(1) First Week

Lecture 1-2 Mostow Rigidity

Luo, Feng

Goal of our lecture: to prove

Thm (Mostow) 1968). If $n \geq 3$, M^n, N^n are two closed connected hyperbolic n-manifolds so that $f: M^n \to N^n$ is a homotopy equivalence (homeomorphism), then $\exists$ an isometry $g: M^n \to N^n$ st $g \simeq f$.

Note $n \geq 4$, there are very few closed hyperbolic mfds. $n=2$. Huge uncountable / iso. $n=3$, Thurston-Perelman. a lot.

Mostow's proof was revolutionary. He introduced the large scale geometry.

We will follow Gromov's proof of it using Gromov norm.

Tools. 1. Hyperbolic geometry

2. quasi-isometry + quasi-isometric equivalence (Large scale geometry)

3. Gromov norm of a closed oriented manifolds

4. Gromov-Thurston's theorem on Gromov norm and hyperbolic volume

5. Gromov's proof

Basic Large Scale Geometry

(X, d), $(Y, \tilde{d})$ two metric spaces $F: X \to Y$ any map not necessary continuous. $N_r(A) = \{x \in X \mid d(x, A) < r\}$ $A \subset X$.

Def If $\exists K > 0$ s.t $\forall x_1, x_2 \in X$

$$\frac{1}{K} d(x_1, x_2) - K \leq \tilde{d}(F(x_1), F(x_2)) \leq K d(x_1, x_2) + K$$

We say F is a K-quasi-isometric embedding (QIE).

If furthermore, $Y = N_K(F(X))$, i.e. $\forall y \in Y$. $\exists x \in X$ s.t $d(y, F(x)) \leq K$

We say F is a K-quasi-isometry (Q.I.). $N_R(A) = \{x \mid d(x, A) \leq R\}$.

Goal: Classify metric spaces up to Q.I. (Large scale geometric classification)

Eg $F = id \Rightarrow F$ is Q.I. clear or F = isometry $\Rightarrow F$ is Q.I.

RM F is K-QI + $K' > K \Rightarrow F$ is K'-Q.I.

Lemma 1. F. G both Q.I $\Rightarrow$ $F \circ G$ is Q.I.

$$X \xrightarrow{G} Y \xrightarrow{F} Z$$

Pf. May assume both K-Q.I.

Now

$$d_Z(F(G(x_1)), FG(x_2)) \leq K d_Y(G(x_1), G(x_2)) + K$$

$$\leq K(K d_X(x_1, x_2)) + K) + K \leq K^2 d_X(x_1, x_2) + (K^2 + K)$$

$(K' = K^2 + K)$

$$d_Z(F(G(x_1)), F(G(x_2))) = \frac{1}{K} d_Y(G(x_1), G(x_2)) - K$$

$$\geq \frac{1}{K}\left(\frac{1}{K} d_X(x_1, x_2) - K\right) - K = \frac{1}{K^2} d_X(x_1, x_2) - (K+1).$$

$K' = \max(K^2, K+1)$

Finally $\forall z \in Z$, $\exists y \in Y$ st

$$d(z, F(y)) \leq K.$$

$+ \exists x \in X$ st $d(y, G(x)) \leq K$

Now $d(z, FG(x)) \le d(z, F(y)) + d(F(y), F(G(x)))$

$$\le K + \left(K\, d(y, G(x)) + K\right)$$

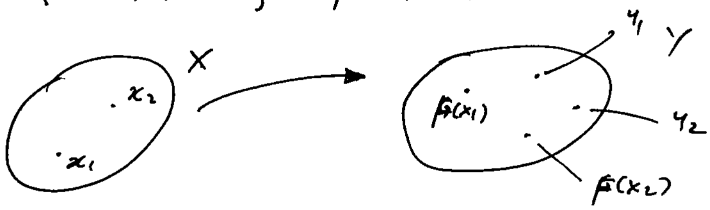
$$\le K + K^2 + K = \underline{K^2 + 2K}$$

i.e. $K' = \max(K^2 + 2K,\ K+1)$ will work

□

Lemma 2 Suppose $F: X \to Y$ is Q.I. Define $G: Y \to X$ to be: $G(y) = x$ $\forall y \in Y$. $x \in X$ s.t. $d(F(x), y) \le K$. Then G is a Q.I.

proof (I am using $d_Y = d$, $d_X = d$)



$\underline{x_i = G(y_i)} \Rightarrow \underline{d(F(x_i), y_i) \le K}$

$\underline{d(F(G(y)), y) \le K}$

To see: $d(G(y_1), G(y_2)) = d(x_1, x_2)$

$$\overset{F}{\le} K\, d(F(x_1), F(x_2)) + K^2$$

$$\le K\left(d(F(x_1), y_1) + d(y_1, y_2) + d(F(x_2), y_2)\right) + K^2$$

$$\le K\, d(y_1, y_2) + 3K^2.$$

Also

$$d(y_1, y_2) \le d(y_1, F(x_1)) + d(F(x_1), F(x_2)) + d(F(x_2), y_2)$$

$$\le 2K + \left(K\, d(x_1, x_2) + K\right)$$

$$= K\, d(x_1, x_2) + 3K$$

Finally: $\forall x \in X$, let $y = F(x)$, consider $G(y)$ satisfies $\underline{d(FG(y), x) \le K}$

$$d(x, G(y)) \le K\, d(F(x), FG(y)) + K^2$$

$$= K\, d(y, FG(y)) + K + K^2$$

$$\le 2K^2 + K.$$

□

Eg 1 Each compact metric space $(X, d) \overset{Q.I}{\sim}$ point.

Pf $K =$ diameter of X $\quad F: X \to \{p\}$ $\quad G: \{p\} \to X$

are Q.I.

Eg 2 $\mathbb{Z} \overset{\text{inclusion}}{\hookrightarrow} \mathbb{R}$ is Q.I. $\quad f(x) = x \quad K = 1.$

(1) $N_1(\mathbb{Z}) = \mathbb{R}$

(2) $d(x_1, x_2) - 1 \leq d(x_1, x_2) = d(x_1, x_2) \leq 1\, d(x_1, x_2) + 1$

Lemma (X, d) metric space $R \subset X$ subset s.t. $N_K(R) = X$

$\Rightarrow$ inclusion $i: R \longrightarrow X$ is a Q.I.

Eg 3 $\mathbb{Z}^n \subset \mathbb{R}^n$ Q.I. (discretize it)

Eg 4. $\mathbb{R}^n$ Not Q.I. to $\mathbb{R}^m$ $n \neq m$.

Easy: F, G Quasi-Isometric embeddings $\Rightarrow$ $F \circ G$ Quasi-Iso embeddings

Eg 5 Geodesic in (M^n, d) QIE (minimizing)

Our main interests Quasi-isometric classifications of groups

Geometry of finitely generated groups

G a group (e.g. $\mathbb{Z}$, $SL(2, \mathbb{Z})$) $S = \{s_1, \cdots, s_n\}$ a finite set of

generators for G s.t. $x \in S \Rightarrow x^{-1} \in S$ + $id \notin S$

Eg $G = \mathbb{Z}$ $\quad S = \{+1, -1\}$

$G = \mathbb{Z}^2$ $\quad S = \{\pm(1,0), \pm(0,1)\}$

$G = SL(2.\mathbb{Z})$ $\quad S = \left\{ \begin{bmatrix} 1 & \pm 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ \pm 1 & 1 \end{bmatrix} \right\}$ (Homework), $G = \underset{\alpha}{\mathbb{Z}} * \underset{\beta}{\mathbb{Z}}$ $\quad S = \{\alpha^{\pm 1}, \beta^{\pm 1}\}$

Fix (G, S), $\forall$ $w \in G - \{id\}$, define

$$|w| \triangleq |w|_S = \min\{k \mid w = x_1 \cdots x_k,\ x_i \in S\}$$

$$|id| = 0$$

Call the word length of w in S

Lemma $|w_1 \cdot w_2| \le |w_1| + |w_2|$ + $|w^{-1}| = |w|$

pf $|w^{-1}| = |w|$ due to $x \in S \Leftrightarrow x^{-1} \in S$ so $w = x_1 \cdots x_k \Leftrightarrow w^{-1} = x_k^{-1} \cdots x_1^{-1}$

Next. say $|w_i| = n_i \Rightarrow$ $w_1 = x_1 \cdots x_{n_1}$, $w_2 = y_1 \cdots y_{n_2}$, $x_i\ y_j \in S$

$\Rightarrow \quad w_1 \cdot w_2 = x_1 \cdots x_{n_1} y_1 \cdots y_{n_2}$

so $|w_1 \cdot w_2| \le n_1 + n_2 = |w_1| + |w_2|$. □

Def The word metric $d_S = d$ on G w.r.t. the symmetric generating set S:

$$d(x, y) = |x^{-1} \cdot y| = |y^{-1} \cdot x| = d(y, x)$$

It is obviously a metric: $d(x,y) \ge 0$ w/ equality iff $x = y$; $d(x,y) = d(y,x)$ and

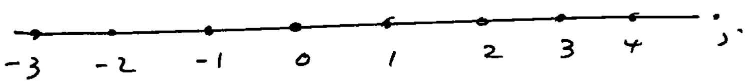
$$d(x,y) + d(y,z) = |x^{-1}y| + |y^{-1}z| \ge |x^{-1}y \cdot y^{-1}z| = |x^{-1}z| = d(x,z).$$

Note. the Cayley graph of G w.r.t. S:

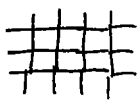
A graph s.t.: vertices $= G$.
edges $x, y \in G \Leftrightarrow x^{-1}y \in S$

Eg $G = \mathbb{Z}$, $S = \{\pm 1\}$ $\Rightarrow$ $d(x,y) = |x - y|$

Cayley graph



$G = \mathbb{Z}^2$ $S = \{\pm(0,1), \pm(1,0)\}$



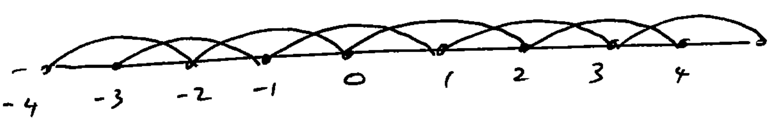
$d((a,b),(c,d)) = |a-c| + |b-d|$

L^1-norm

Cayley graph:

Eg $G = \mathbb{Z}$ $S = \{\pm 1, \pm 2\}$

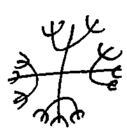
Cayley graph



$d(x,y) =$

$|2n| = |n|$
$|2n+1| = |n| + 1$ $\quad n \ge 0$

Eg $G = \mathbb{Z} * \mathbb{Z}$, $S = \{\alpha^{\pm 1}, \beta^{\pm 1}\}$ α, β generators



universal 4-valent tree

Corollary $\varphi : H \hookrightarrow G$ finite index injective group homomorphism

$\Rightarrow$ φ is Q.I. (ie $H < G$ finite index $\Rightarrow$ $H \overset{Q.I.}{\sim} G$)

Proof We already know it is Q.I. Embedding

(1) ~~$d(\varphi(H))$~~ $N_K(\varphi(H)) = G$

Why: $G/\varphi(H)$ finite $\Rightarrow$ $\forall$ $g \in G$ $\exists$ $h \in \{h_1, \ldots, h_k\}$ s.t.

$g h^{-1} \in \varphi(H)$

$\Rightarrow$ $g \cdot x^{-1} \in H$ & $|g \cdot x^{-1}| \leq K$ some fixed K

$= \max \{ |h_i| \}$.

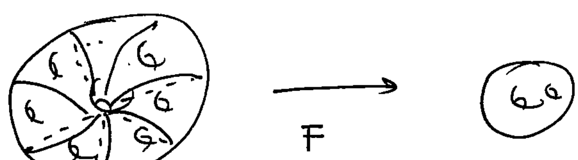
(2). the rest are the SAME

Eg $\Gamma_g = \pi_1(\Sigma_g)$ Σ_g closed surface of genus $g \geq 1$.

Then for $g \geq 2$ $\Gamma_g \underset{Q.I.}{\sim} \Gamma_2$

Indeed $\Gamma_g \hookrightarrow \Gamma_2$ is a finite index subgroup

due to the finite covering map



finite degree covering map

$\Rightarrow$ $F_* : \pi_1(\Sigma_g) \longrightarrow \pi_1(\Sigma_2)$ finite index injective homo.

Q. Why Q.I embedding ?

HW show that $\mathbb{Z} \oplus \mathbb{Z}$ Not Q.I. to $\pi_1(\Sigma_2)$.

HW F_2 Q.I F_n $n \geq 3$.

$\forall\ w \in G,\ \exists!\ w = \alpha^{n_1}\beta^{m_1}\alpha^{n_2}\beta^{m_2}\cdots\alpha^{n_k}\beta^{m_k}$ where $m_1, n_2, m_2, \cdots n_k \neq 0$

$|w| = \Sigma(|n_i| + |m_i|)$.

