Lecture 2. Riemann surfaces, More examples

Recall $(\Sigma^2, \mathcal{C})$ $\mathcal{C}$ special collection of charts covering Σ^2. R-surface if $\phi_\alpha \circ \phi_\beta^{-1}$ analytic $\forall \alpha\beta$

- (1) If $\phi_\alpha \circ \phi_\beta^{-1}$ smooth $\forall \alpha, \beta$, we call $(\Sigma^2, \mathcal{C})$ a <u>smooth</u> surface
- (2) If $\phi_\alpha \circ \phi_\beta^{-1}$ orientation preserving (for smooth $\det(D(\phi_\alpha \circ \phi_\beta^{-1})) > 0$ $\forall \alpha\beta$).
  $(\Sigma^2, \mathcal{C})$ orientable surface

Eg. 1 $(S^2, \mathcal{C})$, $\underset{\text{open}}{\Omega \subset \mathbb{C}}$, $\mathbb{C}/\mathbb{Z}$ are smooth orientable surfaces

Proposition 1. $\Omega \subset \mathbb{C}$ open $F(x,y) = (u(x,y), v(x,y)) : \Omega \to \mathbb{C}$ smooth s.t. $\det DF \neq 0$ $\forall (x,y)$. Then $F : \Omega \to \mathbb{C}$ analytic $\Leftrightarrow$ DF preserves orientation + angle

Furthermore, in this case $v = \alpha + i\beta \cong \begin{bmatrix}\alpha\\ \beta\end{bmatrix}$. $DF \cdot \begin{bmatrix}\alpha\\ \beta\end{bmatrix} = F'(z) \cdot (\alpha + i\beta)$

and $\det(DF) = |F'(z)|^2 > 0$ $\Leftrightarrow$ DF complex linear.

Pf <u>Easy fact</u>. For $A = \mathbb{R}^2 \to \mathbb{R}^2$ linear w/ $\det(A) \neq 0$, then A preserves angles and orientation $\Leftrightarrow$ $A = \lambda \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix} = \begin{bmatrix}a & -b\\ b & a\end{bmatrix}$ $a^2+b^2>0$

(A rotation + scaling) $\qquad := \begin{bmatrix}a & c\\ b & d\end{bmatrix}$ $\boxed{\begin{matrix}a=d\\ b=-c\end{matrix}}$ (Hw)

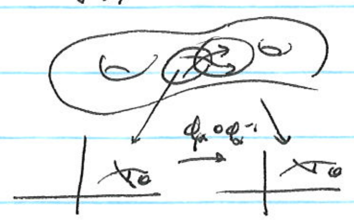
Now $DF = \begin{bmatrix}u_x & u_y\\ v_x & v_y\end{bmatrix}$ $\det DF \neq 0$ $\Leftrightarrow$ $\underbrace{\begin{cases}u_x = v_y\\ u_y = -v_x\end{cases}}_{\text{Cauchy-Riemann.}}$, $\dfrac{u_x + i v_x \neq 0}{F'(z) = u_x + i v_x}$

Next $A = \begin{bmatrix}a & -b\\ b & a\end{bmatrix}$ $v = \begin{bmatrix}\alpha\\ \beta\end{bmatrix}$ then $A \cdot v = \begin{bmatrix}a & -b\\ b & a\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \begin{bmatrix}a\alpha - b\beta\\ b\alpha + a\beta\end{bmatrix}$

$$F'(z) \cdot v = (a+ib)(\alpha + i\beta) = (a\alpha - b\beta) + i(a\beta + b\alpha) \quad . \quad (F' = a + ib) \qquad \square$$

Corollary (1). All Riemann surfaces are orientable (Möbius band  No)

(2). The notion of angle well defined on a R-surface $(X, \mathcal{C})$.



Cauchy-Riemann Eq $u_x = v_y$ $u_y = -v_x$ $\Leftrightarrow$ $DF \circ i = i \circ DF$

RM $DF = \begin{bmatrix}a & -b\\ b & a\end{bmatrix}$ $\Leftrightarrow$ $DF : \mathbb{C} \to \mathbb{C}$ complex linear $DF(i \cdot v) = i(DF(v))$ $\forall v$

$i = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$ $\quad \begin{bmatrix}a & c\\ b & d\end{bmatrix}\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}a & c\\ b & d\end{bmatrix}$ $\Leftrightarrow$ $\begin{matrix}a=d\\ b=-c\end{matrix}$ (Hw)

Lecture 2 Examples of Riemann surf

Algebraic Curves like $\{(z,w)\in\mathbb{C}^2 \mid z^4+w^4=1\}$ or $\{(z,w)\in\mathbb{C}^2 \mid w^2=\prod_{i=1}^{n}(z-a_i)\}$ $a_i\neq a_j$ are Riemann surfaces

Prop Let $P(z,w)$ polynomials in z,w, X a connected component of

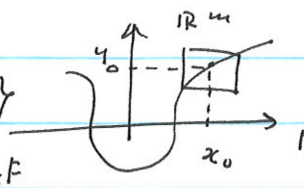
$$\Sigma=\{(z,w)\in\mathbb{C}^2 \mid P(z,w)=0,\quad |P_z(z,w)|+|P_w(z,w)|\neq 0\}.$$ Then

X is a Riemann surf s.t $p_1(z,w)=z$ $p_2(z,w)=w$: $\Sigma\to\mathbb{C}$ are analytic.

Proof Take $(z_0,w_0)\in X$ say $P_w(z_0,w_0)\neq 0$

IFThm $\Omega_1\subset\mathbb{R}^n$, $\Omega_2\subset\mathbb{R}^m$, $F:\Omega_1\times\Omega_2\to\mathbb{R}^m$ smooth and $F(x_0,y_0)=0$. If $D_yF(x_0,y_0)$ invertible, then $\exists$ nbhds U of x_0, and V of y_0 and a smooth $y=y(x): U\to V$ s.t

$$\Sigma\ \{F(x,y)=0\}\cap U\times V=\{(x,y(x))\mid x\in U\}$$



(Graph of a function) Furthermore $DY(x)=-(D_yF)^{-1}\circ D_xF$.

Corollary: $n=m=2$ $F=P(z,w)\Rightarrow w=w(z)$ analytic in z.

polynomial

~~($P_w(z_0,w_0)\neq 0 \Rightarrow D_wF(z_0,w_0)$ invertible) (thm)~~

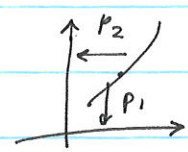
~~Use $\frac{\partial}{\partial\bar z}P=0$ $P(z,w(z))=0$ take $\frac{\partial}{\partial\bar z}$~~ Justify

~~+ chain rule: $P_{\bar z}(z,w(z))+\frac{\partial z}{\partial\bar z}+P_w\cdot\frac{\partial w}{\partial\bar z}=0\Rightarrow\frac{\partial w}{\partial\bar z}=0$~~

Even better: $F(x,Y(x))=0$

Thus by IFThm. $\exists$ nbhds U,V of z_0 + w_0 + analytic $w=w(z)$ st

$$\{P(z,w)=0\}\cap U\times V=\{(z,w(z))\mid z\in U\}$$



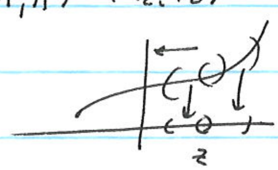
The the analytic charts for X at (z_0,w_0): $(X\cap U\times V,\ p_1)$

Clearly these charts cover X

Claim Transition functions $\phi_1\circ\phi_2^{-1}$ analytic (U_1,ϕ_1) (U_2,ϕ_2)

