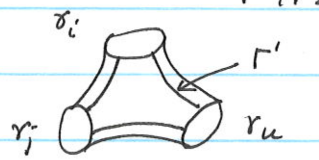
Lecture 15 The Fenchel-Nielsen Coordinate

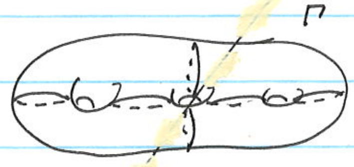
S a closed surface of genus $g \geq 2$

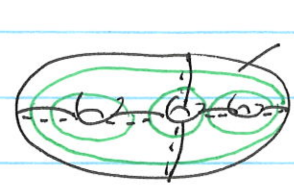
$\mathrm{Teich}(S) = \{ [(X, d, \varphi)] \mid d$ hyperbolic metric $\varphi: S \to X$, "homotopy class" of orientation preserving homeo $\}$

Now. fix a 3-holed sphere decomp $\Gamma = \{\gamma_1, \dots, \gamma_{3g-3}\}$ of S, let Γ' be a set of disjoint s.c.c called "seams" s.t for each 3-holed sphere $\Sigma_{0,3} \subset S - \Gamma$

$\Gamma' \cap \Sigma_{0,3}$ consists of 3 arcs joining different comp.







with (S, Γ, Γ') we can associate the FN coordinate,

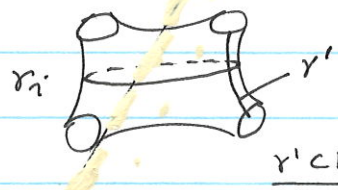
$$FN: \mathrm{Teich}(S) \longrightarrow (\mathbb{R}_{>0} \times \mathbb{R})^{3g-3} : [(X, d, \varphi)] \mapsto (l_i, t_i)$$

$(l_1, \dots, l_{3g-3}) \in \mathbb{R}_{>0}^{3g-3}$ length coord $\quad (t_1, \dots, t_{3g-3})$ twist-coord

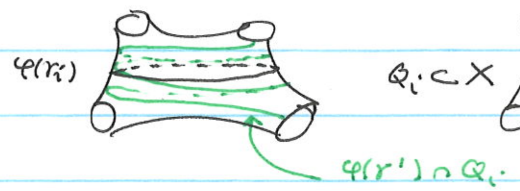
The length. $\forall i$ let $\varphi(\gamma_i)^*$ be the geod $\simeq \varphi(\gamma_i)$ in (X, d) $\quad l_i = \mathrm{length}(\varphi(\gamma_i)^*)$

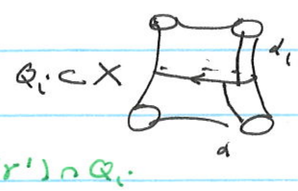
We may assume, for (X, d) that $\varphi(\gamma_i) = \varphi(\gamma_i)^*$ after a homotopy

For each γ_i. let Q_i = union of two hyperbolic 3-holed spheres in X adjacent to $\varphi(\gamma_i)^*$ "$P_i' \cup P_i''$"



$\xrightarrow{\varphi}$



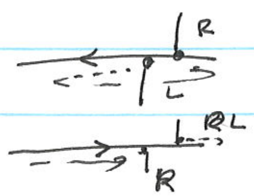


Now $\quad \varphi(\gamma') \cap Q_i \simeq \alpha_1 * \alpha_2 * \alpha_3$ rel (∂Q_i) $\quad \partial Q_i$ - invariant.

where α_1, α_3 are the shortest geod in P_i' + P_i'', $\alpha_2 \subset \varphi(\gamma_i)$

Define t_i = the signed distance of $\alpha_2 \in \mathbb{R}$.
(length)

Signed distance Fix any orientation on $\varphi(\gamma_i)$, use orientation of $X \Rightarrow$ left + right side of $\varphi(\gamma_i)$.



t_i = from left end pt, to right end pt of $\alpha_i \cap \varphi(\gamma_i)$ $\rightarrow$ -

Lecture 15. F-N coordinate

<u>Thm</u> (Fenchel-Nielsen) Fix (S, Γ, Γ') the map

(sketch) $FN: \text{Teich}(S) \longrightarrow \mathbb{R}^{3g-3}_{>0} \times \mathbb{R}^{3g-3}$ is 1-1 onto

<u>Pf</u>: Well defined <u>o.k.</u> (it is independent of the choice of <u>two</u>. γ_j', γ_k').

<u>Onto</u> clearly from the construction.

Given $(l, t) \to$ produce hyperbolic pants of given lengths

Use t to glue them isometrically. $\Rightarrow (X, d)$. Use $t_i \to$ map

$h: S \to X$. sending $h(\gamma_j')$ to a curve homotopic to $d_1 * d_2 * d_3$.

<u>1-1</u>: If (X, d, φ), (X', d', φ') two marked hyperbolic surfaces of the same FN coordinates $\Rightarrow (X, d, \varphi) \cong (X', d', \varphi')$

By the construction $l_i + t_i \Rightarrow \exists$ an isometry $h: X \to X'$ sending geodesics $\varphi(\gamma_i)$ to $\varphi'(\gamma_i)$. SAME twisting $\Rightarrow$ glued nicely

The homotopy $\Rightarrow$

$$\begin{tikzcd} & X \arrow[d, "h"] \\ S \arrow[ur, "\varphi"] \arrow[r, "\varphi'"'] & X' \end{tikzcd}$$

s.t $h \circ \varphi$, φ' homotopic on $\gamma_i + \gamma_j'$

$\Rightarrow h\circ\varphi \simeq \varphi'$. <u>A topological fact</u>

$\square$

<u>Key Topological Fact</u>: $h: S \to S$ o.p. homeo $h(\gamma_i) \simeq \gamma_i$ $h(\gamma_j') \simeq \gamma_j'$

$\Rightarrow h \simeq id$. (genus $g \geq 2$) $\quad$ preserve orientation of γ_i

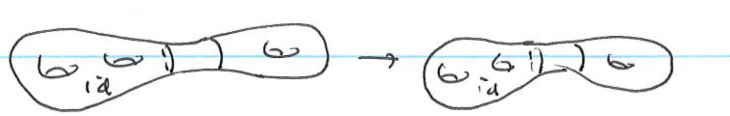
<u>Pf</u>: (sketch)

<u>Dehn twists</u> $\cdot F: S^1 \times [0,1] \to S^1 \times [0,1]$ $\quad F|_\partial = id$ $\quad F(e^{i\theta}, t) = F(e^{i\theta + 2\pi i t}, t)$

F

If $\alpha \subset S$ is a simple closed curve $\varphi: N(\alpha) \to S^1 \times [0,1]$ o.p. <u>homeo</u>. Then the Dehn twists along α, $D_\alpha: S \to S$ <u>s.t.</u> $D_\alpha(x) = x$ $\quad x \notin N(\alpha)$

$D_\alpha(x) = \varphi^{-1}(F \circ \varphi(x))$

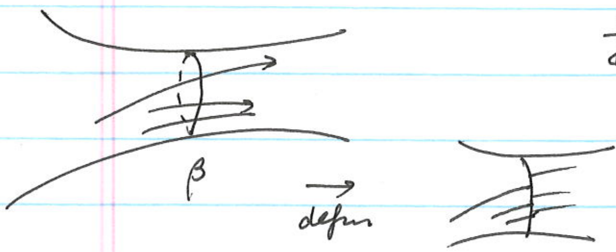


<u>Now</u>: $h(\gamma_i) \simeq \gamma_i \ \forall i \Rightarrow h \simeq D_{\gamma_1}^{a_1} \circ \cdots \circ D_{\gamma_N}^{a_N}$ Dehn twists along γ_i's

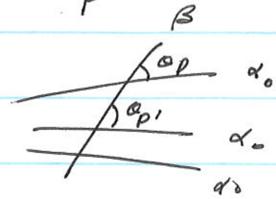
$h(\gamma_j') \simeq \gamma_j' \Rightarrow a_i = 0. \Rightarrow h \simeq id.$

Lecture 15 Wolpert's Cosine law.

