Goal Head toward Fenchel-Nielsen coordinates

Lecture 8. Hyperbolic Geometry

Conclusion All hyperbolic geometric invariant of (Σ, ds^2) are invariant of the Riemann surface $\mathbb{H}/\Gamma$. In particular, the length of the shortest geodesics.

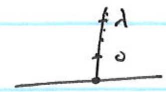
Def Geodesic $\alpha : (a,b) \to (M, g) \Leftrightarrow$ locally distance minimizing R_λ path. closed geod $f: S^1 \to (M,g)$

Eg The hyperbolic annulus $X_\lambda = \mathbb{H}/z \sim \lambda z = \{\mu < |z| < 1\}$, $-\log\lambda \log\mu = 2\pi^2$

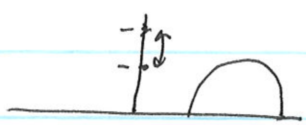
Q What is the length of the shortest geodesics in R_μ. ANS $\boxed{\log\lambda}$ $\boxed{\lambda > 1}$

Solution:

What are the closed geodesics in X_λ? ANS $\pi[\,y\text{-axis}\,]$

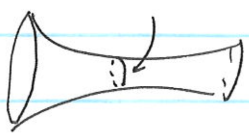


Indeed if L is a closed geodesics in $X_\lambda \Rightarrow \pi^{-1}(L)$ is a geodesic in $\mathbb{H}$ invariant under $\gamma(z) = \lambda z$



$\pi^{-1}(L)$ has end pts $\{a, b\}$ $a \neq b$ $a, b \in \mathbb{R} \cup \{\infty\}$

$\{\lambda a, \lambda b\} = \{a, b\}$ $\lambda > 1 \Rightarrow a, b = 0, \infty$



Now $i \sim i\lambda \Rightarrow$ of distance $\log\lambda$

$\Rightarrow$ result.

$\exists!$ unique closed geodesic in $\mathbb{H}/z \sim \lambda z$.

Conclusion If X_λ is biholomorphic to $X_{\lambda'}$ $\lambda, \lambda' > 1 \Rightarrow \log\lambda = \log\lambda' \Rightarrow \lambda = \lambda'$

or $\{\mu < |z| < 1\}$ biholomorphic to $\{\mu' < |z| < 1\} \Rightarrow \mu = \mu'$

or $\text{Mod}(\{1 < |z| < 2\}) \cong [0, 1) \cup \{\infty\}$.

(Goal understand Moduli space through geometry)

Homework Use Schwarz lemma to show. if $\varphi : \Sigma_1 \to \Sigma_2$ is analytic between two hyperbolic surfaces, Then φ decreases hyperbolic distances

Next goal: understand $\text{mod}(\Sigma)$, $\text{Teich}(\Sigma)$ through the use of Poincaré metric.

Hw 1. Compute the length of a circle of radius r in hyperbolic space.

2. Show that a circle in $(\mathbb{H}, ds_h^2)$ is a Euclidean circle in $\mathbb{C}$ $(\mathbb{D}^2, d_h^2)$

3. Show Gauss-Bonnet theorem

4. Show that the area of a hyperbolic triangle is less than the length of its edge. (Gromov)

$\left|\lg(\text{Eigenvalues}(A))\right| =$ length of geodesic corresponding to it.

Lecture 9. Basic geometry of geodesics

(M, g) Riemannian metric. $\forall\, p \in M$. the ball $B_r(p) = \{x \in M \mid d(x,p) < r\}$

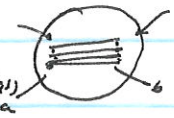
Geodesic $\gamma : [a,b] \to M$: locally the shortest path.

Key Fact. $\forall\, p \in M$, $\exists\, r_p > 0$ small s.t $B_{r_p}(p)$ is convex and

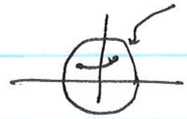
$\forall\, x_1, x_2 \in B_r(p)$. $\exists !$ shortest geodesic $G(x_1, x_2)$ in $B_r(p)$ from x_1 to x_2.