Pf (1) ϕ_1,ϕ_2 both p_1 (or both p_2)

$$\phi_2\circ\phi_2^{-1}(z)=z$$



(2) $\phi_1=p_2$, $\phi_2=p_1$ $\phi_1\circ\phi_2^{-1}(z)=\phi_1(z,w(z))=w(z)$ analytic

$\square$

$\boxed{\text{RM}}$ Algebraic geometry if $P(z,w)$ irreducible poly $\Rightarrow\Sigma$ connected (Not $P(z,w)=zw$)

Lecture 2 Examples, Riemann surfaces

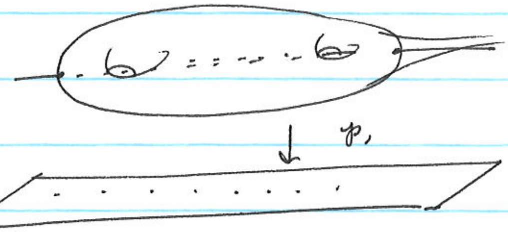
Eg 2. Fermat curve $P_z = P_w = 0 \Rightarrow (z, w) = (0,0) \notin F_n$. o.k. Why is F_n connected?
(Riemann $p_1 : F_n \to \mathbb{C}$ projection to $\mathbb{C}$ image connected)
(Hw)

Eg 3. Hyperelliptic $zw^2 = \prod_{i=1}^{n} (z - a_i)$ $a_i \neq a_j$ (Hy_n) $\boxed{n \geq 2}$

$P_w = 0 \Rightarrow w = 0 \Rightarrow z = a_i$ some i Parts $(a_i, 0)$

$P_z = 0$ $\sum_{j=1}^{n} \prod_{j \neq i} (z - a_j) = 0$. For $(a_i, 0) \Rightarrow \prod_{j \neq i} (a_j - a_i) \neq 0$.

So it is a topological surface:



Hy_n

p_1

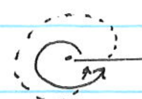
Q. What is the genus of Hy_n $(n-1)(n-2)/2$ (Why is Hy_n connected)

Q genus $\frac{n-1}{2}$, $\frac{n-2}{2}$ depends on the parity

Consider $w^2 = z(z-1)(z-2)$ or $w = \sqrt{z(z-1)(z-2)}$

or even $w^2 = z(z-1)$ $w = \sqrt{z(z-1)}$ what is it?

or even $w^2 = z$ $\boxed{w = \sqrt{z}}$



two sheets

Eg (Riemannian metric + R-surface) Suppose (Σ, ds^2) is an orientable surface w/ a Riemannian metric. Then Σ is automatically a R-surface

Two metrics (Σ, ds^2) (Σ, dt^2) defines the same R-surface

$\Leftrightarrow$ $\exists$ a function $u : \Sigma \to \mathbb{R}$ smooth st

$$dt^2 = e^{2u} ds^2 \quad \text{(Conformal)} \qquad (1)$$

$\boxed{(1)}$ Follows from simple linear algebra if $\langle , \rangle$ and $(,)$ are two inner product on $\mathbb{R}^n$ defining the same notion of angle $\Rightarrow \exists$ constant $k > 0$ st

$$\langle u, v \rangle = k (u, v) \quad \forall u, v \in \mathbb{R}^n \qquad (2)$$

Obviously $\cos\theta = \frac{\langle u, v \rangle}{\langle u, u \rangle^{1/2} \langle v, v \rangle^{1/2}} = \frac{(u, v)}{(u,u)^{1/2} (v,v)^{1/2}}$ (3). If (2) holds $\Rightarrow$ SAME angs
If (3) holds $\forall u, v$. $\Rightarrow$ (2). (Hw)

Lecture 2. Riemann Surfaces, Examples

Indeed, by Rieze rep: $\exists$ linear A, $\langle Ax, y\rangle = (Ax, y)$ $\Rightarrow A = A^T$

so we claim: $A = k \cdot Id$.

If A has two distinct eigenvectors v_1, v_2 for $\lambda_1 \neq \lambda_2$ of norm 1 in $(\,,\,)$

then $\exists$ s.t s.t $u = v_1 + t v_2$, $v = v_1 + s v_2$ satisfies $(u, v) = 0$, $\langle u, v\rangle \neq 0$,

$1 + st = 0$. $\underline{\lambda_1 + \lambda_2 st \neq 0}$

May assume $(\,,\,)$ standard, $\langle\,,\,\rangle$ diagonalizable $\langle x, y\rangle = (Ax, y)$ $A = A^T$. posi def

Choose a basis for $\mathbb{R}^2$ st $\left(\binom{x}{y}, \binom{x}{y}\right) = x^2 + y^2$ $\left\langle \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix} \right\rangle = \lambda x^2 + \mu y^2$. Claim $\lambda = \mu$

<u>If Not</u>: $\alpha = \begin{bmatrix} a \\ b \end{bmatrix}$ $\hat{\beta} = \begin{bmatrix} -b \\ a \end{bmatrix}$ $(\alpha, \beta) = 0$ $\left\langle \begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} -b \\ a \end{bmatrix} \right\rangle = -\lambda ab + \mu ab$, $\forall$ <u>ab</u> $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

<u>Lemma</u> $(\,,\,)$, $\langle\,,\,\rangle$ inner products on $\mathbb{R}^2$ <u>st</u> the same angles $\Rightarrow \exists \lambda$ st $\langle u, v\rangle = \lambda (u, v)$ $\forall u v$

<u>pf.</u> • May assume $(\,,\,)$ standard. (why?)

• May write $\langle u, v\rangle = \langle Au, v\rangle$ for some $A: \mathbb{R}^2 \to \mathbb{R}^2$ symmetric $A = A^T$, positive definite $\Rightarrow$ A diagn orthogonal diag. $\Rightarrow \exists$ orthogonal basis —

$\square$

<u>Corollary</u> Two R-metrics g_1 g_2 on Σ^2 conformal iff $g_1 = e^{2u} g_2$ $u: \Sigma \to \mathbb{R}$

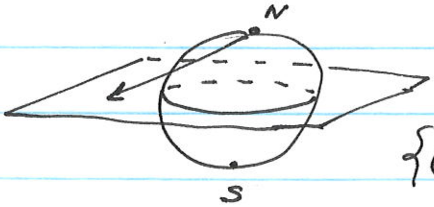
<u>Pull back metrics.</u>

Examples of Riemann surfaces

1. $\mathbb{C}$ $\{(\mathbb{C}, id)\}$

2. $(\Sigma, \mathcal{C})$ R-surface, $X \subset \Sigma$ connected open subset $\Rightarrow$ X naturally R-surf

3. $\hat{\mathbb{C}} = \mathbb{S}^2$, the 2-sphere, two charts $N = (0,0,1)$ $S = (0,0,-1)$



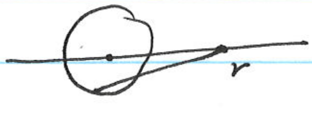
$\overline{\phi_N}: \mathbb{S}^2 - N \to \mathbb{C}$ stereographic

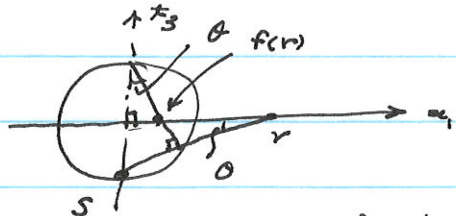
$\overline{\phi_S}: \mathbb{S}^2 - S \to \mathbb{C}$ stereo + conjugate

$\{(U_N, \overline{\phi_N}), (U_S, \overline{\phi_S})\}$ analytic charts

Reason: $\phi_N \circ \phi_S^{-1}(z) = \frac{1}{\bar{z}}$: $\phi_N \circ \phi_S^{-1}(\cancel{\;}\, re^{i\theta}) = f(r)e^{i\theta}$

Some f. $f(1) = 1$.





