Jan 3. 2013

# Lectures on Riemann surfaces + 3-manifold

Abstract: Riemann surfaces, hyperbolic geometry, Teichmuller space, Uniformization theorem, moduli space of surfaces, matrix module mapping class groups, triangulated objects, linear representations

3-manifolds: hyperbolic gluing equation, simplicial Chern-Simons. Haken's theory, Casson invariants, + TQFT.

Other topics: discrete Riemann surfaces, quantum Teichmuller spaces + the work of Kashaev on TQFT + quantum dilogarithms

Not a systematic course on Riemann surfaces or 3-manifolds.

Selected topics of my interests. Will provide some background if needed.

# Lecture 1. Introduction to 2-, 3-dim topology & geometry

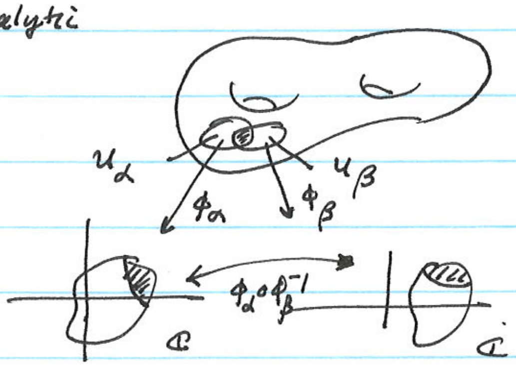
<u>Def</u> An $n$-manifold $M^n$ is a Hausdorff topological space with countable basis s.t. $\forall\ x \in M^n$, $\exists$ open nbhd $U$ of $x$ and $\phi: U \to \phi(U) \underset{\text{open}}{\subset} \mathbb{R}^n$ which is a homeomorphism. $(U, \phi)$ — a topological chart.

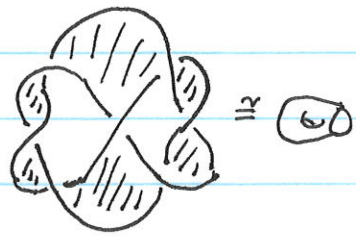
A Riemann surface is a connected 2-manifold $M^2$ together w/ a special collection of charts $\mathcal{C} = \{(U_\alpha, \phi_\alpha) \mid \alpha \in A\}$ s.t.

(1) $M^2 = \bigcup_\alpha U_\alpha$ and

(2) the transition function $\phi_\alpha \circ \phi_\beta^{-1}: \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$

is complex analytic



$\boxed{\text{chart } (U, \varphi) \text{ at } x: \ x \in U}$

<u>RM</u> (2) replaced by $\phi_\alpha \circ \phi_\beta^{-1}$ $C^\infty$ smooth $\Rightarrow$ $(M^2, \mathcal{C})$ smooth. or $\phi_\alpha \circ \phi_\beta^{-1}$ orient$^t$ pres $\Leftrightarrow$ $(M, \mathcal{C})$ orientable

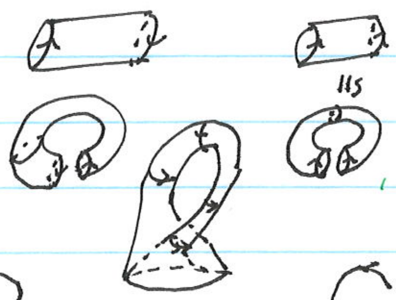
<u>BIG Goal</u>: classify surfaces up to homeomorphism, classify 3-manifolds up to homeomorphism, & classify all Riemann surfaces up to "biholomorphism".

<u>Examples</u>: <u>Möbius thm</u>: All compact surfaces are

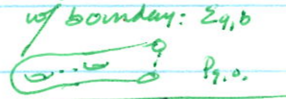

<u>or</u> $\mathbb{RP}^2$, $\mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2$ $P_{g,0}$

$\mathbb{RP}^2 \# \mathbb{RP}^2$ Klein bottle

$\mathbb{D}^2$

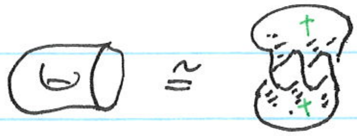


Compact surface w/ boundary: $\Sigma_{g,b}$

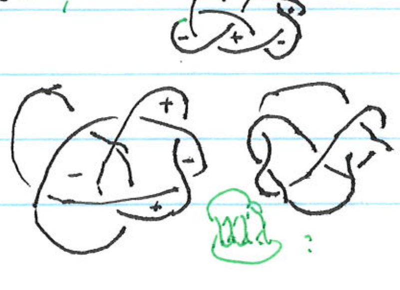



$\mathbb{D}^2 = \{ |z| \le 1 \}$ <u>disk</u>

Homework <u>HW</u>: show that



what is



?

