2 Lecture 21-22

Mostow Rigidity

(Mostow)

Thm If M^n, N^n two closed connected orientable hyperbolic n-manifolds and $f: M^n \to N^n$ a homeomorphism, then $f \simeq g: M^n \to N^n$ st g is an isometry

RM We can relax $f \to$ homotopy equivalence

Proof: Orginal Mostow: introduced large scale geometry.

Gromov: introduced the notion of norm.

Basic ideas: $\Gamma_1 = \pi_1(M)$, $\Gamma_2 = \pi_1(N)$ act on $\mathbb{H}^n$ $\tilde{f}: \mathbb{H}^n \to \mathbb{H}^n$ the lift

There is an isomorphism $\varphi: \Gamma_1 \to \Gamma_2$ induced by f.

st $\tilde{f}(\gamma \cdot x) = \varphi(\gamma)\tilde{f}(x)$, $\gamma \in \Gamma_1$

Step 1. $\tilde{f}: \mathbb{H}^n \to \mathbb{H}^n$ is a quasi-isometry (Q.I)

Step 2. $\tilde{f}$ extends to $\bar{\mathbb{H}}^n \to \bar{\mathbb{H}}^n$ continuously + $\tilde{f}|_{\partial\bar{\mathbb{H}}^n}$ homeo

Step 3 Gromov norm $\Rightarrow$ $\underbrace{\tilde{f}|_{\partial\mathbb{H}^n}}_{=\tilde{g}|}$ $\in \mathrm{Iso}(\mathbb{H}^n)$. $\tilde{g} \in \mathrm{Iso}(\mathbb{H}^n)$

$\Rightarrow$ $\tilde{g}(\gamma \cdot x) = \varphi(\gamma)\tilde{g}(x)$

$\Rightarrow$ $\tilde{g}$ induces iso $g: M^n \to N^n$.

4-4-2013

Basic large scale Geometry

(X, d), (Y, d') two metric spaces $F: X \to Y$ (Not necessary cont)

Def If $\exists$ $K>0$ s.t.

$$\frac{1}{K} d(x_1, x_2) - K \le d'(F(x_1), F(x_2)) \le K\, d(x_1, x_2) + K$$

We say F is a <u>K-quasi-isometric</u> embedding,

If furthermore $Y = N_K(F(X)) = \{ y \in Y \mid \exists\, x \text{ s.t. } d'(y, F(x)) \le K \}$

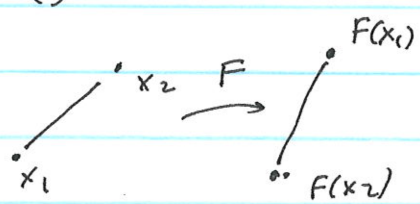
We say F is a quasi-isometry. (Q.I)

Obviously F, G Q.I $\Rightarrow$ $F \circ G$ <u>Q.Iso</u>

Eg F Q.I, $G: Y \to X$ $G(y) = x$ $d'(y, F(x)) \le K$

Then G is a Q.I

(i)



To see

$$d(G(y_1), G(y_2)) = d(x_1, x_2)$$
$$\le K^2 + K\, d(F(x_1), F(x_2))$$
$$\le K^2 + K\left[d(F(x_1), y_1) + d(y_1, y_2) + d(y_2, F(x_2)) \right]$$
$$\le K\, d(y_1, y_2) + (K^2 + 2K).$$

The other way around is fine:

$$d(y_1, y_2) \le d(y_1, F(x_1)) + d(F(x_1), F(x_2)) + d(F(x_2), y_2)$$
$$\le 2K + (K\, d(x_1, x_2) + K)$$ ◻

Corollary Q.I is an equivalence relation among metric spaces

Eg Each compact space is Q.I. to a point!

Eg $\mathbb{Z} \underset{Q.I}{\sim} \mathbb{R}$ $f(x) = x$ $\mathbb{Z} \hookrightarrow \mathbb{R}$. $K=1$

$\mathbb{Z}^2 \underset{Q.I.}{\cong} \mathbb{R}^2$ $f(x) = x$ inclusion $K=1$

$\mathbb{Z}^n \underset{Q.I.}{\cong} \mathbb{R}^n$

Eg $K \subset X$ subset $N_R(K) = X \Rightarrow i: K \hookrightarrow X$ Q.Iso.

Easy F, G Q.Isometric embedding $\Rightarrow$ $F \circ G$ Q.Iso embedding.

Large Scale Geometry

Metrics on finitely generated groups

G group $\quad S = \{s_1, \cdots, s_n\}$ generates G s.t. $x \in S \Rightarrow x^{-1} \in S$

Eg $\quad G = \mathbb{Z} \quad S = \{+1, -1\}$. $\quad id \notin S$

Given $w \in G$, $\quad |w| = |w|_S = \min\{k \mid w = x_1 \cdots x_k,\ x_i \in S\}$

minimal number of generators to produce w.

Eg $\mathbb{Z}$. $\{\pm 1\}$. $\quad \{\pm 1 \pm 2\}$. $\quad$ $|id| = 0$.

Lemma 1. $\quad |\gamma \cdot \delta| \le |\gamma| + |\delta| \quad$ & $\quad |w^{-1}| = |w|$.

Define $\quad d(x,y) = |x^{-1}y| = d(y,x) = |y^{-1}x|$

Triangle inequality $\quad d(x,y) + d(y,z) = |x^{-1}y| + |y^{-1}z| \le |x^{-1}z| = d(x,z)$

Prop If T is another symmetric generating set, then $\exists\, k \ge 0$ st

$$|w|_T \le k|w|_S \quad \& \quad |x|_S \le k|x|_T.$$

Proof Let $K = \max\{|t_i| \mid t_i \in T\}$ Then

$|w|_T = n$

$\Rightarrow \quad w = x_1 \cdots x_n \quad x_i \in T$

$\Rightarrow \quad |w|_S \le |x_1|_S + \cdots + |x_n|_S \le K \cdot n$ $\quad \square$

Corollary 2. (a) The Q.I. class of G does not depend on the choice of S.

(b) If $\varphi: G \to H$ is a group homomorphism, then φ is a (quasi isometric embedding) Lipschitz map.

Pf (b) . $\quad |\varphi(x)| \le k|x| \quad$ same proof.

(c) If $\varphi: G \to H$ group isomorphism $\Rightarrow$ φ is Q.I.

Eg $(\mathbb{Z}, \{\pm 1\})$ the metric $\underset{d}{\Rightarrow}$ $d(n,m) = |m-n|$. so $(\mathbb{Z}, d) \hookrightarrow (\mathbb{R}, d)$ Q.I.

Large Scale Geometry

Prop Let (M, g) be a closed connected Riemannian w/ universal cover $(\tilde{M}, \tilde{g})$ s.t. $\Gamma = \pi_1(M)$ acts isometrically on $\tilde{M}$ as deck transf. Then

$$\Phi : \Gamma \to \tilde{M} \qquad \gamma \mapsto \gamma(b) \qquad (b \in \tilde{M} \text{ fixed})$$

is a Q.I.

Proof (Say $S \subset \Gamma$ is a) Let $\Omega \subset \tilde{M}$ be a compact connected set s.t.

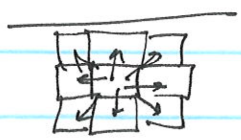
(1) $\tilde{M} = \bigcup_{\gamma \in \Gamma} \gamma(\Omega)$

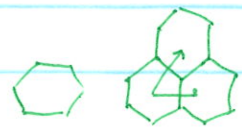
(2) $\gamma(\mathring{\Omega}) \cap \mathring{\Omega} = \phi \quad \forall\, \gamma \in \Gamma - \{id\}$

Ω - fundamental domain, Ω exists. $\Omega = \{x \in \tilde{M} \mid d(x, b) \le d(\gamma x, b)\}$

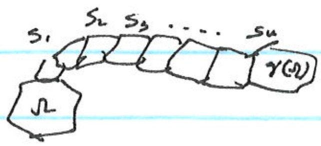
Es $\mathbb{Z} \curvearrowright \mathbb{R}$ $\quad n_* x = x + n \quad \Omega = [0, 1]$

$\mathbb{Z}^2 \curvearrowright \mathbb{R}^2$ translations $\quad \Omega = [0, 1] \times [0, 1]$





$S = \{ \gamma \in \Gamma \mid \Omega \cap \gamma(\Omega) \ne \phi,\ \gamma \ne id \}$. Note $\Omega \cap \gamma(\Omega) \ne \phi \Leftrightarrow \Omega \cap \gamma^{-1}(\Omega) \ne \phi$



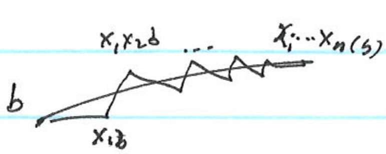
HW S generates Ω.

Fix $b \in \Omega$. Let $K = \max\{ d(b, s(b)) \mid s \in S \}$

Note $\quad d(\Phi(\gamma_1), \Phi(\gamma_2)) = d(\Phi(\gamma_1^{-1}\gamma_2), \Phi(id)) \qquad (\gamma \text{ acts iso})$

and $d(\gamma_1, \gamma_2) = d(\gamma_1^{-1}\gamma_2, id)$

Thus, it suffices to prove: $\forall\, \gamma \in \Gamma$



$$\frac{1}{K} d(\gamma b, b) - K \le |\gamma| \le K\, d(\gamma b, b) + K$$

(1) Write $|\gamma| = n \qquad \gamma = x_1 \cdots x_n \qquad x_i \in S \qquad \gamma_i = x_1 \cdots x_i,\ \gamma_0 = id$

$$d(\gamma b, b) \le \sum_{i=0}^{n-1} d(\gamma_i b, \gamma_{i+1} b) \le \sum_{i=0}^{n-1} d(\gamma_i(b), \gamma_i(x_{i+1} b))$$

$$= \sum_{i=0}^{n-1} d(b, s(b)) \le nK = |\gamma| \cdot K.$$

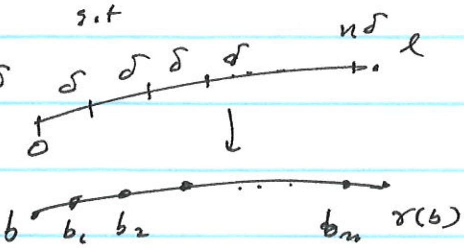
Let $\delta = \frac{1}{2}\min\{ d(\Omega, \gamma(\Omega)) \mid \gamma \in \Gamma \ \ \Omega \cap \gamma\Omega = \phi \} > 0$

(2) $\forall\ \gamma \in \Gamma$. let $\alpha : [0, l] \longrightarrow \tilde{M}$ $\alpha(0) = b$, $\alpha(l) = \gamma(b)$

be the shortest geodesic w/ $l = d(b, \gamma(b))$.