Prop. Suppose T is another symmetric finite generating set for the group G, then $\exists\ K > 0$ s.t $\forall\ w \in G$

$$|w|_T \le k\,|w|_S \quad + \quad |w|_S \le k\,|w|_T.$$

Proof Let $K = \max\{\, |t_i|_S \mid t_i \in T \,\}$

Suppose $|w|_T = n$ s.t

$w = x_1 \cdots x_n \quad x_i \in T$

Then $|w|_S = |x_1 \cdots x_n|_S \le \sum_{i=1}^{n} |x_i|_S \le K \cdot n = K \cdot |w|_T$. □

Corollary (a) The identity map $id : (G, T) \to (G, S)$ $(G, d_T) \to (G, d_S)$

is a quasi-isometry., i.e. the quasi-isometry class of G is independent of the choices of the generating set.

pf $d_T(x, y) = |x^{-1}y|_T \le k|x^{-1}y|_S = k\, d_S(x, y)$.

(In fact Lipschitz) □

Corollary (b). If $\varphi : G \to H$ is ~~a~~ ~~quasi-isometry~~ group ~~homomorphism~~ isomorphism, then φ is a quasi-isometri ~~embedding. In particular if φ is a group isomorphism then φ is a Q.I.~~

pf (b) The same $|\varphi(w)|_T \le K|w|_S$ the same proof.

$K = \max\{\, |\varphi(s)|_T \mid s \in S \,\}$. □

Goal Classify finitely generated groups up to quasi-isometry.

L3. Basic Large Scale Geometry

Thm (Milnor–Svarc) Let (M, g) be a closed connected Riemannian mfd w/ universal cover $(\tilde{M}, \tilde{g})$ s.t $G = \pi_1(M)$ acts isometrically on $\tilde{M}$ as the deck transformation group for $\pi : \tilde{M} \to M$. (ie $\pi(x) = \pi(y)$ iff $y = gx$ + G acts properly freely discontinuously.)
Then $\forall\, b \in \tilde{M}$, $\Phi : G \longrightarrow \tilde{M} : g \longmapsto g(b)$ is a quasi-isometry.

Conclusion:
$$\pi_1(M) \underset{Q.I}{\sim} \tilde{M}$$

Proof Choose a cpt connected fundamental domain Ω for $\pi : \tilde{M} \to M$ s.t

(1) $$\tilde{M} = \bigcup_{\gamma \in G} \gamma(\Omega)$$

(2) $$\gamma(\mathring{\Omega}) \cap \mathring{\Omega} \neq \phi \implies \gamma = id$$

Let $S = \{ \gamma \in G \mid \gamma(\Omega) \cap \Omega \neq \phi \}$. Evidently $|S| < +\infty$ + $x \in S \Leftrightarrow x^{-1} \in S$.

If $K_1 = \operatorname{diam}(M) \Rightarrow \tilde{M} = N_{K_1}(\Phi(G))$: $\forall\, x \in \tilde{M}$ is within distance K of $\Phi(G)$

We will show that S generates G and Φ satisfies

$$\frac{1}{K} d(\gamma_1, \gamma_2) - K \leq d(\Phi(\gamma_1), \Phi(\gamma_2)) \leq K d(\gamma_1, \gamma_2) + K \qquad (1)$$

as follows.

First that that $d(\gamma x_1, \gamma x_2) = d(x_1, x_2)$ since G acts isometrically, we may assume in (1) that $\gamma_2 = id$:

$$d(\Phi(\gamma_1), \Phi(\gamma_2)) = d(\gamma_1 b, \gamma_2 b) = d(b, \gamma_1^{-1}\gamma_2 b)$$
$$\text{Also } d(\gamma_1, \gamma_2) = d(1, \gamma_1^{-1}\gamma_2) = d(id, \gamma_1^{-1}\gamma_2 b)$$

Next: We will show: S generates G + $d(\Phi(id), \Phi(g)) \geq \frac{1}{K} d(1, g) - K$
$$d(b, g(b)) \geq \frac{1}{K} |g| - K$$
as follows: Let $\delta = \frac{1}{2} \min \{ d(\Omega, \gamma(\Omega)) \mid \gamma \in \Omega \text{ st } \Omega \cap \gamma(\Omega) = \phi \}$

So $\underline{\hat{f}(\gamma(a)) = f_*(\gamma) \cdot \hat{f}(a)}$. ☐

Eg. Homework. Show directly that $F_2 \underset{QI}{\longrightarrow} F_3$ $F_3 \to F_2$ QI.

by constructing an explicit group homomorphism.

Now $\tilde{f}\circ\gamma$ is another lift of $f\circ\pi_1$

$$\begin{array}{ccc} (\tilde{M}, a) & \xrightarrow{\tilde{f}\circ\gamma} & (\tilde{N}, a'') \\ & \searrow^{f\circ\pi_1} & \downarrow \pi_2 \\ & & N \end{array} \qquad \pi_2\,\tilde{f}\circ\gamma = f\,\pi_1\circ\gamma = f\pi_1$$

s.t. $\quad \tilde{f}\circ\gamma(a) = \underline{f_*(\gamma)\tilde{f}(a)} \quad \Rightarrow$ both $\tilde{f}\circ\gamma$ and $f_*(\gamma)\tilde{f}$ are lifts of $f\circ\pi_1$ w/ the identical action on a

They are equal.

Using the lemma, we see that the diagram commutes

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\tilde{f}} & \tilde{N} \\ \Phi_1 \uparrow & & \uparrow \Phi_2 \\ \pi_1(M) & \xrightarrow{f_*} & \pi_1(N) \end{array}$$

i.e. $\quad \forall\ \gamma\in\pi_1(M) \qquad \tilde{f}\gamma(a) = f_*(\gamma)(\tilde{f}(a))$

Now all Φ_1, Φ_2, f_* are Q.I $\Rightarrow \tilde{f}$ is Q.I

Note $\tilde{f}$ is a homeomorphism:

Let f^{-1} be the inverse of f and $\tilde{g}$ be a lifting of f^{-1} from $(\tilde{N}, a') \to (\tilde{M}, a)$

$$\begin{array}{ccccc} (\tilde{M}, a) & \xrightarrow{\tilde{f}} & (\tilde{N}, a') & \xrightarrow{\tilde{g}} & (\tilde{M}, a) \\ \pi_1 \downarrow & & \downarrow & & \downarrow \\ (M, b) & \xrightarrow{f} & (N, b') & \xrightarrow{f^{-1}} & (M, b) \end{array} \quad \Longrightarrow \quad \begin{array}{ccc} (\tilde{M}, a) & \xrightarrow{\tilde{g}\circ\tilde{f}} & (\tilde{M}, a) \\ & \searrow^{\pi_1} & \downarrow \pi_1 \\ & & (M, b) \end{array}$$

$\Rightarrow$ uniques of lifting $\Rightarrow \tilde{g}\circ\tilde{f} = id$

the SAME $\Rightarrow \tilde{f}\circ\tilde{g} = id$

$\square$

L2 Gromov Hyperbolicity and Hyperbolic Groups

Luo, Feng

Length spaces (X, d) metric space, $\gamma: [a, b] \to X$ continuous path.

the length $L(\gamma)$ of γ is:

$$L(\gamma) = \sup \left\{ \sum_{i=0}^{n} d(\gamma(t_i), \gamma(t_{i+1})) \,\middle|\, a = t_0 < \cdots < t_{n+1} = b \right\} \geq d(\gamma(a), \gamma(b))$$

A **Length space** is a metric space s.t. (X, d) st

$\forall\, p, q \in X \quad d(p, q) = \inf \{ L(\gamma) \mid \gamma: [0,1] \to X \text{ path } \gamma(0) = p, \gamma(1) = q \}$

Eg Connected Riemannian mfd, polyhedral spaces, connected graphs

Def A geodesic $\alpha: [a, b] \to X$ (X, d) metric is a path s.t.

$\forall\, t < t' \in [a, b] \quad d(\alpha(t), \alpha(t')) = \text{length}(\alpha|_{[t, t']})$.

Def A geodesic space (X, d) = length space s.t. $\forall\, (p, q) \in X$

$d(p, q) = \text{length}(\alpha)$ α a geod. from p to q.
$\leq \text{length}(\gamma)$ $\forall$ path γ from p to q

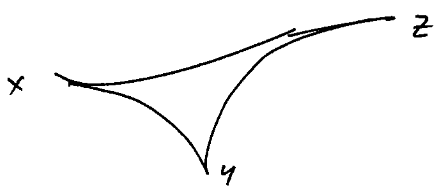
Eg Complete Riemannian manifolds (connected)

Compact connected polyhedra

G group w/ finite symmetric generating set S, Cayley graph G_S is a length space where each edge has length $= 1$

Notation In a geodesic space (X, d), $\forall\, p, q \in X$ $[p, q]$ denotes a geodesic (shortest path) from p to q. It may not be unique (e.g. S^2).

Def (Gromov) A geodesic space (X, d) is hyperbolic (or δ-hyperbolic) if $\exists\, \delta \geq 0$ s.t. $\forall\, x, y, z \in X$



$[y, z] \subset N_\delta([x, y] \cup [y, z])$

Eg. A tree is 0-hyperbolic $\delta = 0$. $\Rightarrow$ F_n $(n \geq 2)$ free group is hyperbolic.

Eg $\mathbb{R}^2$ is Not hyperbolic, $\mathbb{R}^n$ not

Eg Every cpt space is δ-hyperbolic

Eg $\mathbb{H}^n$ $(n \geq 2)$ is δ-hyperbolic ($\Leftarrow$ $\mathbb{H}^2$ δ-hyperbolic)

We need a quick introduction to $\mathbb{H}^n$ Here.

I will focus on $\mathbb{H}^3$. See extra

Eg Universal cover of a closed sectional curvature negative Riemannian mfd.

Goal: $F: \mathbb{H}^n \to \mathbb{H}^n$ a quasi-isometric homeomorphism, then

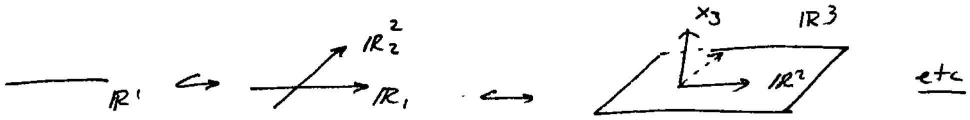
F extends continuously to the compact closure $\hat{F}: \overline{\mathbb{H}^n} \to \overline{\mathbb{H}^n}$

as a homeomorphism $\overline{\mathbb{H}^n} = \mathbb{H}^n \cup S^{n-1}_{\infty}$ sphere at infinity.

Introduction of Hyperbolic Geometry from Inversions

Convention $\mathbb{R}^n = \{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid x_i \in \mathbb{R}\}$ $\quad \bar{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$

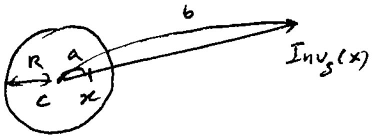
We always consider, identify $\mathbb{R}^n \hookrightarrow \mathbb{R}^{n+1}$ by $\mathbb{R}^n = \{(x_1, \cdots, x_n, 0) \mid x_i \in \mathbb{R}\}$



A sphere $S = \{x \mid |x-c| = R\} \subset \mathbb{R}^n$ produces an inversion

$$Inv_S(x) = c + \frac{x-c}{|x-c|^2} \cdot R^2 \qquad c \leftrightarrow \infty$$

$a \cdot b = R^2$



Reflection about $(x-a)\cdot n = 0 \quad |n| = 1$

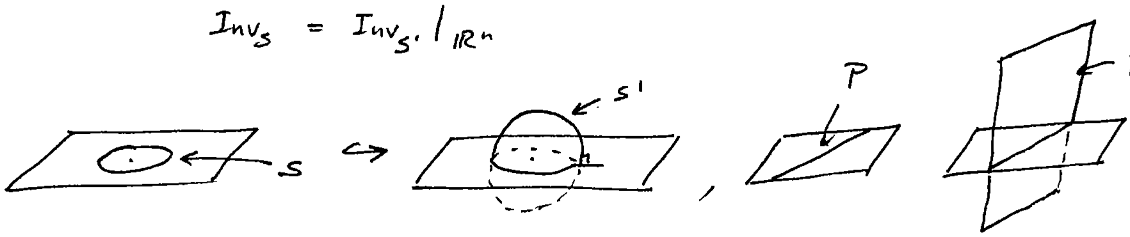
$x \mapsto x - 2((x-a)\cdot n)\,n$

Eg $S = \{|x| = 1\}$. Then $Inv_S(x) = \frac{x}{\|x\|^2}$. For $n=2$ $\quad Inv_{S^{n-1}}(z) = \frac{1}{\bar{z}}$

Lemma For $S = \{x \in \mathbb{R}^n \mid |x-c| = R\}$, let $S' = \{x \in \mathbb{R}^{n+1} \mid |x - (c,0)| = R\}$

i.e. $S' \cap \mathbb{R}^n = S$ and $S' \perp \mathbb{R}^n$. Then

$$Inv_S = Inv_{S'}|_{\mathbb{R}^n}$$



Def A Möbius transformation of $\mathbb{R}^n$ = a composition of inversions and reflections

Let $\text{Möb}(\mathbb{R}^n)$ = the group of all Möbius transformations.

