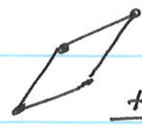
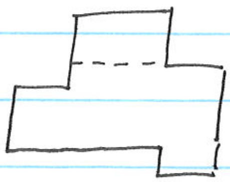
Lecture 5 P.L. Surfaces

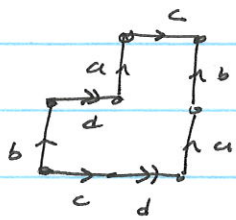


torus

Eg. (Translation surfaces)

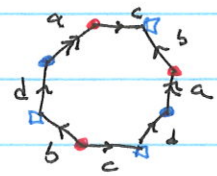
Take a Euclidean polygon P and identity pairs of edges of P by translations





The quotient space is a Riemann surface

Topologically:



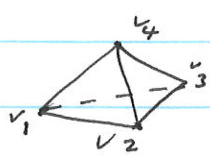
$\chi = 3 - 4 + 1 = 2 = 2 - 2 \cdot 1$

$\Rightarrow$ Homework For any $g \geq 1$. show that there exists a translation Riemann surface of genus g.

It turns out: Every closed Riemann surface is iso biholomorphic to a translation surface.

Bers Conjecture Every closed Riemann surface is biholomorphic to a polyhedral surface in $\mathbb{R}^3$.

Eg Tetrahedron

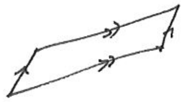


$\underset{\text{biho}}{\overset{\Phi}{\cong}}$ $\hat{\mathbb{C}}$

$\phi(v_1) = 0$
$\phi(v_2) = 1$
$\phi(v_3) = \infty$
$\phi(v_4) = z$

Conclusion: Each tetrahedron has a conformal invariant z.

No one knows how to compute this z in terms of the 6 lengths

Translation surf $\mathbb{C}/\mathbb{Z}w_1+\mathbb{Z}w_2 =$ 

Lecture 7. Hyperbolic Geometry and Uniformization thm

Recall. 2-dim Riemannian manifold (Σ^2, ds^2).

$\Rightarrow$ distance $d(p,q) = \inf \{ \text{length}(\gamma) \mid \gamma \text{ smooth path from } p \text{ to } q \}$

Curvature $K: \Sigma \to \mathbb{R}$

$$K_p = \lim_{r \to 0^+} 12 \left(\frac{\pi r^2 - A(r)}{\pi r^4} \right)$$

$A(r) = $ area of $\{ z \in \Sigma \mid d(z,p) \le r \}$.

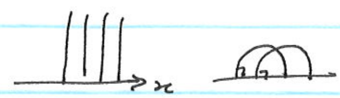
Key fact $\phi: (\Sigma, ds^2) \circlearrowleft$ isometry $\Rightarrow d(\phi(p), \phi(q)) = d(p,q)$. $K_{\phi(p)} = K_p$

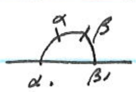
Homework $f(z) = \dfrac{z-i}{z+i} : \left(\mathbb{H}, \dfrac{dx^2+dy^2}{y^2}\right) \longrightarrow \left(\mathbb{D}^2, \dfrac{4\,dx^2+dy^2}{(1-x^2-y^2)^2}\right)$ is an isometry.

Eg Poincaré disk model $(\mathbb{D}^2, ds^2_h)$ $ds^2 = 4\dfrac{dx^2+dy^2}{(1-x^2-y^2)^2}$ has $K_p = -1$.

But before doing that. we need.

Thm For the hyperbolic plane $(\mathbb{H}, ds^2_{\mathbb{H}})$

(1) All geodesics are $x = a$ or semi-circles $\perp$ x-axis 

(2) dist $d(\alpha, \beta) = \ln(\alpha, \beta, \beta', \alpha')$  cross ratio

(3) All orientation preserving isometries are $PSL(2,\mathbb{R})$ (Möbius transf)

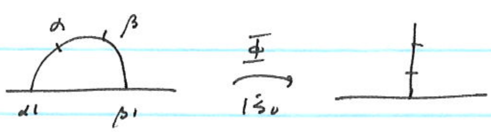
(4) $(\mathbb{H}, ds^2_{\mathbb{H}})$ is a complete ~~hyperbolic~~ metric of curvature $= -1$.