Furthermore if $a, b \in B_r(p)$ $\lim_{\substack{x \to a \\ y \to b}} G(x,y) = G(a,b)$

length$(G(x,y)) \to$ length$(G(a,b))$

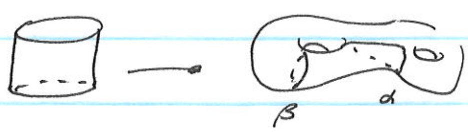


Eg. True in $(\mathbb{H}, ds_h^2)$ or $(\mathbb{D}, ds^2)$:



geodesics circs in

Basic Thm. (M, g) closed Riemannian mfd $\alpha : S^1 \to M$ so that $\alpha \not\simeq p$.
(α continuous + Cannot be extended to $\mathbb{D}^2$). Then $\exists$ a geodesic closed $\beta : S^1 \to M$ so that $\beta \simeq \alpha$: ie $\exists$ continuous $H : S^1 \times [0,1] \to M$
s.t $H(t,0) = \alpha(t)$ $H(t,1) = \beta(t)$ $\forall t$:



Pf. Note if $\alpha \simeq pt$, $\beta =$ constant geodesic

Pf

Lemma 1. $\exists\, \delta > 0$ s.t $\forall\, p \in M$, $B_\delta(p)$ $\subset B_{r_q}(q)$ is geodesic convex and is contractible.

Pf Lebesgue lemma from point set topology $\mathcal{C} = \{B_{r_p}(p) \mid p \in M, r_p$ the radius in key fact$\}$.

□

Lemma 2. If $\alpha_1, \alpha_2 : S^1 \to M$ s.t $d(\alpha_1(t), \alpha_2(t)) \le \delta/2$ $\forall t$. Then $\alpha_1 \simeq \alpha_2$.

Pf. $\alpha_2(t) \in B_{\delta/2}(\alpha_1(t))$. $H(x,s)$: the geodesic path, parameterized propositional to arc length ($\|\frac{\partial}{\partial s} H(x,s)\| =$ const) from $\alpha_1(x)$ to $\alpha_2(x)$.
Continuity follows from the solutions of ODE depends on initial point. □

Now the proof. Let $L = \inf\{\text{length}(\gamma) \mid \gamma \simeq \alpha \text{ in } M\}$

Then $L \ge \delta/2$. since $L < \delta/2 \Rightarrow \gamma \in B_\delta(p) \Rightarrow \gamma \simeq pt \Rightarrow \alpha \simeq pt$

Lecture 9. Basic geometry of geodesics

Let γ_n be a seq. of piecewise smooth loops $\gamma_n \simeq \alpha$, $\lim_{n\to\infty} \text{length}(\gamma_n) \to L$ and $\text{length}(\gamma_n) \leq 2L$

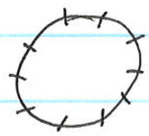
Modify γ_n s.t

(1) $\|\gamma_n'(t)\| = \text{length}(\gamma_n)/2\pi$ (reparametrization) (Each path $\gamma \to |\gamma'(t)|=1$)

$\Rightarrow$ for $t, t' \in S^1$

$$d(\gamma_n(t), \gamma_n(t')) \leq \text{length}(\gamma_n|_{[t,t']}) \leq \frac{2L}{2\pi} d(t,t')$$

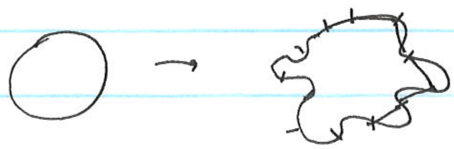
(2) Take $N \gg 1$ s.t $\frac{2L}{N} \leq \delta/2$. and t_i = equal distance partition of S^1 w/ $d(t_i, t_{i+1}) = 2\pi/N$



$\Rightarrow$

$$d(\gamma_n(t_i), \gamma_n(t_{i+1})) \leq \delta/2 \qquad \gamma_n([t_i, t_{i+1}]) \subset B_{\delta/2}(\gamma_n(t_i))$$

(3) Replace γ_n making making $\gamma_n[t_i, t_{i+1}]$ the shortest geodesic joining them in $B_{\delta/2}(\gamma_n(t_i))$. using key fact 1.



(4) May assume $\forall i$, $\lim_{n\to\infty} \gamma_n(t_i) = \beta(t_i)$. since M is compact.

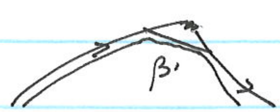
Now use key fact to define $\beta: [t_i, t_{i+1}]$ to be the $\lim_{n\to\infty} \gamma_n|_{[t_i, t_{i+1}]}$.

$$\Rightarrow \quad \beta = \lim_{n\to\infty} \gamma_n \quad \text{uniformly} \quad \text{s.t } \text{length}(\beta) = \lim_{n\to\infty} \text{length}(\gamma_n) = L$$

We claim β is a closed geodesic s.t $\beta \simeq \alpha$.

