9-12-2013 $\to$ 9-13-2013

Berlin School of Math   Berlin Lectures

Introduction to Teichmuller theory

Lecture 1. Hyperbolic Geometry

Let $\mathbb{H}^2 = \{ z \in \mathbb{C} \mid y > 0 \,,\ z = x + iy \}$ upper half plane

The Riemannian metric

$$ds_h^2 = \frac{dx^2 + dy^2}{y^2} = \frac{|dz|^2}{y^2} \quad \text{on } \mathbb{H}^2$$

$v \in T_{\cancel{(a,b)}(x,y)} \mathbb{H}^2$

$|v|_{\mathbb{H}^2} = |v|_E / y^2$

- Angles in ds_h^2 = angles in $\mathbb{C}$. conformal
- length of $\gamma(t) = (x(t), y(t))$ $\quad t \in [a,b]$:

$$L(\gamma) = \int_a^b |\gamma'(t)|_{ds^2} \, dt$$

$$\gamma'(t) = (x'(t), y'(t))$$

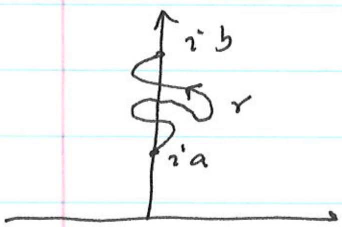
$$|\gamma'(t)|_{ds^2} = \sqrt{x'(t)^2 + y'(t)^2} \Big/ y(t) \geq \frac{|y'(t)|}{y(t)}$$ equality iff $x' = 0$

so $$L(\gamma) = \int_a^b \frac{\sqrt{x'(t)^2 + y'(t)^2}}{y(t)} \, dt$$

Lemma 1. The positive y-axis is a geodesic in $\mathbb{H}^2 = (\mathbb{H}^2, ds^2)$ s.t.

$$d(ia, ib) = \left| \ln \frac{b}{a} \right|$$

Pf Say $\gamma(a) = ia$, $\gamma(b) = ib$ $\quad a < b$ real $\quad, y(t) > 0$,



$$L(\gamma) \geq \int_a^b \frac{\sqrt{y'(t)^2}}{y(t)} \, dt = \int_a^b \frac{|y'(t)|}{y(t)} \, dt$$

$$\geq \left| \int_a^b \frac{y'(t)}{y(t)} \, dt \right| = \left| \ln y(t) \Big|_a^b \right| = \ln \frac{b}{a}$$

Equality holds iff $x'(t) \equiv 0$, $y'(t) \geq 0$ ie $\gamma([a,b]) \subset$ Y-axis + monotonic.

$\square$

Lemma 2. $f(z)=\frac{az+b}{cz+d}$, $\begin{bmatrix}a&b\\c&d\end{bmatrix}\in SL(2,\mathbb{R})$, $\mathbb{H}\to\mathbb{H}$ are isometries

Pf. f is a composition of $z\mapsto z+b$, $z\mapsto \lambda z$ $\lambda\in\mathbb{R}$ and

$z\mapsto -\frac{1}{z}=w$. Obviously $f(z)=\lambda z+b\in Iso(\mathbb{H}^2)$. Now:

for $w=-\frac{1}{z}$ $\quad dw=\frac{1}{z^2}dz \Rightarrow \qquad Im(w)=\frac{1}{2i}\left(\frac{1}{\bar z}-\frac{1}{z}\right)$

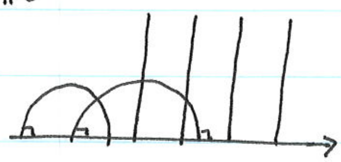
$$w^*(ds^2)=\frac{|dw|^2}{Im(w)^2}=\frac{\frac{1}{|z|^4}|dz|^2}{\left(\frac{1}{2i}\right)^2\left(-\frac{1}{z}+\frac{1}{\bar z}\right)^2}$$

$$=\frac{\frac{1}{|z|^4}|dz|^2}{\frac{1}{|z|^4}\cdot\frac{1}{4}|\bar z-z|^2}=\frac{|dz|^2}{y^2}$$

□

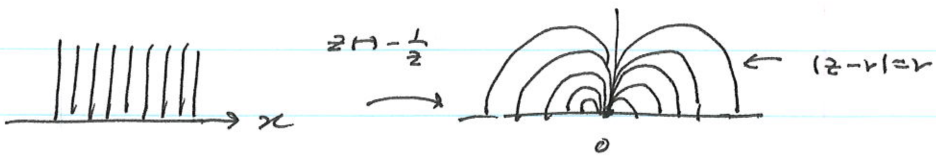
Corollary 3. All geodesics in $\mathbb{H}$ are, either $Re(z)=c$ or $|z-a|=r$

$a, c\in\mathbb{R}$



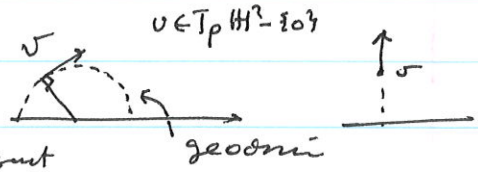
Pf. y-axis geodesic + $z\mapsto z+b$ iso $\Rightarrow$ $Re(z)=c$ geod.

$Re(z)=c$ geodesic + $z\mapsto -\frac{1}{z}$ iso $\Rightarrow$ $|z-r|=r$ geodesic



Now $z\mapsto z+b$ iso $\Rightarrow$ $Re(z)=c$ & $|z-a|=r$ geodesic

These are all geodesics:



Geodesic determined by ONE tangent □

Homework: Show that $PSL(2.\mathbb{R}) = SL(2.\mathbb{R})/\pm Id \cong Isom^+(\mathbb{H}^2)$

hint: $\forall\ \gamma \in Isom^+(\mathbb{H}^2)$, find $f \in PSL(2.\mathbb{R})$ s.t.

(1) $f(i) = \gamma(i)$

(2) $f'(i) = \gamma'(i)$.

Cross ratio: $a, b, c, d \in \hat{\mathbb{C}}$ distinct define

$$(a, b, c, d) = \frac{a-c}{a-d} : \frac{b-c}{b-d}$$

We known $f \in PSL(2.\mathbb{C})$, Möbius

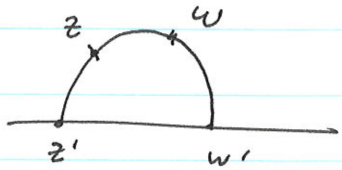
$$f(z) = \frac{\alpha z + \beta}{\gamma z + \delta} \quad \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in SL(2.\mathbb{C}) \Rightarrow$$

$$\big(f(a), f(b), f(c), f(d)\big) = (a, b, c, d)$$

$a<b$

Also $d(ia, ib) = \ln\left(\frac{b}{a}\right) = \ln(ia, ib, \infty, 0)$ (1)

Corollary 4. If $z, w \in \mathbb{H}$, then $d(z.w) = \ln(z, w, w', z')$.

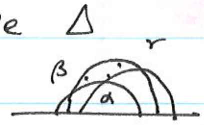


Pf: Let $f \in PSL(2.\mathbb{R})$ sending $f(z) = ia$ $f(w) = ib$, then $f(z') = 0$ $f(w') = \infty$. done

$$d(z.w) \overset{f\ iso}{=} d(ia, ib)$$
$$= \ln(ia, ib, \infty, 0)$$
$$\overset{f\ inv}{=} \ln(z, w, w', z')$$

$\square$

HW Gauss-Bonnet: The area of a hyperbolic triangle Δ of angles α, β, γ is $\pi - \alpha - \beta - \gamma$



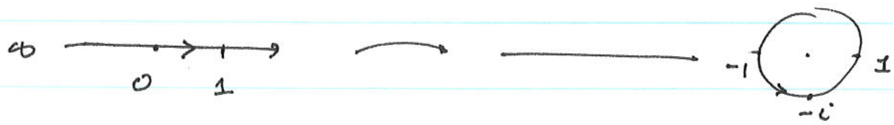
The ball model.