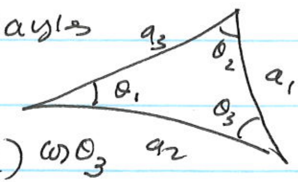
Thm (Wolpert) If d_t is a smooth family of hyperbolic metrics on Σ obtained by t-twist along a simple closed geodesic β_0 in d_0 and α_t is the closed geodesic in d_t homotopic to α_0, then



$$\frac{d}{dt}\, l_{d_t}(\alpha_t) = \sum_{p \in \alpha_0 \cap \beta} \cos\theta_p$$

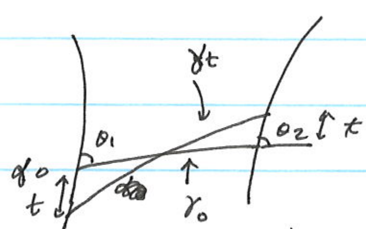


Proof Uses the cosine law: for hyperbolic triangles



$$\cosh(a_3) = \cosh a_1 \cosh a_2 - \sinh(a_1)\sinh(a_2)\cos\theta_3 \qquad \text{(HW)}$$

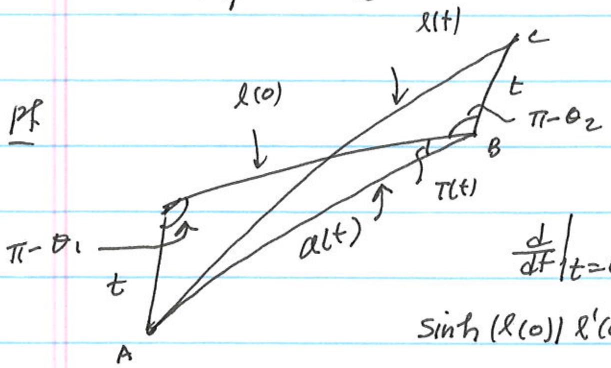
Lemma 1



in $\mathbb{H}$

$$\left.\frac{d}{dt}\right|_{t=0} l(\gamma_t) = \cos\theta_1 + \cos\theta_2$$

pf



ΔABC

$$\cosh l(t) = \cosh(a(t))\cosh t - \sinh(a(t))\sinh t \cos(\pi-\theta_2+\tau(t))$$

$\left.\frac{d}{dt}\right|_{t=0} \Rightarrow$

$$\sinh(l(0))\, l'(0) = \sinh(a(0))\, a'(0) - \sinh(a(0))\cos(\pi-\theta_2)$$

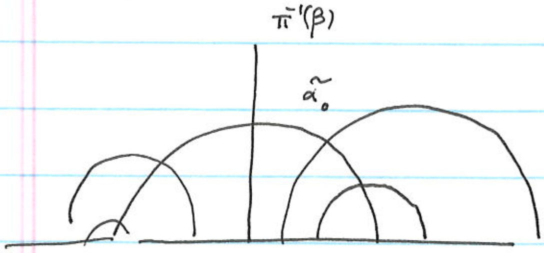
$$\Rightarrow \quad l'(0) = a'(0) + \cos(\theta_2)$$

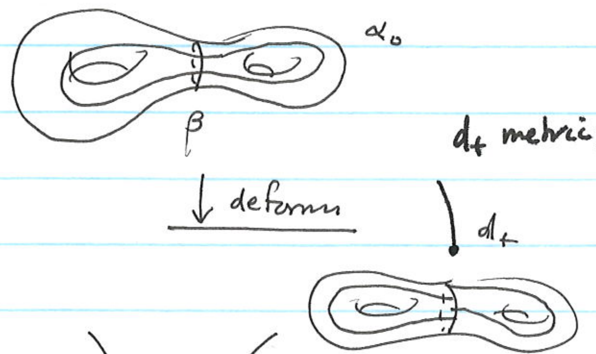
Now $\quad \cosh(a(t)) = \cosh t \cosh(l(0)) - \sinh t (\sinh l(0)) \cos(\pi-\theta_1)$

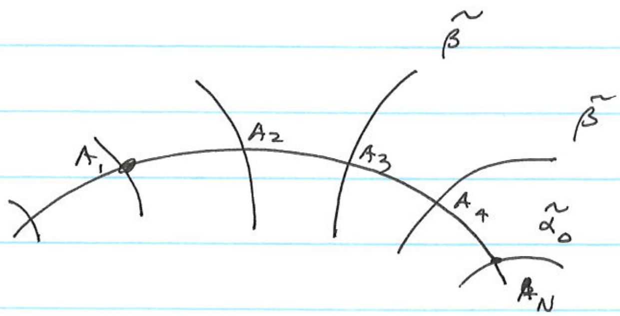
Take $\left.\frac{d}{dt}\right|_{t=0}$ $\quad \sinh(a(0))\, a'(0) = \underset{0}{\underset{\parallel}{\cosh l(0)}} + \sinh l(0) \cos(\theta_1)$

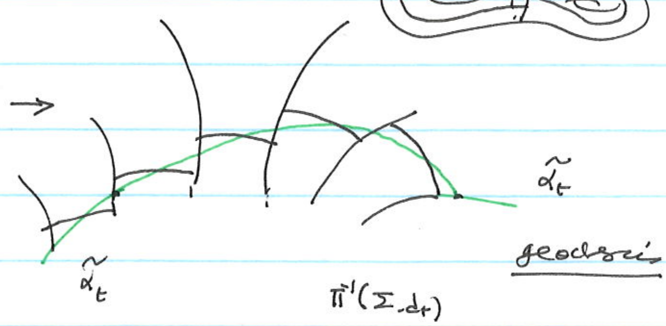
$$\Rightarrow \quad a'(0) = \cos\theta_1 \qquad \Rightarrow \quad l'(0) = \cos\theta_1 + \cos\theta_2.$$

Proof of thm lifts (Σ, α_0) to the universal cover: β, α_0







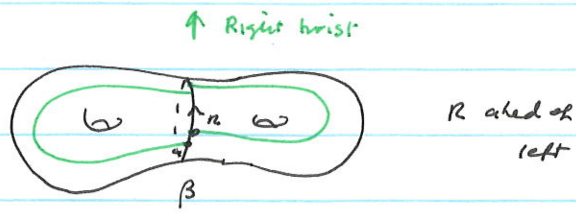


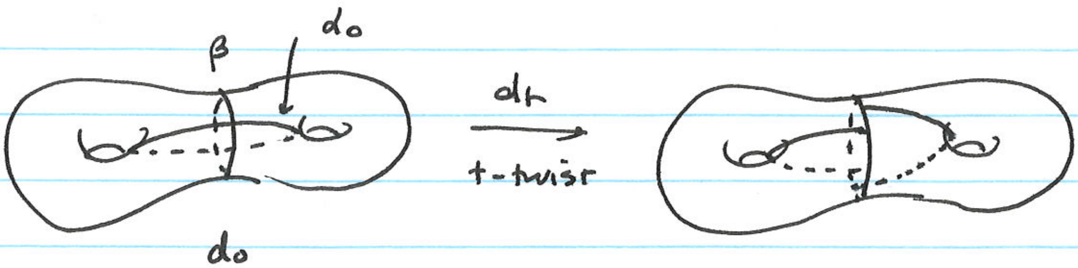
$\ell_t(\alpha_0) = \boxed{\cancel{\text{dist}(A, B)}}$ $\quad \mathbb{H} = \pi^{-1}(\Sigma, \alpha_0)$

Thurston: earthquake.

$\ell_t(\alpha_t) = \text{dist}(A_t, B_t) = \sum \Rightarrow$ result.

$\square$





Lecture 16. The Mapping Class Group

Back to $\mathrm{Mod}(S)$ and $\mathrm{Teich}(S)$. <u>S orientable surface</u>

<u>Def</u> The mapping class group $\Gamma(S) = \mathrm{Home}^+(S)/\text{homotopy}$

$\mathrm{Home}^+(S)$ group of orientation preserving homeo. $f \simeq g$ homotopic

<u>Note</u> $\Gamma(S)$ is a group $[f\circ g] = [f]\circ[g]$ is well defined,

<u>ie</u> $f \simeq f'$ $g \simeq g' \Rightarrow f\circ g \simeq f'\circ g'$ Basic fact from <u>Top</u>.