<u>First</u>: $\exists N \gg 1$ st $d(\beta(t), \gamma_n(t)) \leq \delta/2 \;\; \forall t \Rightarrow \beta \simeq \gamma_n \simeq \alpha$.

<u>Next</u> β is geodesic except possibly at $\beta(t_i)$'s



β must be smooth at $\beta(t_i)$

since otherwise $\Rightarrow \beta \simeq \beta'$ s.t $\text{length}(\beta') < L$!

$\square$

<u>Homework</u> Write down carefully the proof of: for any complete Riemannian mfd (M, g) and $\alpha: ([0,1], 0, 1) \to (M, p, q)$, $\exists$ a geodesic path $(\beta: [0,1], 0, 1) \longrightarrow (M, p, q)$ st $\alpha \simeq \beta$ rel $\{0,1\}$.

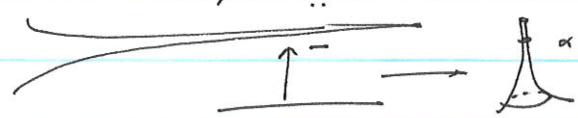
Lecture 9. Basic Geodesics

Where do you use the completeness?

RM1 The geodesic β may not be unique: $S^1 \times S^1$ flat torus $\mathbb{C}/\mathbb{Z}+i\mathbb{Z}$ 

RM2 If M is not compact, even (M, g) is complete, β may not exist.

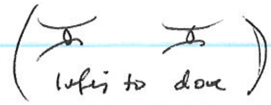
Eg $\Sigma = \mathbb{H}/(z \sim z+1) \cong \mathbb{D}-\{0\}$ hyperbolic α the quotient of $a(t) = i + t, 0 \le t \le 1$.



First $L = 0$ now since $d(N_i, N_{i+1}) = \frac{1}{N} \to 0$.

Next, Σ contains NO closed geodesics at all. Indeed if $g: S^1 \to \Sigma$ is a closed geodesic then $\pi^{-1}(g)$ is a geodesic in $\mathbb{H}$ invariant under $z \sim z+1$. But there is no such. (Not short loop)

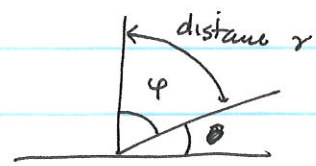
Proposition If (Σ, g) is hyperbolic and $\alpha \simeq \beta$ are two homotopic closed geodesics, then $\alpha = \beta$. (uniqueness of) (True also for neg curv)

Pf. We need the basic information. Easy proof  (left to done)

Lemma 1 Let l be the positive y-axis in $\mathbb{H}$. Then for any $r \ge 0$

$$N_r(l) = \{x \in \mathbb{H} \mid d(x, l) \le r\} = \{z \in \mathbb{H} \mid \pi - \theta \le \operatorname{Arg}(z) \le \theta\}$$

where or $\boxed{\sinh(r) = tg(\varphi)}$



Proof The isometries $z \mapsto \lambda z$ leave l invariant $\forall \lambda > 0 \Rightarrow$

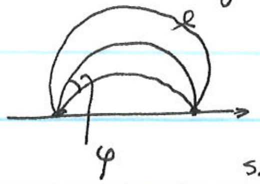
$\partial N_r(l)$ is invariant under $z \mapsto \lambda z \Rightarrow N_r(l) = \{\frac{\pi}{2} - \varphi \le \operatorname{Arg}(z) \le \frac{\pi}{2} + \varphi\}$

The actual calculation is homework. $\alpha(t) = (\cos t, \sin t)$

$$r = \int_{\frac{\theta}{2}-\varphi}^{\frac{\pi}{2}} \frac{dt}{\sin t} = \ln tg \frac{t}{2} \Big|_{\frac{\pi}{2}-\varphi}^{\frac{\pi}{2}} = -\ln tg\left(\frac{\pi}{4} - \frac{\varphi}{2}\right) \quad \square$$

Lecture 9. Basic Geodesics

<u>Corollary</u>: (1) For any geodesic ℓ in $\mathbb{H}$ or $\mathbb{D}^2$. For any r. $N_r(\ell)$ is



$tg(\varphi) = \sinh(r)$

s.t $\partial N_r(\ell)$ circles. union of two circular arcs

(2) if ℓ_1, ℓ_2 are two geodesics s.t $\ell_1 \subset N_r(\ell_2)$ for some $r \Rightarrow \ell_1 = \ell_2$.