Note: $f(z) = (z, a, b, c)$ Möbius

$$= \frac{z-b}{z-c} : \frac{a-b}{a-c} \quad \text{sends} \quad \begin{array}{l} b \mapsto 0 \\ c \mapsto \infty \\ a \mapsto 1 \end{array}$$

So $f(z) = (z, 1, i, -1) = \dfrac{z-i}{z+1} : \dfrac{1-i}{2}$ $\quad 0 \mapsto$

Eg $\varphi(z) = \dfrac{z-i}{z+i}$: $\begin{array}{l} i \mapsto 0 \\ -i \mapsto \infty \\ 0 \mapsto -1 \\ \infty \mapsto 1 \end{array}$ and $1 \mapsto -i$

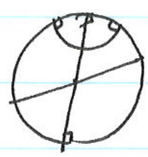


$\varphi : (\mathbb{H}) = \mathbb{D} = \{ |z| < 1 \}$

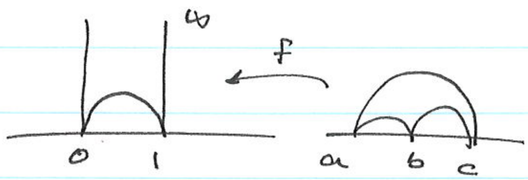
The metric on $\mathbb{D}$ making φ isometry: $\dfrac{4(dx^2+dy^2)}{(1-x^2-y^2)^2} = \dfrac{4|dz|^2}{(1-|z|^2)^2}$

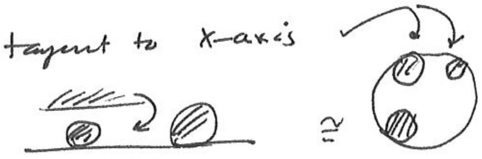
Isometries: Möbius transf. preserving $\mathbb{D}$: $z \mapsto e^{i\theta}\left(\dfrac{z-a}{\bar{a}z-1}\right)$

Geodesics: lines and circles $\perp \partial\mathbb{D}^2$



Eg Any two ideal triangles in $\mathbb{H}$ are isometric: all isometric to $(0, 1, \infty)$ due to $f(z) = (z, a, b, c)$. $a, b, c \in \mathbb{R}$ cross ratio (order)



horoballs Euclidean discs tangent to x-axis or $\mathrm{Im}(z) \geq c$ 

Lecture 2 Hyperbolic Structures on Surfaces

Uniformization thm Σ surface, connected, $\chi(\Sigma) < 0$, then $\forall$ Riemannian metric g on Σ, $\exists!\ u: \Sigma \to \mathbb{R}$ s.t.

$$(\Sigma, e^{u} g) \text{ is a complete hyperbolic metric.}$$

(Gaussian curvature $= -1$).

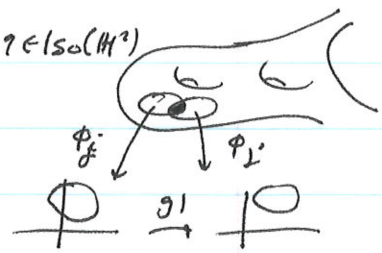
$\Rightarrow$ We should study hyperbolic metrics on surfaces & organize them.

Def 1. A hyperbolic structure on surface Σ: special collection of charts $\{(U_i, \varphi_i) \mid i \in I\}$ s.t.

(1) $\Sigma = \bigcup_i U_i$

(2) $\varphi_i: U_i \to \mathbb{H}^2$ 1-1 continuous

(3) $\varphi_i \circ \varphi_j^{-1} = g|_{\varphi_j(U_i \cap U_j)}$ $g \in \mathrm{Isom}(\mathbb{H}^2)$

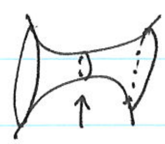


The structure is complete if each geodesic extends to ∞.

Geodesics, angles in (Σ, φ) $\Leftarrow$ using charts

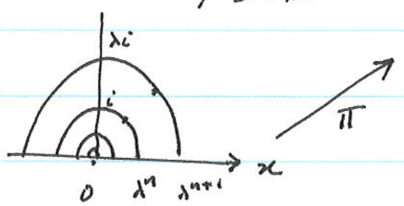
Eg 2. $(\mathbb{H}^2, ds^2)$ complete hyperbolic $d(ia, it) \to +\infty$ $t \to +\infty$ or $t \to 0^+$

Eg 3. $\gamma(z) = \lambda z$, $\lambda > 1$ acts on $\mathbb{H}^2$ generates a group $\mathbb{Z} = \langle \gamma^n \rangle$, the quotient space

Eg 3' $\mathbb{H}/z \sim z+1$ cusp $\qquad \mathbb{H}^2/z \sim \lambda z$ =  a hyperbolic annulus

length of the shortest geod $\boxed{\ln \lambda}$

non-isometric



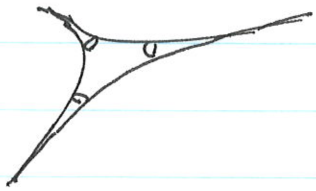
$\Rightarrow \exists$ many distinct hyperbolic structures on $S^1 \times \mathbb{R}$.

Eg 4. $\Gamma(2) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2,\mathbb{Z}) \,\middle|\, \begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \bmod 2 \right\} / \pm \mathrm{Id}$

L 2

$\Gamma(2)$ acts on $\mathbb{H}^2$ freely properly discontinuously w/ quotient

$\mathbb{H}^2/\Gamma(2) \cong$



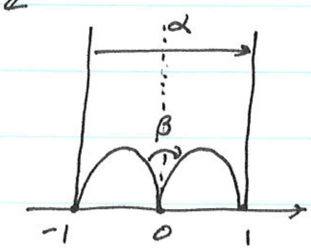
metric double of an ideal triangle

$\cong \mathbb{C}-\{0,1\}$

Sketch:

$\Gamma(2)$ generated by $\alpha(z) = z+2 \Leftrightarrow \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and

$\| \quad \mathbb{Z}*\mathbb{Z}$ $\quad \beta(z) = \dfrac{z}{2z+1}$. $\quad \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$.

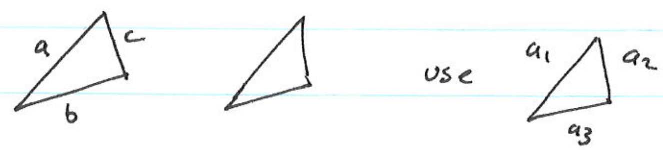


$\beta: 0 \mapsto 0$
$\quad -1 \mapsto 1$

$\Rightarrow$ Schottky group $\Rightarrow$ result.

What is Teichmuller theory?: space of all equivalence classes hyperbolic metrics on Σ.

Eg 5. Let $M(\Delta)$ = space of all triangles in $\mathbb{E}^2$ modulo isometries



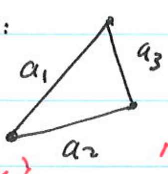
Q Is $M(\Delta) = \{(a,b,c) \in \mathbb{R}^3_{>0} \mid a+b>c,\ b+c>a,\ c+a>b\}$?

ANS: is No: $M^{(\Delta)} \neq \{(a_1, a_2, a_3) \in \mathbb{R}^3_{>0} \mid a_i + a_j > a_k\}$.

In fact $M(\Delta)$ is the "moduli" space (Riemann)

$T(\Delta)$ = space of all labelled trigs in $\mathbb{E}^2$ modulo isometries preserving labelling

$T(\Delta)$ = Teichmuller space.



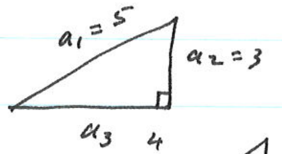
We know first second third edge $v_1\ v_2\ v_3$)

so what space $\{(a_1, a_2, a_3) \in \mathbb{R}^3_{>0} \mid a_i + a_j > a_k\} \longleftrightarrow$ marked trigs (verts

modulo isometry preserving marking

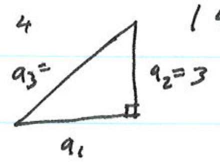
$M = T/S_3$ S_3 the permutation group

i.e.



$(5, 4, 3) \in T$ &

$(4, 3, 5) \in T$ Are different



But they are the same in M: isometric

The ultimate goal: (Riemann's moduli space)

$$Mod(\Sigma) = \{ (\Sigma, d) \mid d \text{ complete hyperbolic metric on } \Sigma \} / \text{isometry}$$

finite area

Very difficult to study.