Let $b_i = \alpha(i\delta)$ $i = 0, 1, \cdots, n$ s.t

$n\delta \le l < (n+1)\delta$



Say $b_i \in \gamma_i(\Omega)$ Then $d(b_i, b_{i+1}) < \min\{d(\Omega, \gamma(\Omega))\}$

$\Rightarrow \gamma_i(\Omega) \cap \gamma_{i+1}(\Omega) \neq \phi$

so $\gamma_i^{-1}\gamma_{i+1} \in S \cup \{id\}$ or $\gamma_{i+1} = x_i \gamma_i$, $x_i \in S$ or $x_i = id$

Also $\gamma = x_{n+1}\gamma_n$ hold

Now

$\gamma = x_n x_{n-1} \cdots x_0$ $x_i \in S \cup \{id\}$

$\Rightarrow |\gamma| \le n+1 = \frac{1}{\delta}(n\delta) + \frac{1}{\delta} \le \frac{1}{\delta} l + \frac{1}{\delta} = \frac{1}{\delta} d(b, \gamma(b)) + \frac{1}{\delta}$

□

This result says: as far as Quasi-isometric property goes, the study of $\pi_1(M)$ is the same as the geometry of $\tilde{M}$.

Eg For any hyperbolic, closed surface Σ_g $g \ge 2$ $\pi_1(\Sigma_g) \underset{\text{Q.Iso}}{\cong} \mathbb{H}^2$.

$\pi_1(S^1) \underset{\text{Q.Iso}}{\cong} \mathbb{R}^1$

$\pi_1(S^1 \times S^1) \underset{\text{Q.I}}{\cong} \mathbb{R}^2$

Corollary. If $f : M^n \to N^n$ homeo (or just any map) induces isomorphism f_* in $\pi_1(M^n) \to \pi_1(N^n)$ then the map lifted to

$\hat{F} : \tilde{M}^n \to \tilde{N}^n$ is a Q.Iso

$$\begin{tikzcd} \tilde{M}^n \arrow[r] & \tilde{N}^n \\ \pi_1(M) \arrow[u, "ev"] \arrow[r, "f_*"] \arrow[d, dash] & \pi_1(N) \arrow[u, "ev"'] \\ \text{Q.I.} & \end{tikzcd}$$

Gromov Hyperbolic Spaces + Gromov product

<u>Length spaces</u> (X,d) metric space $\gamma: [0,1] \to X$ continuous. The length $l(\gamma)$ of γ

$$l(\gamma) = \sup \left\{ \sum_{i=0}^{n} d(\gamma(t_i), \gamma(t_{i+1})) \;\middle|\; 0 = t_0 < \cdots < t_{n+1} = 1 \right\} \geq 0$$

A metric space (X,d) is a <u>length space</u> if

$$d(x,p) = \inf \{ l(\gamma) \mid \gamma: [0,1] \to X \quad \gamma(0)=p,\ \gamma(1)=q \}$$

<u>Eg</u> Connected Riemannian manifold, polyderal spaces, connected graphs of given edge lengths are length spaces

<u>Def</u> $\alpha: [a,b] \to (X,d)$ a <u>geodesic</u> (segment) if $d(\alpha(t_1), \alpha(t_2)) = \text{length}(\alpha|_{[t_1,t_2]})$

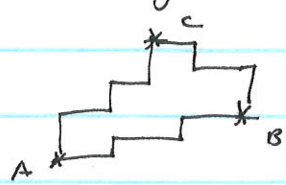
<u>ie</u> α is an isometric embedding $= |t_1 - t_2|$.

A <u>geodesic space</u> (X,d) = length space st $\forall\, p, q \in X$ $d(p,q) = \text{length}(\alpha)$.

<u>Eg</u> G. gp finite symmetric generating set S, Form the Cayley graph

$\Gamma(G,S)$: vertices G. $g_1, g_2 \in G$ are joint by an edge $g_1 = s g_2$ $s \in S$

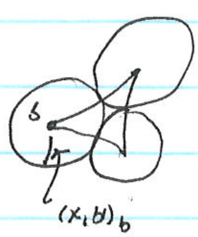
$\boxed{\text{So}}$ $\mathbb{Z}^2$ $\{\pm(1,0), \pm(0,1)\}$.

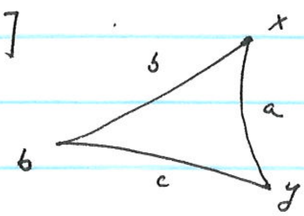


A triangle in a length space,

<u>Def</u> (X,d) metric space, $b \in X$. the Gromov product of $x, y \in X$ w.r.t b

$$(x,y)_b = \frac{1}{2}\left[d(x,b) + d(y,b) - d(x,y) \right]$$

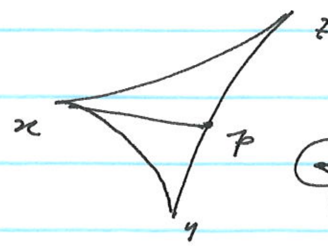




<u>Notation</u>: (X,d) Length space $p, q \in X$, $[p,q]$ = Any geodesic segment from p to q

Prop $\forall\, x, y, z \in X$ $\quad (X, d)$ length space

$$(y, z)_x \le d(x, [y, z])$$



Pf. Let $p \in [y, z]$ s.t. $d(x, [y, z]) = d(p, x)$

$\Rightarrow d(y, z) = d(y, p) + d(p, z)$

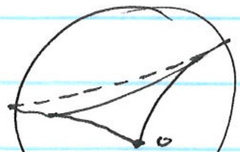
Now $2(y, z)_x = d(x, y) + d(x, z) - d(y, p) - d(p, z) \le d(x, p) + d(x, p) = d(x, [y z])$.

Def A geodesic space (X, d) is hyperbolic (δ-hyperbolic) if

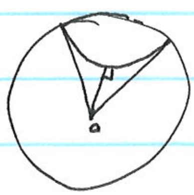
$\forall\, x, y, z \in X$ $\quad [y, z] \subset N_\delta([y, x] \cup [x, z])$

Eg Hyperbolic space $\mathbb{H}^2$ is δ-hyperbolic.

Pf

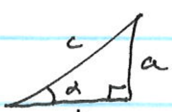


Try to prove $\Rightarrow$



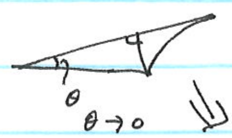
Compute the distance

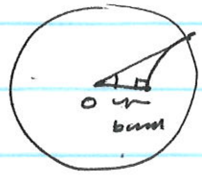
$\theta \ge \pi/2$



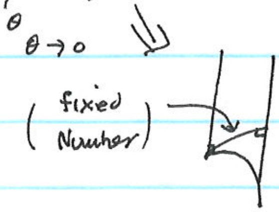
① $\begin{cases} \sinh(a) = \sinh(c) \sin(\alpha) \\ \cos(\alpha) = \dfrac{\tanh(b)}{\tanh(c)} \end{cases}$

$\theta \to 0$

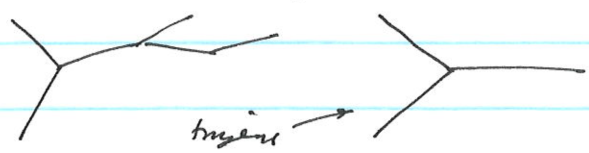




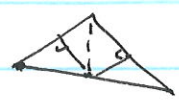
$\Rightarrow$



Eg. A tree Gromov 0-hyperbolic



Eg $\mathbb{R}^2$ Not Gromov hyperbolic



far

Def A quasi-geodesic in (X, d) is quasi-isometric embedding

$\gamma : [a, b] \to X$:

$$\frac{1}{K}|t - t'| - K \le d(\gamma(t), \gamma(t')) \le K|t - t'| + K \qquad \forall\, t, t' \in [a, b]$$

Lecture 23–24

Geodesic Spaces + Hyperbolicity

<u>Def.</u> (X, d) metric s.t $\forall p, q \in X$. $\exists$ (geodesic) $\alpha : [0, \ell] \to X$ cont s.t $\alpha(0) = p$, $\alpha(\ell) = q$ + $\forall s, t, \in [0, \ell]$

$$d(\alpha(s), \alpha(t)) = |s - t|$$

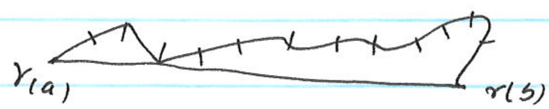
In particular: ① $d(p, q) = \ell$.

② $\text{length}(\alpha) = \ell$

Indeed:

<u>Eg</u> $\gamma : [a, b] \to X$ Any path, continues $\Rightarrow$ $\text{length}(\gamma)$

$$= \sup \left\{ \sum_{i=0}^{n-1} d(\gamma(t_i), \gamma(t_{i+1})) \;\middle|\; a = t_0 < \cdots < t_n = b \right\} \geq d(\gamma(a), \gamma(b))$$



$\text{length}(\gamma) \geq d(\gamma(a), \gamma(b))$

<u>To our case</u> for α $\displaystyle\sum_{i=0}^{n-1} d(\alpha(t_i), \alpha(t_{i+1})) = \sum_{i=0}^{n-1} |t_i - t_{i+1}| = |0 - \ell| = |\ell| = d(\ldots)$.