<u>Fact 1</u> $\Gamma(S)$ acts on $\mathrm{Teich}(S) = \{ [(S, \mathcal{C})]_T \mid \mathcal{C} \text{ complex structure} \}$

$[f] * [(S, \mathcal{C})] \triangleq [(S, f^*(\mathcal{C}))]$ $f^*(\mathcal{C})$ complex structure

st $f : (S, \mathcal{C}) \to (S, f^*(\mathcal{C}))$ <u>biho</u>

<u>By definition</u> if $f \simeq f' \Rightarrow (S, f^*(\mathcal{C})) \underset{T}{\cong} (S, f'^*(\mathcal{C}'))$

$f'\circ f^{-1} \simeq id$

$\boxed{\text{Fact}}$ · <u>Eg</u> For torus $S^1\times S^1$, $\Gamma(S^1\times S^1) \cong SL(2\cdot\mathbb{Z})$, the

Indeed, $\mathrm{home}^+(S^1\times S^1) \longrightarrow \mathrm{Aut}(\mathbb{Z}\oplus\mathbb{Z}) = GL(2,\mathbb{Z})$

$h \longmapsto h_*$ $(\det A = \pm 1)$

orientation preserving $\Rightarrow h_* \in SL(2\cdot\mathbb{Z})$

<u>It is 1-1</u>: we proved a in lecture 13

<u>It is onto</u>: direct construction: $A\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} : \mathbb{R}^2 \to \mathbb{R}^2$

sending $\mathbb{Z}\oplus\mathbb{Z}i$ to $\mathbb{Z}\oplus\mathbb{Z}i \Rightarrow$ induces a map

$\tilde{A} : \mathbb{R}^2/\mathbb{Z}\oplus\mathbb{Z}i \to \mathbb{R}^2/\mathbb{Z}\oplus\mathbb{Z}i$ whose lift is A.

<u>Fact 2</u> $\mathrm{Mod}(S) = \mathrm{Teich}(S)/\Gamma(S)$

<u>Pf</u> Define $\Phi : \mathrm{Teich}(S) = \{[(S, \mathcal{C})]_T\} \twoheadrightarrow \mathrm{Mod}(S) = \{[(S, \mathcal{C})]_R\}$

Obviously $\Phi([f]\cdot[(S, \mathcal{C})]_T) = \Phi([(S\,\mathcal{C})]_T)$ definition.

Thus Φ induces an onto map

$\mathrm{Teich}(S)/\Gamma(S) \longrightarrow \mathrm{Mod}(S)$

<u>Claim: it is 1-1</u>

Indeed, if $\Phi([S, \mathcal{C}_1)] = \Phi[(S. \mathcal{C}_2)] \Rightarrow \exists$ biho $h : (S\ \mathcal{C}_1) \to (S, \mathcal{C}_2)$

$\Rightarrow \mathcal{C}_2 = h^*(\mathcal{C}_1) \Rightarrow$ done $\square$

Lecture 16 The Mapping Class Group

Eg $\mathrm{Mod}(S^1\times S^1) = \mathrm{Teich}(S^1\times S^1)/PSL(2,\mathbb{Z}) = \mathrm{Teich}(S^1\times S^1)/SL(2,\mathbb{Z})$

Proved before.

Question How do you understand the group $\Gamma(S)$?

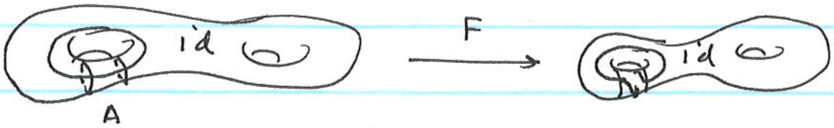
Def The Dehn twist along an annulus $D_{st}: S^1\times[0,1]\to S^1\times[0,1]$ is an homeomorphism s.t (1) $D_{st}|_{\partial(S^1\times[0,1])} = id$

(2) $D_{st}(e^{i\theta}, t) = D(e^{i\theta+2\pi i t}, t)$

Def S is a surface, $A\subset S$ is an annulus $h: S^1\times[0,1]\to A$ orientation preserving homeo, Then a dehn twist along A (a) is $D_a: S\to S$ homeo s.t (1) $D_a|_{S-A} = id$

(2) $D|_A = h\circ D_{st} h^{-1}$

Eg



S cpt orientable

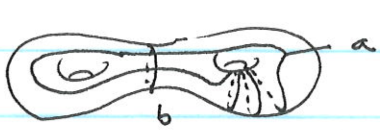
Eg $S^1\times S^1$ $\begin{bmatrix}1&1\\0&1\end{bmatrix}, \begin{bmatrix}1&0\\1&1\end{bmatrix}$ are Dehn twists

Thm (Dehn-Lickorish) 1. Each $h: S\to S$ orientation preserving s.t $h(b)=b$ for all boundary component $b\subset\partial S$ is homotopic to

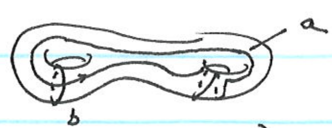
$$D_{\alpha_1}^{\pm}\cdots D_{\alpha_k}^{\pm}$$ for a finite set of Dehn twists.

Notation $a\subset A$ central curve D_a = the Dehn twist

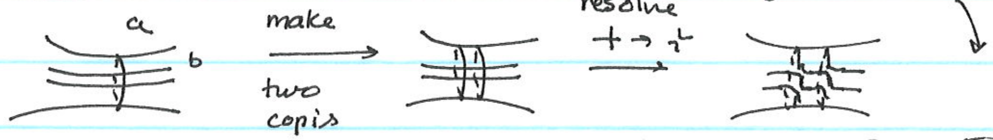
Eg. How to compute $D_a(b)$ where a, b s.c.c's st $|a\cap b|\geq 1$



Find $D_a(b)$. or

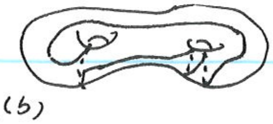


Basic rule



Resolution

$\cong$



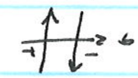
$D_a(b)$

Calculations on Simple loops

Some simple calculations, after homotopy (isotopy)

Notation 1. All intersections of curves are transverse  Not 

2. $a \perp b \Leftrightarrow |a \cap b| = 1 \quad a \pitchfork b$

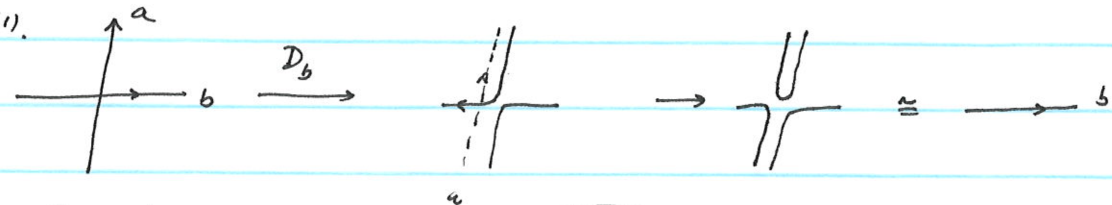
3. $a \perp_0 b \Leftrightarrow |a \cap b| = 2$, $a \pitchfork b$ two pts of different signs 

Lemma (1) $a \perp b \Rightarrow \quad D_a D_b(a) \simeq b$ + $\cancel{D_a D_b D_b D \;\; D_a D_b}$

$D_b D_a D_a D_b(a) \simeq a$ which reverses orientation

(2) If $a \perp_0 b$ in $\Sigma_{0,N}$, ~~then either $a \simeq b$ or a, b bounds~~ and a, b bound the same boundary component $\Rightarrow a \simeq b$

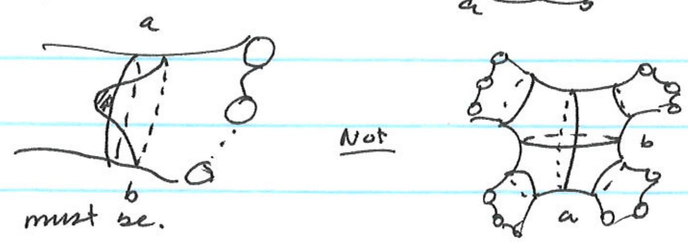
Pf (1).



surface level. 

cannot be any boundary.

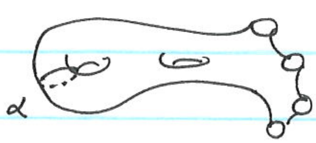
(2).