Teichmuller space

$$T(\Sigma) = \{ (\Sigma, d) \mid (\Sigma, d) \underset{T}{\cong} (\Sigma, d') \text{ if } \exists \text{ isometry } h \simeq id \}$$

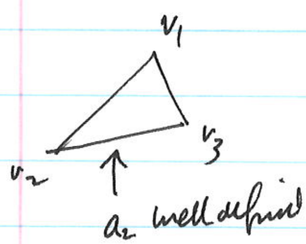
$$= \{ (\Sigma, d) \mid d - \} / \text{ isometry homotopic to } id.$$

Why $T(\Sigma)$?: Just like $T(\Delta)$. We can define the length of a loop (edge). The length of the 2nd edge function does not make sense in $M(\Delta)$!

So $\forall$ loop $\alpha \subset \Sigma$

$$\ell_\alpha : T(\Sigma) \longrightarrow \mathbb{R} \quad (\Sigma, d) \longmapsto \text{length of the shortest path } \alpha' \simeq \alpha \text{ in } d$$

Why Teich(Δ)? We can talk about the i-th edge lengths



$$a_i : \text{Teich}(\Delta) \longrightarrow \mathbb{R}_{>0}$$

No such function on $Mod(\Delta)$? $\longrightarrow \min\{a_1, a_2, a_3\}$

Only the minimal length. C^0

Not Smooth

Lecture 2 Berlin 9-13-2013

(1)

Σ orientable

(1)

$d =$ complete finite area hyperbolic

Main goal Σ topological surface ~~w/ complete hyperbolic structure~~ finite area

$$\mathrm{Mod}(\Sigma) = \{(\Sigma, d) \mid d \text{ complete hyperbolic finite area}\}/\mathrm{iso}$$

What is the space? dim? connected? Topology? Geometry?

$$T(\Sigma) = \{(\Sigma, d) \mid d. \;—\; \}/\text{isometry homotopic to id}$$

i.e. $(\Sigma, d) \underset{\text{Teich}}{\sim} (\Sigma, d')$ if $\exists$ iso $h: (\Sigma, d) \to (\Sigma, d')$ s.t. $h \simeq id$

So $\mathrm{Mod}(\Sigma) = T(\Sigma)/MCG(\Sigma)$

$$MCG = \{\text{orientation preserving homeo } h\}/\{h \mid h \simeq id\}$$

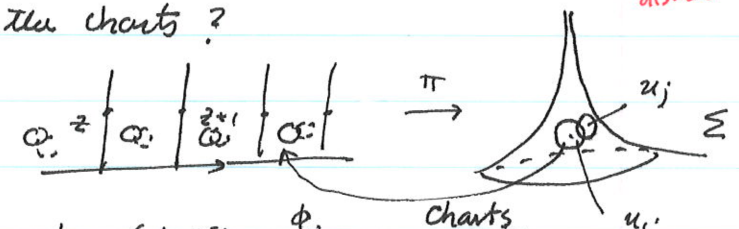
This plays the role of changing of marking S_3 for Δ.

Key Fact: Each loop $\alpha \subset \Sigma$ $\exists$ lengths $\ell_\alpha : T(\Sigma) \to \mathbb{R}$, $[d] \to$ length of the shortest $\beta \simeq \alpha$

(2) Many one words about hyperbolic st. on $\mathbb{H}/\Gamma$ $\Gamma < \mathrm{Isom}^+(\mathbb{H})$ discrete

How to see the charts?

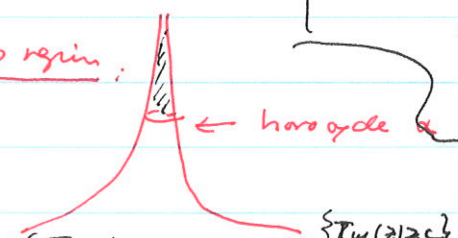
Eg. $\Sigma = \mathbb{H}/z \sim z+1$



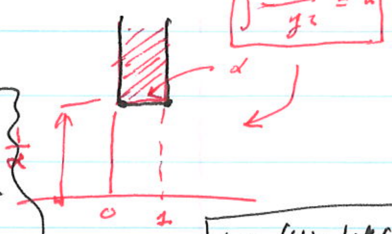
$$\phi_i = (\pi|_{V_i})^{-1}, \quad \phi_j = (\pi|_{V_j})^{-1} \qquad \phi_i \circ \phi_j^{-1} \in \langle \gamma \rangle!$$

More generally: $\mathbb{H}/\Gamma$ has charts $\{(U_i, \phi_i) \mid i \in I\}$ st $\phi_i \circ \phi_j^{-1} \in \Gamma$!

Finite area of cusp region:



$$\boxed{\int \frac{dx\,dy}{y^2} = \alpha}$$



~~A cusp~~ A horoball in (Σ, d) $\cong$ isometric to

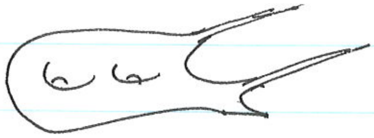
$\boxed{\mathrm{Area}(H) = \mathrm{length}(\partial H)}$

S_g closed surface, oriented. $V = \{v_1, \dots, v_n\} \subset S$. $\Sigma = S_g - V \cong \Sigma_{g,n}$, $\chi(\Sigma) < 0$

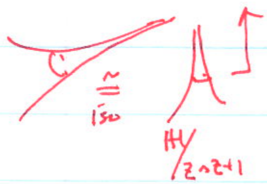
L3 Topological Triangulations

Q. How to construct all complete finite area hyperbolic structures on Σ which is non-closed? w/ $\chi(\Sigma) < 0$. ? Known: finite area $\Rightarrow$ cusp end!

$\Sigma = \Sigma_g - \{v_1, \dots, v_n\}$ $\underline{n \geq 1}$ $\chi(\Sigma) < 0 \Leftrightarrow$ $g \geq 1$ or $g = 0$ $n \geq 3$.



$V = \{v_1, \dots, v_n\}$



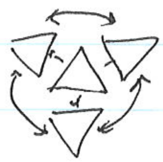
ANS Use triangulations

2-Dimensional triangulations.

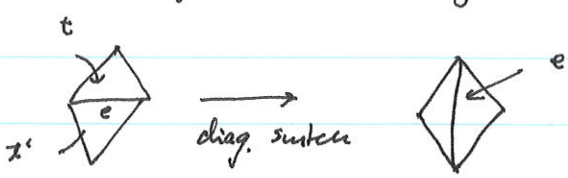
Take a finite collection of disjoint oriented triangles $\Delta_1, \dots, \Delta_k$, identify pairs of edges in $\sqcup \Delta_i$ by orientation reversing homeomorphisms. The quotient space $S = \sqcup \Delta_i / \sim$ is a compact oriented surface w/ a triangulation $\mathcal{T}$: simplexes in $\mathcal{T} \Leftrightarrow$ quotient of simplexes in $\sqcup \Delta_i$.

Let V, E be the sets of vertices and edges respectively in $\mathcal{T}$.
$V = V(\mathcal{T})$, $E = E(\mathcal{T})$

We say $\mathcal{T}$ an <u>ideal triangulation</u> of $S - V$.

Eg  double of a triangle, or  ∂(tetra). 

Def Diagonal switch: If $e \in E$ is adjacent to two triangles t, t', then



$\mathcal{T} \to \mathcal{T}'$ by replacing e by the other diagonal e' in $t \cup_e t'$.

Thm If $\mathcal{T}, \mathcal{T}'$ two triangulations of S with the same set of vertices $V(\mathcal{T}) = V(\mathcal{T}')$, then $\mathcal{T}$ and $\mathcal{T}'$ are related by a sequence of diagonal switches.

Each $\Sigma = \Sigma_{g,n}$ $2-2g-n<0$ has an ideal triangulation $(n>0)$

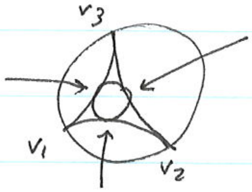
Q Suppose $(\Sigma, \mathcal{T})$ ideal triangulated surface, How to use $\mathcal{T}$ to produce all hyperbolic metrics on Σ ?

