4] = \frac{z_1 - z_3}{z_1 - z_4} : \frac{z_2 - z_3}{z_2 - z_4}$ . In particular, if the cross ratio tends to 0, 1, or  $\infty$ , then the volume tends to 0. In our case, by the assumption, we see that the cross ratio of  $(w_1(m), w_2(m), w_3(m), w_4(m))$  tends to the cross ratio of  $u_1, u_2, u_3, u_4$  which is 0, 1, or  $\infty$ . Thus, the volume  $\text{vol}_3(C(v_1(m), v_2(m), v_3(m), v_4(m)))$  tends to 0.

In the second case that one of the points of  $\{u_1, u_2, u_3, u_4\}$  is in  $D^3$ , by the above claim, we may assume that  $u_1 = u_2$  is in  $S^2$ . Furthermore, by the claim, we may assume that  $u_3 \neq u_4$  and  $u_3 \in D^3$ . Note that  $u_4 \neq u_1$ . Let  $w_1(m), \dots, w_4(m)$  be four points in  $S^2$  constructed as in the previous paragraph. By the construction  $C(w_1(m), \dots, w_4(m))$  contains the tetrahedron  $C(v_1(m), v_2(m), v_3(m), v_4(m))$ . Furthermore, we have  $\lim_m w_1(m) = \lim_m w_2(m) = u_1$ , and  $\lim_m w_3(m) = w_3$  and  $\lim_m w_4(m) = w_4$  both exist so that the cross ratio of  $\{u_1, u_1, w_3, w_4\}$  is 0, 1, or  $\infty$ . Thus by case 1, we see that the volume of  $C(w_1(m), \dots, w_4(m))$  tends to zero. This in turn implies that the volume of  $C(v_1(m), \dots, v_4(m))$  tends to zero. This finishes the proof.

#### 4.4. A proof of Proposition 4.2

Suppose otherwise, there is  $\epsilon_0 > 0$ , a sequence of hyperbolic  $n$ -simplices  $\sigma_m = C(v_1(m), \dots, v_{n+1}(m))$  with center 0 in  $D^n$  so that the radius of  $\sigma_m$  is at most  $1/m$  and its volume  $\text{vol}(\sigma_m) \geq \epsilon_0$ . By taking a subsequence if necessary, we may assume that the limit  $\lim_{m \rightarrow \infty} v_i(m) = u_i$  exists in  $\overline{D^n}$ . Then by Lemma 4.5 and Proposition 4.1, we conclude that  $\sigma = C(u_1, \dots, u_{n+1})$  is a non-degenerate generalized  $n$ -simplex. In particular, these vectors  $u_1, \dots, u_{n+1}$  are linearly independent in  $\mathbf{R}^{n+1}$ . On the other hand, since the radius of  $\sigma_m$  tends to zero, we see that all codimension-1 totally geodesic surfaces  $\text{sp}\{u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_{n+1}\}$  contain 0. This is impossible for a non-degenerated simplex.  $\square$

#### 4.5. A proof of Lemma 4.3

Suppose otherwise, there exist  $\delta_0 > 0$ ,  $r_0 > 0$ , a sequence of  $n$ -simplices  $\{\sigma_m \mid m \in \mathbf{Z}_{\geq 1}\}$  of radius at least  $\delta_0$ , and a point  $x_m \in \sigma_m$  so that

(4.4)  $x_m$  is within  $r_0$  distance to each codimension-1 totally geodesic surface containing a codimension-1 face of  $\sigma_m$ , and,

(4.5)  $d(x_m, c_m) \geq m$  where  $c_m$  is the center of  $\sigma_m$ .

Let us put the center  $c_m$  of  $\sigma_m$  to be the origin 0 of  $D^n$ . By choosing a subsequence if necessary, we may assume that  $\sigma_m = C(v_1(m), \dots, v_{n+1}(m))$  so that  $\lim_{m \rightarrow \infty} v_i(m) = u_i$  exists in  $\overline{D^n}$ , and  $\lim_{m \rightarrow \infty} x_m = x$  also exists in  $\overline{D^n}$ . Since the radius of  $\sigma_m$  is bounded away from zero, we apply Lemma 4.5 and Proposition 4.1 to conclude that the simplex  $\sigma = C(u_1, \dots, u_{n+1})$  is non-degenerated whose center is 0. Since  $d(x_m, 0) \geq m$ , it follows that  $x$  has to be one of the vertex, say  $u_1$  of  $\sigma$ . Now,