<u>Basic Fact</u> geodesic $=_{\text{globally}}$ distance minimizing path in (X, d) $\Rightarrow \alpha$ is 1-1

<u>Eg</u> $\alpha(t) = e^{it}$ $0 \leq t \leq 1.5\pi$ $\alpha : [0, 1.5\pi] \to S^1$ is <u>Not</u>! a geodesic

But $\beta(t) = e^{it} : [0, \ell] \to S^1$ <u>$\ell \leq \pi$ is</u>

(This is different from diffeo. geometry)

<u>Notation</u> $p, q \in X$ $[p, q]$ denote a geodesic from p to q. May be many, say on S^2.

<u>Eg</u> Connected graph, polyhedral, complete Riemannian mflds are geod. spaces

<u>Def</u> A geod space (X, d) is δ-hyperbolic if $\forall$ geodesic trigs $\Delta p, q, r$ in X $[p, q] \subset N_\delta([p, r] \cup [r, q])$

<u>Eg</u> $\mathbb{H}^2$ + hence $\mathbb{H}^n$ is δ-hyperbolic

Geodesic Space + Hyperbolicity

Key lemma 1. (X, d) δ-hyperbolic $\gamma : I \to X$ rectifiable $(\ell(\gamma) < +\infty)$ from p to q

Then $\forall\ x \in [p, q]$

$$d(x, \gamma(I)) \leq 1 + \delta \cdot n \quad \text{where} \quad n = [\ln \ell(\gamma)].$$

the largest integer $\leq \ell(\gamma)$. (In particular $\ell(\gamma) \geq 1 \Rightarrow d(x, \gamma(I)) \leq 1 + \delta|\ln \ell(\gamma)|$)

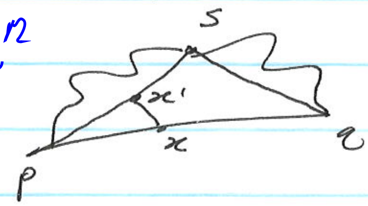
Proof Induction on n

$n = 1$ $\ell(\gamma) \leq 1 \Rightarrow d(p, q) \leq 1 \Rightarrow$ done

Suppose it holds for n.

For $n+1$, let s be the midpoint of γ $s = \gamma(c)$

$\Rightarrow \ell(\gamma|_{[a,c]}) = \ell(\gamma|_{[c,b]}) = \ell(\gamma)/2 \leq \cancel{n}$ 2^n



Induction $\Rightarrow \exists\ x' \in [p, s] \cup [s, q]$ $x' \in [p, s]$

say x' st $d(x', x) \leq$ δ

Induction $\Rightarrow d(x', \gamma([a, c])) \leq 1 + \delta n$

$\Rightarrow d(x, \gamma(I)) \leq d(x, x') + d(x', \gamma(I)) \leq 1 + \delta(n+1).$ □

Recall a quasi-geodesic $\alpha : I \longrightarrow (X, d)$: $\exists\ \lambda$ st (λ-quasi)

$$\frac{1}{\lambda}|s-t| - \lambda \leq d(\alpha(s), \alpha(t)) \leq \lambda|s-t| + \lambda$$

Is a q.i embedding

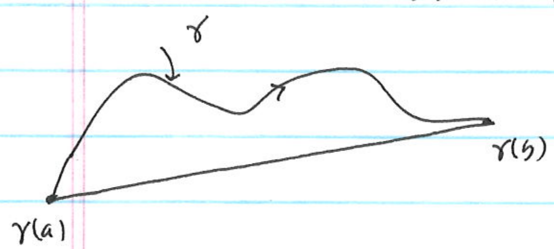
Goal

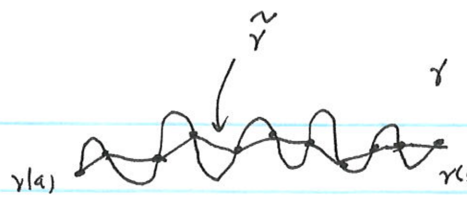
Thm (X, d) δ-hyperbolic space. Then $\exists\ R = R(\delta, K)$ s.t $\forall$ K-quasi-geod

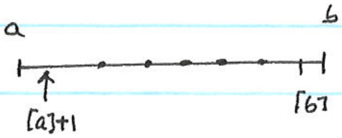
$\gamma : I \to X$ $(\gamma : [a, b] \to X)$ satisfies

$$\gamma([a, b]) \subset N_R([\gamma(a), \gamma(b)]).$$

and $[\gamma(a), \gamma(b)] \subset N_R(\gamma[a, b])$







Proof (γ may not be continuous)

step 1

Lemma 1 Define $\tilde{\gamma}: I=[a,b] \to X$ to be: $\tilde{\gamma}(x)=\gamma(x)$ $x \in \{a,b\} \cup (\mathbb{Z} \cap [a,b])$

and $\tilde{\gamma}|_{[n,n+1]}$, $\tilde{\gamma}|_{[a,[a]+1]}$, $\tilde{\gamma}|_{[[b],b]}$ be the shortest geodesic between its end pts. $\tilde{\gamma}$ is rectifiable s.t

(1) $\gamma(I) \subset N_{2k}(\tilde{\gamma}(I))$ & $\tilde{\gamma}(I) \subset N_{2k}(\gamma(I))$.

(2) $\forall\ s,t \in [a,b]$, $\ell(\tilde{\gamma}([s,t])) \leq 2k|s-t| + 4k$

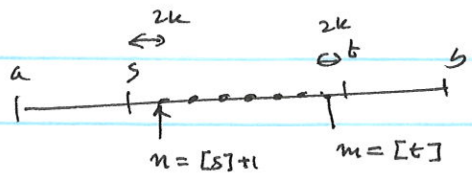
Pf

(1) $\tilde{\gamma}(I) \subset N_{2k}(\gamma(Z))$ $Z=\{a,b\} \cup (\mathbb{Z} \cap [a,b])$

due to $d(\gamma(n),\gamma(n+1)) \leq k|n+1-n| + k \leq 2k$.

Also $\gamma(I) \subset N_{2k}(\gamma(Z))$.

(2) Say



Then $\ell(c|_{[s,t]}) \leq 2k(m-n+1) + 2k \leq 2k|s-t| + 4k$.

Conclusion We may assume $\gamma: I \to X$ rectifiable (tame) k-quasi-geodesic

s.t $\ell(\gamma|_{[s,t]}) \leq \lambda d(\gamma(s),\gamma(t)) + \lambda$ $\lambda = \lambda(k)$

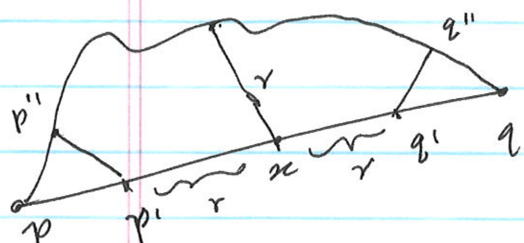
step 2 $\exists\ R_1 = R_1(k)$ s.t $[p,q] \subset N_{R_1}(\gamma(I))$ $\gamma(a)=p$, $\gamma(b)=q$

Find $x \in [p,q]$ s.t $r = d(x, \gamma(I))$ is the largest among all $t \in [p,q]$

Find $p' \in [p,x]$ st $d(p',x)=r$ ($p'=p$ if $d(x,p)<r$)

Find $q' \in [x,q]$ st $d(q',q)=r$

Find $p'', q'' \in \gamma(I)$ $d(p',p'') = d(q',q'') \leq r$



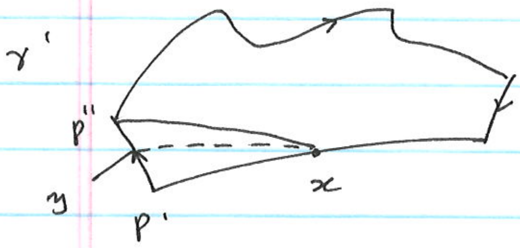
quasi-geod

tame geod

Then $\quad l(\gamma|_{p''\to q''}) \leq \lambda\, d(p'', q'') + \lambda \leq \lambda\big(d(p'', p') + d(p', x) + d(x, q') + d(q', q'') + 1\big)$

$\leq 4\lambda r + \lambda$

Let γ' = path from p' to q' : $[p', p''] * \gamma|_{(p''\to q'')} * [q'', q']$

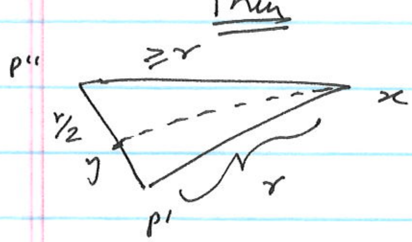


Claim $\quad d(x, [p', p'']) \geq r/2$

Indeed, $\quad d(x, p'') \geq r$.

Say $\quad d(x, [p', p'']) = d(x, y) \quad$ st $\quad d(y, p'') \leq \frac{d(p', p'')}{2} \leq \frac{r}{2}$

Then $\quad d(x, y) \geq d(x, p'') - d(p'', y) \geq r/2$



□

Conclusion $\quad d(x, \gamma') \geq r/2$

By the key lemma $\Rightarrow$

$$\frac{r}{2} \leq 1 + \delta\, |\ln_2(l(\gamma'))| \leq 1 + \delta\big(\ln_2(4\lambda r + \lambda)\big)$$

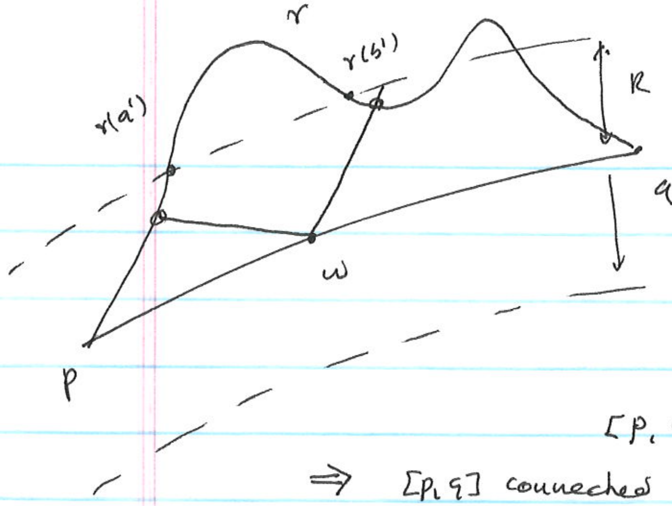
$$\Rightarrow \quad r \leq R(K, \delta) \qquad \big(x \leq a + \ln(bx + c) \Rightarrow x \text{ bounded} \big).$$