Thurston's work

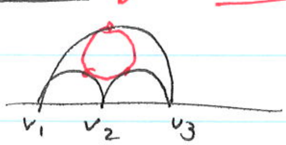
Ideal triangle: Convex hull of 3 points $v_1, v_2, v_3 \in \partial \mathbb{H}^2$

the midpt

it has 3 mid pts



or

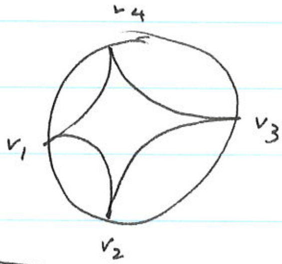


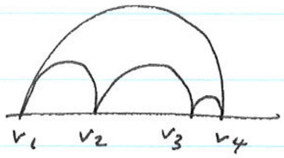
ANY two of them are isometric

$f(z) = (z, v_1, v_2, v_3)$

drop

Ideal quadrilateral: Convex hull of 4 points $v_1, v_2, v_3, v_4 \in \partial \mathbb{H}^2$

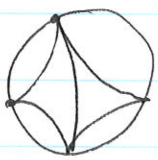




Not all are isometric since the cross ratio (v_1, v_2, v_3, v_4) is an isometry invariant.

A marked ideal quad δ: ~~an ideal quad + diagonal~~ e

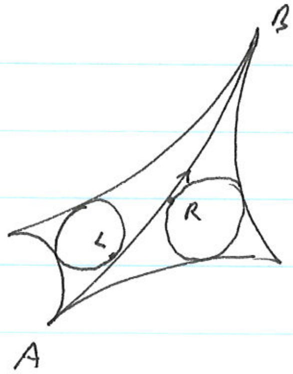
= union of two ideal triangles along an edge e.



Every surface is oriented

Def. Thurston's shear coordinate $d(\delta)$ of an oriented marked ideal quad δ is the real number $d(\delta)$ = signed distance from L to R along the diagonal.

L3. Thurston's Shear Coordinate

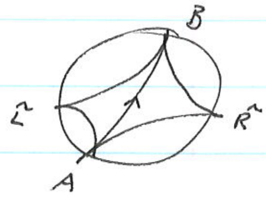


orient e : $A \to B$: $L \to R$

so $d(\delta) = R - L$ computed along e

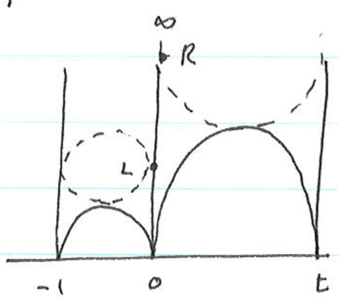
Note independent of the choice of orientation on e (depending only on the orientation of δ!)

Lemma 1. Suppose $\delta = [A, \tilde{R}, B, \tilde{L}]$ as shown



then $d(\delta) = \ln[-(A, B, \tilde{R}, \tilde{L})]$

Proof May assume after a Möbius transf $A = 0$, $B = \infty$, $\tilde{R} = t > 0$, $\tilde{L} = -1$



$$(A, B, \tilde{R}, \tilde{L}) = (0, \infty, t, -1) = \frac{0-t}{0+1} : \frac{\infty - t}{\infty - 1} = -t$$

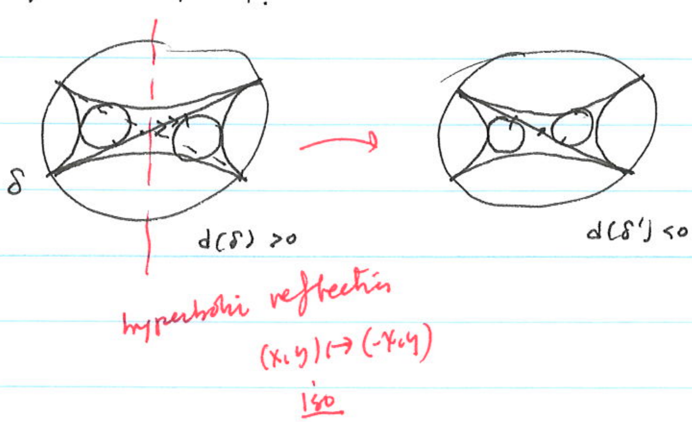
$L = i \quad R = it$

so $d(\delta) = \ln(t)$

□

Lemma 2. Suppose δ' is obtained from δ by a diagonal switch $\Rightarrow d(\delta') = -d(\delta)$

Pf Follows from the basic property of cross ratio. Or a geometric proof:



$|d(\delta)| = |d(\delta')|$.

□

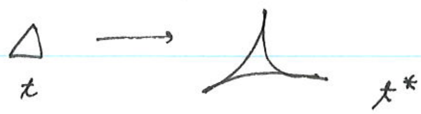
L4 The shear coordinate of Thurston

Assume $\Sigma = \Sigma_g - \{v_1, \dots, v_n\}$ $n \geq 1$, $\chi(\Sigma) < 0$

$(\Sigma, \mathcal{T})$ ideal triangulation  $\boxed{\text{Fixed}}$ $\mathcal{T}$

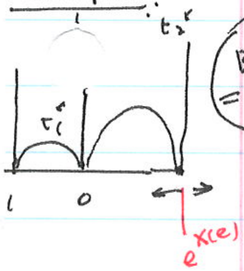
For each $x \in \mathbb{R}^{E(\mathcal{T})}$ i.e. $x : E(\mathcal{T}) \to \mathbb{R}$, one produce a (possibly incomplete) hyperbolic structure $\pi(x)$ on Σ as follows

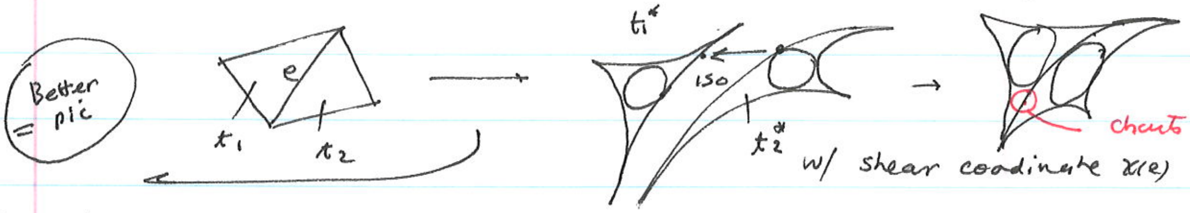
(1) replace each triangle $\Delta \in \mathcal{T}$ by an ideal hyperbolic triangle



(2) each edge $e \in E(\mathcal{T})$ with $x(e)$ assigned, glue isometrically t_1^* t_2^*

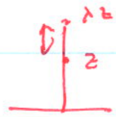
More pict





the corresponding edge by the isometry s.t the resulting shear coordinate at e is $x(e)$. There is a unique way to glue.

$\boxed{\text{infinite geodesic}}$ $L = \{ \mathrm{Re}(z) = 0 \}$



translation

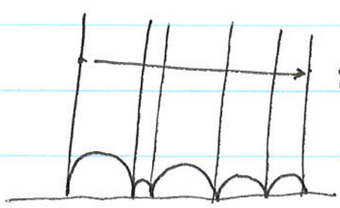
$f_\lambda(z) = \lambda z : L \to L$ $\boxed{d(z, f_\lambda(z)) = \ln \lambda}$

Lemma $\pi(x)$ is a complete finite area metric $\Leftrightarrow$ $\forall\, v \in V = \{v_1, \dots, v_n\}$

$$\sum_{e > v} x(e) = 0$$

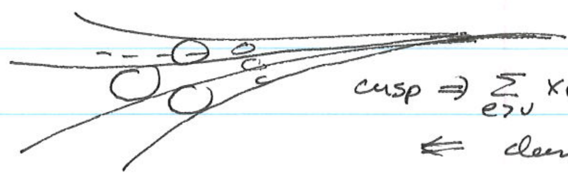
$\Leftarrow$ cusp end

Proof At v_1 say put it at ∞



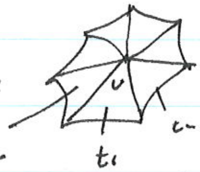
glued isometrically $\Leftrightarrow$ $\sum_{e>v} x(e) = 0$

to produce a cusp end to be complete



cusp $\Rightarrow$ $\sum_{e>v} x(e) = 0$

$\Leftarrow$ clear.