$a \not\simeq b \Rightarrow$

□

Strategy of the proof of thm 1. First $T_0(S) = \{ h \in \mathrm{Home}^+(S) \mid h(b) = b$ for each boundary $b \subset \partial S \}/$homotopy. $S = \Sigma_{g,n}$, we use induction on $3g + n = \|S\|$. Simple fact: S cut open along an essential s.c.c $\alpha \subset S \Rightarrow \|S - \alpha\| < \|S\|$ (S disjoint $= \coprod S_i$, $\|S\| \triangleq \sum \|S_i\|$)

Es 

$(g, n) \longmapsto (g-1, n+2)$

$3g+n \longrightarrow 3g-3+n+2 = 3g+n-1$ reduce

$(g, n) \longmapsto (g_1, n_1) \cup (g_2, n_2) \quad \begin{cases} g_1 + g_2 = g \\ n_1 + n_2 = n + 2 \end{cases}$

$\longrightarrow$ SAME

$3g + n \qquad 3g_1 + 3g_2 + n_1 + n_2$

(No Good)

Keep cutting

Lecture 16. Dehn twists

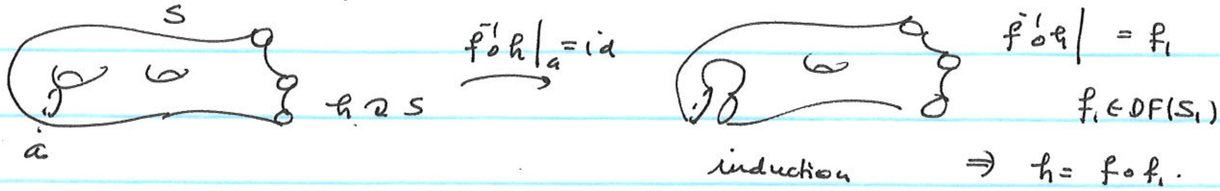
Let $D\Gamma(S)$ be the subgroup of $\Gamma_0(S)$ generated by Dehn twists. Goal $D\Gamma = \Gamma_0$

<u>Def</u> Two s.c.c $\alpha, \beta \subset S$ are equivalent $\alpha \sim \beta$ if $\alpha = f(\beta)$ $f \in D\Gamma$.

<u>Eg</u> $a \perp b \Rightarrow a \sim b$

<u>Main idea</u> $\forall$ $[h] \in \Gamma_0(S)$. take s.c.c $a \subset S$, we want $a \sim h(a)$. say
$h(a) = f(a)$ $[f] \in D\Gamma \Rightarrow (f^{-1} \circ h) a = a$. Assume orientati

Now cut S open along a + use induction



<u>So key step</u> is $b = h(a) \sim a$

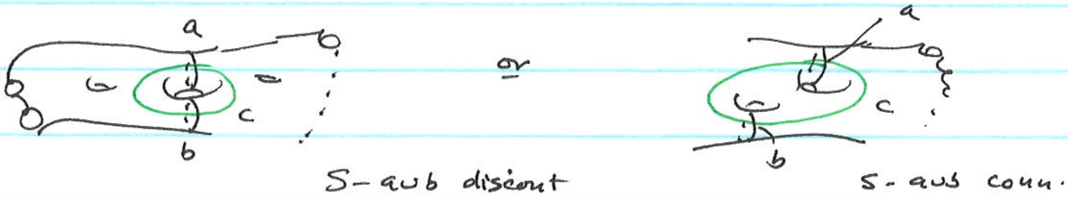
<u>Case 1</u> genus$(S) > 0$. a non-sep. s.c.c.

<u>Lemma</u> a, b non-separating $\Rightarrow a \sim b$.

<u>Pf</u> Induction on $|a \cap b|$

(1) $|a \cap b| = 1 \Rightarrow a \perp b \Rightarrow a \sim b$

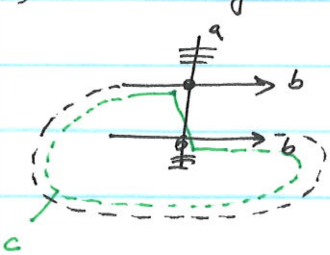
(2) $|a \cap b| = 0 \Rightarrow$ s.c.c c st $a \perp c$ and $b \perp c$



$\Rightarrow$ $a \sim c \sim b \Rightarrow a \sim b$

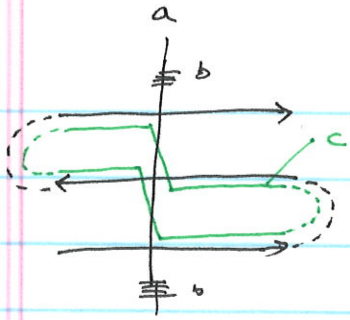
(3) $|a \cap b| \geq 2$

(3.1) $\exists$ two adjacent pts $p, q \in a \cap b$ along a of the same signs



$\Rightarrow \exists$ s.c.c c s.t
$c \perp b$ + $|c \cap a| < |b \cap a|$
($\Rightarrow c$ non-sep). $\Rightarrow b \sim c \sim a$.

(3.2) $\exists$ 3 adjacent pts $p, q, r \in a \cap b$ along a of alternating signs

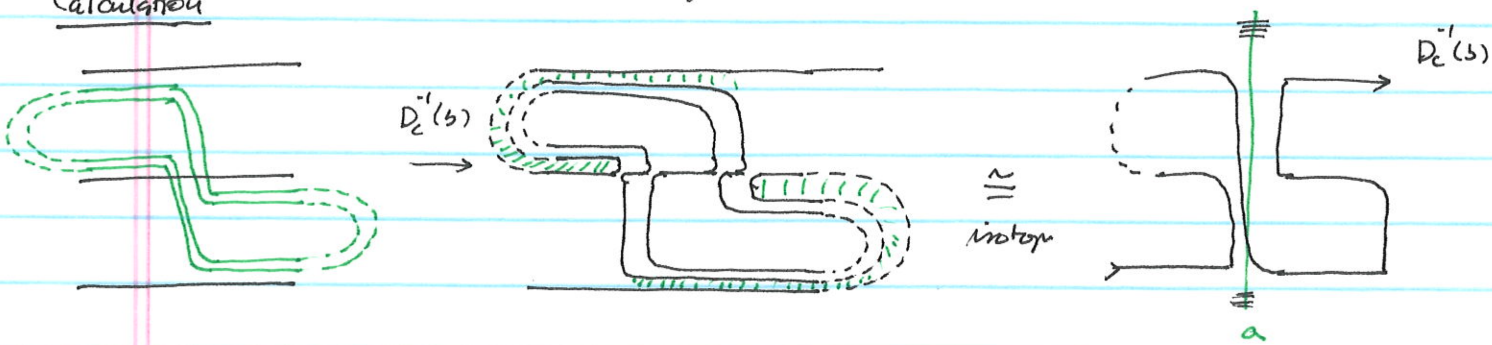


Consider c, then $b \sim D_c^{-1}(b)$

and $|D_c^{-1}(b) \cap a| < |b \cap a| \Rightarrow D_c^{-1}(b) \sim a$

Done

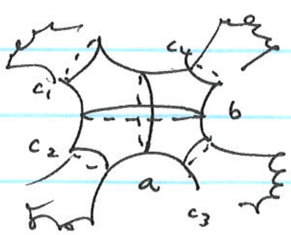
Calculation



So $|D_c^{-1}(b) \cap a| \leq |b \cap a| - 2.$

(3.3) The only case left: $|a \cap b| = 2$ of the same diff signs

Then one of $c_1, \cdots c_4$ of $\partial N(a \cup b)$ is non-separating.



$\Rightarrow$ say c_i

$\Rightarrow$ $|a \cap c_i| = |b \cap c_i| = 0$

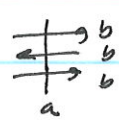
$a \sim c_i \sim b \Rightarrow$ Done.

Next if genus$(S) = 0$, a, $b = h(a)$ s.c.c bound the same boundary components $\Rightarrow$ $|a \cap b| =$ even

(i) $|a \cap b| = 0 \Rightarrow a \cong b$ done

(ii) $|a \cap b| = 2 \Rightarrow a \cong b$ done

(iii) $|a \cap b| \geq 4 \Rightarrow$ must



due to non-sep.

$\Rightarrow$ done

$\square$

Pisa lecture

Thm 1 M^3 closed orientable $\Rightarrow \exists$ a link $L \subset S^3$ + Dehn surgery on L with surgery coeff ± 1 s.t $M^3 \cong S^3(L)$

The proof uses

Thm 2 (Dehn Lickorish) Σ^3 closed orientable surface. Then $\forall\, h: \Sigma \to \Sigma$ orientation preserving $\Rightarrow$ $h \cong$ product of Dehn twists

Pf Use thm 2 $\Rightarrow$ thm 1 **Goal** $\exists$ link $L' \subset M$ st $M - L' \cong S^3 - L$.

Step 1. $M = H_g \cup_{h_1} H_g$ $\exists$ Heegaard splitting H_g = handlebody of genus g

$\Sigma_g = \partial H_g$ boundary surface $h_1\ \Sigma_g \to \Sigma_g$ o.p. homeo (Heegaard Sp)

$S^3 = H_g \cup_{h_2} H_g$ $h_2,\ \Sigma_g \to \Sigma_g$ o.p. home

Step 2 if $K \subset \mathring{H}_g$ and $F: H_g - K \to H_g - K'$ extending homeomorphism

s.t $F|_{\partial \Sigma_g} = h_2 \circ h_1^{-1}$ $F \circ h_1 = h_2$

Then $\exists$ homeo $H_g \cup_{h_1} (H_g - K) \cong H_g \cup_{h_2} (H_g - K')$

$M - K$ $S^3 - K$

$$H_g \sqcup (H_g - K) \longrightarrow H_g \sqcup (H_g - K')$$

$$x \longmapsto x$$

$$y \longmapsto F(y) \qquad \underline{y \in H_g - K}$$

$$x \sim h_1(x) \longrightarrow x \sim F h_1(x) = h_2(x)$$

So it descends to a homeo in the quotient.