$\Rightarrow$ $\Delta\, NO f(r)$: $f(r) = tg\theta$, $\Delta\, SOr$: $r = \cot(\theta)$ $\Rightarrow f(r) = \frac{1}{r}$.

RM This proof works in any dim. $\mathbb{S}^n$ n-manifold. transition $x \mapsto \frac{x}{\|x\|^2}$.

4' $\mathbb{C}/\mathbb{Z} = X$ top^{l} surface

4. Let $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ $\omega_1/\omega_2 \notin \mathbb{R}$ a lattice in $\mathbb{C}$.

$$T^2 = \mathbb{C}/L = \{ [z] \mid z \in \mathbb{C} \quad z \sim z' \text{ iff } z - z' \in L \}$$

In the quotient topology $\mathcal{T}$: $\pi: \mathbb{C} \to T^2$ $\pi(z) = [z]$ s.t. $U \subset T^2$ open iff $\pi^{-1}(U)$ open in $\mathbb{C}$.

HW. (A) $(T^2, \mathcal{T})$ is a topological 2-manifold

(B) $(T^2, \mathcal{T})$ is a R-surface s.t. $\pi: \mathbb{C} \to T^2$ is analytic map s.t. transition maps are $z \mapsto z + \lambda$ $\lambda \in L$.

Def $(X, \mathcal{C}), (Y, \mathcal{C}')$ two Riemann surfaces $F: X \to Y$ is analytic if $\forall$ $x \in X$ and analytic $(U_\alpha, \phi_\alpha) \in \mathcal{C}$ $x \in U_\alpha$ and $(V_\beta, \psi_\beta) \in \mathcal{C}'$ $f(x) \in V_\beta$

$$\begin{tikzcd} U_\alpha \arrow[r, "F"] \arrow[d, "\phi_\alpha"] & V_\beta \arrow[d, "\psi_\beta"] \\ \mathbb{C} \arrow[r, "\psi_\beta \circ F \circ \phi_\alpha^{-1}"] & \mathbb{C} \end{tikzcd}$$

$\psi_\beta \circ F \circ \phi_\alpha^{-1}$ analytic

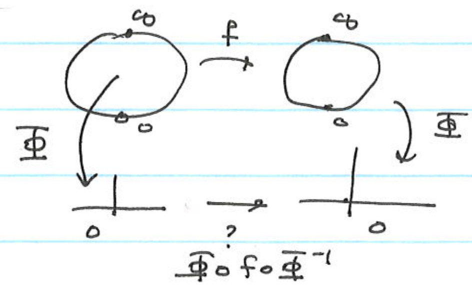
(Define biholomorphism)

Lecture 3 Analytic Maps

Eg $f(z) = a_n z^n + \cdots + a_0 \quad a_n \neq 0: \mathbb{C} \to \mathbb{C}$ extends to an analytic map $f: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ $(f(\infty) = \infty)$ $\quad \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ the Riemann sphere ($\cong$ image of S^2 under stereographic) $\{(\mathbb{C}, id), (\mathbb{C}^* \cup \{\infty\}, \Phi)\}$ $\phi(w) = \frac{1}{w}$

Pf: $z_0 \in \mathbb{C} \qquad f(z_0) \in \mathbb{C}$ clear

$z_0 = \infty$ use chart



$$\Phi \circ f \circ \Phi^{-1}(z) = \frac{1}{f(\frac{1}{z})}$$

$$= \frac{1}{\frac{a_n}{z^n} + \cdots + \frac{a_1}{z} + a_0} = \frac{z^n}{a_n + a_{n-1} z + \cdots + a_0 z^n}$$ Analytic at $\underline{z = 0}$!

$\square$

$\boxed{\text{HW}}$ 1. Every rational function $R(z): \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is analytic.

2. Every analytic $\varphi: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is rational $\varphi = \frac{P}{Q}$, P, Q poly.

Lecture 3. Group Actions + Riemann Surf

<u>Prop 3.1</u> $\Omega \subset \mathbb{C}$ open connected Γ group acting on Ω s.t

(1). $\forall\ \gamma \in \Gamma \quad \gamma: \Omega \to \Omega$ analytic

(2). (free) $\forall\ \gamma \in \Gamma - \{id\}, \quad \gamma(z) \neq z \quad \forall\ z \in \Omega$

(3). (properly discontinuous) $\forall$ cpt $K \subset \Omega \quad \{\gamma \in \Gamma \mid K \cap \gamma(K) \neq \phi\}$ is finite.

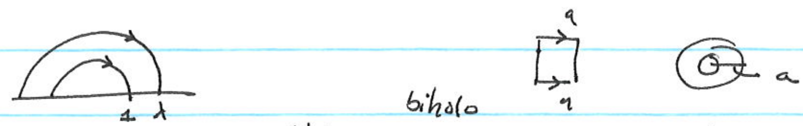
Then $X = \Omega/\Gamma$ is a R-surface s.t the quotient map

$\pi: \Omega \to X$ is analytic

Eg BACK Page

<u>Eg</u> $\lambda \in \mathbb{R}_{>1} \quad \gamma(z) = \lambda z$ acts on $\mathbb{H} = \{ z \mid \mathrm{Im}(z) > 0 \}$ $\quad \mathbb{Z}(\gamma) = \Gamma$

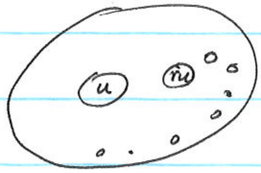
acts — (1), (2), (3). The quotient space is homeo to $\{r < |z| < 1\}$.



<u>Homework</u> $\qquad \mathbb{H}/\Gamma \overset{\text{biholo}}{\cong} \{ r < |z| < 1 \}$ find r.

<u>Pf</u>: <u>Claim</u> $\forall\ x \in \Omega, \exists$ nbhd U of x s.t $\gamma(U) \cap U = \phi \quad \forall\ \gamma \in \Gamma - \{1\}$

<u>$\gamma(U) = \{\gamma(x) \mid x \in U\}$</u>



<u>Done last time</u>

Take W nbhd of x s.t $\overline{W}$ cpt

$\Rightarrow \{ \gamma \in \Gamma \mid \gamma \overline{W} \cap \overline{W} \neq \phi \} = \{ id, \gamma_1, \ldots, \gamma_n \}$

Choose U nbhd of x in W st $\gamma_i U \cap U = \phi$ (due to $\gamma_i x \neq x$)

<u>Done</u> $\qquad U$ — good open sets

<u>Now</u>: nbhd of $[z]$ $\quad \pi(U)$, U good <u>nbhd</u>

<u>St</u> $\qquad \pi|_U : U \to \pi(U)$ is 1-1, onto, continuous

<u>Hw</u>: $\pi|_U : U \to \pi(U)$ is a homeomorphism

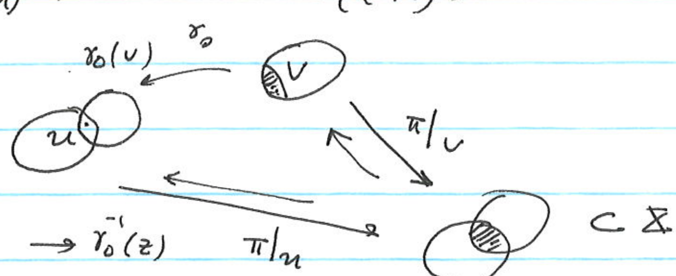
We take $(\pi(U), (\pi|_U)^{-1})$ as analytic charts for X.