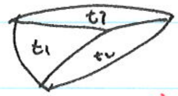
Pf Sketch. Topology 

at v

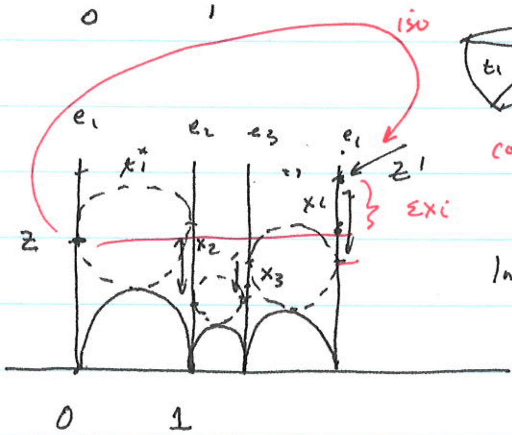
Geometry: 

idealized

z' at height

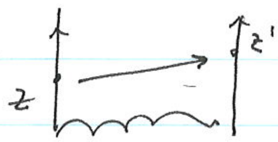


corresponding pts



$z \sim z'$ by the construction

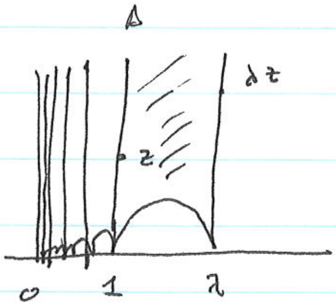
$\operatorname{Im} z' / \operatorname{Im} z = e^{x_1 + \cdots + x_3}$



must be horocyclic in order to produce cusp

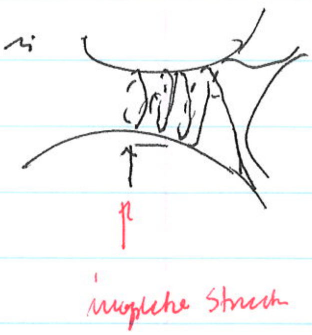
$\Rightarrow \boxed{\Sigma x_i = 0}$

Eg: The image of the ...



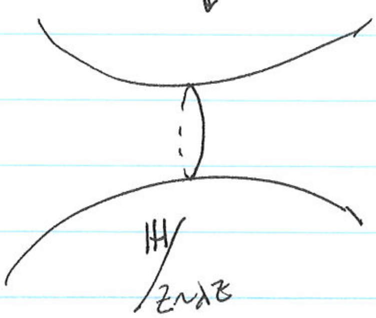
$\lambda > 1$

is $\mathbb{H} / z \sim \lambda z$ is

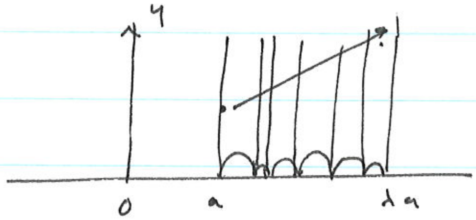


$\downarrow \pi$ quasi-image of A

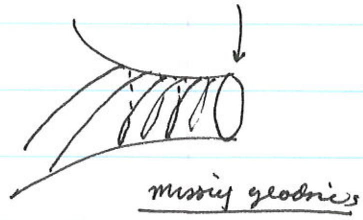
spiral



Note if $\sum_{e>v} x(e) \neq 0$, the gluing is



⇒ the infinity



incomplete structure

□

Let $\mathbb{R}^E_p = \{ x \in \mathbb{R}^E \mid \forall v \in V \ \sum_{e>v} x(e) = 0 \}$ (linear subspace)

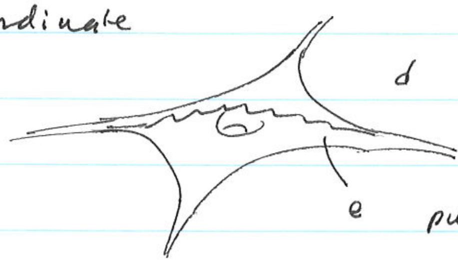
<u>Thm</u> (Thurston) The map $\Phi_{\mathcal{T}} : \mathbb{R}^E_p \longrightarrow T(\Sigma)$ $\quad x \mapsto [\pi(x)]$

is a ~~homeomorphism~~ 1-1 onto map. $\Phi(x)$ has <u>shear coordinate x</u> w.r.t. $\mathcal{T}$.

<u>Proof</u> Φ is onto: If $[d] \in T(\Sigma)$ + $\mathcal{T}$ ideal triangulation $\mathcal{T}$

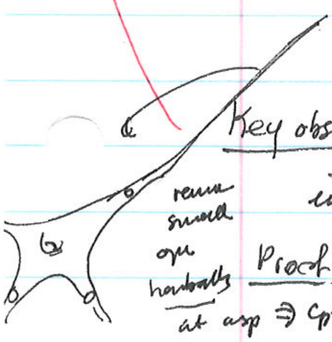
We can isotopy $\mathcal{T}$ to be a geodesic triangulation of (Σ, d)

where all edges are infinite simple geodesics from cusp to cusp ⇒ x = shear coordinate



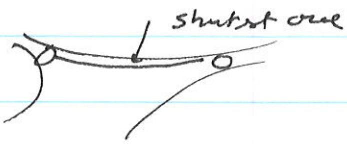
pull tight ⇒ <u>done</u>

" <u>Φ is 1-1</u>": if $\phi(x) = \phi(x')$ ⇒ they have the same shear coordinate. $\boxed{\text{How to see it?}}$



<u>Key observation</u> For (Σ, d), Each edge $e \in E(\mathcal{T})$ is homotopic in (Σ, d) to a unique infinite geodesic e^*.

<u>Proof</u> Pull tight by producing horocycles



□

$\boxed{h(e) \simeq e}$

$\underline{\Phi \text{ is } 1\text{-}1}$: $\exists$ an isometry

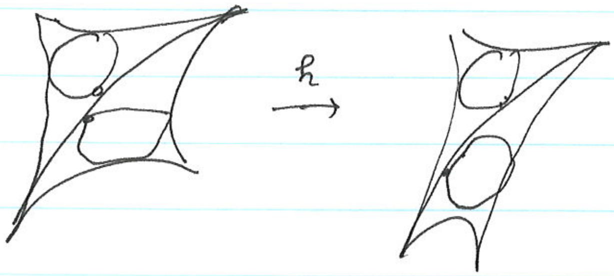
$h : (\Sigma, \phi(x)) \to (\Sigma, \phi(x'))$

$\Rightarrow$ $\underline{h(\mathcal{T}_x) = \mathcal{T}_{x'}}$ (∪

$h \simeq id$ ⇓ & circles

$\mathcal{T}_x, \mathcal{T}_{x'}$ geodesic triangulations in $\phi(x)$ & $\phi(x')$ $\underline{\text{metrics}}$)

$\underline{\text{properly}}$



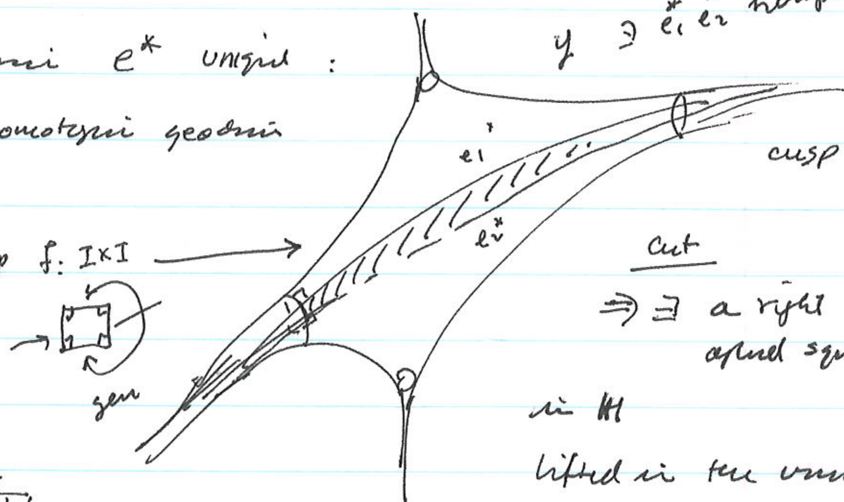
must preserve the distance

The point is that $\mathcal{T}_x$ is $\underline{\text{unique}}$!