consider the upper-half space model  $U^n$  for the hyperbolic space so that  $u_1 = x$  is the infinity and the totally geodesic codimension-1 surface containing  $u_2, \dots, u_{n+1}$  is the unit upper-hemisphere  $S_+^{n-1} = \{(t_1, \dots, t_n) \in \mathbf{R}^n \mid \sum_{i=1}^n t_i^2 = 1, t_n > 0\}$ . Let the center of the simplex  $\sigma$  in this model be  $C$  and the point of the shortest distance to  $C$  in  $S_+^{n-1}$  be  $P$ . We claim that the angle  $\angle PCu_1$  at  $C$  is at least  $\pi/2$ . This follows from the Gauss–Bonnet theorem. Let  $U^2$  be the unique 2-dimensional hyperbolic plane containing  $C$  and  $u_1$  so that  $U^2$  is perpendicular to  $S_+^{n-1}$ . Let  $Q = [0, \dots, 0, 1]^t$  be the north pole in  $S_+^{n-1}$ . Then by the construction,  $Q \in U^2$  and  $P \in U^2$  due to the orthogonality. If  $P = Q$ , then the angle  $\angle PCu_1$  is  $\pi$ . The claim follows. If otherwise, consider the hyperbolic quadrilateral  $QPCu_1$  in  $U^2$ . The angle of the quadrilateral at  $Q$ ,  $P$  and  $u_1$  are  $\pi/2$ ,  $\pi/2$  and  $0$ , respectively. On the other hand, since  $C$  is the center of the simplex  $\sigma$ , the complete geodesic from  $u_1$  to  $C$  intersects the hemisphere  $S_+^{n-1}$  at some point, say  $R$ . Thus, the quadrilateral  $QPCu_1$  is inside the hyperbolic triangle  $\Delta u_1QR$  whose inner angles are  $\pi/2, 0, \theta$ . In particular, the area of this triangle is less than  $\pi/2$  by the Gauss–Bonnet formula. This implies that the area of the quadrilateral  $QPCu_1$  is at most  $\pi/2$ . By the Gauss–Bonnet formula, we conclude that the angle  $\angle PCu_1$  at  $C$  is at least  $\pi/2$ .

On the other hand, we will derive from (4.4) and (4.5) that the angle  $\angle PCu_1$  is strictly less than  $\pi/2$ . Thus, we arrive a contradiction. To see this, let  $P_m$  be the point in the totally geodesic codimension-1 surface  $\text{sp}(C(v_2(m), \dots, v_{n+1}(m)))$  which is closest to the center  $c_m$  of  $\sigma_m$ . By the construction, the limit of the angle  $\angle P_m c_m x_m$  is equal to  $\angle PCu_1$ . To estimate the angle  $\angle P_m c_m x_m$ , consider the 2-dimensional totally geodesic plane  $D_m$  which contains  $c_m$  and  $x_m$  so that  $D_m$  is perpendicular to  $\text{sp}(C(v_2(m), \dots, v_{n+1}(m)))$ . By the construction  $P_m$  is in the plane  $D_m$ . Let  $R_m$  be the point in  $\text{sp}(C(v_2(m), \dots, v_{n+1}(m)))$  of the shortest distance to  $x_m$ . Then, we again have  $R_m$  is in  $D_m$ . Consider the quadrilateral  $P_m R_m x_m c_m$  in the plane  $D_m$ . The angles at the vertices  $P_m$  and  $R_m$  are  $\pi/2$ . The distances  $d(c_m, P_m) \geq \delta_0$ ,  $d(x_m, R_m) \leq r_0$  and  $d(c_m, x_m) \geq m$ . Thus, as  $m$  becomes large, the quadrilateral is tending to a right-angled hyperbolic triangle with one vertex at infinity (corresponding to  $R_m$  and  $x_m$ ). There is an edge of the triangle having finite length which is at least  $\delta_0$  (corresponding to the edge between  $c_m$  and  $P_m$ ). The acute angle at the end point of this finite length edge is at most  $\theta = \arcsin(1/\cosh(\delta_0)) < \pi/2$  by the cosine law. Thus, as  $m$  tends to infinity, the angle  $\angle P_m c_m x_m$  tends to a number less than or equal to  $\theta$ . In particular, the angle  $\angle P_m c_m x_m$  is strictly less than  $\pi/2$  for  $m$  large. This contradicts the previous conclusion.

## Acknowledgment

We would like to thank Z.-C. Han, Daniel Ocone and Saul Schleimer for discussions. We thank Professor Nick Higham for directing our attention to the results on matrices and Igor Rivin for informing us that the continuity was conjectured by Milnor and that the volume formula (1.1) was actually known to Kneser.

### Appendix A. Proof of Lemma 3.2

We give a proof of the following lemma used in the paper.