Corollary: There exists a natural injective homomorphism $\text{Möb}(\mathbb{R}^n) \hookrightarrow \text{Möb}(\mathbb{R}^{n+1})$

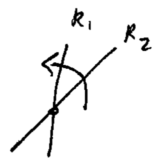
by $Inv_S \longleftrightarrow Inv_{S'}$ $\quad Ref_P \longmapsto Ref_{P'}$

i.e. Each Möbius $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ can be naturally extends to $\tilde{\varphi} : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$

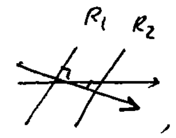
$\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R} \qquad \mathbb{R}^n \times \mathbb{R}$.

Eg $Iso(\mathbb{R}^n)$ = generated by reflections about planes

$n=2$



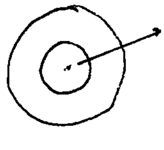
rotation $= R_1 \circ R_2$



translation

$z \mapsto \lambda z \quad \lambda > 0$ scaling

$Inv_{S_1} \circ Inv_{S_2}$



$\Rightarrow \quad Iso(\mathbb{R}^n) \subset \text{Möb}(\mathbb{R}^n)$

Also: Any $Inv_S = \varphi \circ Inv_{S^{n-1}} \circ \varphi^{-1}$, $\quad \varphi(x) = \lambda A(x) + b \quad A \in O(n) \quad \lambda > 0$

$b \in \mathbb{R}^n$

Corollary: Möbius($\mathbb{R}^n$) generated by Iso($\mathbb{R}^n$) + $\frac{x}{|x|^2}$ + $x \mapsto \lambda x$.

Q. Is it true that Möb($\mathbb{R}^n$) generated by Iso($\mathbb{R}^n$) and $\frac{x}{|x|^2}$? (Yes)

Eg (Homework) Möb($\mathbb{C}$) consists of $\frac{az+b}{cz+d}$ and $\frac{a\bar{z}+b}{c\bar{z}+d}$, $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2,\mathbb{C})$

Lemma. For a sphere $S \subset \mathbb{R}^n$ $Inv_S \circ Inv_S = id$ and Inv_S sends the set of $(n-1)$-dim spheres and $(n-1)$-planes to itself.

Pf Check it for $I = \frac{x}{|x|^2}$

The equations of $(n-1)$-spheres + planes are

$$A\, x\cdot x + B\cdot x + C = 0 \qquad A, C \in \mathbb{R} \quad B \in \mathbb{R}^n$$

Let $x = \frac{y}{|y|^2}$ $\Rightarrow$

$$A\,\frac{y\cdot y}{|y|^4} + \frac{B\cdot y}{|y|^2} + C = 0$$

Multiply by $|y|^2$ $\Rightarrow$ $A + B\cdot y + C\, y\cdot y = 0$ Another eq. of sphere + plane.

Corollary. Let $\mathcal{C}$ be the set of all circles and lines in $\mathbb{R}^n$. Then $\forall \varphi \in$ Möb($\mathbb{R}^n$)

$$\varphi\cdot\mathcal{C} = \mathcal{C}$$

Pf circle = intersection of $(n-1)$-spheres.
line = ... $(n-1)$-planes

It suffices to check it for $x/|x|^2$. True by the lemma. □

The upper half space model of hyperbolic space $\mathbb{H}^n$.

Let $U^n = \{(x_1,\dots,x_{n-1},x_n) \in \mathbb{R}^n \,/\, x_n > 0\}$ w/ Riemannian metric

$$ds^2 = \frac{dx_1^2 + \cdots + dx_n^2}{x_n^2}$$

Propositions (Isometry of U^n)

(1) $\frac{x}{|x|^2}$ preserves ds^2

(2) $\lambda \in \mathbb{R}_{>0}$ λx preserves ds^2

(3) $A \in O(n-1)$, $b \in \mathbb{R}^{n-1}$, then $x = (y, t) \in \mathbb{R}^{n-1} \times \mathbb{R}_{>0} \longmapsto (Ay+b, t)$ preserves ds

pf (2), (3) obvious

To see (1), let $y = (y_1, \cdots, y_n) = \frac{x}{|x|^2}$ $\quad y_i = \frac{x_i}{|x|^2}$, $y_n^2 = \frac{x_n^2}{|x|^4}$

$$\Rightarrow dy_i = \frac{dx_i}{|x|^2} + x_i \, d\left(\frac{1}{|x|^2}\right)$$

$$= \frac{dx_i}{|x|^2} - \frac{2x_i}{|x|^4} \sum_j x_j dx_j$$

So $$dy_i^2 = \frac{dx_i^2}{|x|^4} - \frac{4x_i dx_i}{|x|^6} \sum_j x_j dx_j + \frac{4x_i^2}{|x|^8} \sum_{j,k} x_j x_k dx_j dx_k$$

So
$$\sum_i dy_i^2 = \frac{\sum dx_i^2}{|x|^4} - \frac{4}{|x|^6} \sum_{i,j} x_i x_j dx_i dx_j + \frac{4 \sum x_i^2}{|x|^8} \sum_{j,k} x_j x_k dx_j dx_k$$

$$= \frac{\sum dx_i^2}{|x|^4}$$

Use $y_n^2 = x_n^2 / |x|^4$ $\Rightarrow$

$$\frac{\sum_i dy_i^2}{y_n^2} = \frac{\sum_i dx_i^2}{x_n^2}$$

The above calculate works for $x_n < 0$.

Corollary $\frac{x}{|x|^2}$ preserves angles in $\mathbb{R}^n$. Thus $\varphi \in \text{Möb}(\mathbb{R}^n)$ preserves angles in $\mathbb{R}^n$.

Pf ds^2 and $dx_1^2 + \cdots + dx_n^2$ define the same notion of angles

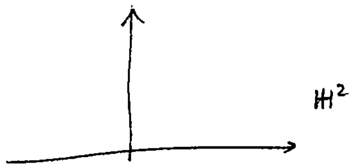
Corollary The extension of $\varphi \in \text{Möb}(\mathbb{R}^{n-1})$ to $\tilde{\varphi} \in \text{Möb}(\mathbb{R}^n)$ preserves ds^2 on U^n

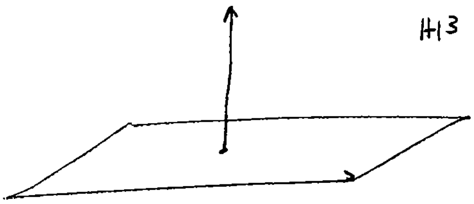
i.e. $\text{Möb}(\mathbb{R}^{n-1}) \subset \text{Iso}(U^n)$

action at infinity of U^n (It turns out to be equal)

Eg $\text{Iso}(U^2) \supseteq PSL(2,\mathbb{R}) = \{ z \mapsto \frac{az+b}{cz+d} \} \cup \{ \frac{a\bar{z}+b}{c\bar{z}+d} \}$

$\text{Iso}(U^3) \supseteq PSL(2,\mathbb{C})$.





Prop. The x_n-axis $l = \{(0,\dots,0,t) \mid t > 0\} \subset U^n$ is a geodesic so that $d(\underset{p}{(0,\dots,0,a)}, \underset{q}{(0,\dots,0,b)}) = |\ln \frac{b}{a}|$.

pf Let $\gamma(t) = (x_1(t), \dots, x_n(t)) : [0,1] \longrightarrow U^n$ be a smooth path from p to q. $b/a > 1$

$$\gamma'(t) = (x_1', \dots, x_n')$$

$$|\gamma'(t)|_{ds} = \sqrt{\sum x_i'(t)^2} \Big/ x_n(t)$$

So the length

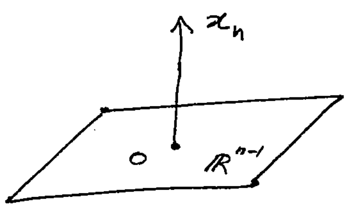
$$L(\gamma) = \int_0^1 |\gamma'(t)|_{ds}\, dt = \int_0^1 \frac{\sqrt{\sum x_i'(t)^2}}{x_n(t)}\, dt$$

$$\underset{x_1' = \dots = x_{n-1}' = 0}{\geq} \int_0^1 \frac{\sqrt{x_n'(t)^2}}{x_n(t)}\, dt$$

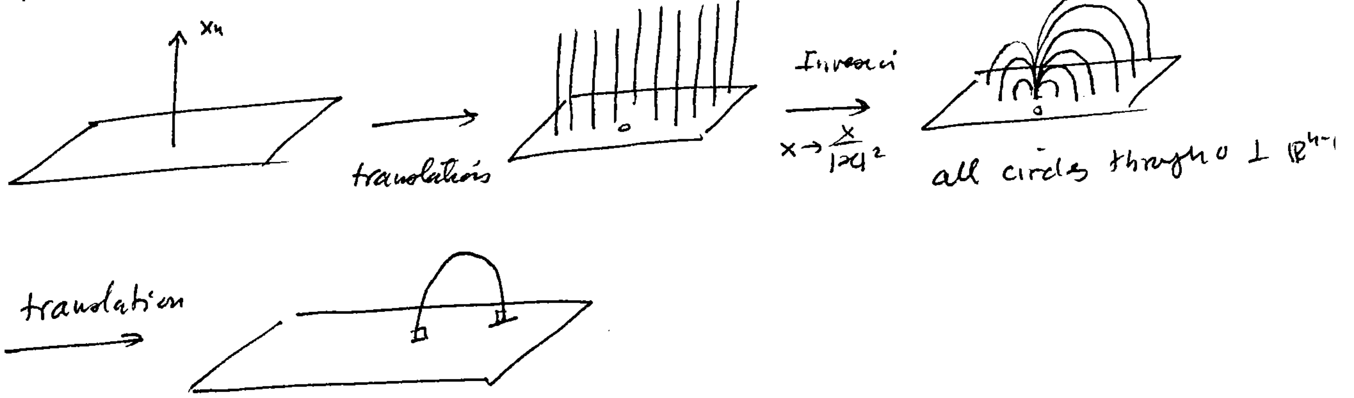
$$\underset{x_n' \geq 0}{\geq} \left| \int_0^1 \frac{x_n'(t)}{x_n(t)}\, dt \right|$$

$$= \left| \ln x_n(t) \Big|_0^1 \right| = \left| \ln \frac{b}{a} \right|$$

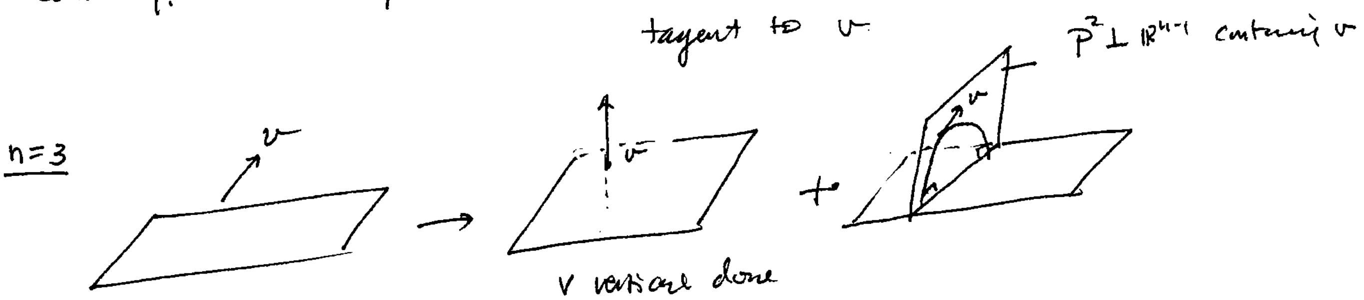
Equality holds iff $x_1' = \dots = x_{n-1}' = 0$ + $x_n' \geq 0$, i.e. γ lies in the l monotonically



All geodesics in U^n are: vertical lines ($\perp \mathbb{R}^{n-1}\times 0$) and semi-circles ($\perp \mathbb{R}^{n-1}\times 0$)
perpendicular to $\mathbb{R}^{n-1}\times 0$



Conversely. $\forall\ v \in T_p U^n - \{0\}$, $\exists !$ line L or semi-circle $C \perp \mathbb{R}^{n-1}$
tangent to v



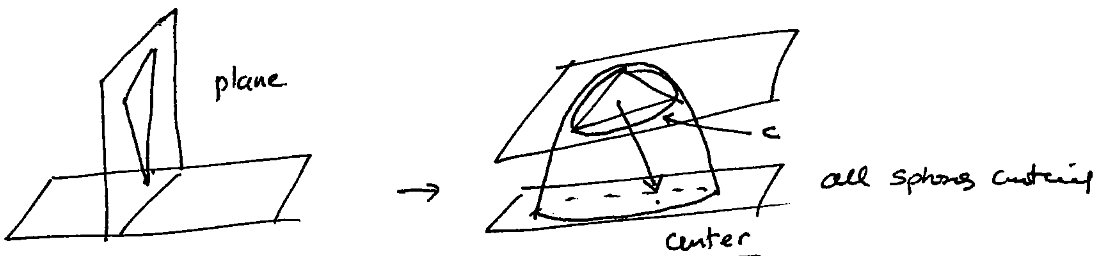
$\forall\ p \neq q\ \ \exists !$ geodesic