<u>General</u> Topological surfaces $\Sigma_{g,n}$, $P_{g,n}$ : interior of a cpt

# R- surface, the 2-sphere

Examples

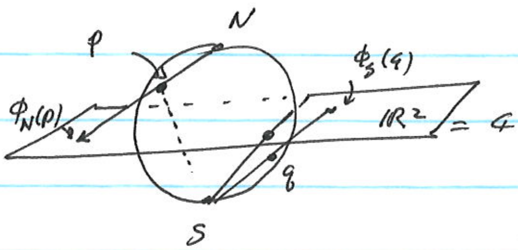
Eg 1. $(\Sigma, \mathcal{C})$ R- surface, $X \subset \Sigma$ open connected $\Rightarrow$ $(X, \mathcal{C}/_X)$ a R-surface

$\mathcal{C}/_X = \{ (U_\cap X, \varphi/) \mid (U, \varphi) \in \mathcal{C} \}$ SAME $\varphi \circ \varphi^{-1}$

$\Rightarrow$ A: $(\mathbb{C}, id)$ R- surface $\Rightarrow$ All open connected $X \subset \mathbb{C}$ a R-surfaces

For instance: $\mathbb{C}$ - {Cantor set} a R- surface. $\mathbb{C}^* = \mathbb{C} - \{0\}$.

Eg 2. The Riemann sphere $S^2 \subset \mathbb{R}^3$ $N = (0,0,1)$, $S = -N$ stereographic



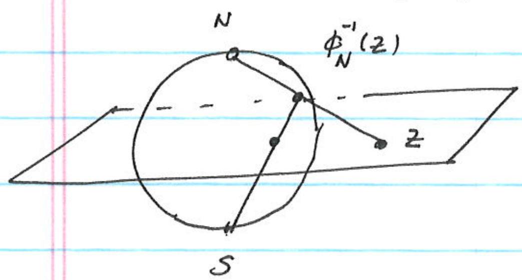

$\phi_N : S^2 - \{N\} \to \mathbb{C}$, $\phi_S : S^2 - \{S\} \to \mathbb{C}$

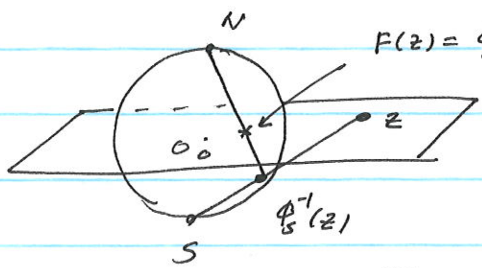
$\phantom{\phi_N :} \underset{\textstyle U_N}{\shortparallel}$ $\qquad\qquad$ $\underset{\textstyle N_S}{\shortparallel}$

Then

$\mathcal{C} = \{ (U_N, \phi_N), (U_S, \phi_S) \}$ analytic chart

What is $\phi_N \circ \phi_S^{-1} = F \quad : \mathbb{C}^* \to \mathbb{C}^*$



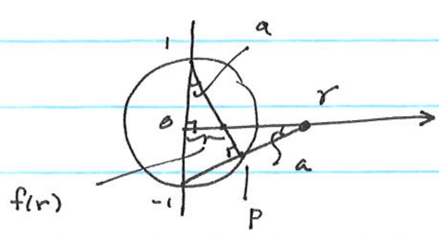

$F(z) = \phi_N \phi_S^{-1}(z)$ $\boxed{\cancel{\lambda z}}$ $\lambda \in \mathbb{R}_{>0}$

$\boxed{F(z) = \frac{1}{\bar{z}}}$

$( \Rightarrow \bar{\phi}_N \circ \phi_S^{-1}(z) = \frac{1}{z} )$

Claim $\phi_N \circ \phi_S^{-1}(re^{i\theta}) = f(r) e^{i\theta}$ SAME argument.

What is $f(r)$ ?