Step 3 $\quad$ We claim that $\quad \cancel{[p, q] \subset N_R(\gamma(I))}$

$$\gamma(I) \subset N_{2R\lambda + R + \lambda}([p, q])$$

Indeed consider a max interval $[a', b'] \subset [a, b]$ st

$$\gamma([a', b']) \cap \mathring{N}_R([p, q]) = \phi$$



$\Rightarrow \quad d(\gamma[a',b'], [p,q]) \geq R$

$\Rightarrow$ ~~But~~ $\quad [p,q] \subset N_R(\gamma(I))$

$\Rightarrow$

$$[p,q] \subset N_R(\gamma[a,a']) \cup N_R(\gamma[b',b])$$

$\Rightarrow$ $[p,q]$ connected

$\exists \quad w \in [p,q] \quad w \in N_R(\gamma[a,a']) \cap N_R(\gamma[b',b])$

say $\quad d(w, \gamma(a'')) \leq R \qquad a'' \in [a,a']$

$\quad d(w, \gamma(b'')) \leq R \qquad b'' \in [b',b]$

$$\Rightarrow \quad \text{length}(\gamma|_{[a',b']}) \leq \text{length}(\gamma[a'',b'']) \leq \lambda\, d(\gamma(a''), \gamma(b'')) + \lambda$$

$$\leq \lambda\left(d(\gamma(a''), w) + d(w, \gamma(b''))\right) + \lambda$$

$$\leq \underline{2R\lambda + \lambda}$$

$$\Rightarrow \quad \gamma[a',b'] \subset N_{2R\lambda+\lambda+R}([p,q])$$

$\square$

Continuous Extension

Thm If $F: \mathbb{H}^n \to \mathbb{H}^n$ is a K-Quasi-isometry, ^lift of homeo^ then F extends continuously to $\bar{F}: \bar{\mathbb{H}}^n \to \bar{\mathbb{H}}^n$

pf

Lemma 1 (HW) ~~If $f: (X, d) \to (Y, d')$ a map s.t. for a dense subset $A \subset X$~~ If A is a dense subset of (X, d) and $f: A \to (Y, d')$ is continuous s.t. $\forall$ sequence $\{a_n\} \subset A$ with $\lim_n a_n = x$, $x \in X$,

$\lim_{n\to\infty} f(a_n)$ exists, then f extends to a continuous map $\bar{F}: X \to Y$.

Pf $\forall\, x \in X$, A dense $\Rightarrow \exists \{a_n\} \subset A$ $\lim_n a_n = x$

Define $f(x) = \lim f(a_n)$

Claim 1. It is well defined, i.e., $f(x)$ independent of the choice of a_n.

if $\{b_n\} \subset A$ and $b_n \to x \Rightarrow \underbrace{\{a_1, b_1, a_2, b_2, \dots\}}_{=\{c_n\}} \subset A$ converge to $x \Rightarrow \lim f(c_n)$ exists $= \lim f(a_n) = \lim f(b_n)$.

Claim 2 f so defined is continuous

If Not. $\exists\, x_n \to x$ in X s.t. $f(x_{2n}) \to p$, $f(x_{2n+1}) \to q$ s.t. $p \neq q$ in Y.

~~Now choose $a_n \in A$ s.t. $d(x_n, a_n) < \frac{1}{2^n}$ s.t.~~ $f(x_n) \leftrightarrow p$

~~$d(f(x_{2n}), d(f(a_{2n})) \leq p$ ... $d(f(a_{2n})$~~

~~May assume~~

Chose $a_n \subset A$ s.t. ① $d(x_n, a_n) < \frac{1}{n}$

② $2d(f(x_{2n}), p) \geq d(f(a_{2n}), p)$

③ $2d(f(x_{2n+1}), q) \geq d(f(a_{2n+1}), q)$

$\Rightarrow a_n \to x$ But $f(a_{2n}) \to p$

$f(a_{2n+1}) \to q$ $\square$

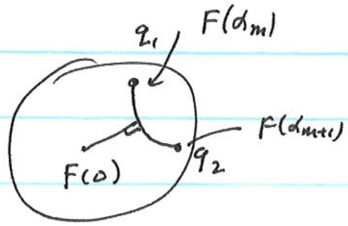
Therefore, it suffices to show that if $d_m \subset \mathbb{H}^n$, $d_m \to p \in \partial\mathbb{H}^n$ then $F(d_m)$ converges.

Suppose otherwise $\exists$ a sequence $\alpha_m \to p^0$ s.t $F(\alpha_{2m}) \to q_1$, $F(\alpha_{2m+1}) \to q_2 \neq q_1$ in $\partial \mathbb{H}^n$.

Known 1. F K-quasi-iso $\Rightarrow$ $\forall$ geodesic $[x,y]$ $F[x,y]$ a K-quas-geod

$$\Rightarrow \quad [F(x), F(y)] \subseteq N_R\big(F([x,y])\big).$$

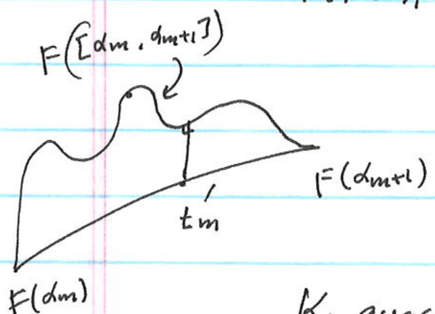
But also, by the construction, $d(0, [\alpha_m, \alpha_{m+1}]) \to +\infty$



$d(F(0), [F(\alpha_m), F(\alpha_{m+1})])$ is bounded

Now find $t_m' \in [F(\alpha_m), F(\alpha_{m+1})]$ st

$$d(F(0), [F(\alpha_m), F(\alpha_{m+1})]) = d(F(0), t_m')$$



But $\exists$ $t_m \in [\alpha_m, \alpha_{m+1}]$ st

$$d(t_m', F(t_m)) \le R$$

$$\Rightarrow \quad d(F(0), F(t_m)) \le \text{bounded} \ \forall m.$$

K-quasi-iso shows

$$d(F(0), F(t_m)) \ge \frac{1}{K} d(0, t_m) - K$$

$$\ge \frac{1}{K} d(0, [\alpha_m, \alpha_{m+1}]) - K \to +\infty$$

a contradiction.

$\square$

Corollary (1) The extension $\bar{F}|_{\partial \mathbb{H}^n} : \partial \mathbb{H}^n \to \partial \mathbb{H}^n$ is a homeomorphism

Pf Let $G = F^{-1}$ be the inverse G extends continuously. Now

$$\underline{F \circ G = id} \quad \Rightarrow \quad \underline{\bar{F} \circ \bar{G} = id} \ + \ \bar{G} \circ \bar{F} = id$$

extends continuously.

Corollary The extension $\bar{F}$ still satisfies $\bar{F}(\gamma x) = f_x(\gamma) \cdot \bar{F}(x)$ $\forall x \in \bar{\mathbb{H}}^N$

Lecture 25

Gromov Norm

§1 Gromov Pseudo-Norm

X topological space

$S_n(X) \triangleq S_n(X, \mathbb{R})$ real coefficient singular chain

$= \{ \sum_{i=1}^{k} x_i \sigma_i \mid x_i \in \mathbb{R},\ \sigma_i : \sigma^n \to X \text{ continuous} \}$

For $\underset{\parallel}{\sum_{i=1}^{k} x_i\sigma_i} = c$, $|c| = \sum |x_i|$.

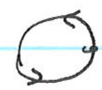
$H_n(X; \mathbb{R}) = \{ [c] \mid \partial c = 0,\ c \sim c' \text{ if } c - c' = \partial d.\ d \in S_{n+1}(X) \}$

Def (Gromov) The pseudo norm $\| [c] \| = \inf \{ |c'| \mid c' \sim c \}$

$\| \lambda \alpha \| = |\lambda| \, \|\alpha\| \quad \lambda \in \mathbb{R}$ clear

Eg 1 $X = S^1$ $\alpha = [a] \in H_1(S^1, \mathbb{Z})$ generator $\subset H_1(S^1, \mathbb{R})$

$\underset{1}{a} : [0,1] \longrightarrow S^1 \qquad a(t) = e^{2\pi i t}$



Claim $\| \alpha \| = 0$

Indeed $a_n : [0,1] \longrightarrow S^1 \qquad a_n(t) = e^{2\pi i n t}$

$a_n / n \sim a_1 \Rightarrow \| [\alpha] \| \le \frac{1}{n} \longrightarrow 0$

Prop: $\alpha, \beta \in H_n(X, \mathbb{R})$ then

(1) $\| \alpha + \beta \| \le \|\alpha\| + \|\beta\|$

(2) $\| k \alpha \| = |k| \, \|\alpha\|$

(3) $f : X \to Y$ continuous $\| f_*(\alpha) \| \le \|\alpha\|$.

Proof Trivial. □

Note $\|\alpha\| = 0 \not\Rightarrow \alpha = 0$. As we have seen above.

$\Rightarrow \| \cdot \|$ pseudo norm on $H_n(X; \mathbb{R})$

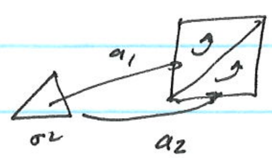
§2.

M^n closed orientable manifold (connected) $\Rightarrow H_n(M, \mathbb{Z}) \cong \mathbb{Z}$.

its generator α_M is the fundamental class of M

Eg $\alpha_{S^1} = [a_1]$

Eg $\alpha_{S^1 \times S^1} = [a_1 + a_2]$.



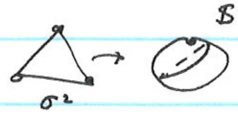
. Basically $a \in \alpha_M$. each point of is covered algebraically exactly once by chains in a.

