Lecture 5

The moduli space problem

Riemann: Σ topological surface $\mathcal{C}$ complex structure on Σ.

$$\mathrm{Mod}(\Sigma) = \{ [(\Sigma, \mathcal{C})] \mid (\Sigma, \mathcal{C}) \sim (\Sigma, \mathcal{C}') \text{ if biholomorphic} \}.$$

Riemann's moduli space of the surface

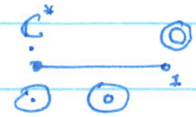
or, classify Riemann surfaces up to analytic homeomorphisms

Uniformization thm $\Rightarrow$

$\mathrm{mod}(S^2) = \{[\hat{\mathbb{C}}]\}$

$\mathrm{mod}(\mathbb{R}^2) \cong \{[\mathbb{C}], [\mathbb{D}]\}$ $\qquad [\mathbb{D}] = [\mathbb{H}]$

$\mathrm{mod}(S^1 \times (0,1)) = \{ [R_\mu] \mid 0 \le \mu < 1 \} \cup \{[\mathbb{C}^*]\}$



What about other surfaces? What is $\dim(\mathrm{mod}(\Sigma))$? ...

Thm $\mathrm{mod}(S^1 \times S^1) = \{ [\mathbb{C}/\mathbb{Z}1 \oplus \mathbb{Z}\tau] \mid \tau \in \mathbb{H} \}$.

Pf It suffices to show that the universal cover $\tilde{X}$ of $X \underset{\text{homeo}}{\cong} S^1 \times S^1$ is $\mathbb{C}$ Not $\mathbb{D}$.

Let $\Gamma = \mathbb{Z}(\gamma_1) \oplus \mathbb{Z}(\gamma_2)$ be the $\pi_1(X)$

First $\tilde{X}$ non-compact $\Rightarrow$ $\tilde{X} \not\cong \hat{\mathbb{C}}$.

Now $\tilde{X} \not\cong \mathbb{H}$: if otherwise, then γ_1, γ_2 are two commuting elements in $PSL(2,\mathbb{R})$ corresponding to

(1) $\gamma_1 \sim A_1 = \begin{bmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{bmatrix}$ $\quad A_2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ or $\gamma_1 \circ \gamma_2 = \gamma_2 \circ \gamma_1$

(2) $\gamma_1 \sim A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\quad A_2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad \gamma_2(z) = \frac{az+b}{cz+d}$

(2) $\gamma_1(z) = z+1$ $\quad \gamma_1 \circ \gamma_2 = \frac{az+b}{cz+d} + 1 = \frac{(a+c)z + (b+d)}{cz+d} = \frac{a(z+1)+b}{c(z+1)+d} = \frac{az+(a+b)}{cz+(c+d)}$

$\Rightarrow \gamma_2(z) = z + b$ translations only $\Rightarrow$ Non-discrete

(2) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \pm \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \Leftrightarrow \begin{bmatrix} a+c & b+d \\ c & d \end{bmatrix} = \pm \begin{bmatrix} a & a+b \\ c & c+d \end{bmatrix}$ $\boxed{HW}$

(+) $\Rightarrow$ $c = 0$ $\Rightarrow$ translation. $\qquad +1 \Rightarrow c = b = 0$

(1) $\begin{bmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \pm \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{bmatrix} \Leftrightarrow \begin{bmatrix} \lambda a & \lambda b \\ \frac{1}{\lambda} c & \frac{1}{\lambda} d \end{bmatrix} = \pm \begin{bmatrix} \lambda a & \frac{1}{\lambda} b \\ \lambda c & \frac{d}{\lambda} \end{bmatrix}$ (-1) same

Lecture 5.

Therefore $\gamma_i(z) = \lambda_i z$ $\quad$ $\underline{\lambda_1, \lambda_2 \neq 0 \text{ real}}$ $\quad$ But again $\Rightarrow$ It is not discrete

Since $\Rightarrow$ $\mathbb{Z} \oplus \mathbb{Z}$ acts on $\mathbb{R}$ by $z \mapsto z + \lg \lambda_i$ (Taking log)

$\Rightarrow$ Conclusion, there is no discrete subgroup of $PSL(2, \mathbb{R})$

acting properly discont. on $\mathbb{H}$.