$\Rightarrow$ $\cancel{\triangle}$ $\triangle$ 0 r -1

$tg a = \frac{1}{r} = f(r)$

$\triangle$ 1 0 $\cancel{p}$ f(r)

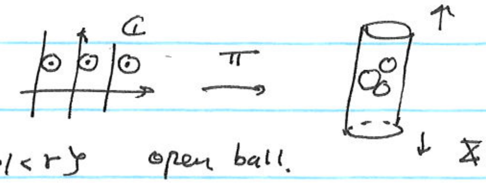
Conclusion $F(z) = \frac{1}{\bar{z}}$.

The SAME works for $S^n$ $\phi_N \circ \phi_S^{-1}(x) = \frac{x}{\|x\|^2}$ : $\mathbb{R}^n - \{0\} \circlearrowleft$.

Eg. $X = \mathbb{C}/\mathbb{Z} = \{ [z] \mid z \sim w \text{ iff } z-w \in \mathbb{Z} \}$. $\pi: \mathbb{C} \to X$, $z \mapsto [z]$

is a R-surface: Topology: quotient top: $U \subset X$ open iff $\pi^{-1}(U)$ open in $\mathbb{C}$

$\Rightarrow \quad \pi: \mathbb{C} \to X$ continuous



Notation $p \in \mathbb{C}$, $r>0$ $\quad B_r(p) = \{ z \mid |z-p| < r \}$ open ball.

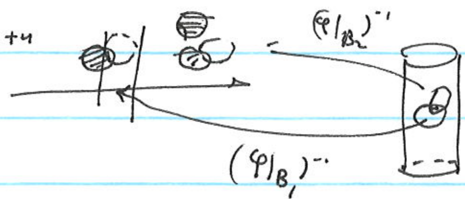
Charts for $X$: $r < \frac{1}{2}$ $\quad B_r(p) \cap B_r(p+n) = \phi \quad n \in \mathbb{Z} - \{0\}$

$$\mathcal{C} = \{ \ ( \pi(B_r(p)), \ (\varphi|_{B_r(p)})^{-1} ) \ \mid r < \tfrac{1}{2}, \ \forall p \}$$

$\varphi|_B : B \to \varphi(B)$ is a homeomorphism (1-1, onto, cont + $\underline{(\varphi|_B)^{-1} \text{ cont}}$ Hw)

Transition function $\quad (\varphi|_{B_1}) \circ (\varphi|_{B_2'})^{-1}: z \mapsto z+n$

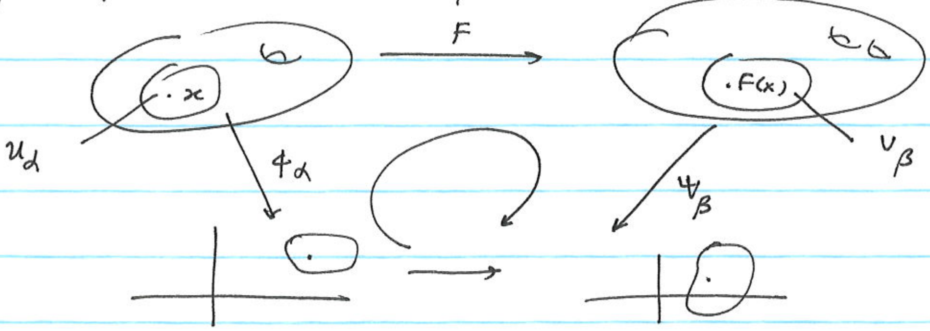
$\underline{n \in \mathbb{Z}}$



Analytic $\Longrightarrow$ $X$ a R-surface

HW Show that $(\varphi|_B)^{-1}$ is continuous in quotient top.