<u>Homework</u> $\quad Iso(U^n) \subset Möb(\mathbb{R}^{n-1})$

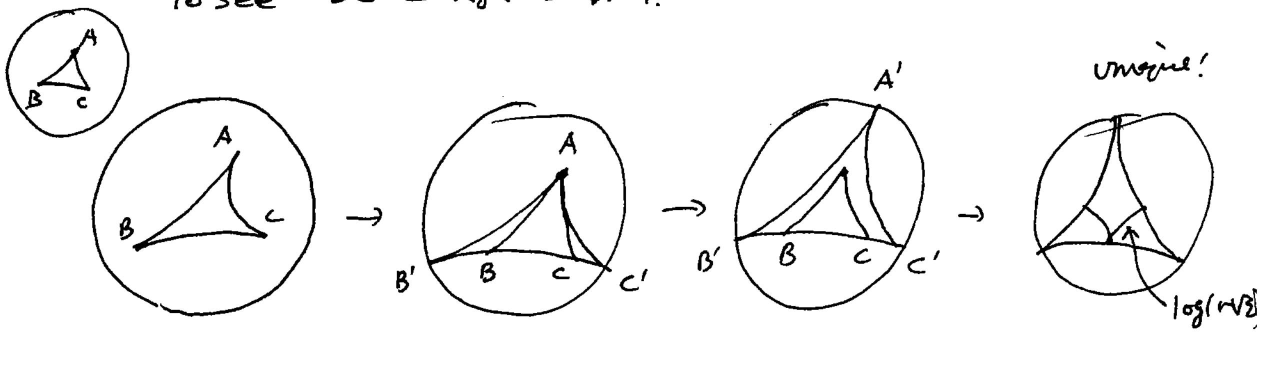
<u>Corollary</u> $\mathbb{H}^n$ is Gromov δ-hyperbolic for $\delta = \ln(1+\sqrt{2})$

<u>pf</u>. It suffices to show it for $\mathbb{H}^2$ (Note: $\forall\ p \neq q \in \mathbb{H}^n\ \exists !$ geodesic from p to q)

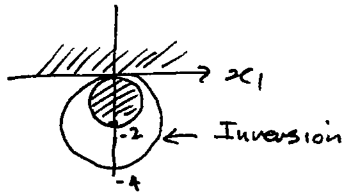
$\forall$ triangle p, q, r $\exists !$ $\mathbb{H}^2 \subset \mathbb{H}^n$ containing p, q, r



So $\qquad$ To see $BC \subset N_\delta(AB \cup AC)$.



The ball model B^n of $\mathbb{H}^n$



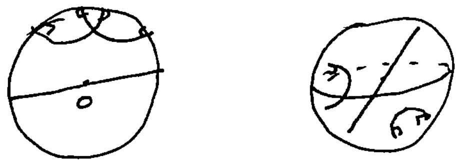
Let $S = \{ x \in \mathbb{R}^n \mid |x - (0, \cdots 0, -2)| = 2 \}$

Then $\mathrm{Inv}_S(U^n) = \{ x \mid |x - (0, \cdots 0, -\frac{1}{2})| < \frac{1}{2} \}$ a unit ball

After a translation by $x \mapsto x + (0, 0 \ldots 0, \frac{1}{2})$, it is

$B^n = \{ x \in \mathbb{R}^n \mid |x| < 1 \}$ — the Poincare model of $\mathbb{H}^n$

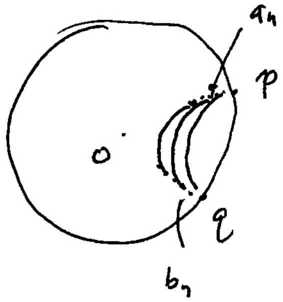
- the metric is $\dfrac{4|dx|^2}{(1-|x|^2)^2}$ conformal to $\mathbb{R}^n$
- geodesics lines or semi circles perpendicular to $\partial \bar{B}^n = S^{n-1}$



- Isometries = Compositions of Inv_S's + reflection P st
  $S \perp S^{n-1}$ $P \perp S^{n-1}$

$\Rightarrow$ Now we can see the Gromov hyperbolicity easily.

Corollary. Suppose $a_n \to p$ $b_n \to q$ $a_n, b_n \in B^n$ $p \neq q \in S^{n-1}$ $(\bar{B}^n)$



$\Rightarrow$ $d(0, [a_n, b_n])$ is bounded.

(In fact $\forall\, r \in B^n$

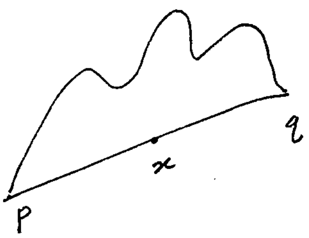
$d(r, [a_n, b_n])$ is bounded.)

Our goal:

Thm If $f: B^n \to B^n$ is quasi-isometric homeomorphism, then f extends continuously to a homeomorphism $\tilde{f}: \bar{B}^n \to \bar{B}^n$.

Geodesics in length spaces

Key lemma (X, d) δ-hyperbolic, $\gamma : I = [a,b] \longrightarrow X$ a rectifiable curve (i.e., $L(\gamma) < +\infty$) from p to q. Then $\forall\ x \in [p, q]$



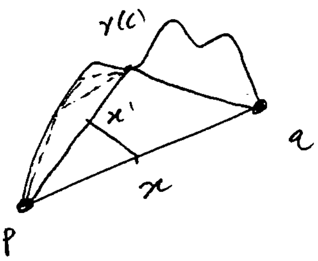
$$d(x, \gamma(I)) \leq 1 + \delta \ln_2 (L(\gamma)) + \delta \quad \text{if } L(\gamma) \geq 1 \text{ or}$$
$$\leq 1 \quad \text{if } L(\gamma) \leq 1$$

Pf. Clearly it holds for $\underline{L(\gamma) \leq 1}$: $d(x, \gamma(I)) \leq d(x, p) \leq d(p, q) \leq 1.$

Induction on $[\ln_2 L(\gamma)]$.

For $L(\gamma) > 1$, Let $c \in [a, b]$ be the point s.t $L(\gamma|_{[a,c]}) = \frac{1}{2} L(\gamma)$

Say I_1, I_2 are $[a, c]$ + $[c, b]$



Hyperbolicity $\Rightarrow$ $\exists\ x' \in [p, \gamma(c)] \cup [\gamma(c), q]$ say $[p, \gamma(c)]$

s.t $d(x, x') \leq \delta$

Now
$$d(x, \gamma(I)) \leq d(x, x') + d(x', \gamma(I_1)) \leq d(x, x') + d(x', \gamma(I_1)) \leq$$
$$\underset{\text{Lemma}}{\overset{\text{induction}}{\leq}} \delta + \left[1 + \delta \ln_2 (L(\gamma)/2)\right] + \delta$$
$$= 1 + \delta \ln_2 L(\gamma) + \delta$$

$\square$

Def A quasi-geodesic $\alpha : I \longrightarrow (X, d)$ is a map s.t $\exists\ K$

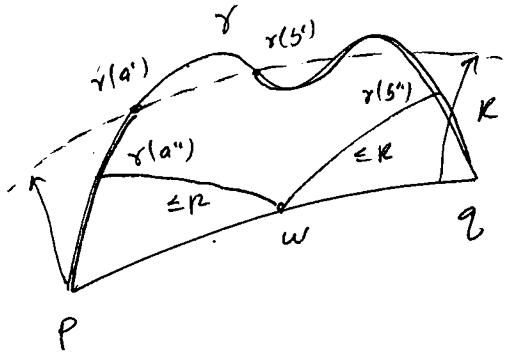
$$\frac{1}{K}|s-t| - K \leq d(\alpha(s), \alpha(t)) \leq K|s-t| + K$$

(i.e. a K-Q.I of interval into X).

Eg $\gamma : I \to (X, d)$ geodesic, $F : (X, d) \to (Y, d')$ K-Q.I

$\Rightarrow$ $F \circ \gamma : I \to Y$ is a K-quasi-geodesic.

To see (2). Take R produced in (1), $\exists R' = R'(\delta, K)$ will do



Consider a maximal interval $[a', b'] \subset [a, b]$

s.t. $\gamma([a', b'])$ lies outside $N_R([p, q])$.

i.e. $d(\gamma[a', b'], [p, q]) \geq R$. Want: $\text{Length}(\gamma[a', b']) \leq C(K, \delta)$.

But $[p, q] \subset N_R(\gamma[a, b])$

$$\Rightarrow [p, q] \subset \underbrace{N_R(\gamma([a, a']))} \cup \underbrace{N_R((\gamma[b', b]))}$$

Both closed $\quad$ $[p, q]$ connected

$$\Rightarrow w \in [p, q] \cap N_R(\gamma[a, a']) \cap N_R(\gamma[b', b]))$$

$$\Rightarrow \exists \quad a'' \in [a, a'] \quad b'' \in [b', b] \quad \text{s.t.}$$

$$d(w, \gamma(a'')) \leq R$$

$$d(w, \gamma(b'')) \leq R$$

Now

$$L(\gamma|_{[a', b']}) \leq L(\gamma|_{[a'', b'']}) \leq C_1(2K)\, d(\gamma(a''), \gamma(b'')) + C_2(2K)$$

(γ quasi; ?)

$$\leq 2K^2 \left(d(\gamma(a''), w) + d(w, \gamma(b'')) \right) + \cdots$$

$$= 2RK^2 + \cdots \text{ polynomial in } K$$

So done

Quasi-Geodesics

Corollary. (X, d) δ-hyperbolic $F: (X,d) \to (X,d)$ K-quasi-isometry. then

$\exists\ R = R(\delta, k)$ s.t $\forall\ p, q \in X$

$$F([p,q]) \subset N_R([F(p), F(q)]) \quad + \quad [F(p), F(q)] \subset N_R(F([p,q])).$$

Proof $F([p,q])$ is a K-quasi-geodesic $\Rightarrow$ Done $\square$

Thm. If $F: \mathbb{H}^n \to \mathbb{H}^n$ $(n \geq 2)$ is a K-quasi-isometric homeomorphism, then F extends continuously to a homeomorphism $\hat{F}: \overline{\mathbb{H}^n} \to \overline{\mathbb{H}^n}$.

Pf Need a lemma

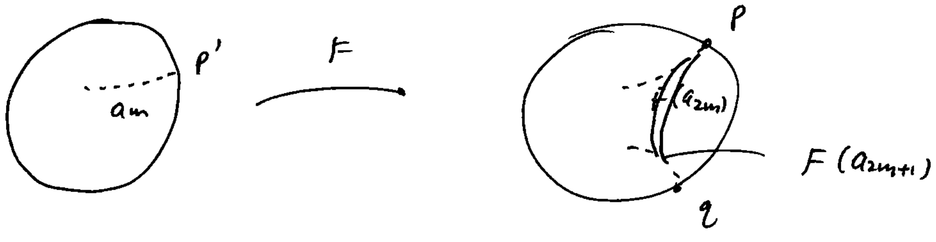
Lemma A a dense subset of a metric space (X, d) $f: (A, d|_A) \to (Y, d')$ is continuous s.t. $\forall$ sequence $\{a_n\}$ in A with $\lim_{n\to\infty} a_n = x \in X$ $\lim_{n\to\infty} f(a_n)$ exists in Y. Then f extends continuously to $\hat{f}: X \to Y$

$$\hat{f}(x) = \lim_{n\to\infty} f(a_n) \qquad \text{where } a_n \to x. \qquad (Hw)$$

Thus, it suffices to show $\forall$ if $\{a_m\} \subset \mathbb{H}^n$ s.t $a_m \to p' \in \overline{\mathbb{H}^n}$

$F(a_m)$ conv to a pt in $\overline{\mathbb{H}^n}$

If Not. $\Rightarrow$ $\exists$ $a_m \to p'$ s.t $F(a_{2m}) \to p$ $F(a_{2m+1}) \to q \neq p$



Now $d(0, [a_m, a_{m+1}]) \to +\infty$

$\Rightarrow$

F quasi-iso $d(F(0), F([a_m, a_{m+1}])) \to +\infty$

by corollary $d(F(0), [F(a_m), F(a_{m+1})]) \to +\infty$

But $d(F(0), [F(a_m), F(a_{m+1})])$ is bounded due to $p \neq q$.