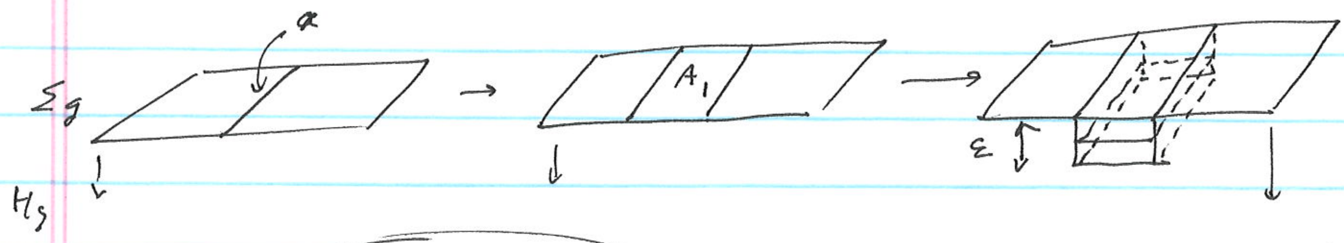
Step 3 It suffices to prove $h = h_2 \circ h_1^{-1}: \Sigma_g \to \Sigma_g$ extends to a homeo $F: H_g - L \to H_g - L$.

Thm 2. $h = D_{a_1}^{\pm} \circ \cdots \circ D_{a_n}^{\pm}$ Dehn twists along s.c.c $a_i \subset \Sigma_g$.

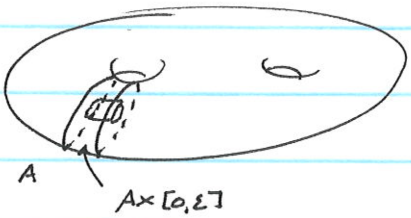
Let us see how to extend $D_{a_i}: \partial H_g \to \partial H_g$.

Let $A = N(a)$ tubular nbhd (annulus) of $a \subset \Sigma_g$ st $D_{a_i}|_{A^c} = id$

$A \times [0, \varepsilon] \subset H_g$ small product region of A



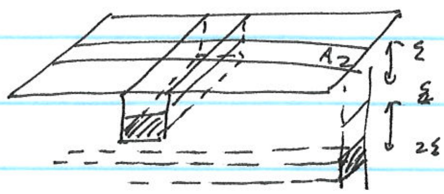
Eg 3-dim



$K_1 = \boxed{A_1 \times [\frac{\varepsilon}{2}, \varepsilon]} = A_1 \times [\frac{\varepsilon}{2}, \varepsilon] \subset \overset{\circ}{H_g}$

<u>Define</u> $F_1 : H_g - K_1 \longrightarrow H_g - K_1$ by $\begin{cases} F(x) = x & x \in H_g - A_1 \times [0, \varepsilon] \\ F(y, t) = (D_a(y), t) & y \in A_1, \frac{\varepsilon}{2} \le t \le \varepsilon \end{cases}$

<u>Now</u> for a_2 dig <u>deeper</u>



1.5ε

$\underline{K_2 = A_2 \times [0, 2\varepsilon]}$

$\underline{F_2}$

$\underline{F = F_1 \circ \cdots \cdots \circ F_k}$

Lecture 17. Ergodicity of Geodesic Flows

G a group acting on a measurable space (X, m) preserving m:

$\forall$ measurable $B \subset X$ $\quad m(g \cdot B) = m(B) \quad \forall\ g \in G \quad$ ($g: X \to X$ measurable)

<u>Def</u> G action is ergodic if $\forall$ measurable set $B \subset X$ s.t $gB = B$ a.e. $\forall\ g \in G \Rightarrow m(B) = 0$ or $m(X - B) = 0$.

$F: (X, m) \to (X, m)$ measurable map is measure pres if $\forall\ B \quad m(B) = m(F^{-1}(B))$

<u>Eg 1</u>. $X = S^1$ Lebesgue m. $G = \mathbb{Z}(\gamma)$, $\varphi \in \mathbb{R} \quad \gamma(e^{i\theta}) = e^{i\theta + i\varphi}$ $\quad \varphi$ rotation

2. $X = \mathbb{R}^n$, Lebesgue m, $G = SL(n, \mathbb{R}) \quad A \cdot x = Ax$

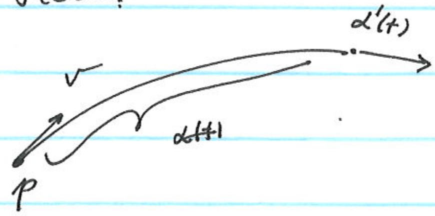
3. (M, g) Riemannian, $G = \mathrm{Iso}(M)$, $m = d\mathrm{vol}_g$.

4. (M^n, g) Complete Riemannian, $X = T^1 M = \{ v \in T_p M \mid \|v\| = 1 \}$.

$G = (\mathbb{R}, +)$, G action: geodesic flow:

$\forall\ t \in \mathbb{R} \quad \forall \quad v \in X \quad v \in T_p M$

$$t * v = g_t(v) = \alpha'(t)$$



$\alpha(s)$ geodesic arc length

$\alpha(0) = p \quad \alpha'(0) = v, \quad t * v = \alpha'(t)$, march t distance

Measure m: Liouville measure: $\underline{d\,\mathrm{vol}_g \wedge ds^{n-1}} \quad ds^{n-1}$ standard $(n-1)$ volume form on the $(n-1)$-dim unit sphere

5. $G = \mathbb{Z}(A) \quad A \in SL(2, \mathbb{Z}) \quad A: \mathbb{R}^2 \to \mathbb{R}^2 \quad \mathbb{Z}^2 \to \mathbb{Z}^2 \Rightarrow A$ induces a map $A: \mathbb{C}/_{\mathbb{Z} + \mathbb{Z}i} \to \mathbb{C}/_{\mathbb{Z} + \mathbb{Z}i}$ preserving the $dx\,dy$.

6. (Gauss Maps) $X = [0, 1)$, $\quad m(B) = \frac{1}{\lg 2} \int_B \frac{dx}{1+x} \qquad m(X) = 1$

$G = \mathbb{Z}(F) \qquad F(x) = \begin{cases} 0 & x = 0 \\ \{\frac{1}{x}\} = \frac{1}{x} - [\frac{1}{x}] & \text{fractional part.} \end{cases}$ (Continuous fraction)

<u>Lemma 1.</u> F preserves the measure. m

<u>Proof</u> $\quad F(\frac{1}{n}) = 0 \qquad x \in (\frac{1}{n+1}, \frac{1}{n}) \Leftrightarrow n < \frac{1}{x} < n+1$

<u>Thus</u> $\quad F(x) = \frac{1}{x} - n \quad$ for $\quad \frac{1}{n+1} < x < \frac{1}{n}$

<u>Thus</u> it suffices to <u>check</u> $\quad B \subset (0, 1)$
$\qquad\qquad\qquad\qquad\qquad\qquad \overset{\shortparallel}{(a, b)}$

$$F^{-1}(B) = \bigsqcup_{n=1}^{\infty} \left(\frac{1}{b+n}, \frac{1}{a+n}\right)$$

Now $\quad m(B) = \frac{1}{\lg 2} \ln\left(\frac{b+1}{a+1}\right)$

$$m(F^{-1}(B)) = \frac{1}{\lg 2} \sum_{n=1}^{\infty} \int_{\frac{1}{b+n}}^{\frac{1}{a+n}} \frac{dx}{1+x} = \frac{1}{\lg 2} \sum_{n=1}^{\infty} \ln\left(\frac{\frac{1}{a+n}+1}{\frac{1}{b+n}+1}\right)$$

$$= \frac{1}{\lg 2} \sum_{n=1}^{\infty} \left[\ln\left(\frac{b+n}{a+n}\right) - \ln\left(\frac{b+n+1}{a+n+1}\right)\right] \xrightarrow{\text{Convergence}} \frac{1}{\ln 2} \ln\left(\frac{b+1}{a+1}\right)$$

<u>Note</u> $\quad m([0,1)) = 1$ $\square$

Related to continue fraction expansion: $x \in \mathbb{R}_{>0} \setminus \mathbb{Q}$ $\quad 0<x<1$:

<u>Poincaré – Recurrence thm</u> $\quad (X, m)$ finite measure $\varphi: (X,m) \to (X,m)$ measure preserving. Then $\forall\ A \subset X$ measurable, for a.e. $x \in A$. $\exists\ n_i \to \infty$ st $\quad \varphi^{n_i}(x) \in A. \Leftrightarrow x \in \varphi^{-n_i}(A)$

<u>Pf</u> <u>Want</u> the set $\{x \in A \mid \forall i, \exists j \geq i \quad x \in \varphi^{-j}(A)\}$ of full measure

$$= \{x \in A \mid x \in \bigcap_{i=1}^{\infty} \bigcup_{j \geq i} \varphi^{-j}(A)\} = A \cap \bigcap_{i=1}^{\infty} \bigcup_{j \geq i} \varphi^{-j}(A)$$

Its complement in A

$$\Leftrightarrow \quad m\left(A \cap \bigcup_{i=1}^{\infty} \bigcap_{j \geq i} \varphi^{-j}(A^c)\right) = 0$$

Fix $m \geq 1$, let $\quad B_m = \left(\bigcap_{j \geq m} \varphi^{-j}(A^c)\right) \cap A \Rightarrow \varphi^{-k}(B_m) = \varphi^{-k}(A) \cap \bigcap_{j \geq m+k} \varphi^{-j}(A^c)$

Then if $\quad k \geq k' + m \Rightarrow$

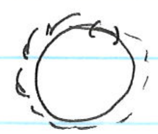
$$\varphi^{-k}(B_m) \cap \varphi^{-k'}(B_m) \subset \varphi^{-k}(A) \cap \varphi^{-k}(A^c) = \phi$$

<u>Thus</u> $\quad \{\varphi^{-jm}(B_m) \mid_{j=1,2\ldots}\}$ is a disjoint measurable set

$\Rightarrow$ measurable, φ^{-jm} measure preserving $\Rightarrow$ $m(B_m) = 0$

$\Rightarrow$ <u>Done</u> $\square$

A deep, elementary thm

<u>Eg</u> Rotation:  . Discuss the application to continuous fraction expansion of $x \in (0,1)$

Continue fraction expansion

Lecture 17 Ergodicity.

$$\sqrt{2} = 1 + (\sqrt{2}-1) = 1 + \frac{1}{\left(\frac{1}{\sqrt{2}-1}\right)} = 1 + \frac{1}{1+\sqrt{2}}$$

$$\frac{1}{\sqrt{2}-1} = 1+\sqrt{2}$$

Repeat if $x > 1 \to x = n + (x-n)$

$$= n + \frac{1}{\left(\frac{1}{x-n}\right)} \quad \text{repeat}$$

$\sqrt{2} = [1, 1, 1, 1, \cdots \cdots, 1, \cdots]$ always obtain by F

Now what does Poincaré thm tell you?

Let $A = (\frac{1}{3}, \frac{1}{2})$, $\Rightarrow$ a.e $x \in A$. has cont. fraction expansion in each 2 appears infinitely many times. $\square$

RM ~~It will be great if your map is measure preserving.~~

~~Back to geodesic flows~~ ~~drop~~

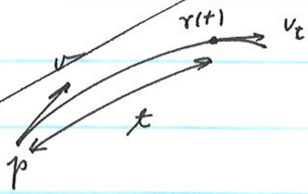
~~Ergodicity of $\mathbb{Z}+\mathbb{Z}\alpha$, $\alpha \notin \mathbb{Q}$, on $\mathbb{R}$ by translations. Dense orbit.~~

~~Unit tangent $T^1M = \{(p,v) \in TM \mid |v| = 1\}$ unit tangent bundle.~~

~~$T^1\mathbb{H} = \{(p,v) \in \mathbb{H} \times \mathbb{C} \mid \|v\|_p = 1\}$~~

~~Geodesic flow $(p,v) \in T^1\mathbb{H} \Rightarrow$ geodesic $\gamma(t)$ st $\gamma(0) = p$, $\gamma'(0) = v$~~

~~Let $v_t = \gamma'(t)$. Then $t * (p,v) \triangleq (\gamma(t), v_t)$.~~



~~Lemma $v_0 = \frac{\partial}{\partial y}\big|_i$ $\uparrow$. The map $\Phi : PSL(2,\mathbb{R}) \to T^1\mathbb{H}$~~

~~$\Phi(g) = g \cdot v_0 \; (\triangleq g'(i) v_0)$ is 1-1 onto~~

~~Pf Given $g(z) = \frac{az+b}{cz+d}$ $ad-bc=1$. $g'(z) = \frac{1}{(cz+d)^2}$ $g'(i) = \frac{1}{(ci+d)^2}$~~

~~Now $\Phi(g_1) = \Phi(g_2) \Leftrightarrow \Phi(g) = v_0$ $g = g_2^{-1} \circ g_1$.~~

~~So it suffices to show: $g(i) = i$ $g'(i)i = i \Rightarrow g = 1$~~

~~$g(i) = i$ $g'(i) = 1 \Rightarrow (ci+d)^2 = 1$ $ci+d = \pm 1 \Rightarrow ai+b = \pm 1$~~

~~$\Rightarrow$ done. (Geometry isometry g: fix i, $Dg(i)\big|_T = id \Rightarrow g = id$)~~

Lecture 17. Geodesic Flows

$(M, \overset{ds^2}{g})$ Riemannian $\quad T^1M = \{ v \in T_pM \mid \|v\|_g = 1 \}$

Unit tangent bundle

Eg $\quad T^1\mathbb{H} = \{ (p, u) \in \mathbb{H} \times \mathbb{C} \mid \|u\|/y = 1 \quad p = (x, y) \}$

Now $\quad PSL(2, \mathbb{R}) = G$ acts isometrically on $\mathbb{H}$.

$\Rightarrow$ G acts on $T^1\mathbb{H}$, still denoted by $g \cdot v$. $\quad g \in G$

(In practice $\quad g(p, u) = (g(p), g'(p)u) \quad \overset{\shortparallel}{(Dg)_p(v_p)} \quad v \in T_p$

Note (1). $g(z) = \dfrac{az+b}{cz+d} \Rightarrow g'(z) = \dfrac{1}{(cz+d)^2}$

(2). $g(i) = i \iff \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \quad g'(i) = e^{2i\theta}$.

In particular $\quad g \cdot (h(v)) = (gh) \cdot v$

$= (Dg)_i(v_0)$

$\Phi(g) = g \cdot v_0$

Proposition The evaluation map $\Phi: PSL(2, \mathbb{R}) \to T^1\mathbb{H}$ is 1-1 onto and equivariant: $\underline{\Phi(g \cdot h) = g\,\Phi(h)}$. $\quad \Phi(g) = g \cdot \Phi(1)$.

Proof $\quad v_0 = \frac{\partial}{\partial y}\big|_i$.

(1) Equivariance: $\quad \Phi(g) = Dg_i(v_0) \quad \Phi(gh) = D(gh)_i\, v_0 = (Dg)_{h(i)}(Dh)_i\, v_0$

$g \cdot \Phi(h) = (Dg)_{h(i)}((Dh)_i \cdot v_0)$

(2). It is 1-1: if $\quad Dg(i) = Dh(i) \Rightarrow g(i) = h(i) \;+\; g'(i) = h'(i)$

Algebraically $\quad g \sim \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad h \sim \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}$

Then $\quad (ci+d)^2 = (c'i+d')^2 \iff ci+d = \pm(c'i+d')$

$\dfrac{ai+b}{ci+d} = \dfrac{a'i+b'}{c'i+d'} \iff a'i+d = \pm(a'i+b')$

$\Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \pm \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}$.

Geometrically $\quad h^{-1}g: \mathbb{H} \to \mathbb{H}$ isometry, orientation preserving $i \mapsto i$ acts identically on $T_i^1\mathbb{H} \Rightarrow$ sending geod γ from i $\; h^{-1}g/_\gamma = id$

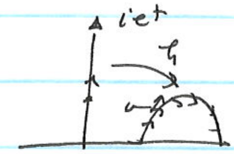
Done

(3) It is onto: Known $PSL(2, \mathbb{R})$ acts transitively on $\mathbb{H}$

Thus $\forall\ v \in T_p \mathbb{H}$, we may assume after composing w/ $g \in PSL(2,\mathbb{R})$ that $v \in T_i \mathbb{H}$. Now for $g = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \Rightarrow Dg'(i) = e^{2i\theta}$ can rotate s.t. $g\left(\frac{\partial}{\partial y}\Big|_i\right) = v$.