Def (Analytic maps) $(X, \mathcal{C})$ and $(Y, \mathcal{C}')$ two R-surfaces. $F: X \to Y$ analytic (or holomorphic) if $\forall x \in X$ and $(U_\alpha, \phi_\alpha) \in \mathcal{C}$ with $x \in U_\alpha$
$\forall (V_\beta, \psi_\beta) \in \mathcal{C}'$ with $F(x) \in V_\beta$



$\psi_\beta \circ F \circ \phi_\alpha^{-1}$ analytic near $\phi_\alpha(x)$

F is <u>biholomorphic</u> if F is a homeomorphism s.t. $F, F^{-1}$ analytic

Eg $\pi: \mathbb{C} \to \mathbb{C}/\mathbb{Z}$ $\quad z \mapsto [z]$ analytic

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\pi} & \mathbb{C}/\mathbb{Z} \\ \downarrow id & & \downarrow (\varphi|_B)^{-1} \\ z & \longmapsto & z+n \end{array}$$

Eg $F: \mathbb{C}/\mathbb{Z} \to \mathbb{C}^*$ $\quad [z] \longmapsto e^{2\pi i z}$ analytic. It is a biholomorphic

$$\begin{array}{ccc} \mathbb{C}/\mathbb{Z} & \to & \mathbb{C}^* \\ \updownarrow & & \downarrow id \\ z & \longmapsto & e^{2\pi i z} \end{array}$$

$F$ is 1-1, onto homeo + inverse $\log z$ analytic.

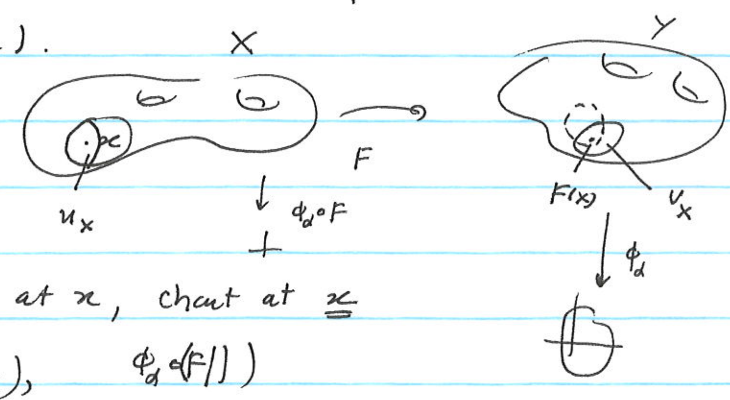
Goal of Riemann surfaces theory: classify Riemann surface up to biholomorphism.

Eg (Riemann mapping thm) $\Omega \subsetneq \mathbb{C}$ simply connected $\Rightarrow$ $\Omega \underset{biho}{\cong} B_1(0) = \{|z|<1\}$

Eg Major difference: $\{1<|z|<2\} \not\cong_{biho} \{0<|z|<2\}$ Even though homeo (Hw)

Eg. Suppose $F: X \to Y$ a local homeomorphism between two surfaces ($\forall x \in X$, $\exists$ nbhds $\mathcal{U}_x$ of $x$ and $V_x$ of $F(x)$ s.t $F|: \mathcal{U} \to V$ homeo). If $(Y, \mathcal{C})$ is a Riemann surface, then $\exists$ analytic charts $\mathcal{C}'$ covering $X$ s.t $F$ is analytic ($\exists!$ way to make $X$ a R-surf s.t $F$ is analytic) (the pull back complex structure).

Proof




Take $\forall x \in X$, and analytic chart $(\mathcal{U}_\alpha, \varphi_\alpha)$ of $Y$ at $x$, chart at $x$

$$\left( (F|)^{-1}(\mathcal{U}_\alpha \cap V_x),\ \phi_\alpha \circ (F|) \right)$$

Transition:

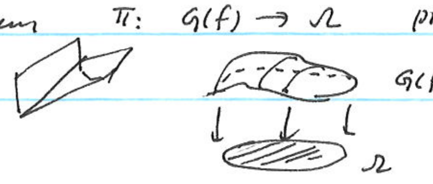
$$(\varphi_\alpha \circ F|)(\varphi_\beta \circ F|)^{-1} = \phi_\alpha \circ (F|) \circ (F|)^{-1} \circ \phi_\beta^{-1} = \varphi_\alpha \circ \varphi_\beta^{-1}$$

analytic + $F$ is the identity map! in the charts $\square$

Hw.

Eg $\forall \underset{\text{open set}}{\Omega \subset \mathbb{R}^2}$ $\quad f: \Omega \to \mathbb{R}$ cont. $\quad G(f) = \{(z, f(z)) \in \Omega \times \mathbb{R} \mid z \in \Omega\}$ is a R-surface with by pullback $\pi: G(f) \to \Omega$ projection

$f(x,y) = |x|$




or $f: \Omega \to \mathbb{C}$ analytic.