Eg 1 α_{S^n}: $a : \sigma^n \to S^n$ s.t. $a(\partial\sigma^n) = N$ the north pole

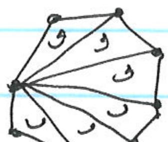
$a : \sigma^n - \partial\sigma^n \to S^n - N$ homeomorphism

Then $\alpha_{S^n} = [a + N]$ if n is even

$= [a]$ if n is odd



Eg Σ_g orientable surface of genus $= g$



4g-2 trigls

$\sum_{i=1}^{4g-2} a_i \in \alpha_{\Sigma_g} \Rightarrow \|\alpha_{\Sigma_g}\| \le 4g-2$

Def. The Gromov norm $\|M\|$ of M is $= \|\alpha_M\|$.

Eg $\|S^1\| = 0$

Recall if M, N orientable conn. n-mfds $f : M \to N \Rightarrow f_* : H_n(M) \to H_n(N)$

$f_*(\alpha_M) = d \cdot \alpha_N$ $\quad d \in \mathbb{Z}$ $\quad d = \deg(f)$, the degree

Prop $|\deg(f)|\,\|N\| \le \|M\|$. It becomes equality if f covering map.

Proof (1) $|d|\,\|N\| = \|f_*(\sigma_N)\| \le \|\sigma_M\| \le \|M\|$ $\quad \in \alpha_M$

(2) $\forall\, c = \sum x_i \sigma_i \in \alpha_N$, we can lift σ_i to d of them $\tau_{i1}, \dots, \tau_{id} \Rightarrow \hat{c} = \sum x_{ij}\tau_{ij}$

$\forall \varepsilon > 0\ \exists c$
$\|N\| \ge \sum|x_i| - \varepsilon$
$= \frac{1}{d}\left(\sum d|x_i|\right) - \varepsilon$
$\ge \frac{1}{d}\|M\| - \varepsilon$

Corollary If M^n admits a self map of $|\deg(f)| \ge 2 \Rightarrow \|M\| = 0$

Eg $\|S^n\| = 0 = \|S^1 \times \cdots \times S^1\|$.

Question. What is $\|\Sigma_g\|$ for $g \ge 2$?

Prop ($g \ge 2$) $\quad \|\Sigma_g\| = (-2)\chi(\Sigma_g)$.

Proof $\|\Sigma_g\| \ge -2\chi(\Sigma_g)$. Use area. Let $c = \sum x_i \sigma_i \in \alpha_{\Sigma_g}$ (straight ones) & a hyperbolic metric with area forms ω: GB: $-2\pi\chi(\Sigma_g) = \int_{\Sigma_g} \omega = \sum_i x_i \int_{\sigma_i} \omega$

Gromov Norm

so $\quad -2\pi\chi(\Sigma_g) \le \sum_i |x_i| \left|\int_{\sigma_i} \omega\right| = \sum_i |x_i| \cdot \text{Area}(\sigma_i) \le \pi \sum |x_i|$.

This proof works after we assume c is represented by straight,

<u>Next</u> we have $\quad \|\Sigma_h\| \le 4g-4 = -2\chi(\Sigma_h)+2$

For any $n \ge 1$, $\exists$ an n-fold cover $F: \Sigma_h \to \Sigma_g$ with $\chi(\Sigma_h) = n\chi(\Sigma_g)$

$\Rightarrow$

$$\deg(F)\cdot\|\Sigma_g\| \le \|\Sigma_h\| \Rightarrow \|\Sigma_g\| \le \frac{1}{n}\cdot\left[-2n\chi(\Sigma_g)+2\right] \xrightarrow{n\to\infty} (-2)\chi(\Sigma_g).$$

$\square$

The same proof shows, for an n-dim hyperbolic M^n of (compact)

$$\|M^n\| \ge \frac{\text{vol}(M^n)}{V_n} \qquad V_n = \sup\{\text{vol}(\sigma^n) \mid \sigma^n \text{ geometric tetra in } \mathbb{H}^n\}$$

<u>Two steps</u> ① Each $c = \sum x_i\sigma_i \rightsquigarrow_{\text{replace}} \sum x_i\hat{\sigma}_i$ $\quad \hat{\sigma}_i$ geometric (need a proof)

② $\sup\{\text{vol}(\sigma^n) \mid \sigma^n \text{ geometric}\} < +\infty$ $\qquad$ <u>$n=2$ known</u> $V_2 = \pi$

<u>Thm</u> (Haagerup-Munkholm) $V_n = \text{vol}(\text{ideal regular } n\text{-simplex} \subset \mathbb{H}^n)$ $\qquad$ <u>$\sigma \subset$ ideal</u>

<u>Let us proof it for $n=3$</u>

<u>Prop</u>. The volume of an ideal hyperbolic tetrahedron of dihedral angles $\alpha\ \beta\ \gamma$ $\quad \alpha+\beta+\gamma=\pi$ is $\Lambda(\alpha)+\Lambda(\beta)+\Lambda(\gamma)$

where $\Lambda(t) = -\int_0^t \ln|2\sin(s)|\,ds.$ $\qquad$ (Lobachevski) $\quad \Lambda$ cont

$\Lambda(\pi+t) = \Lambda(t)$

$\Lambda(-t) = -\Lambda(t)$

<u>Defer</u>

<u>Proof</u> $\quad F(x,y) = \Lambda(x)+\Lambda(y)-\Lambda(x+y) \qquad x,y\ge 0 \quad x+y\le\pi$

$$\partial F/\partial x = -\ln|2\sin(x)| + \ln|2\sin(x+y)| = \ln\left[|\sin(x+y)|/|\sin(x)|\right]$$

<u>Note</u> $F \ge 0$ on Ω $\quad F=0$ on $\partial\Omega$

Thus the max pt $\Leftarrow$ $\quad \partial F/\partial x = \partial F/\partial y = 0 \Rightarrow |\sin(x)| = |\sin(y)| = |\sin(x+y)|$

$x = y = \pi/3$!

$\square$

<u>This section could expand using Milnor</u>

Gromov Norm

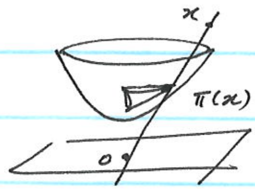
Straight triangles + simplexes

$\sigma^i = \{(x_0, \dots, x_i) \in \mathbb{R}^{i+1} \mid x_j \geq 0,\ \sum_0^i x_j = 1\}$ standard i-simplex

A singular simplex is $f: \sigma^i \to X$. continuous

Def. A straight i-simplex in $\mathbb{H}^n$: $\quad \mathbb{H}^n$ = hyperboloid model.

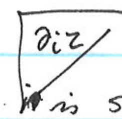
τ: has vertices $v_0, \dots, v_i \in \mathbb{H}^n$. $\mathrm{Affine}(v_0, \dots, v_i) = \{\sum x_j v_j \mid x_j \geq 0, \sum x_j = 1\}$ $\subset \mathbb{R}^{n+1}$. Let $\pi: \mathrm{Affine}(v_0, \dots v_i) \to \mathbb{H}^n$ sending $x \longmapsto \dfrac{x}{\sqrt{-\langle x, x\rangle}}$ Minkowski



$\tau = \pi \circ \mathrm{Standard} = \pi \circ St$

St Standard : $(x_0, \dots, x_i) \longmapsto \sum x_j v_j$

Es Good to know the spherical case



Key Fact: If $\partial_i \tau$ is the i-th face of a straight τ it is straight. Also if $g \in \mathrm{Iso}(\mathbb{H}^n)$, $\Rightarrow g \circ \tau$ is straight.

Def A singular i-simplex $f: \sigma^i \to$ Hyperbolic mfd is straight if its lift is straight.

Lemma (Straightening) Each singular n-simplex $f: \sigma^i \to \Gamma \backslash \mathbb{H}^n$ has a straight representative $\hat{f}: \sigma^i \to \Gamma \backslash \mathbb{H}^n$ s.t. if $c = \sum x_i f_i$ cycle $\hat{c} = \sum x_i \hat{f}_i$ $\quad c - \hat{c} = \partial d$.

Pf Lift f to the universal cover, replace it by its straight ones with the same vertex set + project down $f \rightsquigarrow \tilde{f} \rightsquigarrow \tilde{f}^* \rightarrow \pi \circ \tilde{f}^* = \hat{f}$.

straight

Lemma The straighten map $\Phi: S_i(X) \longrightarrow S_i(X)$ $\quad f \mapsto \hat{f}$ is a chain homotopy. i.e. $\forall$ chain c $\quad \exists$ homomorphism $\Psi: S_i \to S_{i+1}$

$$c - \Phi(c) = (\partial\Psi - \Psi\partial)(c)$$

Ψ is given by the natural homotopy: $f \simeq \hat{f}$ $\qquad f(x) \quad \hat{f}(x)$

unique geodesic

$\Psi(x, t)$ = is the geodesic path from $f(x)$ to $\hat{f}(x)$ at time t $\quad t \in [0,1]$.

In particular if $c = \sum x_i \sigma_i \in \alpha_M \Rightarrow \hat{c} = \sum x_i \hat{\sigma_i} \in \alpha_M$.

Proposition. If (M^n, d) is a hyperbolic n-manifold, closed. $\Rightarrow \|M\| \geq \text{vol}(M)/v_n$

Proof $\forall \varepsilon > 0, \exists \quad c = \sum x_i \sigma_i \in \alpha_M$ st $\|M\| \geq \sum |x_i| - \varepsilon$.

Now $\hat{c} = \sum x_i \hat{\sigma_i} \in \alpha_M$ w/ $\text{vol}(\hat{\sigma_i}) \leq v_n$, ω = volume form in d

So.
$$\text{vol}(M) = \int_M d\omega l = \sum_i x_i \int_{\hat{\sigma_i}} d\omega l \leq \sum_i |x_i| \, \text{vol}(\hat{\sigma_i}) \leq v_n \sum_i |x_i|$$
$$\leq v_n (\|M\| + \varepsilon)$$
□

Corollary For $g \geq 2$ $\|\Sigma_g\| = (-2)\chi(\Sigma_g) = 4g-4$. (It can never be achieved by $\mathbb{Z}$.)