A contradiction. Apply it to F^{-1} $\Rightarrow$ the extension is homeo $\square$

Third Week

Lecture 3. Gromov Norm

We need some basic algebraic topology + homology of manifolds

Luo, Feng

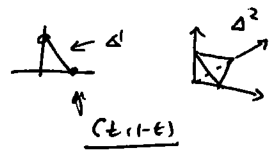
Recall singular homology theory

X top. space

$S_n(X) \triangleq S_n(X, \mathbb{R})$ real coefficient singular chains

$= \{ c = \sum_{i=1}^{k} a_i \sigma_i \mid a_i \in \mathbb{R} \quad \sigma_i : \Delta^n \to X$ continuous map

$\Delta^n = \{ (x_0, \dots x_n) \in \mathbb{R}^{n+1} \mid x_i \geq 0, \sum x_i = 1 \}$



Define $|c| = \sum |a_i|$ the L^1-norm

$H_n(X, \mathbb{R}) \triangleq \{ [c] \mid \partial c = 0, c \in S_n(X, \mathbb{R}).\ c \sim c'$ iff $c - c' = \partial f, f \in S_{n+1}(X, \mathbb{R}) \}$

Def (Gromov) The pseudo norm $\| [c] \| = \inf \{ |c'| \mid c' \sim c \}$ $\quad [c] \in H_n(X, \mathbb{R})$.

Clearly $\quad \|\lambda \alpha\| = |\lambda| \|\alpha\| \quad + \quad \|\alpha + \beta\| \leq \|\alpha\| + \|\beta\|$.

Eg 1. $X = S^1 \quad H_1(X, \mathbb{Z}) \cong \mathbb{Z} \subset H_1(X, \mathbb{R})$ the fundamental class $[S^1] = \alpha$

generator of $H_1(X, \mathbb{Z})$ is represented by $\sigma_1 : [0,1] \longrightarrow X$

$\sigma_1(t) = e^{2\pi i t}$.

so $|\sigma_1| = 1. \qquad \Rightarrow \quad \|\alpha\| \leq 1$.

But $\quad \frac{1}{n} \sigma_n \in \alpha \qquad \sigma_n(t) = e^{2\pi i n t}$

HW. show that $\frac{1}{n} \sigma_n \in \alpha$ i.e. $\frac{1}{n} \sigma_n \sim \sigma_1$

$\Rightarrow \quad \|\alpha\| \leq \frac{1}{n} \qquad \Rightarrow \| [S^1] \| = 0$

Prop $\quad \alpha, \beta \in H_n(X, \mathbb{R}) \Rightarrow$

(1) $\|\alpha + \beta\| \leq \|\alpha\| + \|\beta\|$

(2) $\|\lambda \alpha\| = |\lambda| \|\alpha\|$

(3) $f: X \to Y$ continuous $\Rightarrow \|f_*(\alpha)\| \leq \|\alpha\|$

Pf (1) (2) definition

To see (3), if $c = \sum a_i \sigma_i \in \alpha \Rightarrow f_\#(c) = \sum a_i f \circ \sigma_i \in f_*(\alpha)$

$\Rightarrow \quad \|f_*(\alpha)\| \leq \|\alpha\|$. $\square$

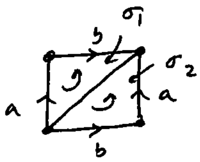
(A.S. Svarc)

Let M^n be a closed connected oriented n-mfd. Then $H_n(M,\mathbb{Z}) \cong \mathbb{Z}$

Its generator, α_M, is called the <u>fundamental class</u> of M

<u>Eg</u> 1 $\alpha_{S^1} = [\sigma_1] = [\frac{1}{n}\sigma_n]$

2. $\alpha_{S^1\times S^1} = [\sigma_1 + \sigma_2]$:



3. α_{S^n} : $\sigma : \Delta^n \to S^n$ s.t $\sigma|_{\partial\Delta^n} = N$ — North pole

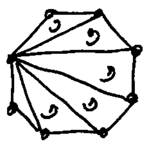
$\sigma : \Delta^n - \partial\Delta^n \to S^n - N$ <u>homeo</u>

then $\alpha_{S^n} = [\sigma + N]$ if n is even

$= [\sigma]$ if n is odd

<u>n=2</u>

4. Σ_g closed orientable surface of genus $g \geq 1$ obtained by $4g$-gon



has a triangulation w/ $4g-2$ triangles

$\alpha_{\Sigma_g} = \left[\sum_{i=1}^{4g-2} \sigma_i\right] \Rightarrow \|\alpha_{\Sigma_g}\| \leq 4g-2$

<u>Basic idea</u>: Fundamental class — n-cycle which covers every point of M exactly once.

<u>Def</u> The Gromov norm of M^n is $\|M^n\| \triangleq \|\alpha_{M^n}\|$. <u>$\alpha_{M^n} \in H_n(M,\mathbb{R})$</u>.

<u>Eg</u> $\|S^1\| = 0$

<u>Prop</u> Suppose M^n, N^n closed oriented $f: M \to N$ continuous of $\deg(f) = k$, i.e.

$f_* : H_n(M,\mathbb{Z}) \to H_n(N,\mathbb{Z})$ is $1 \longmapsto k$

Then (1) $|\deg(f)|\,\|N\| \leq \|M\|$

(2) $|\deg(f)|\,\|N\| = \|M\|$ if f is a covering map

<u>Pf</u>. Let $d = \deg(f)$

<u>Now</u> $f_*(\alpha_M) = \pm d\cdot\alpha_N$ by definition

So $|d|\,\|\alpha_N\| = \|f_*(\alpha_M)\| \leq \|\alpha_M\|$

Gromov Norm

To see (2), $\forall\, \varepsilon > 0$, $\exists\ c \in \alpha_N$ $\quad c = \sum a_i \sigma_i$ st

$\|\alpha_N\| \geq |c| - \varepsilon$.

Now each $\sigma_i : \Delta^n \to N$ can be lifted to d n-simplexes

$\tilde{\sigma}_{ij} : \Delta^n \to M$ w/ $f \circ \tilde{\sigma}_{ij} = \sigma_i$.

so $\quad \tilde{c} = \sum_{i,j} a_i \tilde{\sigma}_{ij} \in \alpha_M \quad$ (why ?)

$\|\alpha_M\| \leq |\tilde{c}| = |d| \sum |a_i| = |d||c| \leq |d| \cdot (\|\alpha_N\| + \varepsilon)$. □

Corollary If $\exists\ f : M^n \to M^n$ of $|\deg(f)| \geq 2 \Rightarrow \|M\| = 0$

Eg. $\|\alpha_{S^n}\| = 0$ since $\exists$ $\deg \geq 2$ maps

$\|\alpha_{S^1 \times S^1}\| = 0$, $\|\alpha_{S^1}\| = 0$ etc

$\|S^1 \times \cdots \times S^1\| = 0$

Question What is $\|\Sigma_g\|$ for $g \geq 2$

Prop If $g \geq 2 \Rightarrow \quad \|\Sigma_g\| = -2\,\chi(\Sigma_g) = 4g-4$.

Proof We will show first $\|\Sigma_g\| \leq 4g-4$

We need to use Geometry to prove $\|\Sigma_g\| \geq 4g-4$ (Later)

First, we know: $\forall\ h \geq 2$

$\|\Sigma_h\| \leq 4h-2$ by explicit construction.

Next. $\forall\ n \geq 1$, let $\Sigma_h \to \Sigma_g$ be an n-fold cover of Σ_g

$\chi(\Sigma_h) = n\,\chi(\Sigma_g) = n(2-2g) = 2n - 2ng$

$\chi(\Sigma_h) = 2-2h$

so $h = ng - n + 1$

Now by the proportion

$$\|\Sigma_g\| = \frac{1}{n}\|\Sigma_h\| \leq \frac{1}{n}(4h-2) = \frac{1}{n}\left[(4ng-4n+4)-2\right]$$

Let $n \to \infty$ $\Rightarrow$ $\|\Sigma_g\| \leq 4g-4$. □

Geometry. Later

Our next goal

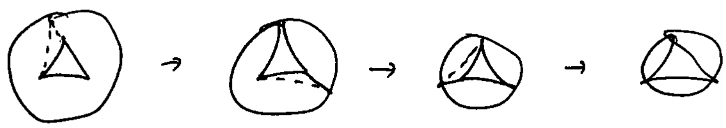
Thm (Gromov–Thurston) If M^n $(n \geq 2)$ is a closed hyperbolic n-manifold, then

$$\|M^n\| = \mathrm{vol}(M)/v_n$$

where $v_n = \sup\{\mathrm{vol}(\sigma) \mid \sigma \text{ hyperbolic } n\text{-simplex}\}$ ($v_2 = \pi$  Gauss-Bonnet)

We need to investigate the volume of hyperbolic n-simplexes. We will focus on $n=3$ for simplicity

Eg Each hyperbolic triangle is contained in an ideal triangle whose area $= \pi$

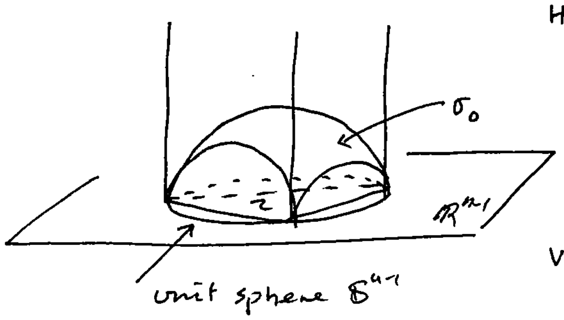


area $= \pi$ done before

Each hyperbolic tetra $\subset$ ideal tetra (SAME true in any dim $\mathbb{H}^n$).

Lemma (Thurston) $\forall n$ $v_n < +\infty$. In fact $(n-1)v_n \leq v_{n-1}$.

Pf. Consider the upper-half-space model of $\mathbb{H}^n$ s.t an ideal n-simplex σ has a vertex at ∞ whose projection to $\mathbb{R}^{n-1}$ is the simplex τ inscribed about the $|x|^2=1$. $x \in \mathbb{R}^{n-1}$



Hyperbolic volume $\dfrac{dx_1 \wedge \cdots \wedge dx_n}{x_n^n}$. $\mathbb{H}^n$

Let the bottom of σ be σ_0

For $x \in \tau$, $h(x) = \sqrt{1 - x \cdot x}$ be the height.

$$\mathrm{vol}(\sigma) = \int_\tau \left(\int_h^\infty \frac{dt}{t^n}\right) dx_1 \wedge \cdots \wedge dx_{n-1} = \frac{1}{(n-1)} \int_\tau \frac{dx_1 \wedge \cdots \wedge dx_{n-1}}{h^{(n-1)}}.$$

Now $\mathrm{vol}(\sigma_0) = \int_\tau dA$ where dA is the area form of $\mathbb{H}^{n-1} = \mathbb{S}^{n-1} \cap \mathbb{H}^n$

Since the model is conformal, $dA = i_V\left(\frac{dx_1\wedge\cdots\wedge dx_n}{x_n^n}\right)$ where V is the hyperbolic unit vectors normal to $\mathbb{S}^{n-1}$.

which can be calculated as $(h = x_n)$

$$V = h\cdot\left(\sum_{i=1}^n x_i \frac{\partial}{\partial x_i}\right) = x_n \cdot \sum_{i=1}^n x_i \frac{\partial}{\partial x_i}.$$

Thus $dA = i_V\left(\frac{dx_1\wedge\cdots\wedge dx_n}{x_n^n}\right) = \frac{1}{x_n^{n-1}}\left(\sum_{i=1}^n (-1)^i x_i\, dx_1\wedge\cdots\wedge \widehat{dx_i}\wedge\cdots\wedge dx_n\right) > \frac{dx_1\wedge\cdots\wedge dx_{n-1}}{x_n^{n-1}}$

Conclusion.

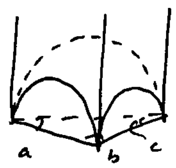
$$\mathrm{vol}(\sigma) = \frac{1}{n-1}\int_\tau \frac{dx_1\wedge\cdots\wedge dx_{n-1}}{x_n^{n-1}} < \frac{1}{n-1}\int_\tau \frac{dx_1\wedge\cdots\wedge dx_{n-1}}{x_n^{n-2}} = \frac{1}{n-1}\mathrm{vol}(\sigma_0)$$

□

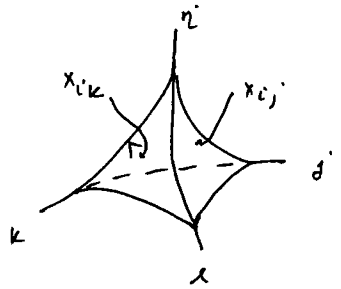
RM. One can shown (homework) for $n=3$.

Prop. The volume of an ideal hyperbolic tetra of dihedral angles a, b, c, $a+b+c=\pi$ is $\Lambda(a)+\Lambda(b)+\Lambda(c)$ where $\Lambda(t) = -\int_0^t \ln|2\sin(s)|\,ds \in C^0(\mathbb{R})$

$\Lambda(-t) = -\Lambda(t)$, $\Lambda(t+\pi) = \Lambda(t)$.