<u>Transition functions</u>: $\quad (\pi(U), (\pi|_U)^{-1})$ + $(\pi(V), (\pi|_V)^{-1})$ Both

st $\qquad \underline{\pi(U) \cap \pi(V) \neq \phi}$.

$\Updownarrow$

$\underline{U \cap \gamma_0(V) \neq \phi}$



$\alpha = (\pi|_U)^{-1} \circ (\pi|_V) : z \longmapsto [z] \to \gamma_0^{-1}(z) \quad \pi|_U$

$\begin{matrix}\alpha(z) \\ \gamma_0(z)\end{matrix} \in U$. Both $\pi|_U$ <u>same</u>.

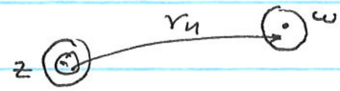
Lecture 3 Group Action

$\Rightarrow$ X locally Euclidean, transition functions analytic.

Claim X is Hausdorff

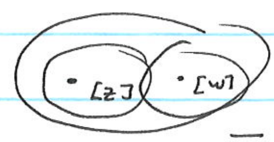
If Not. $\exists\ [z] \neq [w]$ in X s.t $\forall$ nbhd $\pi(U)$, $\pi(V)$ of $[z]$ & $[w]$ intersect.

$B_r(p) = \{z \mid d(z,p)<r\}$ open ball



Let $U_n = B_{\frac{1}{n}}(z)$, $V_n = B_{\frac{1}{n}}(w)$

two open balls of small radii. $(n \gg 1)$



$\overline{U_n}\cup\overline{V_n}$ cpt

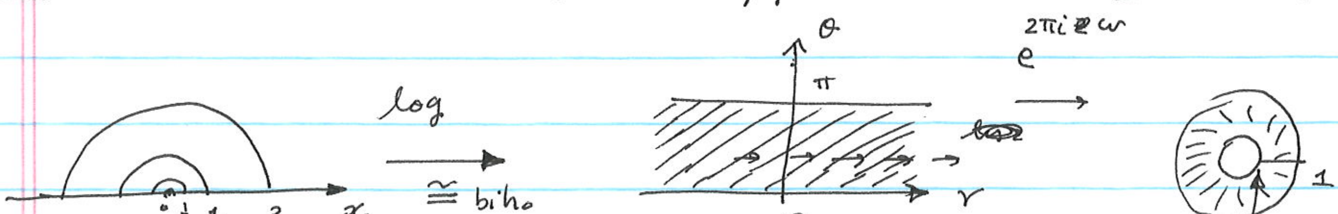
Now $\pi(U_n)\cap\pi(V_n) \neq \phi \Rightarrow \exists\ x_n \in U_n,\ \gamma_n \in \Gamma - \{1\},\ \gamma_n x_n \in V_n$

There are infinitely many such $\{\gamma_n\}$ is infinite

$$K = \overline{U_N}\cup\overline{V_N} \Rightarrow \gamma_n K \cap K \ni \boxed{x_n} = \phi \quad \text{for } n \gg 1$$

A contradiction. $\square$

Eg What is the Riemann surface $\mathbb{H}/_\Gamma$ $\quad \Gamma = \mathbb{Z}(\gamma)$ $\quad \boxed{\gamma(z) = e z}$



$$\boxed{\log(re^{i\theta}) = \ln r + i\theta \quad 0<\theta<\pi}$$

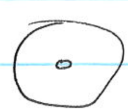
$\log\gamma(z) = \log(z) + 1$

(Similar to $\mathbb{C}/\mathbb{Z}$! But $\{x+iy \mid 0<y<\pi\}/\mathbb{Z}$) $\quad \theta = 0$

$$w = r+i\theta \qquad e^{2\pi i w} = e^{2\pi i (r+i\theta)} = e^{-2\pi\theta}\cdot e^{2\pi i r}$$

Conclusion $\quad \mathbb{H}/_{z\sim ez} \underset{\text{biholo}}{\cong} \{e^{-2\pi^2} < |z| < 1\}$

Eg (HW) $\quad \mathbb{H}/_{z\sim z+1} \underset{e^{2\pi i z}}{\cong}$



Thm 1 $\Rightarrow$ Thm 2 $\forall$ Riemann surface $X \cong \hat{\mathbb{C}}, \mathbb{C}, \mathbb{C}^*, \mathbb{C}/\mathbb{Z}\omega_1+\mathbb{Z}\omega_2$ or $\mathbb{C}/\Gamma$ $\Gamma < PSL(2,\mathbb{R})$.

Lecture 3 the upper half plane $\mathbb{H} = \{ z \in \mathbb{C} \mid \mathrm{Im}(z) > 0 \}$

SL(2.$\mathbb{R}$) and Möbius transformation

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2.\mathbb{R})$, $ad-bc=1$ define $f_A(z) = \dfrac{az+b}{cz+d} : \mathbb{H} \to \mathbb{H}$ bijection

$f_A(z) = z \ \forall z \Leftrightarrow A = \pm Id$ Möbius transfunction

Lemma $f_{AB} = f_A \circ f_B$

Pf $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $B = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}$ $AB = \begin{bmatrix} aa'+bc' & ab'+bd' \\ ca'+dc' & cb'+dd' \end{bmatrix}$

$$f_A(f_B(z)) = \frac{a\left(\frac{a'z+b'}{c'z+d'}\right)+b}{c\left(\frac{a'z+b'}{c'z+d'}\right)+d} = \frac{a(a'z+b')+b(c'z+d')}{c(a'z+b')+d(c'z+d')} = \underline{\qquad\qquad} f_{AB}(z)$$

$\square$

So $PSL(2.\mathbb{R}) = SL(2.\mathbb{R})/\pm I$ acts on $\mathbb{H}$ as analytic action

Lemma $\varphi: \mathbb{H} \to \mathbb{H}$ analytic 1-1 onto $\Rightarrow$ $\varphi = f_A$ for $A \in SL(2.\mathbb{R})$

pf (Sketch): $\mathbb{D} = \{ |z|<1 \}$ Same $\varphi: \mathbb{D} \to \mathbb{D}$ analytic 1-1 onto $\Rightarrow$

$\varphi(z) = e^{i\theta}\left(\dfrac{z-a}{\bar{a}z-1}\right)$ $\underline{\theta \in \mathbb{R},\ |a|<1}$ $\left(\mathbb{H} \cong \mathbb{D} \quad z \mapsto \dfrac{z-i}{z+i} \right)$

$\quad = g_{\theta a}$

Now: $g_{\theta,a}(a) = 0$

So for $\varphi: \mathbb{D} \to \mathbb{D}$ sending $0 \mapsto a$. Let $h = g_{1,a} \circ \varphi$

$h: \mathbb{D} \to \mathbb{D}$ 1-1, onto, analytic $h(0)=0$

Schwartz $\Rightarrow$ $|h(z)| \le |z|$ + $|h^{-1}(w)| \le |w|$

$\Downarrow$

$|w| \le |h(w)|$

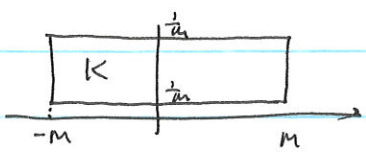
$\Rightarrow |h(z)| = |z| \Rightarrow h(z) = e^{i\theta} z \Rightarrow$ Done. $\square$

Corollary. Γ group acting on $\mathbb{H}$ analytically and faithfully ($\forall \gamma \in \Gamma - \{1\}$ $\exists z,\ \gamma(z) \neq z$) $\Rightarrow$ $\Gamma \subset PSL(2,\mathbb{R})$ action on $\mathbb{H}$.