Pf. (1) Key fact $PSL(2,\mathbb{R})$ acts isometrically on $\mathbb{H}$.

(1) $x = 0 \Rightarrow x = a$ $(z \mapsto z+a) \Rightarrow$ semi-circles $\perp x$ at $0 \Rightarrow$ all.

(3) Φ orientation preserving iso $\Rightarrow$ Φ orientation preserving angle preserving $\Rightarrow$ Möbius

(2) $d(i\alpha, i\beta) = \log(i\alpha, i\beta, \infty, 0)$ $\Rightarrow$ Any. by isometry



$$d(\alpha,\beta) \underset{\Phi}{=} d(i a, i\beta) = \lg(i a, i\beta, \infty, 0)$$

$$= \lg(\alpha, \beta, \beta', \alpha') \quad // \text{ Möbius inv.}$$

Eg $d(0, R) = \lg\left(\dfrac{1+R}{1-R}\right)$

Lecture 7 Hyperbolic Geometry

<u>Completeness</u> geodesics extends to infinity, First constant curvature since $PSL(2,\mathbb{R})$ acts transitively on $\mathbb{H}$.

<u>Curvature calculation</u>

<u>Eg</u> $K_0 = -1$ for $\left(\mathbb{D}^2, \frac{4(dx^2+dy^2)}{(1-x^2-y^2)^2}\right)$

<u>Sol</u>. Take $W(R) = \{ |z| \le R \} \subset \mathbb{D}^2$ $R \in (0,1)$. Its hyperbolic area

<u>metric</u> $A\,dx^2 + 2B\,dxdy + C\,dy^2$ area form $\sqrt{AC-B^2}\,dxdy$, $\frac{4\,dxdy}{(1-x^2-y^2)^2}$

$$\Rightarrow \text{Area} = \int_{W(R)} \frac{4\,dxdy}{(1-x^2-y^2)^2} \underset{\text{polar}}{=} \int_0^R \int_0^{2\pi} \frac{4\rho\, d\rho\, d\theta}{(1-\rho^2)^2}$$

$$= 4\pi \int_0^R \frac{d\rho^2}{(1-\rho^2)^2} = 4\pi \frac{R^2}{1-R^2}$$

Now the hyperbolic radius r of $W(R)$ is $\log(0,R,1,-1) = \log\left(\frac{1+R}{1-R}\right)$

$$\Leftrightarrow \quad R = (e^r-1)/(e^r+1)$$

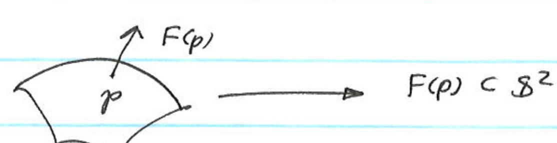
$$\text{so } K_0 = \lim_{r \to 0} 12 \, \frac{\pi r^2 - 4\pi \frac{(e^r-1)^2}{4e^r}}{\pi r^4} = -1. \qquad \square$$

<u>Best way to calculate Gaussian</u> curvature K_p for $\Sigma \subset \mathbb{R}^3$.

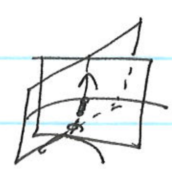
say Σ is $z = f(x,y)$ s.t. $f_x(p) = f_y(p) = 0$, then

$$K_p = \det \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} \Bigg|_p. \qquad (1)$$

<u>Hw</u> Use (1) to show: $K_p = \det(DF)$: $F: \Sigma \to S^2$ Gaussian map



<u>or</u> $K_p = K_1 \cdot K_2$ K_1, K_2 principal curvatures at p



Lecture 7. Hyperbolic Metrics

Uniqueness

<u>Thm</u> (Cartan-Hadamard) Any two complete, constant curvature -1 metrics on simply connected surfaces are isometric.