τ = inner angles a, b, c



RM Opposite edges have the same angles. for ideal tetra

Known $\forall\ i$ $x_{ij} + x_{ik} + x_{il} = \pi$ (i)

$(1)+(2)-(3)-(4)$

$$x_{12}+x_{13}+x_{14}+x_{21}+x_{23}+x_{24}-x_{31}-x_{32}-x_{34}-x_{41}-x_{42}-x_{43}=0$$

$$\Rightarrow \quad x_{12} = x_{34}.$$

Thm. (Haagerup–Munkholm). $v_n = \mathrm{vol}(\text{regular ideal } n\text{-simplex})$ + is unique

pf $n=3$: $F(x,y) = \Lambda(x)+\Lambda(y)-\Lambda(x+y)$ $\{x, y\geq 0,\ x+y\leq\pi\} = P$

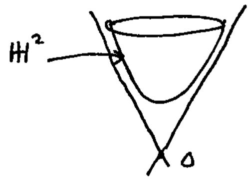
maximize it in P: $\frac{\partial F}{\partial x} = -\ln|2\sin(x)| + \ln|2\sin(x+y)| = \ln\left|\frac{\sin(x+y)}{\sin(x)}\right|$

$F|_{\partial P} = 0$ $\Rightarrow \frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = 0 \Rightarrow |\sin(x)| = |\sin(y)| = |\sin(x+y)| \Rightarrow x = y = \pi - (x+y) = \pi/3$

□

Gromov Norm + Volume, Straight Simplex

We consider the Minkowski (hyperboloid model) of $\mathbb{H}^n$



$= \{ x \in \mathbb{R}^{n+1} \mid x_{n+1} > 0, \quad x_1^2 + \cdots + x_n^2 - x_{n+1}^2 = -1 \}$ $\quad$ $n=2$

totally geodesic subspaces in $\mathbb{H}^n$ = $\mathbb{H}^n \cap L$, L linear subspace

Def A straight i-simplex in $\mathbb{H}^n$, with vertices $v_0, \cdots, v_i \in \mathbb{H}^n$:

is the composition $\sigma = \pi \circ \text{Affine} : \Delta^i \longrightarrow \mathbb{H}^n \cup$ light cone

where $\pi(x) = x / \sqrt{-\langle x, x \rangle}$ the radial projection

and $\text{Affine} : \Delta^i \longrightarrow$ Convex hull $\{v_0, \cdots v_i\}$

$(t_0, \cdots, t_i) \longmapsto \sum t_i v_i$ $\quad$ (standard affine simplex w/ vertices $v_0, \cdots, v_n$)

It is determined by $(v_0, \cdots, v_i)$.

Lemma (1) If σ is straight $\Rightarrow \partial_i \sigma$, its i-th face, is again straight.
(2) if σ is straight, + $\gamma \in \text{Iso}(\mathbb{H}^n) \Rightarrow \gamma \circ \sigma$ is straight.

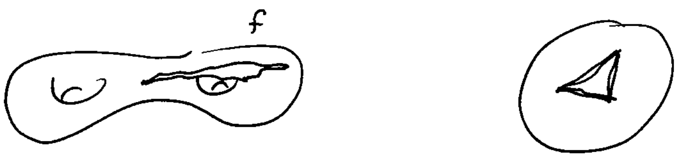
pf (1) $\partial_i \sigma = \pi(\partial_i \text{ affine})$. $\square$

(2) $\gamma \circ \pi = \pi \circ \gamma$, $\gamma \in$ linear map.

Def. A singular i-simplex $f : \Delta^i \longrightarrow$ hyperbolic manifold M is called straight if a lift $\hat{f} : \Delta^i \to \tilde{M} = \mathbb{H}^n$ of f is straight.

Lemma. Each singular n-simplex $f : \Delta^i \to M^n$ has a unique straight representative $\hat{f} : \Delta^i \to M^n$ s.t if $c = \sum a_i \sigma_i \in S_i(M, \mathbb{R})$, then its straightened $\hat{c} = \sum a_i \hat{\sigma}_i$ satisfies $c - \hat{c} = \partial d \quad d \in S_{i+1}(M, \mathbb{R})$.

pf $\hat{f}$: lift f to the univ. cover, replace it by the straight one

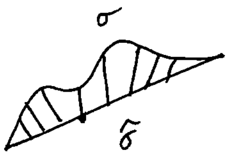


w/ the same vertices and project it down to M. $\hat{f} = \pi \circ (\tilde{f})_{\text{straight}}$.

Prop. For hyperbolic n-manifold M^n, the straighten map $\Phi : S_i(M) \to S_i(M)$ sending σ to $\hat{\sigma}$ is a chain homotopic map.

pf. For each σ, there exists a natural ~~chain~~ homotopic map between σ & $\hat{\sigma}$

$\Psi(x,t) = \pi$ (the geodesic in $\mathbb{H}^n$ from $\sigma(x)$ to $\hat{\sigma}(x)$)



s.t $\Psi|_{\partial_i \Delta^n}$ is the natural homotopy from $\partial_i \sigma$ to $\partial_i \hat{\sigma} = \widehat{\partial_i \sigma}$.

Now the standard homotopy $\Rightarrow$ chain homotopy applies. (Hw)

□

Prop. If (M^n, d) is closed orientable hyperbolic $\Rightarrow$ $\|M\| \geq \mathrm{vol}(M)/v_n$.

pf $\forall \varepsilon > 0$, $\exists$ chain $c = \sum a_i \sigma_i \in \alpha_M$ s.t

$$\|M\| \geq |c| - \varepsilon = \sum |a_i| - \varepsilon.$$

Now $\hat{c} = \sum a_i \hat{\sigma}_i \in \alpha_M$ since $c - \hat{c} = \partial d$

where $\hat{\sigma}_i$ is straight. $\Rightarrow$ $\mathrm{vol}(\hat{\sigma}_i) \leq v_n$. $(< v_n)$

Now

$$\mathrm{vol}(M) = \int_M d\mathrm{vol} = \int_{\sum a_i \hat{\sigma}_i} d\mathrm{vol} = \sum_i a_i \int_{\hat{\sigma}_i} d\mathrm{vol} = \sum_i a_i \,\mathrm{vol}(\hat{\sigma}_i)$$

$$\leq \sum |a_i| \, v_n \leq v_n \cdot (\|M\| + \varepsilon)$$

□

Corollary $\|\Sigma_g\| = 4g - 4$ $(g \geq 2)$

since $\|\Sigma_g\| \geq \mathrm{vol}(\Sigma_g)/\pi = [-2\pi \chi(\Sigma_g)]/\pi = 4g-4$.

Our next goal

Thm (Gromov-Thurston) M^n closed hyperbolic $\Rightarrow$ $\|M\| \leq \mathrm{vol}(M)/v_n$

ie $\|M\| = \mathrm{Vol}(M)/v_n$

Kuiper's Proof

We will show: $\forall \varepsilon > 0 \quad vol(M) \geq (v_n - \varepsilon)\, \|M\|$

<u>Lemma</u>. It suffices to show: $\forall \varepsilon > 0$, $\exists \; c = \sum a_i \sigma_i \in \lambda \alpha_M$, $a_i > 0$, [illegible] $\lambda > 0$,

s.t. $vol(\sigma_i) \geq v_n - \varepsilon$.

<u>Pf</u> If so, $\Rightarrow$

$$vol(M) = \sum \frac{a_i}{\lambda} vol(\sigma_i) \geq (v_n - \varepsilon) \sum \frac{a_i}{\lambda} = (v_n - \varepsilon) \left\| \frac{c}{\lambda} \right\| \geq (v_n - \varepsilon) \|M\|. \quad \square$$

[illegible]

Fix $\varepsilon > 0$, we will produce such c on M. as follows

<u>Let us work for $n = 3$</u> (for simplicity)

<u>Lemma</u> $\forall \varepsilon > 0$, $\exists \; R_0 > 10$ s.t. if $R > R_0$ and σ is a straight tetra whose edge lengths are in $[R-2, R+2]$, then $vol(\sigma) > v_3 - \varepsilon$.

<u>Pf</u> In this case all angles of σ are close to $\pi/3$ $\square$

Produce a cell decomposition $X_1 \sqcup \cdots \sqcup X_K$ of M^3 s.t.

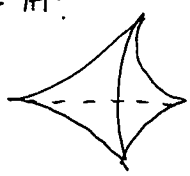
(1). Each X_i is simply connected, $diam(X_i) \leq \min\{1, \frac{1}{10} diam(M)\}$

(2) $X_i \cap X_j = \phi \quad i \neq j$

(3). Find $q_i \in \overset{\circ}{X_i} \quad \forall i$.

Let $\pi : \mathbb{H}^3 \to M$ univ. cover, $\pi^{-1}\{X_1, \cdots X_K\} = \{\Omega_j\text{'s}\}$ $\pi^{-1}\{q_1, \cdots q_K\} = P$ which are decomposition of $\mathbb{H}^3$ invariant under $\Gamma = \pi_1(M)$ action.

Fix $R > R_0$ and a regular straight tetra $\tau_R = [u_1, u_2, u_3, u_4]$ of edge length $= R \subset \mathbb{H}^3$.



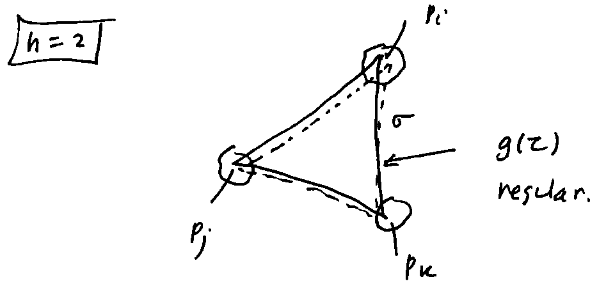
Kuiper's Proof

Now fix ε, R

Def A good tetra $\sigma: \Delta^3 \longrightarrow \mathbb{H}^3$ is a straight simplex (oriented) s.t

(1). its vertices $p_1, p_2, p_3, p_4 \subset P$ $\quad p_i \in \Omega_i$

(2). $\exists$ $g \in \mathrm{Iso}^+(\mathbb{H}^3)$ s.t $\quad g(u_i) \in \Omega_i$



Note $vol(\sigma) > v_3 - \varepsilon$ by definition

(Any left inv 3-form is also right invt)

For a good tetra σ, define $\quad m$ — Haar measure on $\mathrm{Iso}^+(\mathbb{H}^3) = PSL(2.\mathbb{C})$

$$a(\sigma) = m\{ g \in \mathrm{Iso}^+(\mathbb{H}^3) \mid g(u_i) \in \Omega_i \}$$

↑ Key Fact left + Right invt (HW)

Key lemma The infinite chain $\tilde{\beta} = \sum_{\sigma \text{ good}} a(\sigma)\,\sigma$ is closed, $\partial\tilde{\beta} = 0$!

and is invariant under the action of Γ, $\forall\, \gamma \in \Gamma$ $\gamma_\# \tilde{\beta} = \tilde{\beta}$.

Proof First, a good triangle is $\tau: \Delta^2 \longrightarrow \mathbb{H}^3$ is straight s.t

(1) its vertices $p_1, p_2, p_3 \subset P$ $\quad p_i \in \Omega_i$

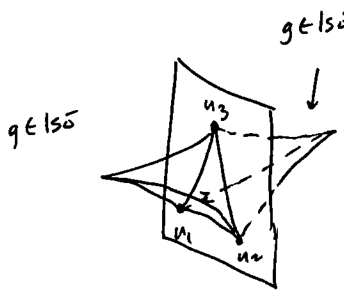
(2) $\exists$ $g \in \mathrm{Iso}^+(\mathbb{H}^3)$ s.t $\quad g(u_i) \in \Omega_i$ $\quad i = 1, 2, 3$.

Thus
$$\partial\beta = \sum_{\tau \text{ good triangle}} \mathrm{Coeff}(\tau) \cdot \tau$$

by definition: $\mathrm{Coeff}(\tau) = m\{ g \in \mathrm{Iso}^+(\mathbb{H}^3) \mid g(u_i) \in \Omega_i \quad i = 1, 2, 3 \}$

$\qquad - m\{ g \in \mathrm{Iso}^-(\mathbb{H}^3) \mid g(u_i) \in \Omega_i \quad i = 1, 2, 3 \}$

$\qquad \triangleq m(R_i) - m(L_e)$



We claim R_i & L_e differ by a right multiplication by $b \in \mathrm{Iso}(\mathbb{H}^3)$ $\Rightarrow$ $m(R_i) = m(L_e)$

Let $\Phi \in \mathrm{Iso}^-(\mathbb{H}^3)$ be the hyperbolic isometric reflection about the plane containing u_1, u_2, u_3, $\Phi(u_i) = u_i$. **Then** $\Phi: R_i \to L_e : g \mapsto g \circ \Phi$ is 1-1 onto $\Rightarrow \partial\beta = 0$

Kuiper's Proof

$\Rightarrow$

Now $\forall\, \gamma \in \Gamma \quad \gamma_* \cdot \beta = \beta$

indeed $\gamma.\beta = \sum_{\sigma \text{ good}} a(\sigma)(\gamma\sigma)$ But $a(\sigma) = a(\gamma\sigma)$

Since $a(\gamma\sigma) = m\{ g \in \mathrm{Iso}^+ \mid g u_i \in \gamma(\Omega_i) \}$

$= m\{ g \in \mathrm{Iso}^+ \mid \gamma^{-1} g(u_i) \in \Omega_i \}$

$\overset{(h=\gamma^{-1}g)}{=} m\left(\gamma\{ h \in \mathrm{Iso}^+ \mid h(u_i) \in \Omega_i \}\right)$

$= m\{ h \in \mathrm{Iso}^+ \mid h(u_i) \in \Omega_i \} = a(\sigma).$

By taking the Γ representations $\Rightarrow \quad \triangleq \sum_{[\sigma]} a(\sigma)\pi(\sigma)$ (where $\sigma \sim \sigma'$ if $\exists\, \gamma \in \pi_1(M)$, $\gamma\sigma = \sigma'$)

$\exists$ a finite cycle $C_R = \sum a_i \sigma_i \in S_3(M, \mathbb{R})$ $\quad a_i \geq 0 \quad \mathrm{vol}(\sigma_i) > v_3 - \varepsilon$

s.t its pull back under $\pi : \mathbb{H}^3 \to M$ is β.