Eg $\mathbb{Z}(\gamma)$ acts on $\mathbb{H}$ $\gamma(z) = z+1$ $\longrightarrow$ $\mathbb{H}/\mathbb{Z} = \mathbb{H}/_{z \sim z+1} \overset{\mathrm{Exp}(2\pi i z)}{\cong} \{0<|z|<1\}$

Eg $\Gamma = SL(2,\mathbb{Z})$ acts on $\mathbb{H}$ properly discontinuously ($\sim$ arithmetic) (Hw)

Pf $\forall$ cpt K. May assume $K_M = [-M, M] \times [\frac{1}{M}, M]$ for $M \gg 1$



Claim

$\{ \gamma \in \Gamma \mid \gamma K_M \cap K_M \neq \phi \}$ is finite

$\quad \| \ \mathrm{Iso}_K$

Say $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2.\mathbb{Z})$ s.t $A(i) \in K_{M'}$ for some M' (?)

$$A(i) = \frac{ai+b}{ci+d} = \frac{(ai+b)(-ci+d)}{c^2+d^2} = \frac{(bd+ac)+i(ad-bc)}{c^2+d^2} = \frac{bd+ac+i}{c^2+d^2}$$

$\Leftrightarrow \quad \frac{1}{M'} \le \frac{1}{c^2+d^2} \le M'$ bounded $\quad bd+ac$ bounded $\quad ad-bc=1$

$\Rightarrow \quad a, b, c, d$ bounded. $\square$

Eg 2 $\quad \Gamma(2) = \{ A \in SL(2.\mathbb{Z}) \mid A \equiv \mathrm{id} \bmod 2 \} / \pm 1 \curvearrowright \mathbb{H}$ freely + properly discontinuously. $\quad \mathbb{H}/\Gamma(2) \underset{\text{bihol}}{\cong} \mathbb{C} - \{0,1\}$ (Elliptic functions + modulo group)

Pf $\quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \ne \pm Id \quad$ If $f_A(z) = z \Leftrightarrow \frac{az+b}{cz+d} = z$

ie $\quad c^2 z + (d-a) z - b = 0$

(i) $c = 0 \quad \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \quad \det A = 1 \Rightarrow a = d \pm = \pm 1 \Rightarrow f_A = z + b \Rightarrow b = 0$

(ii) $c \ne 0 \quad z = \frac{(a-d) \pm \sqrt{(a+d)^2 - 4}}{2c} \in \mathbb{H}$

$\Rightarrow \quad (a+d)^2 - 4 < 0 \Leftrightarrow |a+d| < 2 \quad$ But $a+d$ = even

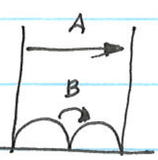
$\Rightarrow \quad a+d = 0 \quad d = -a$

Now $\quad ad - bc = 1 \Rightarrow -a^2 - bc = 1$ or

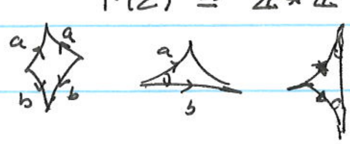
$$a^2 \equiv -1 - bc \equiv (-1) \bmod 4 \equiv 3 \bmod 4$$
$$\| $$
$$(2n+1)^2 \equiv 1 \bmod 4 \qquad \text{Impossible.}$$

$\square$

Geometrically



$\Gamma(2) = \mathbb{Z} * \mathbb{Z}$ free group $\quad A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \quad z \mapsto z+2$



$\cong \mathbb{C} \setminus \{0,1\} \quad B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

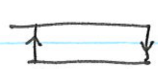
Classical Elliptic Function theory $\Rightarrow \quad \pi : \mathbb{H} \longrightarrow \mathbb{H}/\Gamma(2) = \mathbb{C} \setminus \{0,1\}$.

Hw $\quad \forall \lambda \in \mathbb{R}_{>1} \quad \mathbb{H}/_{z \sim \lambda z} \underset{\text{biho}}{\cong} \{ r < |z| < 1 \}$ Find r in terms of λ.

Eg Möbius band $\quad$ Möb =

$\overset{\Delta}{=} \{ (x,y) \mid 0 < y < 1 \} / (x,y) \mapsto (x+1, -y)$

$\gamma(x,y) = (x+1, -y) \quad \gamma$ acts on —



Lecture 3

1906

Uniformization Thm (Poincaré-Koebe): $\forall$ Riemann surface X is biholomorphic to

(1) $\hat{\mathbb{C}}$ or

(2) $\mathbb{C}^*$ or $\mathbb{C}/(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2)$ $\quad \mathbb{C}^* = \mathbb{C}/\mathbb{Z}$, or $\mathbb{C}$

(3) $\mathbb{H}/\Gamma$ where Γ a subgp of $PSL(2.\mathbb{R})$ acting freely and properly discontinuously on $\mathbb{H}$.

In particular: if X is simply connected: $X \underset{biho}{\cong} \hat{\mathbb{C}}$, or $\mathbb{C}$ or $\mathbb{H}$.

Futhermore Γ is isomorphic to the fundamental group of X. $\pi_1(X.a)$

$\Rightarrow$ Riemann mapping thm: $X \subsetneq \mathbb{C}$ simply connected $X \underset{biho}{\cong} \mathbb{H} \cong \mathbb{D}$

Poincaré-Köbe Unif. X simply connected Riemann surface. then X is biholomorphic to $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ or $\mathbb{C}$ or $\mathbb{D}$. (pairwise nonholo $\mathbb{C} \underset{biho}{\not\cong} \mathbb{D}$ Liouville)

Corollary Above:

Σ Riemann surface $\pi: \tilde{\Sigma} \to \Sigma$ universal covering map $\Gamma = \pi_1(\Sigma)$.

Given $\tilde{\Sigma}$ the pull back complex st s.t π analytic $\Rightarrow$ $\tilde{\Sigma}$ simply connected Riemann surface on which Γ acts freely + properly discont.

Claim Γ action on $\tilde{\Sigma}$ is analytic

Pf $\forall \gamma \in \Gamma$ $\quad \pi \circ \gamma = \pi$ $\quad \pi$ local homeo $\Rightarrow$ $\gamma|_U = (\pi|_{\gamma(U)})^{-1} \circ (\pi|_U)$ analytic.

Now (1) $\tilde{\Sigma} = \mathbb{H}$ $\Rightarrow$ $\Gamma \subset PSL(2.\mathbb{R})$ (Schwarz lemma)

(2) $\tilde{\Sigma} = S^2$ $\Rightarrow$ $\Gamma = \{id\}$ $\quad \gamma \in \Gamma - \{1\}$ $\gamma: S^2 \to S^2$ orientation preserving (Σ orientable) $\Rightarrow$ $\gamma(x) = x$ some x.

(3) $\tilde{\Sigma} = \mathbb{C}$ $\Rightarrow$ $\gamma \in \Gamma \Rightarrow \gamma(z) = az + b$ linear (HW) $a = 1 \Rightarrow$

$\gamma(z) = z + b$ translation $\Rightarrow$ $\Gamma = \{1\}$, or $\mathbb{Z}$ or $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$

$\square$

<u>Proposition</u>: Suppose a group Γ acts on an open set $\Omega \subset \mathbb{C}$ s.t

(1) $\forall\ \gamma \in \Gamma$, $\gamma : \Omega \to \Omega$ analytic homeomorphism.