Thm (Gromov) If M^n is a closed hyperbolic manifold, then
$$\|M^n\| = \text{vol}(M)/v_n$$

Pf. It suffices to show $\|M^n\| \leq \text{vol}(M)/v_n$.

Lecture 26

Straight Simplex and Gromov Norm

$\mathbb{H}^n$ hyperboloid model $= \{ x \in \mathbb{R}^{n+1} \mid \langle x, x\rangle = -1,\ x_{n+1} > 0 \}$

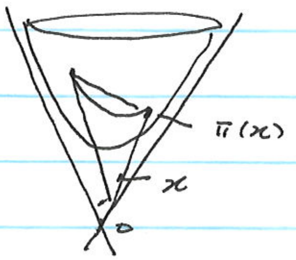
$\uparrow$ Minkowski $\sum_{i=1}^{n} x_i^2 - x_{n+1}^2$

$\pi : \{ x \mid \langle x, x \rangle < 0 \}_{x_{n+1}>0} \to \mathbb{H}^n \qquad x \mapsto x/\sqrt{-\langle x, x\rangle}$ (light cone)

$\sigma_m = \{ (x_0, \cdots, x_m) \in \mathbb{R}^{m+1} \mid x_i \geq 0,\ \sum x_i = 1 \}$ standard simplex

$\sigma_m^{(0)}$ vertex set.

__Def__ A straight simplex $\tau : \sigma_m \to \mathbb{H}^n$ is $\tau = \pi \circ A$ $A : \sigma_m \to \mathbb{R}^{n+1}$ affine w/ $A(\sigma_m^{(0)}) \subset \mathbb{H}^n$, A straight simplex determined by its vertices: edges geodesics $\underline{\mathrm{vol}(\tau_n) \leq C_n}$



If $M = \mathbb{H}^n/\Gamma$ hyperbolic, $\tau : \sigma_m \to M$ straight if a lift of it is straight.

__Lemma 1.__ τ straight $\Rightarrow$

(a) $\partial_i \tau$ straight $\qquad \partial_i \tau = \pi \circ \partial_i A = \pi \circ A|_{\partial_i \sigma_m}$

(b) if $g \in \mathrm{Iso}(\mathbb{H}^n)$ $(\Rightarrow g \in GL(n+1, \mathbb{R})\ \langle gx, gy\rangle = \langle x, y\rangle) \Rightarrow g\tau$ straight

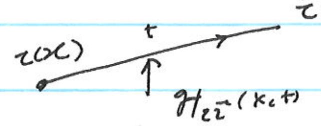
(c) τ determined by its vertices. + $\mathrm{Vol}(\tau) \leq C_m$.

__Pf__ (b): $g \pi A = \pi(gA)$ __or__ $g\pi = \pi g$ $\Rightarrow$ result since $\underline{gA \text{ affine}}$

Indeed $\quad g\pi(x) = g\left(\frac{x}{\sqrt{-\langle x, x\rangle}}\right) = \frac{g(x)}{\sqrt{-\langle x, x\rangle}} = \frac{g(x)}{\sqrt{-\langle gx, gx\rangle}} = \pi(g(x)).$

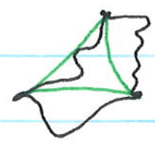
$\square$

__Lemma 2__ (a) Each singular simplex $\tau : \sigma_m \to \mathbb{H}^n$ is naturally homotopic to a straight $\hat\tau : \sigma_m \to \mathbb{H}^n$ $\quad \tau(\sigma_m^{(0)}) = \hat\tau(\sigma_m^{(0)})$ by a homotopy $H_{\tau\hat\tau}(x, t)$ = the geod path from $\tau(x)$ to $\hat\tau(x)$ at time $t \in [0,1]$ with parameter proportional to the arc length



__st__ $\forall g \in \mathrm{Iso}(\mathbb{H}^n)$ $\quad g H_{\tau\hat\tau} = H_{g\tau\, g\hat\tau}$.

(b) Each $\tau : \sigma_m \to M$ is naturally homotopic to $\hat\tau$ a straight (unique) $\hat\tau : \sigma_m \to M$ (carry from the quotient)



(c) Furthermore $H_{\tau\hat{\tau}}\big|_{\partial_i\tau\times I} = H_{\partial_i\tau\,\widehat{\partial_i\tau}}$.

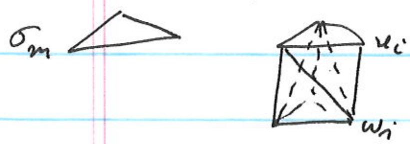
__Proposition__ The straighten map, extended to $\Phi: S_m(M) \to S_m(M)$ linearly is chain homotopic to id: $\exists\ F: S_m(M) \longrightarrow S_{m+1}(M)$ st

$$Id - \Phi = \partial F + \partial F \qquad \underline{c - \hat{c} = \partial F(c) + F\partial(c)}$$

__Proof__ (Sketch)

Recall the standard homotopy construction..

$\sigma_m = [v_0, \cdots v_m]$ $\qquad u_i = v_i \times 0,\quad w_i = v_i \times 1$, Then



$\sigma_m \times I$ is triangulated by $(m+1)$-simplices

$[u_0, \cdots u_i w_i \cdots w_m]$ $\quad$ __s.t__

$$G(\sigma_m \cancel{\times I}) = \sum_{i=0}^{m} (-1)^i [u_0, \cdots, u_i w_i \cdots w_m] \quad c\ (m+1)\text{ chain}$$

__Satisfies__

$$\partial G(\sigma_m) = \sigma_m \times 0 - \sigma_m \times 1 + \sum_{i=0}^{m} (-1)^i G(\partial_i \sigma_m)$$

$$= \sigma_m \times 0 - \sigma_m \times 1 + G(\partial \sigma_m)$$

Now apply the homotopy to it

$\cancel{F(\sigma_m) \partial \in}$ $\qquad F(\tau) = H_{\tau\hat{\tau}\,\#}(G(\sigma_m)) \Rightarrow$ result $\square$

Notation $c = \sum a_i \tau_i$ chain $\Rightarrow \hat{c} = \sum a_i \hat{\tau}_i$ its straighten one

__Corollary__ 1 If $c \in d_M \Rightarrow \hat{c} \in d_M$ s.t $|c| = |\hat{c}|$

Thus $\|M\| = \inf \{ \sum a_i \tau_i \in d_M \mid \tau_i \text{ straight} \}$

__Corollary 2.__ M closed hyperbolic $\Rightarrow$ $\|M\| \geq \mathrm{Vol}(M)/c_n$

__Proof__ $\forall$ straight $c = \sum a_i \hat{\tau}_i \in d_M$

$$\mathrm{Vol}(M) = \int_M d\mathrm{vol} = \sum_i a_i \int_{\hat{\tau}_i} d\mathrm{vol} = \sum_i a_i \mathrm{vol}(\hat{\tau}_i) \leq c_n \sum a_i$$

$$\leq c_n \sum |a_i| \qquad \text{Now take inf to it.} \quad \square$$

<u>Next Goal</u>: $\mathrm{vol}(M) \geq c_n \|M\|$ <u>i.e.</u> $\forall \varepsilon > 0$ $\underline{\mathrm{vol}(M) \geq (c_n - \varepsilon)\|M\|}$

<u>Lemma</u>: It suffices to show: $\forall \varepsilon > 0$, $\exists$ $c = \sum a_i \tau_i \in [M]$, $a_i \geq 0$, straight, st $\mathrm{vol}(\sigma_i) \geq c_n - \varepsilon$

<u>Proof</u> If so $\Rightarrow$

$$\mathrm{vol}(M) = \sum a_i \,\underline{\mathrm{vol}(\tau_i)} \geq \sum a_i (c_n - \varepsilon) \geq (c_n - \varepsilon)\|M\|. \qquad \square$$

<u>RM</u> We don't even need c be <u>straight</u>.

We will focus on M^3 to simplify notations

<u>Lemma</u>. $\forall \varepsilon > 0$. $\exists$ $R_0 > 10$ s.t if $R \geq R_0$ and σ is a straight tetra whose edge lengths $\subset [R-2, R+2]$, then $\mathrm{vol}(\sigma) \geq c_3 - \varepsilon$.

This is due to the continuity of volume in dihedral angles

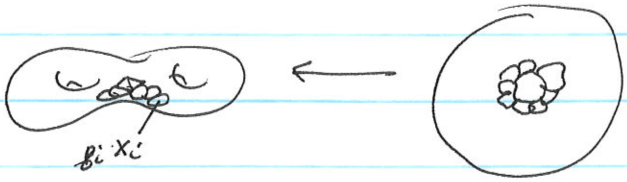
Next, we will produce a chain $c \in [M]$.

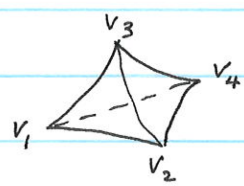
Produce a finite cell decomp $X_1 \sqcup \cdots \sqcup X_k$ of M st $\pi : \mathbb{H}^3 \to M$ cover

(1) X_i simply connected, $\mathrm{diam}(X_i) \leq \min\{1, \frac{1}{10}\mathrm{diam}(M)\}$, X_i measurable

(2) $X_i \cap X_j = \emptyset$ $i \neq j$

(3) Fix $q_i \in X_i$ for each i. let $\pi^{-1}\{X_1, \ldots, X_k\} = \{\Omega_j\text{'s}\}$ associated decomp of $\mathbb{H}^3$ invariant under $\Gamma = \pi_1(M)$ and $\pi^{-1}\{q_1, \ldots, q_k\} = \{p_j\text{'s}\} =: P$





Let us fix $R \geq R_0$ and $\tau_R = [u_1, u_2, u_3, u_4]$ be fixed straight regular edge length $= R$ tetra, <u>oriented</u>

Kuiper's Proof

(Fix $\varepsilon > 0$, $R > 0$)

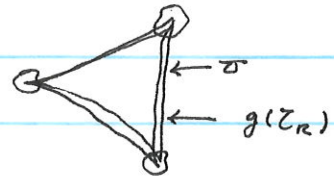
Def A good tetra $\sigma: \sigma^3 \to \mathbb{H}^3$ is a straight st (oriented)