<u>Now the proof of uniqueness</u> $\quad = \mathbb{H}/\Gamma$

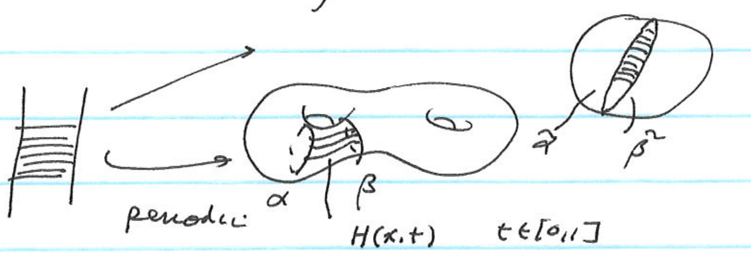
suppose $\alpha, \beta : S^1 \to \Sigma$ two homotopic geodesic s.t $H: S^1 \times I[0,1] \to \Sigma$ is the smooth homotopy. Define $F: \mathbb{R} \times [0,1] \to \Sigma$ by

$F(s,t) = H(e^{2\pi i s}, t)$: st $F(s,0)$, $F(s,1)$ are geodesics

<u>Lifting Thm</u> If $\pi: X \to Y$ is a covering map (i.e $\pi: \mathbb{H} \to \mathbb{H}/\Gamma$) and $f: A \to Y$ is a continuous map from a simply connected manifold, then $\exists$ a continuous $\tilde{f}: A \to X$ s.t $\pi \circ \tilde{f} = f$ ($\tilde{f}$ is called a lifting).

Let $\tilde{F}: \mathbb{R} \times [0,1] \longrightarrow \mathbb{H}$ be a lifting of F, s.t $\underbrace{\tilde{F}(s,0)}_{=\tilde{\alpha}}$, $\underbrace{\tilde{F}(s,1)}_{=\tilde{\beta}}$ are two geodesics in $\mathbb{H}$.

projecting down to α, β.



Now let $r = \max\{\text{length}\{H(x,t)\} \mid x \in S^1\} < +\infty$ <u>compactness</u>

$\Rightarrow$

$\forall\ s \qquad d(\tilde{\alpha}(s), \tilde{\beta}(s)) \le \text{length}_t (H(e^{2\pi i s}, t)) \le r$

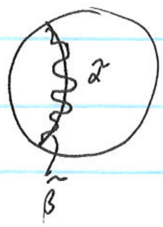
i.e $\tilde{\alpha} \subset N_r(\tilde{\beta}) \quad \Rightarrow \tilde{\alpha} = \tilde{\beta}$! by the corollary

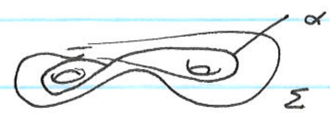
$\Rightarrow \alpha = \beta$.

$\square$

Lecture 9. Geodesics

<u>Corollary</u> Suppose α is an essential smooth loop in a closed hyperbolic surface Σ homotopic to the closed geodesic β. If $\tilde\beta$ is a lift of β to $\mathbb{H}$, then there exists a lift $\tilde\alpha$ of α s.t. $\tilde\alpha \subset N_r(\tilde\beta)$ for some $r>0$.

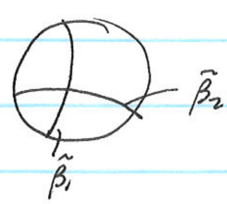




simple embedding of S^1

<u>Thm</u> If α is a simple essential loop homotopic to a closed geodesic β in $\mathbb{H}/\Gamma$ then β is simple

<u>Proof</u> If β is not simple, $\Rightarrow \exists$ two lifts $\tilde\beta_1, \tilde\beta_2$ of β s.t $\tilde\beta_1 \cap \tilde\beta_2 \neq 0$ in $\mathbb{H}$:



. Let $\tilde\alpha_i$ be the lift of α st $\tilde\alpha_i \subset N_{r_i}(\tilde\beta_i)$

$\Rightarrow \tilde\alpha_1 \cap \tilde\alpha_2 \neq \phi$ transversely intersecting

$\Rightarrow \alpha$ is not simple. $\square$

<u>RM</u>. The same argument shows if $\alpha_1\ \alpha_2$ two disjoint simple essential loops $\underbrace{}_{\alpha_1 \not\simeq \alpha_2}$ homotopic to geodesic ~~loops~~ $\beta_1, \beta_2 \Rightarrow \beta_1 \cap \beta_2 = \phi$.