□

Lemma 1 The geodesic flow G_t on $T^1\mathbb{H}$ start at $v = h\cdot v_0$ is $h \cdot A_t \cdot v_0$
$h\cdot\Phi(A_t)$.
where $A_t = \begin{bmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{bmatrix}$.



Proof The geodesic flow start at $\frac{\partial}{\partial y}\Big|_i$ is $A_t\left(\frac{\partial}{\partial y}\Big|_i\right)$

$\gamma_t(z) = e^t z$ geodesic $\gamma_t(i) = ie^t$ $\gamma_t'(\cdot) = ie^t$. $\Rightarrow$.

Now h isometry $\Rightarrow$ $h\cdot A_t\cdot(v_0)$ the geodesic flow from $h(v_0)$.

□

Corollary **Lemma 2.** The action of $A = \left\{ \begin{bmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{bmatrix} \,\middle|\, t \in \mathbb{R} \right\}$ on the right on $PSL(2,\mathbb{R})$ corresponds to geodesic flow

$$\Phi(h\cdot A_t) = h\cdot\Phi(A_t).$$

Let $\Gamma < PSL(2,\mathbb{R})$ act discretely faithfully on $\mathbb{H}$. $\Gamma\backslash\mathbb{H} = \{\Gamma x \mid x \in \mathbb{H}\}$ is the space of left cosets = Hyperbolic surfaces (I switched notation)

Lemma. $T^1(\Gamma\backslash\mathbb{H}) \cong \Gamma\backslash PSL(2,\mathbb{R})$ s.t. the geodesic flow is the right action of A_t on $PSL(2,\mathbb{R})$.

Pf Let $\pi: \mathbb{H} \longrightarrow \Gamma\backslash\mathbb{H}$ $\quad \pi(\gamma\cdot x) = \pi(x)$ quotient map

π induces onto: $\pi: T^1\mathbb{H} \longrightarrow T^1(\Gamma\backslash\mathbb{H})$ (use the same)

Def $F: PSL(2,\mathbb{R}) \longrightarrow T^1(\Gamma\backslash\mathbb{H})$: $\pi\circ\Phi$.

Clearly F is onto since both Φ & π are

F 1-1: if $F(g_1) = F(g_2) \Rightarrow \Phi(g_1) = \gamma\,\Phi(g_2)$ $\quad \gamma \in \Gamma$

$= \Phi(\gamma g_2) \Rightarrow g_1 = \gamma g_2$ ✓

Last part definition.

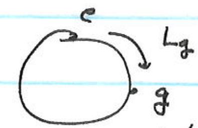
Lecture 17

Let m be the left invariant Haar measure on $PSL(2, \mathbb{R})$. It turns out to be right invariant as well. $\mathfrak{m}$ is $\phi^*(m)$ on T^1M is called the Liouville measure on T^1M invariant under the geodesic flow.

Lecture 17. Haar Measure

The Haar measure on $SL(2,\mathbb{R}) = \left\{ \begin{bmatrix} x & y \\ z & w \end{bmatrix} \middle| \ xw - yz = 1 \right\} \subset \mathbb{R}^4$

<u>Haar measure</u> on any Lie group G^n: Take n-form $\omega_e \neq 0$ in T_eG and define left invariant n-form $\omega_p = (L_{g^{-1}})^* \omega_e$



$L_g(x) = gx$

<u>Key Fact</u> For $SL(2,\mathbb{R})$, the Haar measure is right-invariant.

$R_g(x) = x \cdot g$ <u>Then</u> $Ad(g) : x \longmapsto R_{g^{-1}}(L_g x) = gxg^{-1}$ <u>$G \to G$</u> $e \mapsto e$

<u>Lemma</u> $\cancel{A}$ $ad(g) = DAd(g) : T_eG \to T_eG$ preserves ω, i.e. is in $SL(3,\mathbb{R})$.

$$\begin{matrix} T_eG & \to & T_eG \\ \| & & \| \\ \mathbb{R}^3 & \to & \mathbb{R}^3 \end{matrix}$$

<u>Proof</u> $\mathfrak{g} = T_eG = \mathfrak{sl}(2,\mathbb{R}) = \mathbb{R}E + \mathbb{R}F + \mathbb{R}H$ $\quad E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$ad(g)(A) = gAg^{-1}$

Say $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $g^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

<u>Let us find</u>:

$$gEg^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} -c & a \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -ac & a^2 \\ -c^2 & ac \end{bmatrix}$$

$$= \boxed{\cancel{ac}} \ a^2E + (-c)^2F - acH$$

$$gFg^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 0 \\ d & -b \end{bmatrix} = \begin{bmatrix} bd & -b^2 \\ d^2 & -bd \end{bmatrix} = -b^2E + d^2F + bdH$$

$$gHg^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ c & -a \end{bmatrix} = \begin{bmatrix} ad+bc & -2ab \\ 2cd & -ad-bc \end{bmatrix} = -2abE + 2cdF + (ad+bc)H$$

So the matrix rep is

$$\alpha = \begin{bmatrix} a^2 & -c^2 & -ac \\ -b^2 & d^2 & bd \\ -2ab & 2cd & ad+bc \end{bmatrix}$$

$g = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}$ $\quad \alpha = \begin{bmatrix} \lambda^2 & 0 & 0 \\ 0 & +\lambda^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ✓

$h \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$ $\quad \alpha = \begin{bmatrix} 1 & 0 & 0 \\ -x^2 & 1 & x \\ -2x & 0 & 1 \end{bmatrix}$ ✓

$L \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix}$ $\quad \begin{bmatrix} 1 & -x^2 & -x \\ 0 & 1 & 0 \\ 0 & * & 1 \end{bmatrix}$ ✓

□

Lecture 17 The Haar Measure

Hw $SL(n,\mathbb{R})$ has left-right invariant Haar measure

(x,y) θ v

Proposition Parametrize $T^1\mathbb{H}$ by (x,y,θ) $\qquad (x,y)\in\mathbb{H},\ 0\le\theta<2\pi$

Then the Haar measure is $\dfrac{dx\,dy\,d\theta}{y^2} = \omega = \underline{dA_{\mathbb{H}^2}\wedge d\theta}$

Proof We show that ω is invariant under $PSL(2,\mathbb{R})$ action $\Rightarrow$ done

(1) clearly invariant under $z\mapsto \lambda z$ or $A=\begin{bmatrix}\sqrt{\lambda} & 0\\ 0 & \frac{1}{\sqrt{\lambda}}\end{bmatrix}$

Since θ not changing + dA invariant

(2) Invariant under $z\mapsto z+a$ $\quad a\in\mathbb{R}$

(3) Invariant under $F(z)=-\frac{1}{z}$, yes, $\dfrac{dx\wedge dy}{y^2}$ invariant, what about θ?

How to check it $\qquad$ Notation

Basic idea $\quad F'(z)(\theta) = \theta + \alpha(x,y)$ $\quad$ some function of x,y

So $\quad \boxed{d\theta = d\theta + d\alpha(x,y)\,!}$

So $\dfrac{dx\wedge dy\wedge d\theta}{y^2} = \dfrac{dx\wedge dy}{y^2}\wedge\left(d\theta + d\alpha(x,y)\right) = !$ $\qquad F(z) \qquad F'(z)v$

Hw Calculate it carefully, find out exactly $\underline{\tilde{\theta} = \theta + \operatorname{Arg} F'(z)}$

So

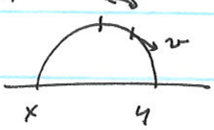
$\square$

Conclusion The Haar measure (or Liouville measure) m on $T^1(\Gamma\backslash\mathbb{H})$ is the wedge product of $\underline{\text{area}\wedge d\theta}$

Conclusion $\operatorname{Area}(\Gamma\backslash\mathbb{H})<+\infty \iff m(\Gamma\backslash PSL(2,\mathbb{R}))<+\infty$.

Hw Show that the 3-form $\eta = \dfrac{dx\wedge dy\wedge dz}{z}$ on $\mathbb{R}^4$ restricted to $PSL(2,\mathbb{R}) = \left\{\begin{bmatrix} w & x\\ y & z\end{bmatrix} \,\middle|\, wz-xy=1\right\}$ is the Haar measure

Hw: $T^1\mathbb{H} \overset{u}{\leftrightarrow}$ $\qquad$ Haar measure is $\dfrac{dx\wedge dy\wedge du}{(x-y)^2}$



Lecture 18 Ergodicity

$\alpha \notin \mathbb{Q}$

<u>Ergodicity</u> $\Gamma = \mathbb{Z} + \mathbb{Z}\alpha$ acts ergodically on $(\mathbb{R}, m)$ by translations

Key Γx dense in $\mathbb{R}$ $\forall x$.