Why is the geodesic e^* unique :

If $\exists$ e_1^* e_2^* homotopic

Two homotopic geodesics



cusp

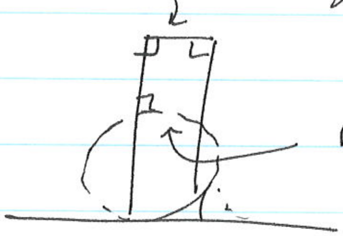
$\exists$ a map $f : I \times I$ →

homotopy

$\underline{\text{cut}}$

$\Rightarrow \exists$ a right angled square in $\mathbb{H}$

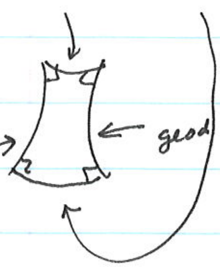
lifted in the universal cover



horocycles $\underline{\text{impossible}}$

universal cover

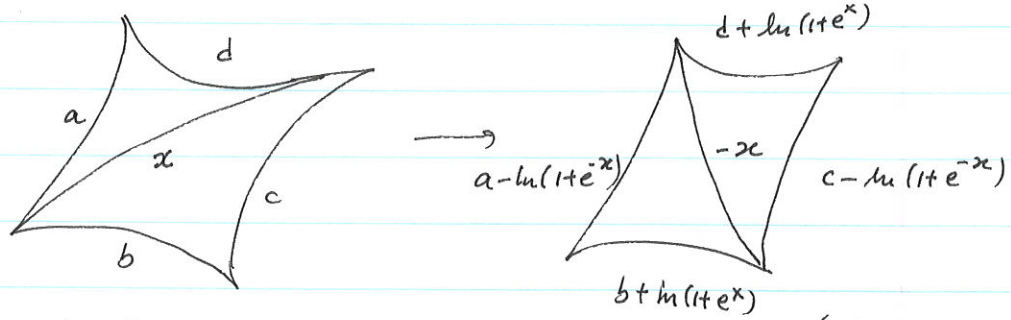
$\Rightarrow \exists$ a right-angled quadrilateral $Q \subset \mathbb{H}$ s.t.



skip it for Now

Thm If $\mathcal{T}, \mathcal{T}'$ two triangulations related by a diagonal switch, then the transition function $\Phi_{\mathcal{T}'}^{-1}\Phi_{\mathcal{T}} : \mathbb{R}_p^{E(\mathcal{T})} \to \mathbb{R}_p^{E(\mathcal{T}')}$ takes the following form.

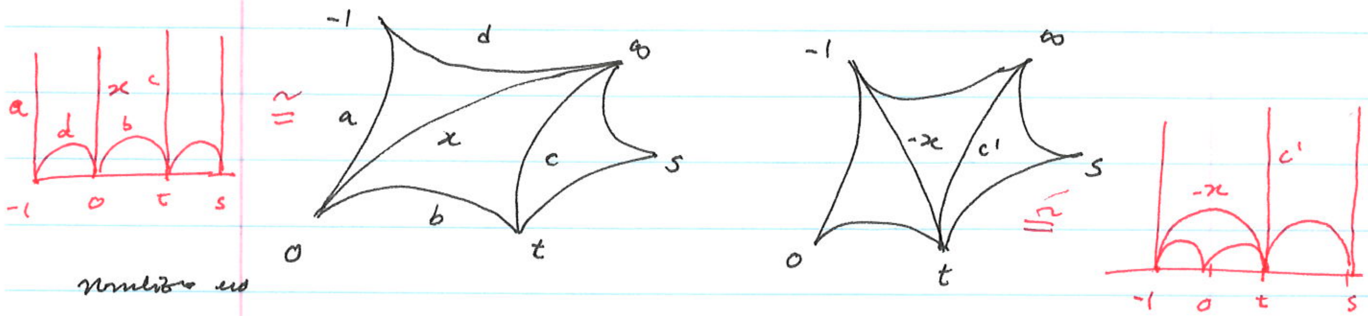
real analytic diffeo



all others places remain the same. (Right-hand orientation)

Proof Follows from the basic cross ratio rule. $\square$

We may assume the end points are: $s > t > 0$



normalize us

Here: $x = \ln t$

$$c = \ln(-1)(t, \infty, s, 0) = \ln\left(\frac{t-s}{t-0}\right)(-1) = \ln\left(\frac{s-t}{t}\right)$$

$$c' = \ln\left[(-1)(t, \infty, s, -1)\right] = \ln(-1)\frac{t-s}{t+1} = \ln\frac{s-t}{(t+1)}$$

$$= \ln\left(\frac{s-t}{t}\right) + \ln\left(\frac{t}{1+t}\right) = c - \ln\left(1+\frac{1}{t}\right) = c - \ln(1+e^{-x})$$

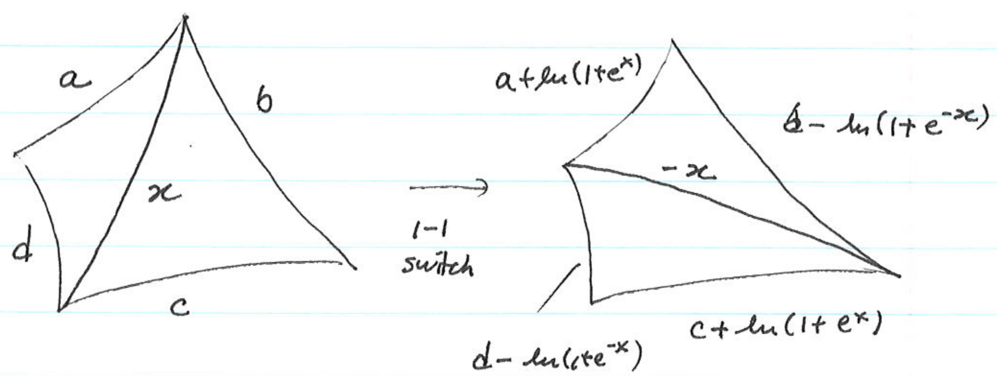
$\square$

RM: Basic rule: Right "shoulder" Add $\ln(1+e^{x})$, left: $-\ln(1+e^{-x})$

Corollary $\Rightarrow$ $\mathcal{T}(\Sigma)$ is a real analytic mfd diffeo $\mathbb{R}^{6g-6+2n}$. $= X - \ln(1+e^{X})$

L5. Penner's λ-length Coordinates, Poisson Structures. Continue.

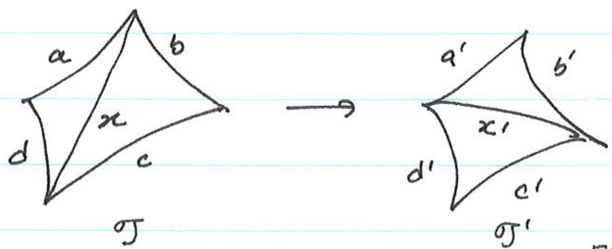
Summary The change of coordinate formula for Thurston shear coord.



Any property invariant under the coordinate change is an invariant of the Teichmüller space.

(HW) <u>Thm</u>. Suppose $f(t), g(t)$ smooth s.t. $f(t)+g(t)=t$

and a coordinate change rule:



with $x' = -x$

$a' = a + f(x)$

$c' = c + f(x)$

$b' = b + g(x)$

$d' = d + g(x)$

Then the Poisson structure on $\mathbb{R}^{E(T)}$

$$\sum_{e \to e'} \frac{\partial}{\partial x_e} \wedge \frac{\partial}{\partial x_{e'}}$$

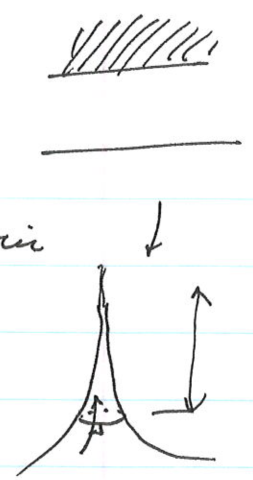
is invariant under coordinate change (the diagonal switch)

where $e \to e'$ means $\exists\ \Delta \in T^{(2)}$ adjacent to both e & e' & $e \to e'$ in the orientation of Δ.