Let us produce F-N coordinates <u>Now</u>

<u>Mention Priyam's Thm</u>

<u>Open Question</u> $\exists$ constant $c>0$ st. $\forall$ closed geodesic α in $(\Sigma, g) = \mathbb{H}/\Gamma$ lifts to a simple closed geodesic in at most $c\,l(\alpha)$ fold cover.

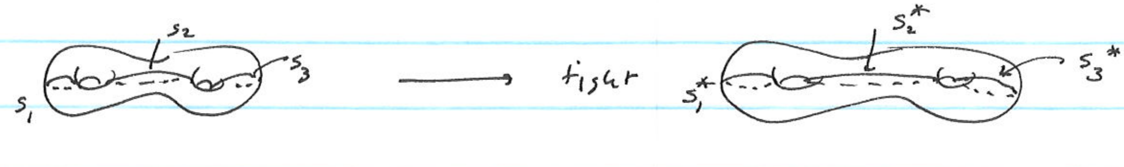
May be some fundamental domain.

Lecture 10. The Fenchel Nielsen Coordinate of Teichmüller Space.

All surfaces are assumed to be orientable.

Thm Suppose (Σ, g) closed orientable hyperbolic surface and $(s_1, \cdots, s_{3g-3})$ is a topological decomposition of Σ into 3-holed spheres by essential loops. Then their geodesic representatives $\{s_1^*, \cdots, s_{3g-3}^*\}$ is also a 3-holed sphere decomposition.

(Why $3g-3$. = $3g-2$ Parts)

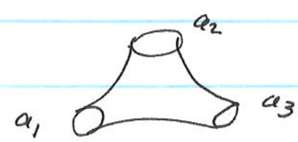
Eg


Pf Each s_i^* is simple since s_i is. Also $s_i^* \cap s_j^* = \phi$ since $s_i \cap s_j = \phi$.

Furthermore $s_i^* \neq s_j^*$. $\Rightarrow$ topological reason that each component X of $\Sigma - \cup s_i^*$ is $\Sigma_{0,3} =$ .

(Homework why ?)

hint: Euler characteristic + classification. $\chi(X) = -1$.

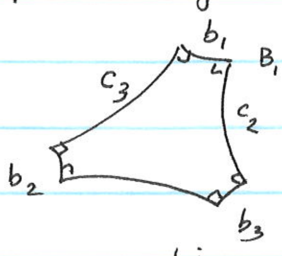


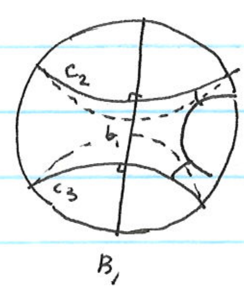
□

Def A hyperbolic pant: hyperbolic metric on $\Sigma_{0,3}$ with geodesic boundary

How to understand them ?

Key lemma $\forall\, b_1, b_2, b_3 > 0$, $\exists!$ right-angled hyperbolic hexagon P whose three non-pairwise adjacent edges have lengths b_1, b_2, b_3



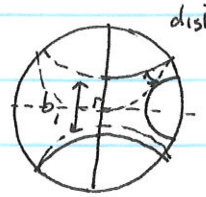


curves of const distance to c_2 c_3

B_3 tangent to c_2, c_3
|
geod

distance b_2

Pf Existence



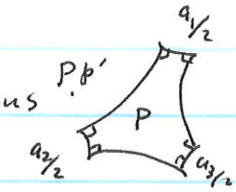
$\Longrightarrow$ existence P

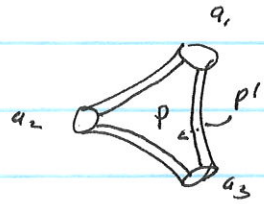
Uniqueness: from the above proof. □

Lecture 10 Fenchel-Nielsen

<u>Corollary</u>. $\forall\ a_1, a_2, a_3 > 0$ $\exists!$ hyperbolic pant where boundary has lengths a_1, a_2, a_3.