$\Rightarrow$ $\tilde{X} = \mathbb{C}$. By the classification before. $\square$

Lecture 5 Riemannian metrics and Riemann surfaces

V n-dim vector space $/\mathbb{R}$ w/ basis $e_1, \cdots, e_n$

V^* dual space w/ dual basis $e_1^*, \cdots, e_n^*$

Eg $\Omega \subset \mathbb{R}^2$ open $V = T_p\Omega$ w/ basis $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ dual dx, dy

A bilinear form $\langle , \rangle : V \times V \to \mathbb{R}$ is written as $\sum \langle e_i, e_j \rangle e_i^* \otimes e_j^*$

w/ coefficient matrix $M = [\langle e_i, e_j \rangle]$. $\langle , \rangle$ positive definite

$$\langle v, v \rangle > 0 \quad \forall\ v \neq 0$$

A Riemannian metric ds^2 on Ω: assigns $\forall\ p \in \Omega$ a positive definite symmetric bilinear form $\langle , \rangle_p = ds^2_p$ s.t. it varies smoothly in p : $\langle \partial/\partial x, \frac{\partial}{\partial y} \rangle_p$, $\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial x} \rangle_p$ $\langle \frac{\partial}{\partial y}, \frac{\partial}{\partial y} \rangle_p$ smooth in p.

$$ds^2 = A\,dx^2 + 2B\,dx\,dy + C\,dy^2 \qquad dx^2 = dx \otimes dx$$

$$A, C > 0, \quad AC - B^2 > 0 \qquad dx\,dy = \tfrac{1}{2}(dx \otimes dy + dy \otimes dx)$$

$M = \begin{bmatrix} A & B \\ B & C \end{bmatrix}$ positive definite matrix.

We define Riemann metric ds^2 on a smooth surface M similarly

Eg 1 $\Sigma \subset \mathbb{R}^3$ smooth $ds^2 = \langle , \rangle_{\mathbb{R}^3}\big|_{T_p\Sigma}$, induced metric (Gauss)

Eg 2 $\Omega = \mathbb{H}$, $ds^2 = \dfrac{dx^2 + dy^2}{y^2}$ $M = \begin{bmatrix} \frac{1}{y^2} & 0 \\ 0 & \frac{1}{y^2} \end{bmatrix}$. $\langle \frac{\partial}{\partial x}\big|_{(x,y)}, \frac{\partial}{\partial x}\big|_{(x,y)} \rangle = \frac{1}{y^2}$.

$\Omega = \mathbb{D}$ $ds^2 = \dfrac{4(dx^2+dy^2)}{(1-(x^2+y^2))^2}$ Hyperbolic metrics

What is the use of metric? : length of $v \in T_p\Omega$ $\|v\| = \langle v, v \rangle_p^{\frac{1}{2}}$

and angle θ between $u, v \in T_p\Omega - \{0\}$ $\cos\theta = \langle u, v \rangle / \|u\|\|v\|$.

Eg 3 (Lemma) If $\gamma(t) = (x(t), y(t))$, $0 \le t \le 1$, is a smooth path in $\mathbb{H}$ with $\gamma(0) = ia$ $\gamma(1) = ib$ $0 < a < b$. then the length of γ in $(\mathbb{H}, ds^2)$

$$\text{length}(\gamma) \ge \log\left(\frac{b}{a}\right) \text{ with equality iff } x(t) = 0 \text{ and } y'(t) \ge 0.$$

Lecture 5 Riemannian metrics

A smooth map $F = (\Sigma_1, ds_1^2) \longrightarrow (\Sigma_2, ds_2^2)$ is an isometry if $DF: T_p\Sigma_1 \to T_{F(p)}\Sigma_2$ is an isometry. $\Leftrightarrow$

$F^*: T^*_{F(p)}\Sigma_2 \to T^*_p\Sigma_1$ $\quad \underline{F^*(ds_2^2) = ds_1^2}$ (Pull back).