**Lemma 3.2.** *Given a symmetric  $n \times n$  matrix  $A$  of signature  $(k, 0)$  or  $(k, 1)$ , and  $\epsilon > 0$ , there exists a positive diagonal matrix  $D$  so that  $|D - Id| \leq \epsilon$  and all non-zero eigenvalues of  $DAD$  are simple.*

**Proof.** For of all, it suffices to find a positive diagonal matrix  $D$  so that all non-zero eigenvalues of  $DAD$  are simple. This is due to the fact from algebraic geometry that an algebraic subvariety in  $\mathbf{R}^m$  is either the whole space or has zero Lebesgue measure. By [3], the set of all diagonal matrices  $D$  so that  $DAD$  has a non-simple non-zero eigenvalue forms an algebraic variety  $X$  in  $\mathbf{R}^m$ . Thus, as long as  $X \neq \mathbf{R}^m$ , we can pick  $D$  in  $\mathbf{R}^m$  within  $\epsilon$  distance to  $[1, \dots, 1]^t$  so that  $D \notin X$ .

Next, we claim that it suffices to prove the lemma for  $n \times n$  matrix  $A$  so that  $\det(A) \neq 0$ . Indeed, if  $k = \text{rank}(A)$ , due to the fact that  $A$  is diagonal,  $A$  has exactly  $k$  non-zero eigenvalues counted with multiplicity and  $A$  has a non-singular principal  $k \times k$  submatrix  $B$  formed by  $(i_1, \dots, i_k)$ th rows and columns of  $A$ . For simplicity, we assume that  $B$  is formed by the first  $k$  rows and columns of  $A$ . Then, by the result for non-singular symmetric matrix, we find a positive diagonal matrix  $D_1 = \text{diag}(a_1, \dots, a_k)$  so that  $D_1 B D_1$  has  $k$  distinct eigenvalues. Consider the  $n \times n$  matrix  $D(t) = \text{diag}(a_1, \dots, a_k, t, t, \dots, t)$  where  $t > 0$ . For  $t$  small, by Theorem 2.7, the eigenvalues of  $D(t)AD(t)$  are close to the eigenvalues of  $D_1 B D_1$  and 0. Since  $D_1 B D_1$  has  $k$  distinct non-zero eigenvalues, this implies that  $D(t)AD(t)$  has  $k$  distinct non-zero eigenvalue for  $t$  small.

Finally, we prove the lemma for non-singular matrices using induction on the size of the matrix. The result clearly holds for  $1 \times 1$  and  $2 \times 2$  matrices. Suppose  $A$  is a non-singular  $n \times n$  matrix for  $n \geq 3$ . Let  $B$  the principal submatrix of  $A$  obtained by removing the last column and the last row. Then, the signature of  $B$  is either  $(n-2, 1)$ ,  $(n-1, 0)$  or  $(n-2, 0)$ . By the induction hypothesis and the argument in the previous paragraph, we find a positive diagonal matrix  $D_1 = \text{diag}(a_1, \dots, a_{n-1})$  so that all  $(n-1)$ -eigenvalues of  $B$  are distinct. Let us denote the eigenvalues of  $B$  by  $\lambda_1 > \dots > \lambda_{n-1}$ . Now, consider the positive diagonal matrix  $D(t) = \text{diag}(a_1, \dots, a_{n-1}, t)$  for  $t > 0$ . For  $t$  small, the eigenvalues  $\mu_1(t) \geq \mu_2(t) \geq \dots \geq \mu_n(t)$  of  $D(t)AD(t)$  is close to  $\{\lambda_1, \dots, \lambda_{n-1}, 0\}$ . We claim that for  $t$  small  $D(t)AD(t)$  has  $n$  distinct eigenvalues. Indeed, if  $B$  is non-singular, i.e. the set  $\{\lambda_1, \dots, \lambda_{n-1}, 0\}$  consists of  $n$  distinct elements, then for  $t > 0$  small,  $\mu_i(t) \neq \mu_j(t)$  for  $i \neq j$ . If  $B$  is singular, then  $B$  is semi-positive definite of rank  $n-2$ . Furthermore, this implies that  $A$  has signature  $(n-1, 1)$ . In particular,  $D(t)AD(t)$  has a negative eigenvalue, i.e.  $\mu_n(t) < 0$ . Now for  $t$  small, we conclude that  $\mu_1(t), \dots, \mu_{n-1}(t)$  are positive and are close to the set of  $n-1$  distinct numbers  $\{\lambda_1, \dots, \lambda_{n-2}, 0\}$  where  $\lambda_i > 0$ . This implies that for  $t > 0$  small, the eigenvalues  $\mu_1(t), \dots, \mu_{n-1}(t)$  are positive and pairwise distinct. Since the smallest eigenvalue  $\mu_n(t) < 0$ , we conclude that  $D(t)AD(t)$  has  $n$  distinct eigenvalues.

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