(2) $\forall\ \gamma \neq id$ in Γ $\quad \gamma(x) \neq x \quad \forall\ x \in \Omega$ (free action) $\leftarrow$

(3) $\forall\ K \subset \Omega$ compact. $\{\gamma \mid \gamma(K) \cap K \neq \phi\}$ is finite. (freely discontinuous)

Then $\Omega/\Gamma = X$ is a Riemann surface s.t the quotient map $\pi : \Omega \to \Omega/\Gamma$ is analytic.

<u>RM</u> $\pi : \Omega \to \Omega/\Gamma$ is a covering map

(3) $\Rightarrow$ (3)$'$ May be replaced by: $\forall\ x \in \Omega$. $\exists$ a nbhd U of x s.t $\gamma(U) \cap U = \phi$ for all $\gamma \neq id$.

$[z] \longmapsto e^{2\pi i z}$

<u>Eg</u>. $\mathbb{Z}$ acting on $\mathbb{C}$ $\quad \gamma^n(z) = z + n \quad \mathbb{C}/\mathbb{Z} \underset{biho}{\cong} \mathbb{C}^*$

$\overset{\shortparallel}{\mathbb{Z}(\gamma)}$ (generator γ) $\mathbb{C}^*$ the Riemann surface of Log.

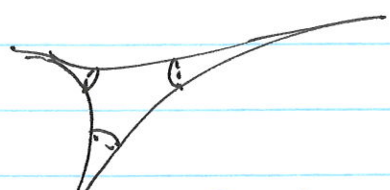
<u>Eg</u>. $\mathbb{Z}$ acting on $\mathbb{C}^*$ $\quad \gamma(z) = \lambda z$, $\lambda \in \mathbb{C}^*$ $|\lambda| \neq 1$

$\overset{\shortparallel}{\mathbb{Z}(\gamma)}$ $\quad \mathbb{C}^*/\mathbb{Z} \underset{top}{\cong} S^1 \times S^1$

<u>Eg</u>. $PSL(2.\mathbb{Z})$ acting on $\mathbb{H} = \{z \in \mathbb{C} \mid Im(z) > 0\}$ analytic.

$$\Gamma(2) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in PSL(2.\mathbb{Z}) \;\middle|\; a, d \equiv 1 \bmod 2 \quad c, b \equiv 0 \bmod 2 \right\}$$

<u>Homework</u> <u>Hw</u>. show that $\Gamma(2)$ action is free and discontinuous $\quad \mathbb{H}/\Gamma(2) \underset{biho}{\cong} \mathbb{C} - \{0,1\}$



elliptic mod

<u>Homework</u> Show that $PSL(2.\mathbb{Z})$ action on $\mathbb{R} \cup \{\infty\}$ is not discontinuous

<u>Eg</u> $\mathbb{Z}_{(\gamma)}$ acting on $\mathbb{H}$ by $\gamma(z) = \lambda z$, $\lambda > 1$.

show that $\quad \mathbb{H}/\mathbb{Z} \underset{biholomorphic}{\cong} \{1 < |z| < R\}$

for some real number R. <u>Hw</u> find R in terms of λ. What is R for $\lambda = 2$.

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2-4-2012

Lecture 4. Uniformization Thm + Its Consequences

Def X a R-surface, $Aut(X)$ = group of self biholomorphisms of X

Eg 1 $Aut(\mathbb{H}) \supset \{ \frac{az+b}{cz+d} \mid a,b,c,d \in \mathbb{R} \quad ad-bc=1 \} \triangleq PSL(2,\mathbb{R})$

Eg 2 $\forall\ |a|<1 \quad \theta \in \mathbb{R} \qquad e^{i\theta}\frac{z-a}{\bar{a}z-1} = g_{a,\theta}(z) : \mathbb{D} \to \mathbb{D}$ in $Aut(\mathbb{D})$

Check $|z|=1 \Rightarrow |g_{a,\theta}(z)|=1. \quad a \mapsto 0, \quad \frac{1}{\bar{a}} \mapsto \infty$

Lemma 1 $Aut(\mathbb{D}) = \{ g_{a,\theta}(z) \mid |a|<1,\ \theta \in \mathbb{R} \}$ + $Aut(\mathbb{H}) = PSL(2,\mathbb{R})$.

Pf. Recall Schwartz lemma: $f: \mathbb{D} \to \mathbb{D}$ analytic $f(0)=0 \Rightarrow |f(z)| \le |z|$ w/ equality (at one $z \neq 0$) iff $f(z) = e^{i\theta} z$.

Now $h \in Aut(\mathbb{D})$ $\quad a = h(0) \Rightarrow f(z) = g_{a,1} \circ h$

$\mathbb{D} \to \mathbb{D}$ $\quad f(0)=0$ and $f^{-1}: \mathbb{D} \to \mathbb{D} \Rightarrow$

$$|f(z)| \le |z| \quad + \quad |f^{-1}(\omega)| \le |\omega| \quad \text{Take } \omega = f(z) \Rightarrow$$

$$|z| \le |f(z)| \le |z| \quad \Rightarrow f(z) = e^{i\theta} z \Rightarrow h = g_{a,1}^{-1}(e^{i\theta} z).$$

□

Note $\mathbb{D} \cong \mathbb{H} \qquad \mathbb{H} \to \mathbb{D} \qquad z \mapsto \frac{z-i}{z+i} \qquad \Rightarrow$ result. □

Corollary Γ acts faithfully and analytically on $\mathbb{H} \Rightarrow \Gamma < PSL(2,\mathbb{R})$.

Lemma Γ acts analytically, freely p.dis on Ω and $h \in Aut(\Omega)$, then $h \circ \Gamma \circ h^{-1}$ acts analytically on Ω freely p. dis s.t $\quad \Omega/_{\Gamma} \to \Omega/_{h\circ\Gamma\circ h^{-1}}$ biholo

$[z] \longmapsto [h(z)]$ $\qquad$ (Check $x = \gamma y \Leftrightarrow h(x) = (h\gamma h^{-1})(h(y))$).

Eg 1 Since $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = A \circ \begin{bmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{bmatrix} \circ A^{-1} \quad \lambda = \left(\frac{3+\sqrt{5}}{2}\right) \Rightarrow \mathbb{H}/_{z \sim \frac{2z+1}{z+1}} \underset{biho}{\cong} \mathbb{H}/_{z \sim \lambda z}$.

Jordan canonical form $\qquad$ Eg $f(z) = z+a$, $a \neq 0$ $\quad g(z) = z+1 \quad h(z) = az \quad h(g(h^{-1})) = f$

Eg $Aut(\mathbb{C}) = \{ A_{a,b}(z) \mid A_{a,b}(z) = az+b,\ a \neq 0,\ b \in \mathbb{C} \}$

pf "$\supset$" clear

"$\subset$" If $\varphi: \mathbb{C} \to \mathbb{C}$ 1-1 onto biholomorphic $\Rightarrow \lim_{z\to\infty} \varphi(z) = \infty$

(If Not. say $\exists\ z_n \to \infty$ st $\underset{w_n}{\underbrace{\varphi(z_n)}} \to a \neq \infty \Rightarrow \varphi^{-1}(w_n) = z_n \to \infty$ $\quad \downarrow \varphi^{-1}(a)$)

~~Really we think of $\varphi: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, $\varphi(\infty) = \infty$), After composing φ by $\varphi(z) - \varphi(0)$~~
~~Now consider $\psi(z) = \frac{1}{\varphi(\frac{1}{z})} : \{0 < |z| < \varepsilon\} \to \{0 < |z| < \varepsilon'\}$ for some $\varepsilon, \varepsilon'$~~