How to compute $\underline{F^* \text{ on tensors}}$?

<u>Formula</u>: $F(x,y) = \overset{(u,v)}{(u(x,y), v(x,y))} : \Omega \to \mathbb{C}$ $\quad \Omega \subset \mathbb{C}$ open

Then $\quad F^*(du) = u_x\,dx + u_y\,dy, \quad F^*(dv) = v_x\,dx + v_y\,dy$

$F^*(du)^2 = (u_x dx + u_y dy)^2$ etc.

<u>Eg</u> $\quad F(z) = -\frac{1}{z} = \frac{-\bar z}{|z|^2} \qquad F(x,y) = \left(\overset{u}{\overset{\shortparallel}{\frac{-x}{x^2+y^2}}}, \overset{v}{\overset{\shortparallel}{\frac{y}{x^2+y^2}}}\right) : \mathbb{H} \to \mathbb{H}.$

Möbius of matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \in SL(2,\mathbb{R})$

Then $du = \frac{(-1)}{(x^2+y^2)^2}\left[(x^2+y^2)dx - x\,d(x^2+y^2)\right] = \frac{1}{|z|^4}\left[(x^2-y^2)\,dx + 2xy\,dy\right]$

$dv = \qquad \frac{1}{|z|^4}\left[(x^2-y^2)dy - 2xy\,dx\right]$

$du^2 + dv^2 = \frac{1}{|z|^8}\left[(x^2-y^2)^2 + 4x^2y^2\right](dx^2+dy^2) = \frac{|z|^4}{|z|^8}(dx^2+dy^2)$

$= \frac{1}{|z|^4}(dx^2+dy^2)$

<u>Now</u> $\quad \frac{du^2+dv^2}{v^2} = \frac{\frac{1}{|z|^4}(dx^2+dy^2)}{y^2/|z|^4} = \frac{dx^2+dy^2}{y^2}$ $\quad$ (Note F is a $\mathbb{H}$ preserving!)

<u>Conclusion</u> $\quad$ It preserves the Riemannian metrics

Eg Easy $\quad F(z) = z + a \quad F(z) = \lambda z \quad$ are isometries of $\left(\mathbb{H}, \frac{dx^2+dy^2}{y^2}\right)$

<u>RM</u> $\quad d\left(\frac{f}{g}\right) = \frac{g\,df - f\,dg}{g^2}$

<u>Prop</u> $\quad PSL(2,\mathbb{R})$ acting on $\mathbb{H}$. preserving the Riemannian metric $\frac{dx^2+dy^2}{y^2}$

(the hyperbolic metric)

Lecture 5. Riemannian metrics

If $\langle , \rangle$, $(,)$ are two inner product on V s.t $\langle u, v\rangle = \lambda (u, v)$ for $\lambda > 0$.
then they define the same notation of angles.
We say $\langle , \rangle$ and $(,)$ are conformal.

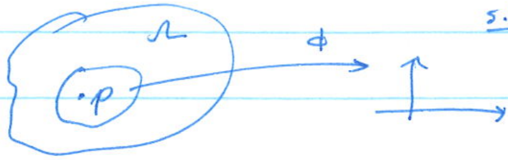
Homework If $\langle , \rangle$, $(,)$ satisfy $\dfrac{\langle u, v\rangle}{(\langle u,u\rangle\langle v,v\rangle)^{\frac12}} = \dfrac{(u,v)}{(u,u)^{\frac12}(v,v)^{\frac12}}$ $\forall u, v \neq 0$

Then $\langle , \rangle, (,)$ are conformal $\langle , \rangle = \lambda (,)$.