Homework

Just do it

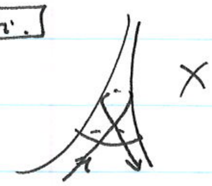
($\Rightarrow$ $T(\Sigma)$ is naturally a symplectic mfd)



Recall d — complete finite area hyperbolic

A horoball $H \subset (\Sigma, d)$ is a subsurface isometric to $\{\mathrm{Im}(z) \geq c\} / (z \sim z+1)$

Fact: (Σ, d) noncpt. ~~nbhd of infinity of~~ ~~near each~~ v_i.



horoballs form nbhds of infinity of Σ

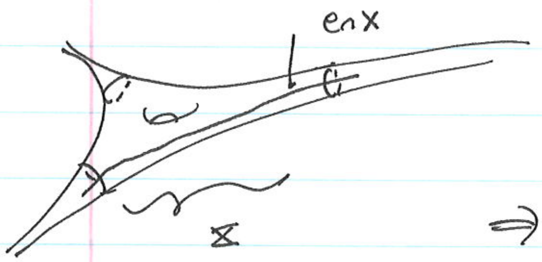
- $\mathrm{Area}(H) = \mathrm{length}(\partial H)$ (Homework)

- (Σ, d) take $H_1, \dots, H_n$ disjoint horoball nbhd of v_i.

$X = \Sigma - \bigcup_i H_i$ is a compact surface w/ ∂H horocycles.

Key lemma (Σ, d), $\mathcal{T}$ ideal triangulation $\forall e \in E(\mathcal{T})$ is homotopic to a unique geodesic e^* in d.

Pf existence Form X + pull tight $e \cap X$



Let α be path in X

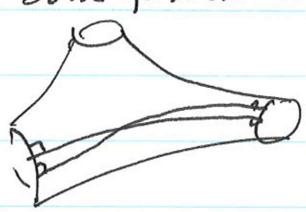
st ① $\alpha \simeq e \cap X$ rel(∂X)

② α has the shortest length

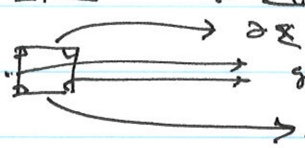
$\Rightarrow$ α geod. + $\alpha \perp \partial X$ (X cpt)

$\Rightarrow$ α extends to a geodesic e^* going to the cusp

Uniqueness if $\exists$ $e_1^* \simeq e_2^*$ homotopic geodesics both go into cusps v_i to v_j $\Rightarrow$ $e_1^* \cap X \simeq e_2^* \cap X$ rel(∂X)



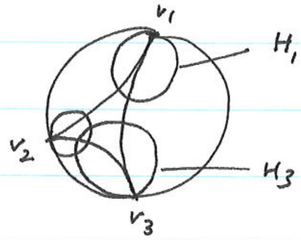
$\Rightarrow \exists\, f : I \times I \longrightarrow \Sigma$, lift it to $\mathbb{H}^2$



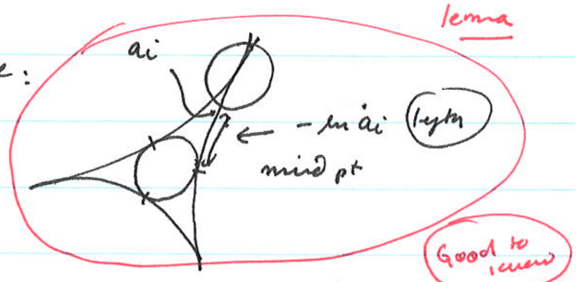
Penner's theory of Decorated Hyperbolic

L5 Penner's λ-coordinate

A decorated ideal triangle τ: ideal triangle w/ vertices $v_1, v_2, v_3 \in \partial \mathbb{H}^2$ s.t each v_i is associated with a horoball H_i at v_i.

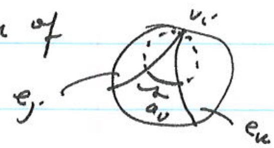


picture:

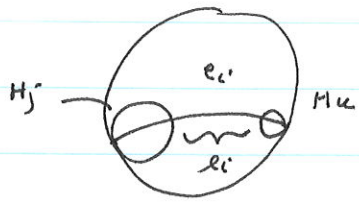


Let the edges of τ be e_1, e_2, e_3 (infinite geodesic) e_i opposite to v_i.

Then the angle a_i of τ at v_i is: the length of

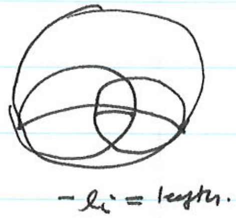


The length l_i of τ at e_i $\quad l_i \in \mathbb{R}$



l_i = length ≥ 0

$H_j \cap H_k = \phi$



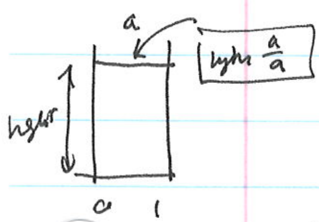
Penner $\quad L_i = e^{\frac{1}{2} l_i} \in \mathbb{R}_{>0}$ the λ-length of e_i

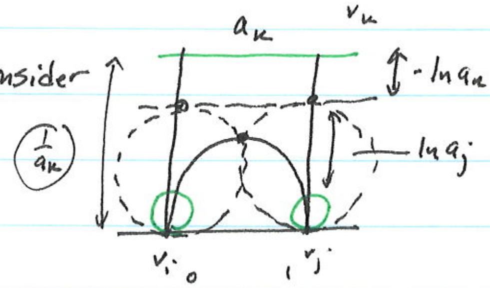
Lemma (1) $\quad a_i = e^{\frac{1}{2}(l_i - l_j - l_k)} = \dfrac{L_i}{L_j L_k}$ $\qquad \ln a_i = \frac{1}{2}\left(\ln l_i - \ln l_j - \ln l_k\right)$

(2) $-\ln a_i$ = dist H_i to mid pt

(3) $\forall\, l_1, l_2, l_3 \in \mathbb{R}$, $\exists!$ decorated ideal tetra of lengths l_1, l_2, l_3 HW

Proof Consider





$\Rightarrow$ $\boxed{+l_i = -(\ln a_j + \ln a_k)}$

$\Rightarrow$ result.

(2) The angles $a_i, a_j, a_k \in \mathbb{R}_{>0}$ can be arbitrary real numbers as shown above $\Rightarrow$ result.

$\square$

λ - length

A decorated hyperbolic metric (d, w) on Σ:

d — complete finite area hyperbolic on Σ.

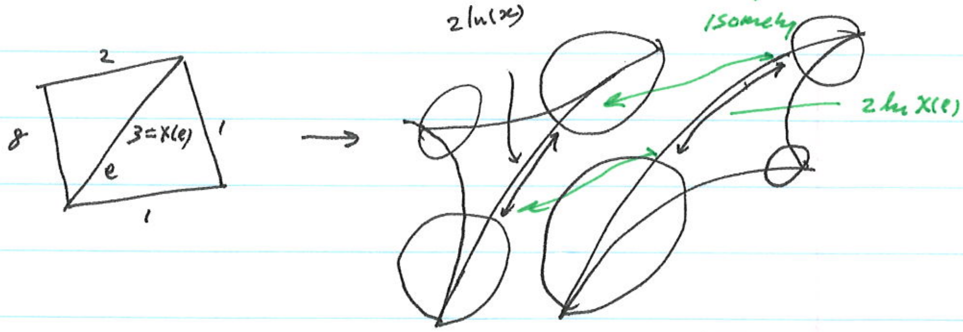
w: each cusp is assigned a horoball H_v centered at the cusp.