<u>Pf</u> If Not $\exists$ Γ invariant measurable sets A st $m(A) > 0$, $m(A^c) > 0$

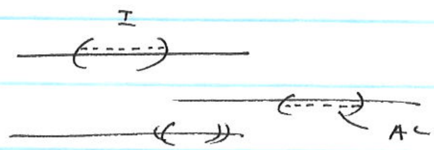
<u>Key Fact</u> If $B \subset \mathbb{R}$, $m(B) > 0$ $\exists$ $x \in B$ s.t. $\displaystyle\lim_{\varepsilon \to 0^+} \frac{m(B \cap (x-\varepsilon, x+\varepsilon))}{m((x-\varepsilon, x+\varepsilon))} = 1$.

(point of density)

Let $x \in A$, $y \in A^c$ be points of density. $\Rightarrow$ $\exists$ two open intervals I, J $m(I) = m(J)$ s.t.

$$m(A \cap I) \geq 0.9\, m(I)$$

$$m(A^c \cap J) \geq 0.9\, m(J)$$



But Γx dense $\Rightarrow$ $\exists$ $\gamma \in \Gamma$ <u>st</u> $m(\gamma I \cap J) \geq 0.9\, m(J)$

$$m(\gamma I \cup J) \leq m(\gamma I) + m(J) - m(\gamma I \cap J) \leq 1.1\, m(J).$$

<u>Now</u> consider

$$m(A^c \cap \underbrace{\gamma(I) \cap J}) = m(\gamma(I) \cap (A^c \cap J))$$

$$= m(\gamma I) + m(A^c \cap J) - m((A^c \cap J) \cup \gamma(I))$$

$$\geq m(J) + 0.9\, m(J) - 1.1\, m(J) = 0.8\, m(J)$$

$$\geq 0.8\, m(\gamma(I) \cap J)$$

<u>SAME</u> $m(\underbrace{A^c \cap \gamma(I)} \cap J) \overset{\gamma}{=\!=} m(\underbrace{A \cap I} \cap \gamma^{-1}(J))$

$$= m(A \cap I) + m(\gamma^{-1}(J)) - m((A \cap I) \cup \gamma^{-1}(J))$$

$$\geq 0.9\, m(I) + m(I) - m(I \cup \gamma^{-1}(J))$$

$$\geq 1.9\, m(I) - 1.1\, m(I \cup \gamma^{-1}(J)) \geq 0.8\, m(\gamma(I) \cap J).$$

<u>Impossible</u>. $\square$

<u>RM 1</u> The same shows if $\Gamma < G$ dense subgroup of a Lie group m - Haar measure, Then Γ acts ergodically on G

<u>RM 2.</u> $e^{2\pi i}$ $\gamma(z) = z e^{i\theta}$, $\theta \notin 2\pi i \mathbb{Q}$ action (S^1, m) is <u>ergodic</u>.

(We will produce a high tech proof)

Lecture 18 Ergodicity

Thm (Ergodicity) If $\Sigma = \Gamma \backslash \mathbb{H}$ finite hyperbolic surface. then the geodesic flow on $T^1\Sigma$ is ergodic w.r.t Liouville measure. $\Leftrightarrow$ $\Gamma \backslash \mathbb{H}$ finite area $vol(\Gamma \backslash PSL(2.\mathbb{R})) < \infty$, then the right multiplication by $A = \{ \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \mid a \in \mathbb{R}_{\neq 0} \}$ is ergodic.

Proof Let $N^+ = \{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{R} \}$, $N^- = \{ \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \mid x \in \mathbb{R} \}$

Lemma 1. $PSL(2.\mathbb{R})$ is generated by $A . H^+ . H^-$

Proof

$$\begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c+xa & d+xb \end{bmatrix} \qquad \text{Row op } \underline{R_2 \to R_2 + xR_1} \quad R_1$$

$$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+xc & d+xd \\ c & d \end{bmatrix} \qquad \text{———} \quad R_2$$

$$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \lambda a & \lambda b \\ c/\lambda & d/\lambda \end{bmatrix} \qquad \text{———} \quad S.$$

Now $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow[R_2]{} \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \xrightarrow[a_1 \neq 0]{R_1} \begin{bmatrix} a_2 & b_2 \\ 0 & c_2 \end{bmatrix} \xrightarrow[c_2 = \frac{1}{a_2}]{S} \begin{bmatrix} 1 & \mu \\ 0 & 1 \end{bmatrix}$ done

HW (Lemma 1') $PSL(2.\mathbb{Z})$ generated by $\overbrace{\begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix}}^{=\alpha}, \overbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}^{=\beta}$ $\hat{G} = \langle \alpha, \beta \rangle$

Given $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2.\mathbb{Z})$ Any all $gB =: B'$ is the

One with $\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}$ One of the non-zero entry has the smallest absolute value

(1) Say a' is: $c' = ka' + r$ $0 \leq r < |a'|$ division rule

$\Rightarrow r = 0$ since is smaller $\Rightarrow \begin{bmatrix} a' & b' \\ 0 & d' \end{bmatrix}$ $a'd' = 1 \Rightarrow a' = d' = \pm 1$

$\Rightarrow \pm \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$.

(2) Say c' is $\Rightarrow a' = 0$ $\begin{bmatrix} 0 & \mp 1 \\ \pm 1 & k \end{bmatrix} \to \pm \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ Row op

Now $\begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix} = \begin{bmatrix} 1+xy & y \\ x & 1 \end{bmatrix}$ $x = -y = 1 \Rightarrow$ done.

Lecture Ergodicity

Lemma 2 If $g = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}$ $a \geq 0$, $h_- = \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix}$, $h_+ = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$

then

$$\lim_{n\to\infty} g^n h_- g^{-n} = \lim_{n\to\infty} \begin{bmatrix} 1 & 0 \\ x \cdot a^{-2n} & 1 \end{bmatrix} = I \quad \text{for } a>1$$

$$\lim_{n\to\infty} g^{-n} h_+ g^{n} = \lim_{n\to\infty} \begin{bmatrix} 1 & x a^{2n} \\ 0 & 1 \end{bmatrix} = I \quad \text{for } a<1$$

Pf

$$g^n h_- g^{-n} = \begin{bmatrix} a^n & 0 \\ 0 & a^{-n} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix}\begin{bmatrix} a^{-n} & 0 \\ 0 & a^n \end{bmatrix} = \begin{bmatrix} a^n & 0 \\ x a^{-n} & a^{-n} \end{bmatrix}\begin{bmatrix} a^{-n} & 0 \\ 0 & a^n \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ x a^{-2n} & 1 \end{bmatrix}$$

$\square$

$\overset{X}{\shortparallel}$

Now suppose $B \subset P \backslash PSL(2.\mathbb{R})$ measurable set, invariant under the action of $A = \{ \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \mid a>0 \}$ then $m(B)$ or $m(B^c) = 0$

Let $f: X \to \mathbb{R}$ be χ_B. $f(x)=1$ $x \in B$ $f(x)=0$ $x \in B^c$

Then $f(x \cdot g) = f(x)$ $\forall g \in A$. (*)

Claim $\forall$ $h_- \in N_-$ $h_+ \in N_+$ $f(x \cdot h) = f(x)$ a.e

$h = h_+$ or h_-

Say $h = h_-$, $g \in A$

Consider $\int_X |f(x\cdot h) - f(x)| dm_x \overset{y = x \cdot g^n}{\underset{dm_y = dm_x}{=\!=}} \int_X |f(y \cdot g^{-n} \cdot h) - f(y \cdot g^{-n})| dy$

$$\overset{(*)}{=\!=} \int_X |f(y \cdot g^{-n} \cdot h \cdot g^{n}) - f(y)| dm_y$$

Now lemma 2 $\Rightarrow$ $\lim_{n\to\infty} f(y \cdot g^{?} \cdot h \cdot g^{-n}) = f(y)$ $\forall y$

pointwise convergence

Also $|f(y \cdot g^n \cdot h \cdot g^{-n}) - f(y)| \leq 2$ $\int_X dm < +\infty$

Lebesque dominance thm

$$\lim_{n\to\infty} \int_X |f(y \cdot g^{n} \cdot h \cdot g^{-n}) - f(y)| dm = 0$$

$\Rightarrow$ $f(x \cdot h) = f(x)$ a.e

Lemma 1 i.e $\forall$ $\gamma \in PSL(2.\mathbb{R})$ $f(x \cdot \gamma) = f(x)$ a.e $\Rightarrow f = \text{const}$ a.e

$\Rightarrow B = X$ or X^c a.e