(1) its vertices are $p_1, p_2, p_3, p_4 \subset P$ $\quad p_i \in \Omega_i$

(2) $\exists$ $g \in \mathrm{Iso}^+(\mathbb{H}^3)$ $\quad g(u_i) \subset \Omega_i$

Note $\underline{\mathrm{vol}(\sigma) \geq c_3 - \varepsilon}$

We write $\boxed{g(\tau_R) \sim \sigma}$



Given a good tetra σ $\qquad m$ — Haar measure on $\mathrm{Iso}(\mathbb{H}^3)$

define $a(\sigma) = m\{ g \in \mathrm{Iso}^+(\mathbb{H}) \mid \underset{g(u_i) \subset \Omega_i}{} \}$ it is finite. (g lives in a cpt set

Key Claim. the infinite chain $\beta = \sum_{\sigma \text{ good}} a(\sigma)\sigma \in S_3(\mathbb{H}^3)$

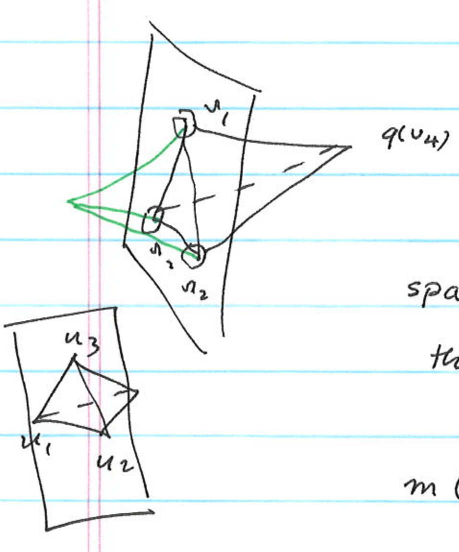
is closed, i.e., $\partial\beta = 0$. This is a countable sum.

Proof. First $\partial\beta = \sum_{\delta \text{ good tris}} \mathrm{coeff}(\delta)\, \delta$ $\qquad p_i \in \Omega_i$

δ: straight tris in $\mathbb{H}$ with vertices $\{p_1, p_2, p_3\} \subset P$ (oriented) s.t. $\exists\ g \in \mathrm{Iso}^+(\mathbb{H}^3)$ st $g(u_i) \in \Omega_i$ $\quad i = 1, 2, 3$

By definition:

$$\begin{aligned} \mathrm{coeff}(\delta) &= m\{ g \in \mathrm{Iso}^+(\mathbb{H}^3) \mid g(u_i) \in \Omega_i,\ i = 1.2.3 \} \\ &\quad - m\{ g \in \mathrm{Iso}^-(\mathbb{H}^3) \mid g(u_i) \in \Omega_i,\ i = 1.2.3 \} \\ &= m(\mathrm{Rig}) - m(\mathrm{lef}) \end{aligned}$$



But if $\phi \in \mathrm{Iso}^-(\mathbb{H}^3)$ reflection about the plane spanned by $u_1 u_2 u_3$, $\phi(u_i) = u_i \Rightarrow$

the map $\Phi^*: \mathrm{Rig} \to \mathrm{lef}$ $\quad g \mapsto g \circ \Phi$

$g \mapsto g \cdot \phi$ $\quad$ is a bijection 1-1 onto

$m(\mathrm{Rig}) = m(\mathrm{Rig} \cdot \phi) = m(\mathrm{lef})$ the bi-invariance of $\underline{m}$!

Kuiper's proof

But $\forall\, \gamma \in \Gamma$ $\quad \gamma \cdot \beta = \beta$ $\quad$ since σ good iff $\gamma\sigma$ good $a(\gamma\sigma) = a(\sigma)$

(right invariance of m

$$a(\gamma\sigma) = m\{\, g \in \mathrm{Isom}^+ \mid g(u_i) \in \gamma\Omega_i \,\}$$

$$= m\{\, g \in \mathrm{Isom}^+ \mid (\gamma^{-1} g)(u_i) \in \Omega_i \,\} = m\left\{\gamma^{-1}\{\, g \in \mathrm{Isom}^+ \mid g(u_i) \in \Omega_i \,\}\right)\Big).$$

Furthermore, $\forall$ R, and $p, q \in M$, $\exists$ only finitely many geodesics from p to q of length $\le R+2$

$\Rightarrow$

$\exists$ a cycle $\quad c_R = \sum a_i \sigma_i \quad \in S_3(M) \qquad a_i > 0, \quad \mathrm{vol}(\sigma_i) \ge c_3 - \varepsilon$

and $\partial c_R = 0$!

$$\Rightarrow \quad c = \frac{c_R}{\sum a_i \,\mathrm{vol}(\sigma_i)} \quad \in \underline{\alpha_M} \qquad \text{with all conditions satisfied.}$$

$$= \sum a_i' \sigma_i \quad \in S_3(M)$$

Lecture 27

Gromov's Proof of Mostow Rigidity

M^3, N^3 closed hyperbolic oriented $f: M \to N$ homeo $\Rightarrow$ $f \simeq g: M \to N$ s.t g is an isometry.

__Step 1.__ $\deg(f) = \deg(f^{-1}) = 1 \Rightarrow \|M\| \geq |\deg(f)| \|N\| \geq \|N\|$ & Conversely

$\|M\| \leq \|N\| \Rightarrow \|M\| = \|N\|$ i.e. (Gromov-Thurston)

$$\mathrm{vol}(M) = \mathrm{vol}(N) \qquad (1)$$

__Step 2__ The lift $\tilde{f}: \mathbb{H}^3 \to \mathbb{H}^3$ of f to the universal cover extends to a continuous $F: \overline{\mathbb{H}^3} \to \overline{\mathbb{H}^3}$ s.t. $F|_{\partial \mathbb{H}^3} = h : S^2 \to S^2$ is a homeomorphism

Furthermore $F(\gamma x) = \rho(\gamma) F(x) \quad \forall\, x \in \overline{\mathbb{H}^3},\ \gamma \in \pi_1(M)$

& $\rho: \pi_1(M) \to \pi_1(N)$ is the isomorphism f_*.

Our goal: $\exists\ \tilde{g} \in \mathrm{Iso}^+(\mathbb{H}^3)$ s.t. $\tilde{g}|_{S^2} = h \Rightarrow$

$$\tilde{g}(\gamma x) = \rho(\gamma)\tilde{g}(x) \quad \forall\, \gamma \in \pi_1(M),\ \forall x \in S^2 \qquad (2)$$

$\Rightarrow$ (2) holds $\forall\, x \in \mathbb{H}^3$ since g Möbius!

Therefore $\tilde{g}$ induces the isometry $g: M \to N$

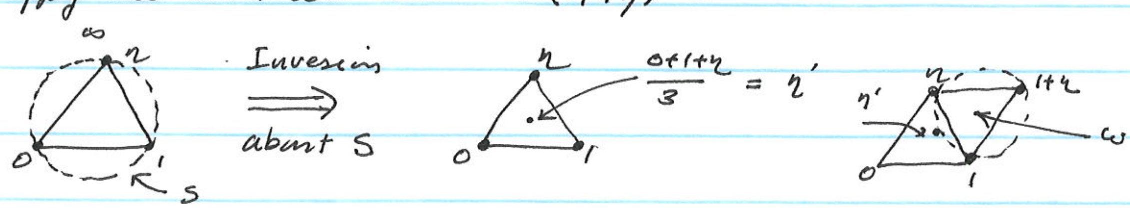
__Homework__ Show that $g \simeq f$ homotopic.

To show $h \in \mathrm{Iso}^+(\mathbb{H}^3)/_{S^2}$, we introduce

__Def__ A regular set $\{v_1, \dots, v_4\} \subset S^2$ if the associated ideal tetra $[v_1, \dots, v_4]$ is regular. If $g \in \mathrm{Iso}(\mathbb{H}^3)$ $\{v_1, \dots, v_4\}$ regular $\Leftrightarrow$ $\{g(v_1), \dots, g(v_4)\}$ regular

Eg 1. $\{0, 1, \eta, \infty\}$ $\eta = e^{\pi i/3}$ is regular. $\Rightarrow$ $\{0, 1, \eta, \frac{0+1+\eta}{3}\}$ regular

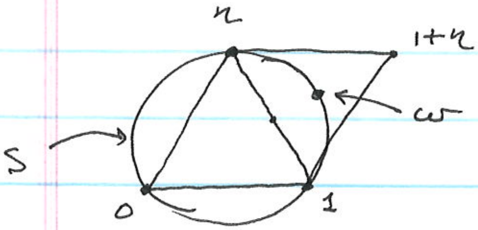
Indeed apply the inversion about $\mathrm{Cir}(0,1,\eta)$ to it



2. Also $\{1, \eta, \eta', \frac{1+\eta}{2}\}$ is regular. It is the image of

Similarly $\{1, \eta, 1+\eta, \omega\}$ regular. Apply I_S to it $\Rightarrow$

Gromov's Proof of Mostow



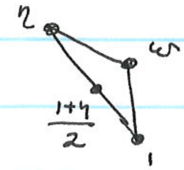
Now same $\Rightarrow$

$\{1, \eta, 1+\eta, \omega\}$ regular $\quad \omega = \frac{1+\eta+(1+\eta)}{3}$

Apply I_S to it, using $\omega \in S$

and $I_S(1+\eta) = \frac{1+\eta}{2}$

$\Rightarrow \{1, \eta, \omega, \frac{1}{2}(1+\eta)\}$ regular



Lemma $h : S^2 \to S^2$ Möbius iff h sends regular sets to regular sets

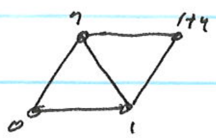
Pf "$\Leftarrow$" ("$\Rightarrow$" trivial)

By composing h by Möbius transformations, we may assume $h(\infty)=\infty$

$h(0)=0 \quad h(1)=1$ and $h(\eta)=\eta$ (Take $\{0,1,\eta,\infty\} \xrightarrow{h} h\{0\,1\,\eta\,\infty\} \xrightarrow{\text{Möb}} \{0\,1\,\eta\,\infty\}$

Now $\{0, 1,$ Goal $\Rightarrow h = id$.