Replace $\varphi(z)$ by $\varphi(z) - \varphi(0)$, may assume $\varphi(0) = 0$

Consider $\psi(z) = \dfrac{1}{\varphi(\frac{1}{z})} : \mathbb{C}^* \to \mathbb{C}^*$ analytic $\lim_{z \to 0} \psi(0) = 0$

$\Rightarrow$ $\psi: \mathbb{C} \to \mathbb{C}$ analytic (Riemann)

$\Rightarrow \exists\ \lambda > 0,\ \underset{\varepsilon > 0}{,}\ + k$ s.t

$$\underline{|\psi(z)| \geq \lambda |z|^k} \quad \text{for} \quad |z| < \varepsilon \quad \Rightarrow \quad \underline{\psi(z) \sim \lambda' z^k \text{ near } 0}$$

$$\left| \frac{1}{\varphi(\frac{1}{z})} \right| \geq \lambda |z|^k \qquad \Big| w = \tfrac{1}{z}$$

$$\Rightarrow \quad \left(\tfrac{1}{\lambda}\right) |w|^k \geq |\varphi(w)| \qquad |w| \geq \tfrac{1}{\varepsilon}$$

Liouville Thm: $\alpha: \mathbb{C} \to \mathbb{C}$ analytic st $|\alpha(z)| \leq K |z|^k \Rightarrow \alpha$ polynomial

$\Rightarrow \ \varphi(z)$ polynomial

But φ is 1-1 $\Rightarrow \varphi(z) = az + b$. $\square$

What Poincaré–Köbe Proved

Uniformization Thm 1 If X is simply connected Riemann surface $\Rightarrow X \underset{\text{biho}}{\cong} \hat{\mathbb{C}}$, or $\mathbb{C}$, or $\mathbb{D}$.

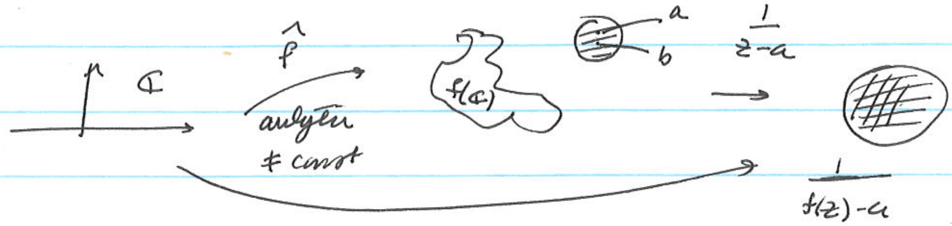
Corollary 1 (Riemann mapping) $\Omega \subsetneq \mathbb{C}$, Ω simply connected $\Rightarrow \Omega \underset{\text{biho}}{\cong} \mathbb{D}$

Pf Obviously $\Omega \not\cong \hat{\mathbb{C}}$ ($\hat{\mathbb{C}}$ cpt). Using Uniformization, it suffices to show $\Omega \underset{\text{biho}}{\not\cong} \mathbb{C}$. Let us assume that $0 \notin \Omega$. Otherwise $f: \mathbb{C} \to \Omega$ biholo

$\Rightarrow f$ lifts to
$$\begin{array}{ccc} & \overset{\hat{f}}{\nearrow} & \mathbb{C} \setminus \{0\} \\ & & \downarrow\ z \mapsto z^2 \\ f: \mathbb{C} & \to & \mathbb{C} \setminus \{0\} \end{array}$$
(Covering space theory)

Now $\hat{f}$ is 1-1 + $\hat{f}(\mathbb{C})$ misses an open set!

$B_r(a)$



$|f(z) - a| \geq r$

$$\left| \frac{1}{f(z) - a} \right| \leq \frac{1}{r}$$

A bounded analytic function contradicting Liouville Again.

Now thm 1 $\Rightarrow$ thm 2 (Complete Version) $X \cong \hat{\mathbb{C}},\ \mathbb{C},\ \mathbb{C}^*,\ \mathbb{C}/_{\mathbb{Z}\omega_1+\mathbb{Z}\omega_2},\ \mathbb{H}/\Gamma \quad \Gamma \subset PSL(2,\mathbb{R})$

Lecture 4

Here is a proof. X Any Riemann surface $\pi: \tilde{X} \to X$ universal cover where π covering map, a local homeo. We can pull back complex structure to $\tilde{X}$ s.t π is analytic

Let $\Gamma = \pi_1(X, a)$ be the deck transformation group for $\pi: \tilde{X} \to X$

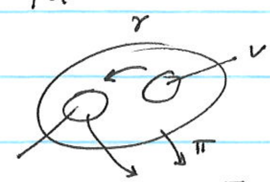
ie $\forall\ \gamma \in \Gamma \quad \pi \circ \gamma = \pi \quad$ + Γ acts freely + properly discontinuously on $\tilde{X}$.

Claim $\gamma: \tilde{X} \to \tilde{X}$ is analytic

Proof $\quad \pi \circ \gamma = \pi \quad$ π local homeo.

$$\pi|_u \circ \gamma|_v = \pi|_v \qquad \pi|_v \quad \pi|_u \text{ local homeo}$$

$$\Rightarrow \gamma|_v = (\pi|_u)^{-1} \circ (\pi|_v) \quad \text{analytic}$$



Thus $\Gamma \subset \mathrm{Aut}(\tilde{X})$ $\square$

Uniformization $\Rightarrow \quad \tilde{X} = \hat{\mathbb{C}} \quad (\Rightarrow \Gamma = \{1_X\}) \quad$ ✓ (γ No Fixed Pt)

$\Rightarrow \quad \tilde{X} = \mathbb{C} \quad \Gamma \subset \mathrm{Aut}(\mathbb{C})$

$\Rightarrow \quad \tilde{X} = \mathbb{H} \quad \Gamma \subset PSL(2, \mathbb{R})$ ✓

Now if $\tilde{X} = \mathbb{C} \quad \gamma \in \mathrm{Aut}(\mathbb{C}) - \{1\}$ st $\gamma(z) \neq z \quad \forall z$

We know $\quad \gamma(z) = az + b \quad \Rightarrow \quad a = 1.$ ie

$$\Gamma \subset \text{Translations} \subset \mathrm{Aut}(\mathbb{C}).$$

+ Γ acts freely discontinuously on $\mathbb{C}$. $\quad \mathbb{C}$ abelian

$\Rightarrow \quad \Gamma \cong \mathbb{Z}^k$

(1) $k \geq 3 \Rightarrow \Gamma$ cannot act properly discontinuously on $\mathbb{C}$. (Hw)

(2) $k = 1. \quad \Gamma = \mathbb{Z}(\gamma) \quad \gamma(z) = z + a \quad a \neq 0 \quad \boxed{z+a \text{ ag } z+1}$

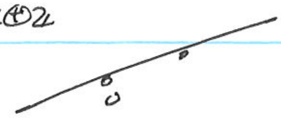
$$\mathbb{C}/_{z \sim z+a} \cong \mathbb{C}/_{z \sim z+1} \cong \mathbb{C}^*$$

$z \mapsto e^{\frac{2\pi i}{a} z}$ biho'

(3) $k = 2 \quad \Gamma = \mathbb{Z}(\gamma_1) \oplus \mathbb{Z}(\gamma_2) \quad \gamma_i(z) = z + \omega_i \quad i = 1, 2$

Claim $\omega_1/\omega_2 \notin \mathbb{R}$ if so $\Rightarrow \Gamma$ action on $\mathbb{R}\omega_1 \cong \mathbb{R}$ by translation

$\mathbb{Z} \oplus \mathbb{Z}$



$\Rightarrow$ Not proper

$n_i \omega_1 + m_i \omega_2 \to 0$

$\square$

Lecture 4 Uniformization thm

Lemma Any subgroup $\mathbb{Z}^2 \subset (\mathbb{R},+)$ is dense in $\mathbb{R}$ ($\Leftrightarrow$ $\mathbb{Z}^2$ cannot act freely and properly discontinuously on $\mathbb{R}$ by translation).