<u>Corollary</u> If ds^2 is a complete, hyperbolic metric on $\mathbb{H}$ conformal (to $\mathbb{H}$), then $ds^2 \equiv ds_h^2$.

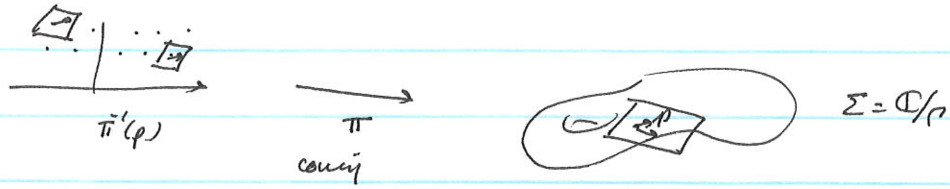
<u>Pf.</u> C-H $\Rightarrow$ $\exists$ isometry $\Phi: (\mathbb{H}, ds_h^2) \to (\mathbb{H}, ds_h^2)$ isometry, may assume Φ preserves orientation by composing it w $z \mapsto -\bar{z}$ if needed.

Now ds^2 conformal $\Rightarrow$ $\Phi: \mathbb{H} \to \mathbb{H}$ 1-1 onto complex analytic. Schwarz $\Rightarrow$ $\Phi \in PSL(2, \mathbb{R})$ $\Rightarrow$ Φ is an isometry of ds_h^2. i.e. $\Phi^*(ds_h^2) = ds^2$ + $\Phi^*(ds_h^2) = ds_2^2$ $\Rightarrow$ result. $\square$

<u>Uniformization Thm in metric Setting</u> If X is a Riemann surface, then $\exists$ a conformal Riemann metric g on X of curvature $= 1, 0,$ or -1. The hyperbolic metric g is unique.

<u>Proof</u> By the uniformization thm $X \underset{biho}{\cong} S^2$ $\Rightarrow$ spherical metric.

(1) $X \cong \mathbb{C}/\Gamma$ $\Gamma <$ translation subgroups. Then $dx^2 + dy^2$ on $\mathbb{C}$ invariant metric on $\mathbb{C}$ $\Rightarrow$ produce a metric on $\mathbb{C}/\Gamma$.



Coming from the quotient

The metric g is not unique since λg <u>also</u> satisfies $K \equiv 0$. $(\lambda > 0)$

(3) $X \cong \mathbb{H}/\Gamma$, g is the quotient of the hyperbolic metric on X.

Curvature $= -1$. conformal.

Lecture 8. Hyperbolic Metric

Uniqueness: suppose $\tilde{g}$ is a second hyperbolic metric on X, complete and conformal $\Rightarrow$ $\pi^*(\tilde{g})$ is a complete hyperbolic metric on $\mathbb{H}$ conformal. By the corollary $\pi^*(\tilde{g}) = ds_h^2$. $\square$

RM Hyperbolic surface = Riemann surface w/ a hyperbolic metric = $\mathbb{H}/\Gamma$ $\quad \Gamma < PSL(2, \mathbb{R})$

A brief recall of covering space theory. + lifting. $\pi: \mathbb{H} \to \mathbb{H}/\Gamma = \Sigma$ hyperbolic surface is an example of covering maps

Recall $F: X \to Y$ covering map $\Leftrightarrow$ $\forall\ p \in Y$ $\exists$ nbhd $U(p)$ of p s.t. $F^{-1}(U) = \bigsqcup_\alpha V_\alpha$ where V_α open and $F|: V_\alpha \to U$ homeo

X, Y, Z manifolds connected

Two key theorems: $F: X \to Y$ covering map

(1). $\varphi: Z \to Y$ continuous and $\pi_1(Z) = 1$, then $\exists\ \tilde{\varphi}: Z \to F$ s.t. $F \circ \tilde{\varphi} = \varphi$

(2). if $\tilde{\varphi}_1, \tilde{\varphi}_2$ are two liftings s.t. $\tilde{\varphi}_1(a) = \tilde{\varphi}_2(a)$ at one $a \Rightarrow \tilde{\varphi}_1 = \tilde{\varphi}_2$

(3). homotopy lifting

Thm If $\varphi: \Sigma_1 \to \Sigma_2$ is an analytic map between two hyperbolic surfaces, then φ decreases hyperbolic distances. In particular, if φ is a biholo $\Rightarrow \varphi$ iso.