It is a finite sum: $\forall\, p, q \in M$, $\exists$ only finitely many geodesics of length $\leq R+2$ from p to q.

$\partial C_R = 0$ due to $\partial\beta = 0$

For $\quad c = \dfrac{\mathrm{vol}(M)\, C_R}{\sum a_i \mathrm{vol}(\sigma_i)} \in d_M \quad (\mathrm{vol}(c) = \|M\|)$

$= \sum a_i' \sigma_i$

it satisfies all the conditions that we are looking for $(a_i' \geq 0)$

$\Rightarrow$ result.

Note $\quad \mathrm{vol}(c) = \dfrac{\mathrm{vol}(M)}{\sum a_i \mathrm{vol}(\sigma_i)}\left(\sum a_i \mathrm{vol}(\sigma_i)\right) = \mathrm{vol}(M) \Longrightarrow c \in d_M.$

Easy fact: if c is a cycle in $S_3(M, \mathbb{R})$ s.t $\mathrm{vol}(c) = \mathrm{vol}(M) \Rightarrow$ $c \in d_M$

Key property: The Haar measure on $\mathrm{Iso}(\mathbb{H}^n)$ is both left + right invariant

Kuiper's Proof

We will show $\forall \varepsilon > 0$, $\quad vol(M) \geq (v_n - \varepsilon) \|M\|$

Lemma 1. It suffices to prove $\forall \varepsilon > 0$, $\exists$ $c = \sum a_i \sigma_i \in \lambda \alpha_M$,

s.t $\quad a_i \geq 0$ and $vol(\sigma_i) \geq v_n - \varepsilon$.

pf If so, due to $\partial c = 0 \Rightarrow \int_c dvol > 0 \Rightarrow \lambda > 0$

$$\Rightarrow \quad c' = \frac{1}{\lambda} c = \sum \left(\frac{a_i}{\lambda}\right) \sigma_i \in \alpha_M$$

Now $\quad vol(M) = \int_{c'} dvol = \sum \frac{a_i}{\lambda} \int_{\sigma_i} dvol = \sum \left(\frac{a_i}{\lambda}\right) vol(\sigma_i)$

$$\geq (v_n - \varepsilon) \sum \left(\frac{a_i}{\lambda}\right) \geq (v_n - \varepsilon) \|M\|.$$

□

Let us work out $n = 3$ case, General case the same.

Lemma 2 $\forall \varepsilon > 0$, $\exists$ $R_0 > 10$ s.t if $R > R_0$ and σ is a straight tetra where edge lengths are $\in [R-2, R+2]$, then $vol(\sigma) > v_3 - \varepsilon$.
This is due to the continuity of the volume on lengths.

Now produce a cell decomposition $X_1 \sqcup \cdots \sqcup X_k$ of M^3 s.t

(1) Each X_i is simply connected $\quad diam(X_i) \leq \min\{1, \frac{1}{10} diam(M)\}$

(2) $X_i \cap X_j = \phi$ if $i \neq j$

(3) Find $q_i \in \mathring{X}_i$ $\quad \forall i$

Let $\pi: \mathbb{H}^3 \to M$ universal cover, $\pi^{-1}\{X_1, \cdots, X_k\} = \{\Omega_j\text{'s}\}$ $\quad \pi^{-1}\{q_i\} = P$

$\pi^{-1}\{X_i\}$ is a decomposition of $\mathbb{H}^3$ invariant under the action of $\pi_1(M)$.

Fix $R > R_0$ + a regular straight tetra $\tau_R = [u_1, \cdots, u_4]$ of edge lengths R $\quad \tau_R \subset \mathbb{H}^3$

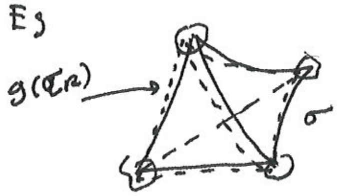
Now fix ε, R

Def A good tetra $\sigma: \Delta^3 \longrightarrow \mathbb{H}^3$ is a straight simplex (oriented)

s.t. (1) its vertices $p_1, \cdots, p_4 \in P$ say $p_i \in \Omega_i$ $i = 1,2,3,4$

(2) $\exists\ g \in \mathrm{Iso}(\mathbb{H}^3)$ s.t. $g(u_i) \in \Omega_i$

Eg



RM. There are only finitely many good tetra up to $\pi_1(M)$ action since there are only finitely many geodesic path in M of length $\leq R+2$ joining q_i to q_j's.

If σ is good, & m is the Haar measure on $\mathrm{Iso}(\mathbb{H}^3)$, Let

$$a(\sigma) = m\{ g \in \mathrm{Iso}(\mathbb{H}^3) \mid g(u_i) \in \Omega_i \quad i = 1,2,3,4 \}$$

Key lemma The infinite chain $\tilde{\beta} = \sum_{\sigma \text{ good}} a(\sigma)\sigma$ is closed, i.e. $\partial\tilde{\beta} = 0$.

It is also invariant under $\pi_1(M)$ action.

Pf Define a good triangle $\delta: \Delta^2 \longrightarrow \mathbb{H}^3$ to be a straight triangle

s.t. (1) its vertices are $p_1, p_2, p_3 \in P$ $p_i \in \Omega_i$ $i = 1,2,3$

(2) $\exists\ g \in \mathrm{Iso}(\mathbb{H}^3)$ s.t. $g(u_i) \in \Omega_i$ $i = 1,2,3$

By definition
$$\partial\tilde{\beta} = \sum_{\tau \text{ good triangle}} \mathrm{Coff}(\tau)\,\tau$$

where
$$\mathrm{Coff}(\tau) = m(A_R) - m(A_L)$$

Let $G = \mathrm{Iso}(\mathbb{H}^3)$, P plane thru $p_1 p_2 p_3$

where
$$A_R = \{ g \in G \mid g(u_i) \in \Omega_i,\ i = 1,2,3,\quad g(u_4) \in P_+ \}$$
$$A_L = \{ g \in G \mid g(u_i) \in \Omega_i,\ i = 1,2,3,\quad g(u_4) \in P_- \}$$

where P_+, P_- are the right & left half spaces bounded by P

We claim that $\exists\ \Phi \in \mathrm{Iso}(\mathbb{H}^3)$, the hyperbolic reflection about the plane through u_1, u_2, u_3 $\Phi(u_i) = u_i$ + reflects it.

Define $F: A_R \longrightarrow A_L$ by $g \longmapsto g \circ \Phi$

It is a bijection $\Rightarrow$ $A_L = A_R \circ \Phi$ $\Rightarrow$ $m(A_L) = m(A_R \cdot \Phi) = m(A_R)$

by the right invariant property of the Haar measure.

$$\Rightarrow \partial \tilde{\beta} = 0.$$

To see $\forall\ \gamma \in \pi_1(M)$ $\gamma_\# \hat{\beta} = \hat{\beta}$. for a good tetra $\sigma = [p_1, \cdots, p_4]$

let $A_\sigma = \{ g \in G \mid g(u_i) \in \Omega_i \}$. Note that $a(\sigma) = m(A_\sigma)$

It is easy to see. $\forall\ \gamma \in \pi_1(M)$

$$\gamma \cdot A_\sigma = A(\gamma\sigma)$$

First σ good $\Rightarrow$ $\gamma\sigma$ is good since $\{\Omega_i\}$ is $\pi_1(M)$ invariant

<u>Next</u> $\gamma \cdot A_\sigma = \{ \gamma g \in G \mid \gamma g(u_i) \in \gamma\Omega_i \} \underset{h = \gamma g}{=} \{ h \in G \mid h(u_i) \in \gamma\Omega_i \}$

$= A_{\gamma\sigma}$ $\Longrightarrow$ $a(\gamma\sigma) = m(A_{\gamma\sigma}) = m(\gamma A_\sigma) = m(A_\sigma) = a(\sigma)$

↑ left inv

Therefore

$$\gamma_\# \hat{\beta} = \sum_{\sigma \text{ good}} a(\sigma)\, \gamma.\sigma = \sum_{\sigma \text{ good}} a(\gamma\sigma)\, \gamma\sigma$$

$$= \sum_{\tau \text{ good}} a(\tau)\, \tau = \hat{\beta}.$$

□

As a consequence, we have produce a finite chain

$$c = \sum_{[\sigma] \text{ good}} a(\sigma)\, [\sigma] \in \lambda\, \alpha_M$$

s.t $\mathrm{vol}[\sigma] > v_3 - \varepsilon$ + $a(\sigma) \geq 0$ + $\exists\ a(\sigma') > 0$

Where $[\sigma]$ is the projection of σ from $\mathbb{H}^3$ to M^3.

By the lemma 1, we are done

To prove Mostow's theorem, we need to organize Kuiper's argument further.

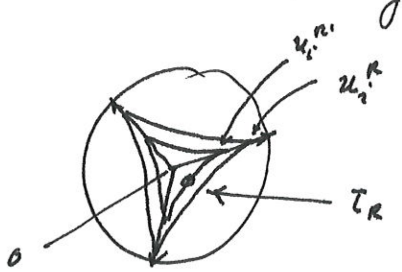
First, we organize the regular length R tetra $\tau_R = [u_1^R, u_2^R, u_3^R, u_4^R]$

<u>s.t</u> $O \in \tau_1$ and is the symmetric center s.t

u_i^R is in the geodesic ray from o to u_i' $i=1,2,3,4$

($\Rightarrow$ O is also the center of τ_R).

<u>$n=2$</u>:



In particular, if $R' > R > 1$

$$\tau_R \subset \tau_{R'}$$

<u>Next</u>, the fundamental cycle produced by Kuiper is

$$C_R = \frac{\mathrm{vol}(M)}{\sum\limits_{\substack{[\sigma] \\ \text{good } R\text{-tetra}}} a(\sigma)\,\mathrm{vol}(\sigma)} \sum_{[\sigma]} a(\sigma)[\sigma] \in \mathcal{C}_M$$

where σ — good R-tetra.

<u>Lemma 3.</u> Fix R, $\sum\limits_{\substack{[\sigma] \\ \text{good } R\text{-tetra}}} a(\sigma) \leq \mathrm{vol}(G/\pi_1(M))$

<u>Pf</u> We claim that if $A_\sigma \cap \gamma A_{\sigma'} \neq \phi \Rightarrow \sigma = \gamma\sigma'$, $\gamma \in \pi_1(M)$

Indeed, if $g \in A_\sigma \cap \gamma A_{\sigma'} \Rightarrow g(u_i^R) \in \Omega_i$ $i=1,2,3,4$

$g(u_i^R) \in \gamma\Omega_i$ $\sigma' = [p_1', p_2', p_3', p_4']$ $p_i' \in \Omega_i$

$\Rightarrow \gamma\sigma' = \sigma$

$\Rightarrow$ The map $G \to G/\pi_1(M)$ send$\,$s A_σ & $A_{\sigma'}$ disjointly for $[\sigma] \neq [\sigma']$

$\Rightarrow \sum\limits_{[\sigma]} a(\sigma) = \sum\limits_{[\sigma]} \mathrm{vol}(A_\sigma) \leq \mathrm{vol}(G/\pi_1(M))$ $\square$

Lecture 4. Gromov's Proof of Mostow Rigidity

We now prove

<u>Thm</u> (Mostow) If $f: M^n \to N^n$ is a homeomorphism between two closed connected hyperbolic manifolds of dim $n \geq 3$, then $\exists$ an isometry $g: M^n \to N^n$ s.t $g \simeq f$.

<u>Proof</u> (Gromov) We may assume $\deg(f) = \deg(f^{-1}) = 1 \Rightarrow$

$$\|M\| \geq |\deg(f)| \, \|N\| \geq |\deg(f)| \, |\deg(f^{-1})| \, \|M\|$$

$$\Rightarrow \qquad \|M\| = \|N\|$$

By Gromov + Thurston $\Rightarrow$ $\mathrm{vol}(M) = \mathrm{vol}(N)$.

Let $\tilde{f}: \mathbb{H}^n \to \mathbb{H}^n$ be a lift of f to the universal cover. s.t

$$\tilde{f}(\gamma(x)) = f_*(\gamma)\tilde{f}(x) \qquad \forall x \in \mathbb{H}^n, \ \gamma \in \pi_1(M) \qquad (1)$$

$f_*: \pi_1(M) \to \pi_1(N)$ is the isomorphism induced by f.