For instance: $\dfrac{dx^2+dy^2}{y^2}$ and dx^2+dy^2 is are conformal. $F: (\Sigma_1 d_1) \to (\Sigma_2 d_2)$ $\downarrow$ conformal

Def Two Riemannian metrics ds_1^2, ds_2^2 are conformal on Σ^2. iff $\exists$
$u: \Sigma \to \mathbb{R}$ smooth st $ds_1^2 = e^{2u} ds_2^2$ (positive multiple)
The calculation $F(z) = -\frac{1}{z}$ $\Leftrightarrow$ F is angle preserving (Another way)

Gauss Question $\forall$ metric ds^2 on $\Omega \underset{\text{open}}{\subset} \mathbb{C}$, $\forall$ $p \in \Omega$.
Is there a nbhd smooth chart (U, ϕ) at p



s.t $\phi: (U, ds^2|) \to (\mathbb{C}, \overset{|dz|^2}{\overset{\shortparallel}{dx^2+dy^2}})$
is angle preserving? (Conformal)

$$\phi^*(dx^2+dy^2) = e^{u(x,y)} ds^2 \quad ?$$

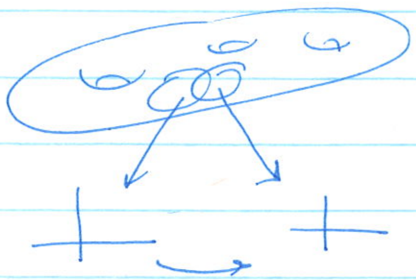
One calls ϕ: isothermal coordinate.
ANS: Yes, ϕ exists, $\uparrow$ Beltrami Equation (Elliptic linear PDE)

Try: $\phi(x,y) = (u, v)$ $\mu \cdot ds^2 = A\,dx^2 + 2B\,dx\,dy + C\,dy^2 = \phi^*(du^2+dv^2)$
$= (u_x dx + u_y dy)^2 + (v_x dx + v_y dy)^2$

Corollary: (Σ, ds^2) smooth oriented surface w/ a Riemannian metric ds^2.
Then $\exists$ collection of charts $\mathcal{C} = \{(U_\alpha, \phi_\alpha) \mid \alpha \in A\}$ covering Σ st
$\phi_\alpha \circ \phi_\beta^{-1}$ analytic and the notion of angles from $\mathcal{C}$ + ds^2 are the same.

Lecture 5 Riemannian metric.

Proof.



ds^2 M

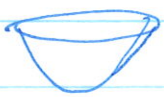
$\forall\, p \in M$, $\exists$ nbhd U_p of p + $\phi_p : U_p \to \mathbb{C}$ which is conformal.

$\phi_p : (U_p, ds^2) \to (\mathbb{C}, dx^2+dy^2)$

$\phi_p \circ \phi_{p'}^{-1} : V_1 \to V_2$ in $\mathbb{C}$ + orientation preserving

orientation preserving + angle preserving $\Rightarrow$ analytic $\square$

Huge list, collection of (surp supply of) Riemann surfaces : $\forall$ smooth $\Sigma \subset \mathbb{R}^3$ is naturally a Riemann surface.

Eg 1. $\Sigma = \{ z = x^2+y^2 \}$  $\underset{\text{b.ho}}{\cong}$ $\mathbb{C}$. what is the map conformal. Can you find it? $\boxed{\text{HW}}$ (Derive the ODE)

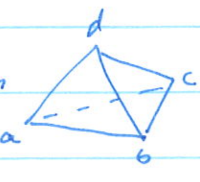
Eg 2 Your face $\underset{\text{biholo}}{\cong}$ $\mathbb{D}^2$  $\underset{\text{b.ho}}{\cong}$ $\mathbb{D}^2$ Can you compute it?

Hint For Eg 1. $(r,\theta) \mapsto (f(r)\cos\theta,\ f(r)\sin\theta,\ f^2(r))$ where $f' = \frac{f}{\sqrt{1+4f^2}}$

or $\sqrt{1+4f^2(x)} - tg^{-1}\left(\frac{1}{\sqrt{1+4f^2(x)}}\right) = x + a.$

Final list of examples:

Eg 3 Polyhedral surfaces

Every tetrahedron 

$\underset{\text{biholo}}{\cong}$ $\hat{\mathbb{C}}$ φ $\downarrow$ Möbius

$\varphi(a) = \infty$
$\varphi(b) = 0$
$\varphi(c) = 1$
$\varphi(d) \in Z$.