Pf
$$\begin{tikzcd}
\mathbb{H} \arrow[r, "\tilde{\varphi}_1"] \arrow[d, "\pi"'] \arrow[dr, "\varphi\circ\pi"] & \mathbb{H} \arrow[d, "\pi"] \\
\Sigma_1 = \mathbb{H}/\Gamma_1 \arrow[r, "\varphi"'] & \mathbb{H}/\Gamma_2
\end{tikzcd}$$

First $\varphi\pi: \mathbb{H} \to \mathbb{H}/\Gamma_2$ cont. $\Rightarrow \exists$ a lifting $\tilde{\varphi}_1$ by looking at the lift of its inverse

$\Rightarrow$ $\tilde{\varphi}: \mathbb{H} \to \mathbb{H}$ is an 1-1 onto analytic $\Rightarrow$ $\tilde{\varphi}$ iso

$\Rightarrow \varphi$ isometry. (Schwarz lemma)

Corollary If $\varphi: \Sigma_1 \to \Sigma_2$ is a biholomorphism between two hyperbolic surfaces, then φ is an isometry.

Pf Let $ds_1^2\ ds_2^2$ be the respective Poincaré metric. Then $\varphi^*(ds_2^2)$ is a conformal complete hyperbolic metric on Σ_1 $\Rightarrow$ $\varphi^*(ds_2^2) = ds_1^2$ $\Rightarrow$ φ isometry. $\square$

Lecture 7. Hyperbolic Geometry.

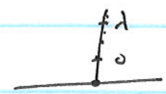
Conclusion All hyperbolic geometric invariant of (Σ, ds^2) are invariant of the Riemann surface $\mathbb{H}/\Gamma$. In particular, the length of the shortest geodesics.

Eg The hyperbolic annulus $X_\lambda = \mathbb{H}/_{z\sim\lambda z} = \{\mu<|z|<1\} = R_\lambda$, $-\log\lambda\log\mu = 2\pi^2$

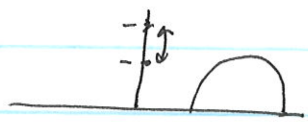
Q What is the length of the shortest geodesic in R_μ. **ANS** $\boxed{\log\lambda}$ $\boxed{\lambda>1}$

Solution:

What are the closed geodesics in X_λ ? **ANS** π[y-axis]

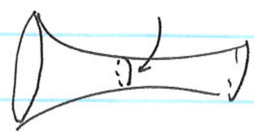


Indeed if L is a closed geodesic in $X_\lambda \Rightarrow \pi^{-1}(L)$ is a geodesic in $\mathbb{H}$ invariant under $\gamma(z)=\lambda z$



$\pi^{-1}(L)$ has end pts $\{a,b\}$ $a\neq b$ $a,b\in\mathbb{R}\cup\{\infty\}$

$\{\lambda a,\lambda b\}=\{a,b\}$ $\lambda>1 \Rightarrow a,b=0,\infty$



Now $\underline{i\sim i\lambda} \Rightarrow$ of distance $\underline{\log\lambda}$

$\Rightarrow$ result

Conclusion If X_λ is biholomorphic to $X_{\lambda'}$ $\lambda,\lambda'>1 \Rightarrow \log\lambda=\log\lambda' \Rightarrow \lambda=\lambda'$

or $\{\mu<|z|<1\}$ biholomorphic to $\{\mu'<|z|<1\} \Rightarrow \mu=\mu'$

or $\mathrm{Mod}(\{1<|z|<2\}) \cong [0,1)\cup\{\infty\}$.

Homework Use Schwarz lemma to show. if $\varphi:\Sigma_1\to\Sigma_2$ is analytic between two hyperbolic surfaces. Then φ decreases hyperbolic distances