Since $\tilde{f}$ is a quasi-isometry $\Rightarrow$ $\tilde{f}$ extends continuously to

$$F: \overline{\mathbb{H}^n} \to \overline{\mathbb{H}^n} \qquad \text{s.t} \quad (1) \text{ still holds}$$

<u>Goal</u>: $\quad F|_{\partial\mathbb{H}^n} = \hat{g} \in \mathrm{Isom}(\mathbb{H}^n)$

Assuming this, $\Rightarrow$ $\quad \hat{g}(\gamma(x)) = f_*(\gamma)\, \hat{g}(x) \qquad \forall x \in \partial\mathbb{H}^n, \ \forall \gamma \in \pi_1(M) \qquad (2)$

$\Rightarrow$ (2) holds for all $x \in \mathbb{H}^n$

$\Rightarrow$ $\hat{g}$, induces an isometry $g: M^n \to N^n$

The fact that $f \simeq g$ follows from (2).

Gromov's Proof

Let us focus on $n=3$ now

To see $\tilde{g} \in Iso(\mathbb{H}^3) \Leftrightarrow \tilde{g}$ is a Möbius transformation.

Lemma (See Bennedetti-Petronio) $h: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ a homeomorphism is a Möbius transformation iff $\forall$ regular set $\{v_1, v_2, v_3, v_4\} \in \hat{\mathbb{C}}$, $\{hv_1, hv_2, hv_3, hv_4\}$ is again regular.

Here $\{v_1, v_2, v_3, v_4\}$ is regular $\Leftrightarrow$ the ideal tetra $[v_1, v_2, v_3, v_4] \in \mathbb{H}^3$ is regular.

We prove by contradiction. Suppose $\hat{g} = F|_{\partial \mathbb{H}^3}$ is not regular.

We will show that $\|M\| > \|N\|$ which contradicts Gromov-Thurston Thm.

To this end, let us go back to Kuiper's proof & his construction.

First $\tilde{g} \notin$ Möbius $\Rightarrow \exists$ regular ideal tetra $[v_1, v_2, v_3, v_4] \in \mathbb{H}^3$ st

$$Vol([g(v_1), \dots g(v_4)]) < V_3 - \delta \quad \text{for some } \delta > 0.$$

(Here we have used the fact that regular ideal tetra achieves the unique maximum point for volume)

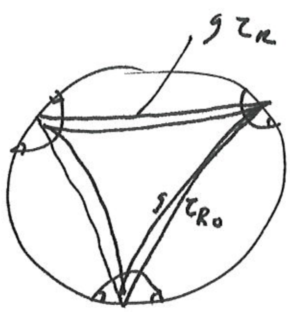
Find open disjoint half-space nbhd U_i of $v_i \in \overline{\mathbb{H}^3}$ s.t

(1) $\forall\ P_1, \dots, P_4 \in \overline{\mathbb{H}^3}$ $P_i \in U_i$ $i=1,2,3,4 \Rightarrow Vol([\hat{g}P_1, \dots, \hat{g}P_4]) < V_3 - \delta$

(2) $\exists\ R_0 > 0$ s.t if $R \geq R_0 \Rightarrow$

$$\phi \neq \{g \in G \mid g(U_i^{R_0}) \in U_i \ \ i=1,2,3,4\} \subset \{g \in G \mid g(U_i^{R}) \in U_i \ \ i=1,2,3,4\}$$

$n=3$



Now fix R_0 and $R > R_0$ as above, let

$$I_R = \{ \text{good } R\text{-tetra } \sigma = [p_1, \cdots p_4] \mid \exists \ \gamma \in \pi_1(M) \text{ st } \tilde{g}(\gamma(p_i)) \in U_i \ \ i=1,2,3,4 \}$$

By the choice of R_0, $I_R \neq \phi$, furthermore

~~$I_{R_0} \subseteq$~~

By the Kuiper's construction, let $\eta = [\sigma]$ be R-good in $M^{n=3}$

$$+ \quad b(\eta) = \frac{\mathrm{vol}(M)}{\displaystyle\sum_{[\sigma]\ R\text{-good}} a(\sigma)\,\mathrm{vol}(\sigma))} \, a(\sigma).$$

then $\displaystyle C_R = \sum_{\eta\ R\text{-good}} b(\eta)\,\eta \in \underline{\alpha_M}$

Consider $\displaystyle \widehat{\tilde{f}_{\#}}(C_R) = \sum_{\eta\ R\text{-good}} b(\eta)\,\eta' \qquad \eta' = \widehat{\tilde{f}\circ\eta}$ straightening

It is in α_N by the assumption.

$$\Rightarrow \mathrm{vol}(N) = \sum_{\eta} b(\eta)\,\mathrm{vol}(\eta')$$

$$= \sum_{I} b(\eta)\,\mathrm{vol}(\eta') + \sum_{II} b(\eta)\,\mathrm{vol}(\eta') \qquad \boxed{\eta' = \pi(\sigma),\ \sigma \in I_R}$$

$$\leq V_3 \sum_{I} b(\eta) + (V_3 - \delta) \sum_{II} b(\eta)$$

$$= V_3 \sum_{I \cup II} b(\eta) - \delta \sum_{II} b(\eta)$$

$$\leq V_3 \sum_{\eta} b(\eta) - \delta \sum_{II} b(\eta) \qquad \text{~~}\eta = \pi(\sigma)\ \ \sigma \in I_{R_0}\text{~~}$$

~~R_0 good~~

But $\mathrm{vol}(N) = \mathrm{vol}(M)$

$$= \sum_{\eta} b(\eta)\,\mathrm{vol}(\eta) \geq (v_3 - \varepsilon) \sum b(\eta)$$

so we conclude that (for any $\varepsilon > 0$) R large

$$\delta \sum_{\substack{\mathrm{II}\ \eta \\ \text{~~R-good~~}}} b(\eta) \leq \varepsilon \sum_{\eta} b(\eta)$$

But

$$b(\eta) = \frac{\mathrm{vol}(M)}{\sum_{[\sigma]\ R\text{-good}} \sigma(\sigma)\,\mathrm{vol}(\sigma)} \; a(\eta)$$

~~$\delta \sum_{\eta}$~~ $\Rightarrow$

$$\delta \left(\sum_{\mathrm{II}} a(\sigma) \right) \leq \varepsilon \sum_{[\sigma]\ R\text{-good}} a(\sigma) \to 0$$

↑ by lemma 3 in Lecture 3

This shows, as $R \to \infty$

$$\sum_{\mathrm{II}} a(\sigma) \longrightarrow 0$$

But $\displaystyle \sum_{\substack{\mathrm{II} \\ R\text{-good}}} a(\sigma) \geq \sum_{\substack{\mathrm{II} \\ R_0\text{-good}}} a(\sigma)$ by the choice of τ^{R}'s

This is impossible.

~~§4. Kremer's Proof of Mostow Theorem~~

§4.1. A characterization of Möbius transformations

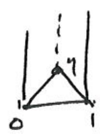
<u>Recall</u>: a Möbius transf of $\hat{\mathbb{C}}$ = compositions of inversions + reflection

It is $g \in PSL(2, \mathbb{C})$ $\frac{az+b}{cz+d}$ or $\frac{a\bar{z}+b}{c\bar{z}+d}$. Möb($\mathbb{C}$) = group of all Möb.

Let $\eta = e^{\pi i/3}$,

<u>Def</u> A set $\{v_1, v_2, v_3, v_4\} \subset \hat{\mathbb{C}}$ is called <u>regular</u> if $\exists$ $g \in$ Möb($\mathbb{C}$) st

$$\{v_1, v_2, v_3, v_4\} = \{g(0), g(1), g(\infty), g(\eta)\}$$



$\Leftrightarrow$ $\{v_1, v_2, v_3, v_4\}$ vertices of an ideal regular hyperbolic tetra

<u>Lemma</u> If $\{a, b, c, d\}$ and $\{a, b, c, d'\}$ are regular and $d' \neq d$, then $d' = Inv_S(d)$ where S is the circle (or line) containing a, b, c

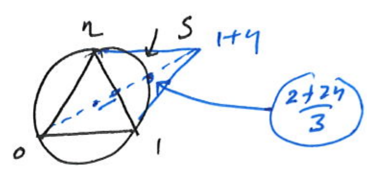
<u>Pf</u>. Indeed, if $h \in$ Möb st $h(0) = 0$ $h(1) = 1$ $h(\infty) = \infty$ $\Rightarrow$ $h = id$ or $h(z) = \bar{z}$. $\square$

<u>Eg</u> if v_1, v_2, v_3 vertices of a regular triangle $\Rightarrow$ $\{v_1, v_2, v_3, \infty\}$ and $\{v_1, v_2, v_3, \frac{v_1+v_2+v_3}{3}\}$ are regular

It is the image of $\{v_1, v_2, v_3, \infty\}$ under inversion about the circle through v_1, v_2, v_3

$|1+\eta| =$

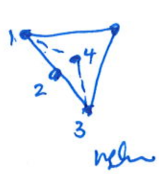
<u>Eg</u> $\{1, \eta, 1+\eta, \frac{1+\eta}{3}\}$ is <u>regular</u>



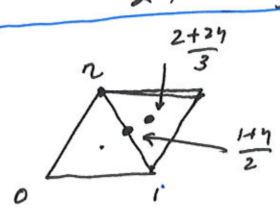
Let S be the circle through $0, 1, \eta$

now $\{1, \eta, 1+\eta, \frac{1+\eta+(1+\eta)}{3}\}$ (center) regular $\Rightarrow$ $Inv_S\{1, \eta, 1+\eta, \frac{2+2\eta}{3}\}$ regular

<u>Which is</u>: $\{1, \eta, \frac{1+\eta}{2}, \frac{2+2\eta}{3}\}$ regular $\left(\frac{2+2\eta}{3} \in S\right)$



$\Rightarrow$



<u>due to</u>: $Inv_S(1+\eta) = \frac{1+\eta}{2}$

<u>Prop</u>. If $h: S^2 \to S^2$ homeomorphism preserving regular sets, then $h \in$ Möb.

Gromov's Proof

h preserves orientation $\quad h(\infty)=\infty \quad h(0)=0, \; h(1)=1, \; h(\eta)=\eta$

+ h preserves regular sets.

<u>Goal</u> $\quad h = id$.

<u>Step 1</u> $\quad h\big|_{\mathbb{Z}+\eta\mathbb{Z}} = id$

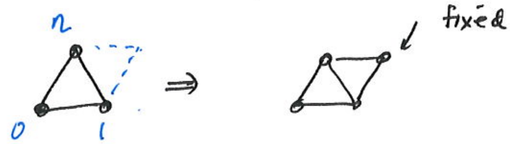
Indeed $\{0, 1, \eta, 1+\eta\}$ regular $\Rightarrow$ $h\{0, 1, \eta, 1+\eta\}$ regular

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ ‖

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ $\{0, 1, \eta, h(1+\eta)\}$

<u>But</u> $\quad h(1+\eta) = 1+\eta$ or 0 due to lemma $\Rightarrow$

$$h(1+\eta) = 1+\eta$$



repeat $\Rightarrow$ $h(n+m\eta) = n+m\eta$

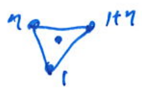
$\forall\, n, m \in \mathbb{Z}$

<u>Step 2</u> $\quad h\big|_{\mathbb{Z}(\frac{1}{2}) + \mathbb{Z}(\frac{\eta}{2})} = id.$

$h(\infty)=\infty$

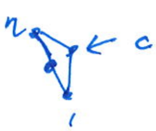
$\Rightarrow h(\frac{2+2\eta}{3}) =$

<u>Note</u> $\quad \{1, \eta, 1+\eta, \frac{1+\eta+(1+\eta)}{3}\}$ regular, apply h and step 1



$\Rightarrow \quad h\left(\frac{1+\eta+(1+\eta)}{3}\right) = \frac{1+\eta+(1+\eta)}{3} = \frac{2+2\eta}{3}$ ✓

<u>But</u>, $\quad \{1, \eta, \frac{1+\eta}{2}, \frac{2+2\eta}{3}\}$ regular $\Rightarrow$ Apply h, SAME



$h(\frac{1+\eta}{2}) = \frac{1+\eta}{2} \quad$ <u>since</u> $\quad h(\frac{1+\eta}{2}) = \begin{cases} \frac{1+\eta}{2} & \text{or} \\ 1+\eta \end{cases}$

$\Rightarrow$

$\Rightarrow \quad h(\frac{1}{2}) = \frac{1}{2} \quad h(\frac{\eta}{2}) = \frac{\eta}{2}$

$\Rightarrow$ the SAME as in step 1

Inductively $\Rightarrow \quad h\big|_{\mathbb{Z}(\frac{1}{2^n}) + \mathbb{Z}(\frac{\eta}{2^n})} = id \quad \Rightarrow$

But $\quad \bigcup_{n \in \mathbb{Z}_{\geq 1}} \left(\mathbb{Z}(\frac{1}{2^n}) + \mathbb{Z}(\frac{\eta}{2^n})\right) \subset \mathbb{C}$ is dense $\Rightarrow h = id$

h continuous. $\quad$ ◻

Key observation :