Step 1. $\Rightarrow h|_{\mathbb{Z}+\mathbb{Z}(\eta)} = id$



: $h(1+\eta) = (1+\eta)$:

$\{1, \eta, 1+\eta, \infty\}$ regular $\Rightarrow \{h(1), h(\eta), h(1+\eta), h(\infty)\}$

$= \{1, \eta, h(1+\eta), \infty\}$ regular

So it must be $\{1, \eta, \infty, 0\}$ or $\{1, \eta, \infty, 1+\eta\}$.

(Given $\{v_1\, v_2\, v_3\}$, $\exists$ exactly two pts v_4, v_4' which are related by the $Inv_{Cir(v_1 v_2 v_3)}$, st $\{v_1\, v_2\, v_3\, v_4$ or $v_4'\}$ is regular).

Repeat this $\Rightarrow h(n \pm m\eta) = n + m\eta$. $\forall\, n, m \in \mathbb{Z}$.

Step 2 $\quad h|_{\mathbb{Z}(\frac{1}{2})+\mathbb{Z}(\frac{\eta}{2})} = id \quad$ Induction $\Rightarrow h|_{\mathbb{Z}(\frac{1}{2^n})+\mathbb{Z}(\frac{\eta}{2^n})} = id \Rightarrow h = id$.

Now $\{1, \eta, 1+\eta, \omega\}$ regular $\quad \omega = \frac{1}{3}(1+\eta+(1+\eta))$

$\Rightarrow \{h(1), h(\eta), h(1+\eta), h(\omega)\} = \{1, \eta, 1+\eta, h(\omega)\}$ regular

$\{1, \eta, 1+\eta, \omega\} + \{1, \eta, 1+\eta, \infty\}$ regular

$h(\omega) = \omega$ or $\infty \Rightarrow h(\omega) = \omega$

Finally $\{1, \eta, \omega, \frac{1+\eta}{2}\}$ regular $\xRightarrow{h} h(\frac{1+\eta}{2}) = \frac{1+\eta}{2} \Rightarrow$ done.

Gromov's Proof of Mostow Rigidity

<u>Recall Kuipers proof:</u> $\|M\| = vol(M)/c_3$. $c_3 = 3\Lambda(\pi/3)$

Produce decomposition $\{X_i\}$, $\{q_i\}$ of $M = \mathbb{H}^3/\Gamma$ $\quad \pi: \mathbb{H}^3 \to M$ univ. cover

Given $R > 0$, let

$$\tilde{\beta} = \sum_{\sigma\ R\text{-good}} a_R(\sigma)\,\sigma \qquad \text{cycle in } \mathbb{H}^3 \qquad \sigma^{(0)} \subset \pi^{-1}(\{q_i\})$$

$\sigma = [p_1, p_2, p_3, p_4]$ $\quad p_i \in \Omega_i$

$$a_R(\sigma) = m\{ g \in Iso^+(\mathbb{H}^3) \mid g(u_i^R) \subset \Omega_i \}$$

$[u_1^R, \dots u_4^R]$ regular edge length R- tetra $\subset \mathbb{H}^3$

Two $\sigma \sim \sigma'$ if $\exists\ \gamma \in \Gamma$ s.t. $\gamma\sigma = \sigma'$

<u>Then</u> $\quad \beta = \sum_{[\sigma]} a_R(\sigma)\,\pi(\sigma)$ is a cycle in M $\quad \partial\beta = 0 \quad$ <u>$a_R(\sigma) \geq 0$</u>

The fundamental cycle $\quad c_R = \dfrac{vol(M)}{\sum a_R(\sigma)\, vol(\pi(\sigma))} \cdot \beta = \sum_{[\sigma]} a_R'(\sigma)\,\pi(\sigma)$

<u>Lemma.</u> $\forall\ R \gg 1$, $\quad \sum_{[\sigma]} a_R(\sigma) \leq vol(Iso^+(\mathbb{H}^3)/\Gamma)$

<u>Proof</u> For σ, let

$$A(\sigma) = \{ g \in Iso^+ \mid g(u_i^R) \subset \Omega_i \} \qquad a_R(\sigma) = m(A(\sigma))$$

Now, if $[\sigma] \neq [\sigma'] \Rightarrow A(\sigma) \cap \gamma A(\sigma') = \phi \quad \forall\ \gamma \in \Gamma$

$\underset{\nexists\ \gamma \quad \gamma(\Omega_i) = \Omega_i'}{\Uparrow} \qquad \overset{\parallel}{\{ h = \gamma g \mid h(u_i^R) \subset \gamma\Omega_i \}}$ But $\gamma\Omega_i \cap \Omega_i' = \phi$

<u>Thus</u>, the projection $\pi A(\sigma) \subset Iso^+(\mathbb{H}^3)/\Gamma$ are pairwise disjoint

$$\Rightarrow \quad \bigsqcup_{[\sigma]} \pi(A(\sigma)) \subset Iso^+(\mathbb{H}^3)/\Gamma \quad \Rightarrow \underline{\text{result}}.$$

On the other hand for $R \gg 1$ $\qquad c_3/2 \leq vol(\pi(\sigma)) \leq c_3 \quad \Rightarrow$

<u>Corollary.</u> For $R \gg 1$, if $c_R = \sum_{[\sigma]} a_R'(\sigma)\,\pi(\sigma)$, then $\quad a_R'(\sigma) \geq a_R(\sigma) \Big/ \dfrac{vol(M)}{c_3\, vol(Iso^+/\Gamma)}$

Gromov's Proof of Mostow

The goal: $F|_{\partial \mathbb{H}^3}$ sends regular sets to regular sets

If not, we will show that $vol(M) > vol(N)$ contradicting that $vol(M) = vol(N)$.

suppose $\sigma_{st} = [v_1, \cdots, v_4]$ is a regular ideal tetra in $\mathbb{H}^3$ st $[\underset{\overset{\shortparallel}{w_1}}{F(v_1)}, \cdots, \underset{\overset{\shortparallel}{w_4}}{F(v_4)}]$ is not regular, so $vol([w_1, \cdots, w_4]) < c_3 - \delta$. $\delta > 0$.

Fact. ① $\exists$ open half spaces $\mathcal{U}_1, \cdots, \mathcal{U}_4$ as nbhd of $v_1, \cdots v_4$ in $\overline{\mathbb{H}^3}$ st $\forall\ p_i \in \mathcal{U}_i$.

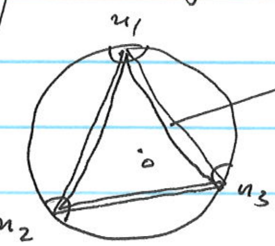
$$vol\,[F(p_1), \cdots, F(p_4)] < c_3 - \delta \qquad \text{(continuity of volume)}$$

② Let $\tau_R = [u_1^R, \cdots, u_4^R]$ regular edge length R tetra in $\mathbb{H}^3$ s.t. $\{u_i^R \mid R \geq 0\}$ is a geodesic ray from o. ($\tau_R \subset \tau_{R'}$ for $R < R'$).

$\boxed{n=2}$

For $R >> 1$ and $R' > R$

$$\{g \in Iso^+ \mid g(u_i^R) \subset \mathcal{U}_i \ \ i = 1,2,3,4\} \subset \{g \in Iso^+ \mid g(u_i^{R'}) \subset \mathcal{U}_i\}$$



$g(\tau_R) \Rightarrow g(\tau_{R'})$ has vertices in $\underline{u_1, \cdots, u_4}$.

Assuming this. Now back to the proof for M, it has f. cycle

$$c_R = \sum_{[\sigma]} a'_R(\sigma)\,\pi(\sigma) \qquad a'_R(\sigma) = a_R(\sigma) \Big/ \left(\frac{vol(M)}{\sum_{[\sigma]} a_R(\sigma)\, vol(\sigma)} \right)$$

By the construction $vol(\pi(\sigma)) \longrightarrow c_3$ as $R \to \infty$ + $\sum_{[\sigma]} a'_R(\sigma) \longrightarrow vol(M)/c_3$

(Kuiper's proof)

Now apply F to it: $\widehat{F}_\#(c_R) = \sum_{[\sigma]} a'_R(\sigma)\pi\underbrace{(\widehat{F(\sigma)})}_{= \tau_\sigma} = \sum_{[\sigma]} a'_R(\sigma)\, \tau_\sigma \in \underline{d_N}$ straight

So $vol(N) = \sum_{[\sigma]} a'_R(\sigma)\, vo(\tau_\sigma) \leq \sum_{[\sigma] \notin I} a'_R(\sigma) \cdot c_3 + \sum_{[\sigma] \in I} a'_R(\sigma)\, vol(\tau_\sigma)$

$[\sigma] \in I \Leftrightarrow$ ~~$\exists g \in Iso^+$, $\exists \gamma \in \Gamma$~~ the $\sigma = [p_1, \cdots, p_4]$ $\exists\, \gamma \in \Gamma$ $\gamma(p_i) \in \mathcal{U}_i$ $\underline{i = 1,2,3,4}$

$\Rightarrow vol(\tau_\sigma) \leq c_3 - \delta$. $\qquad \leq \left(\sum_{[\sigma]} a'_R(\sigma)\right) \cdot c_3 - \delta \sum_{[\sigma] \in I} a'_R(\sigma)$

Gromov's Proof

$$\leq \sum_{[\sigma]} a_R'(\sigma)\, c_3 - \left[\frac{\mathrm{vol}(M)}{\cancel{\sum_{[\sigma]} a(\sigma)}\; c_3\, \mathrm{vol}\left(|so^{+}(\mathbb{H}^3)/\Gamma\right)} \sum_{[\sigma] \in I} a_R(\sigma) \right]$$

$$\leq \sum_{[\sigma]} a_R'(\sigma)\, c_3 - \lambda \cdot \sum_{[\sigma] \in I} a_{R_0}(\sigma)$$

$\boxed{a_R(\sigma) \geq a_R(\sigma_0) \text{ by Fact \# 2!}}$

<u>Let $R \to \infty$</u> $\Rightarrow$ $\mathrm{vol}(M) \leq \mathrm{vol}(M) - \text{const } c$ $\quad$ <u>$c > 0$</u>.