$Z \in \mathbb{C} - \{0,1\}$ is an invariant of the tetra. No one knows how to compute z. + its properties in terms of lengths.

Bers Conjecture Every closed Riemann surface is b.holo to a Euclidean polyhedral surface in $\mathbb{R}^3$.

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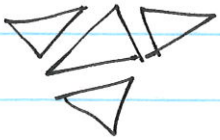
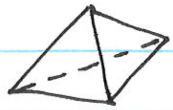
Eg Polyhedral surfaces: take a collection of disjoint Euclidean triangles



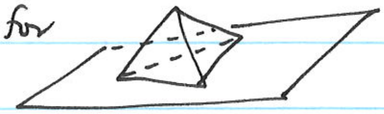
identify pairs of edges by isometries, the quotient is a polyhedral surface:



(with a polyhedral metric).

Prop Each oriented polyhedral surface is a R-surface.

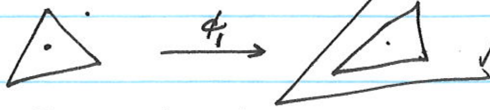
Proof Let us prove it for



Topology o.k a surface. (All maps orientation pres)

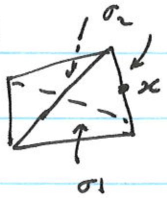
Analytic charts: $\forall\ x \in \Sigma$

(i) $x \in \mathrm{int}(\Delta)$



$\mathcal{U} = \mathrm{int}(\Delta)$ $\quad \phi : \mathcal{U} \to \mathbb{C}$ isometry

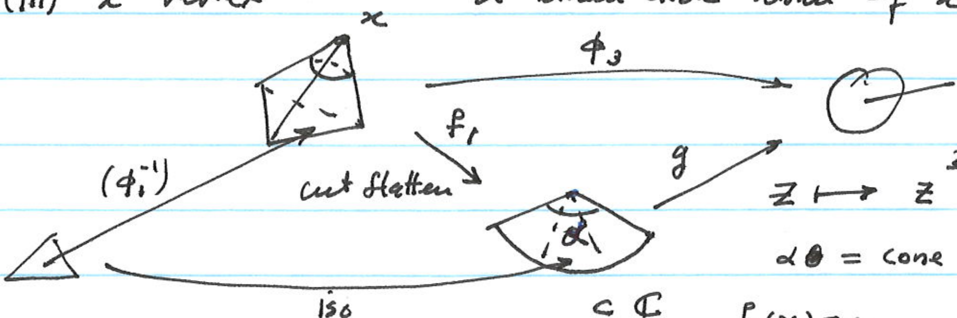
(ii) $x \in \mathrm{int}(e)$



$\mathcal{U} = \mathrm{int}(\sigma_1) \cup \mathrm{int}(\sigma_2) \cup \mathrm{int}(e)$

$\phi_2 : \mathcal{U} \to \mathbb{C}$ $\quad$ flatten, iso

(iii) x vertex $\quad$ $\mathcal{U}$ small disk nbhd of x



$z \mapsto z^{\frac{2\pi}{\alpha}}$ $\quad$ $re^{i\theta} \mapsto r^{\frac{2\pi}{\alpha}} e^{i\frac{2\pi\theta}{\alpha}}$

$\alpha\theta$ = cone angle at x

$f_1(x) = 0$

Note f_1 Not well defined on $\mathcal{U}$ but $\underline{g \circ f}$ is!

We claim $\quad \phi_i \circ \phi_j^{-1}$ analytic $\quad$ clear isometry of $(\mathbb{C}, |dz|^2)$ if ϕ_i, ϕ_j (i) (ii)

$z \mapsto e^{i\theta} z + b$ $\quad$ analytic

$\phi_i \circ \phi_j^{-1}$: composition of an isometry and $z \mapsto z^{\mu}$.

branch of $\quad$ $\mu \in \mathbb{R}_{>0}$