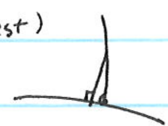
<u>Proof</u> <u>Existence,</u> two two copies of hexagons P, P' + double it

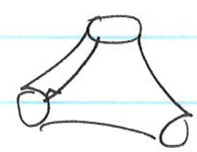




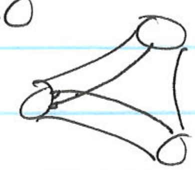
<u>Uniqueness</u> suppose X is a hyperbolic pants of geodesic boundary components B_1, B_2, B_3. Let c_i be the shortest path (geodesic) from B_{i+1} to B_{i+2} ($B_4 = B_1$, $B_5 = B_2$). Then

(1) $c_i \perp B_j$ $j \neq i$ (shortest)





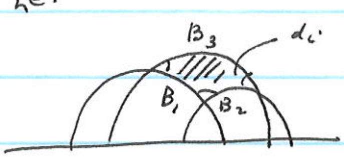
(2) <u>$c_i \cap c_j = \phi$</u> $i \neq j$



Indeed if so, there will exist a hyperbolic triangle of inner angles $\frac{\pi}{2}, \frac{\pi}{2}, \alpha$

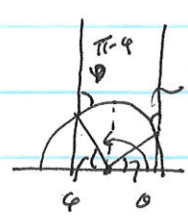
<u>Thm</u> (Gauss-Bonnet) If Δ is a hyperbolic triangle in $\mathbb{H}$ of inner angles $\alpha_1, \alpha_2, \alpha_3$, then $\text{area}(\Delta) = \pi - \alpha_1 - \alpha_2 - \alpha_3$. In particular $\alpha_1 + \alpha_2 + \alpha_3 < \pi$.

<u>Proof</u> Let



$$\text{Area}(\Delta) = \int_\Delta \frac{dx\,dy}{y^2} = \int_\Delta \cancel{d\left(\frac{dx\,dy}{y^2}\right)} \int_\Delta d\left(\frac{dx}{y}\right)$$

<u>stocks thm</u> $$\oint_{\partial\Delta} \frac{dx}{y} = \sum_{i=1}^{3} \int_{B_i} \frac{dx}{y}$$



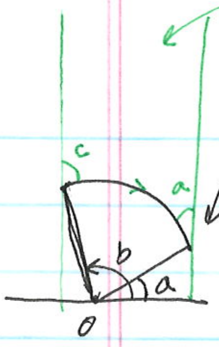
Now B_i part of a circle $\perp$ x-axis :

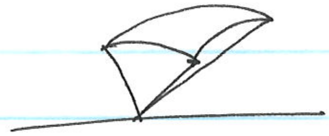
Equation $\begin{cases} x = r\cos t + a \\ y = r\sin t \end{cases}$ t $\quad \frac{dx}{y} = \frac{-r\sin t\,dt}{r\sin t} = -dt$

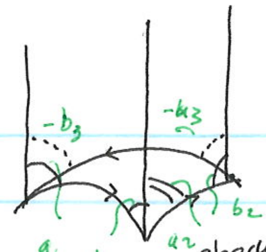
so $\int_{B_i} \frac{dx}{y} = \underline{\theta - \varphi}$. <u>Now</u> you <u>homework</u> to finish $\square$

$\pi - b - a$ $\boxed{a+c}$

Lecture 10 Fenchel-Nielsen

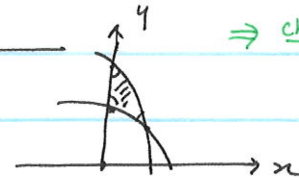






check it

⇒ check Where is $\pi - \theta$



□

RM After a $PSL(2,\mathbb{R})$, alway may assume:

It is better to calculate. □

Corollary. Suppose (Σ, P) is a closed topological surface w/ a pants decomp. Then all hyperbolic metrics on (Σ, P) are obtained by isometrical gluing of hyperbolic pants along their boundary.

Pf (1).

For a hyperbolic metric g on Σ ⇒ making all ∂P geodesic ⇒ done

(2) If each pants hyperbolic, gluing isometry ⇒ hyperbolic metric.

□

The Fenchel-Nielsen coordinate for Teichmuller space

By the uniformization thm For a closed surface Σ w/ $\chi(\Sigma) < 0$

the Teichmüller space $T(\Sigma) = \{ [(\Sigma, g)] \mid g$ hyperbolic metric

$(\Sigma, g_1) \sim (\Sigma, g_2)$ if there exists an isometry $h\,(\Sigma, g_1) \to (\Sigma, g_2)$

st $h \simeq id$, homotopic $\}$.

F.N coordinate: Fix a pants decomposition by $3g-3$ loops, then

$$T(\Sigma) \underset{\text{homeo}}{\cong} \mathbb{R}^{6g-3} = (\mathbb{R}_{>0} \times \mathbb{R})^{3g-3}.$$