Pf Let $\mathbb{Z}^2 = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 = \{ n\omega_1 + m\omega_2 \mid n, m \in \mathbb{Z} \}$. where $\omega_1/\omega_2 \notin \mathbb{Q}$.

Known $\exists$ p_i/q_i $(p_i, q_i) = 1$ $|q_i| \to \infty$ st

$$\left| \frac{\omega_1}{\omega_2} - \frac{p_i}{q_i} \right| \le \frac{1}{q_i^2}$$

$$\Leftrightarrow \quad |q_i \omega_1 - p_i \omega_2| \le \left| \frac{\omega_2}{q_i} \right| \to 0.$$

To see that, fix N. for each $i = 1, 2, \cdots, N+1$, $\exists$ j s.t, $0 < i\frac{\omega_1}{\omega_2} - j < 1$

By the Pegion hole principle $\Rightarrow$ $\left| \left(i\frac{\omega_1}{\omega_2} - j\right) - \left(a\frac{\omega_1}{\omega_2} - b\right) \right| \le \frac{1}{N}$ $\Rightarrow$ result.

$$\left| p\frac{\omega_1}{\omega_2} - q \right| \le \frac{1}{N} \qquad \left| \frac{\omega_1}{\omega_2} - \frac{q}{p} \right| \le \frac{1}{Np} \le \frac{1}{p^2}$$

□

Thm Similary $\mathbb{Z}^k \subset \mathbb{R}^n$ $k > n$ is not a discrete subset

Now $\Gamma \subset (\mathbb{C}, +)$ subgroup $\Rightarrow$ $\Gamma \cong \mathbb{Z}^k$. But $k \le 2$

(1) $k=1$. $\Gamma = \mathbb{Z}(\gamma)$ $\gamma(z) = z + a$ $a \neq 0$.

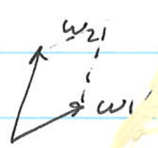
$$\text{so } X \underset{\text{biho}}{\cong} \mathbb{C}/_{z \sim z+a} \underset{\text{biho}}{\cong} \mathbb{C}^* \qquad [z] \mapsto e^{\frac{2\pi i}{a} z} \qquad (a=1 \text{ before})$$

Another way: $f(z) = z + a$ and $g(z) = z+1$ are conjugate in $\mathrm{Aut}(\mathbb{C})$

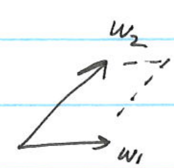
For $h(z) = az$ $h^{-1}(z) = \frac{1}{a} z$ $h \circ g \circ h^{-1}(z) = a\left(g\left(\frac{z}{a}\right)\right) = a\left(\frac{z}{a} + 1\right) = z + a = f$.

$$\text{so } \mathbb{C}/_{z \sim z+a} \underset{\text{biho}}{\cong} \mathbb{C}/_{z \sim z+1} \cong \mathbb{C}^*$$

(2) $k = 2$ $\Gamma \cong \mathbb{Z}(\gamma_1) + \mathbb{Z}(\gamma_2)$ $\gamma_i(z) = z + \omega_i$ Then $\omega_1/\omega_2 \notin \mathbb{R} \Rightarrow$ lattice



$$\mathbb{C}/_{\mathbb{Z}(\gamma_1) + \mathbb{Z}(\gamma_2)} \underset{\text{biho}}{\cong} \text{torus}$$



After a conjugation $h(z) = \frac{z}{\omega_2}$ may assume $\omega_1 = 1$ $\omega_2 \notin \mathbb{R}$

Any torus, $\underset{\text{biho}}{\cong} \mathbb{C}/_{\mathbb{Z}1 + \mathbb{Z}\omega}$ $\boxed{\omega \in \mathbb{H}}$

Lecture 4. Uniformization Thm

<u>Corollary</u> Every R-surface X homeomorphic to a ring $\{1<|z|<2\}$ is biholomorphic to $\mathbb{C}^*$ or $\mathbb{D}-\{0\}$ or $R_r = \{r<|z|<1\}$ $\quad 0<r<1$

<u>Proof</u> Let $\tilde{X}$ be the univ. cover of X

(1) $\tilde{X} \neq \hat{\mathbb{C}}$ $\qquad \pi_1(X) \cong \mathbb{Z} = \mathbb{Z}(\gamma) = \Gamma$

(2) $\tilde{X} = \mathbb{C}$, $\Gamma \cong \mathbb{Z}$ $\qquad \gamma(z) = z+a \Rightarrow X \cong \mathbb{C}^*$

(3) $\tilde{X} = \mathbb{H}$, $\gamma(z) = \dfrac{az+b}{cz+d}$ s.t. $\gamma(z) \neq z \quad \forall z \in \mathbb{H}$ No fixed pt

<u>Lemma</u> $\gamma(z) \neq z \Leftrightarrow A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has real eigenvalues $\Leftrightarrow A = B\begin{bmatrix} \lambda_1 & 0 \\ 0 & \frac{1}{\lambda_1} \end{bmatrix}B^{-1}$ or $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$\Leftrightarrow |\mathrm{tr} A| \geq 2$ $\qquad$ ↑ Jordan Canonical form

<u>Pf</u> $\gamma(z) = z$ in $\mathbb{C}$

$\Leftrightarrow (az+b)/(cz+d) = z$ in $\mathbb{C}$

$\Leftrightarrow cz^2 + (d-a)z - b = 0$

$\Leftrightarrow$ has no roots in $\mathbb{H}$ $\cdot (d-a)^2 + 4bc \geq 0$ $\qquad bc = ad-1$

$\Leftrightarrow (a-d)^2 + 4ad - 4 \geq 0$

$\Leftrightarrow |a+d|^2 \geq 4 \Leftrightarrow |\mathrm{tr} A|^2 \geq 4$

Now characteristic polynomial of A $\left|\begin{bmatrix} \lambda-a & -b \\ -c & \lambda-d \end{bmatrix}\right| = (\lambda-a)(\lambda-d) - bc$

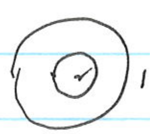
$= \lambda^2 - (a+d)\lambda + ad - bc = \lambda^2 - (a+d)\lambda + 1$, discriminant $(a+d)^2 - 4 \geq 0$. $\square$

Thus, may assume $\gamma(z) = \lambda z$ $\quad \lambda > 1$ or $\gamma(z) = z+1$

<u>ie</u> $X \underset{\text{biho}}{\cong} \mathbb{H}/_{z \sim \lambda z} \underset{\text{biho}}{\cong} \{\mu < |z| < 1\}$ where $\log\lambda \log\mu = -2\pi^2$

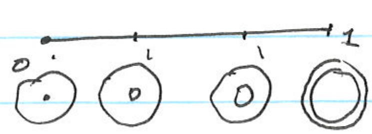
or $X \underset{\text{biho}}{\cong} \mathbb{H}/_{z \sim z+1} \underset{e^{2\pi i z}}{\cong} \{0<|z|<1\}$ $\square$

<u>Note</u>



$R_r \not\cong R_{r'}$ ($r \neq r'$ biholomorphic)

(Moduli space)



<u>all different!</u>

<u>Problem</u> Not easy $F_n \cong \mathbb{H}/\Gamma$ what is Γ?