$w = (w_1, \cdots, w_n) \in \mathbb{R}^n_{>0}$. w_i = length of ∂H_i

$T_D = \{[(d, w)] \mid (d, w)$ decorated metric on $\Sigma\} /$ isometry $\simeq$ id preserving horoballs

$= T(\Sigma) \times \mathbb{R}^n_{>0}$

Now fix a $\mathcal{T}$ of Σ. For any $x \in \mathbb{R}^{E(\mathcal{T})}_{>0}$ one constructs a decorated metric $\varphi(x) \in T_D$ as follows.



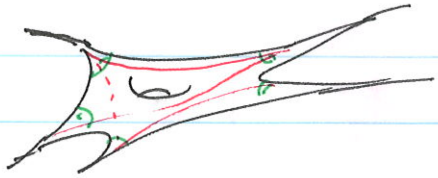
make each $\Delta \in \mathcal{T}^{(2)}$ a decorated ideal tetra of λ-length $x(e)$.
Now glue these (them) isometrically along edges preserving decorations

$\Rightarrow$ complete hyperbolic metric of finite area (d, w)

The horoballs — gluing of the portion of the horoballs.

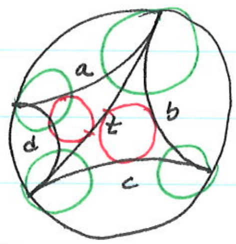
<u>Thm</u> (Penner) $\Phi_{\mathcal{T}}: \mathbb{R}^E \longrightarrow T_D(\Sigma): x \mapsto \varphi(x)$ is a homeomorphism.

<u>Proof</u> Onto: pull $\mathcal{T}$ tight. 1-1 definition.



Relationship between shear and λ-coordinate:

<u>Lemma</u>: Consider $a, b, c, d, t \in \mathbb{R}$ the length coord.

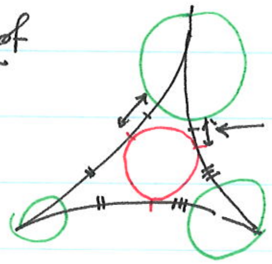


then the shear coordinate x at e

$$x = \cancel{(a+c+b+d)/2} \left(\frac{b+d-a-c}{2}\right)$$

From Penner → shear

<u>Proof</u>



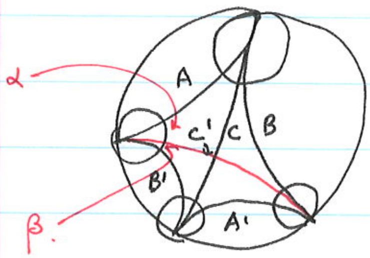
$\frac{-c+b+t}{2}$ by definition.

so $$x = \left(\frac{-c+b+t}{2}\right) - \left(\frac{-d+a+t}{2}\right) = \frac{1}{2}(b+d-a-c)$$

$\square$

<u>Corollary</u>: (Penner's Ptolemy). Let A, A', B, B', C, C' be the λ-lengths of decorated ideal quad. Then

$$CC' = AA' + BB'$$



<u>Proof</u> Let α, β be the angles as shown

By the cosine law:

$$\alpha = \frac{B}{AC'} \qquad \beta = \frac{A'}{B'C'}$$

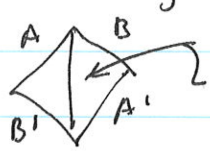
But $\alpha + \beta = \frac{C}{AB'}$ $\Rightarrow$

$$\frac{B}{AC'} + \frac{A'}{B'C'} = \frac{C}{AB'} \iff AA' + BB' = CC'$$

$\hat{\Phi}_T = \frac{\Phi}{2}(\Phi_T)$ $\square$

$\Rightarrow$ the change of coordinate formula $\tilde{\Phi}_\sigma \circ \tilde{\Phi}_{\sigma'}^{-1}$: even better the λ-lengths

$\longrightarrow$

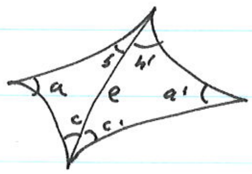


$\frac{AA'+BB'}{C}$ real analytic

Using a convex variational principle R. Guo + myself proved the following thm which generalizes Penner's previous work

(Guo-L)

Thm For each $x \in \mathbb{R}^{E(T)}$, let $\psi_0(x) \in \mathbb{R}^E$ be:

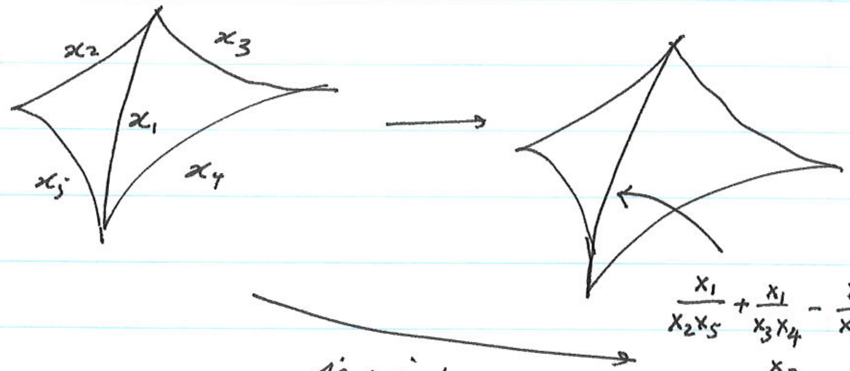


$$\psi_0(x)(e) = a + a' - b - b' - c - c'$$

(curvature)

Then the map: $x \mapsto \psi_0(x)$ is a smooth embedding.

i.e.:



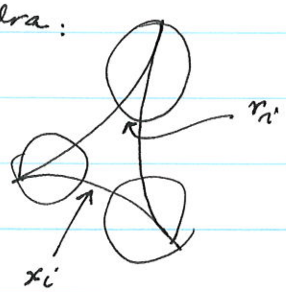
$$\frac{x_1}{x_2x_5} + \frac{x_1}{x_3x_4} - \frac{x_4}{x_1x_3} - \frac{x_3}{x_1x_4} - \frac{x_2}{x_1x_5} - \frac{x_5}{x_1x_2}$$

is injective

$\square$

$\boxed{\text{May skip to the last page}}$

Proof We study the function from $x_i \to r_i$ of decorated ideal tetrahedra:



Known,

$$r_i = e^{\frac{1}{2}(x_i - x_j - x_k)}$$

Let $t_i = \frac{1}{2}x_i$ for simplicity

so $r_i = e^{t_i - t_j - t_k}$

Let $y_i = r_j + r_k - r_i = e^{t_j - t_k - t_i} + e^{t_k - t_i - t_j} - e^{t_i - t_j - t_k}$

<u>Lemma</u> (1) $\dfrac{\partial y_i}{\partial t_i} = - e^{t_j - t_k - t_i} - e^{t_k - t_i - t_j} - e^{t_i - t_j - t_k}$

$$= (-1)\, e^{-t_1 - t_2 - t_3} \left[e^{2t_j} + e^{2t_i} + e^{2t_k} \right]$$

(2) $\dfrac{\partial y_i}{\partial t_j} \underset{i \neq j}{=} e^{t_j - t_i - t_k} - e^{t_k - t_i - t_j} + e^{t_i - t_j - t_k}$

$$= \left[e^{-t_1 - t_2 - t_3} \right] \left[e^{2t_i} + e^{2t_j} - e^{2t_k} \right]$$

As a consequence, $\dfrac{\partial y_i}{\partial t_j} = \dfrac{\partial y_j}{\partial t_i}$. Furthermore the

matrix $\left[\dfrac{\partial y_i}{\partial t_j} \right]_{3\times 3} = (-1) e^{-t_1 - t_2 - t_3} \begin{bmatrix} s_1+s_2+s_3 & s_1+s_2-s_3 & s_1+s_3-s_2 \\ s_1+s_2-s_3 & s_1+s_2+s_3 & s_2+s_3-s_1 \\ s_1+s_3-s_2 & s_2+s_3-s_1 & s_1+s_2+s_3 \end{bmatrix}$

is strictly negative definite.

<u>Pf</u> (1) (2) follows. <u>Note</u>: $s_1, s_2, s_3 > 0$

We claim that Note diagonally dominated matrix! But

$$s_1+s_2+s_3 > |s_1+s_2-s_3| + |s_1+s_3-s_2| \quad \underline{\text{etc}}$$

<u>It is still</u> positive definite

~~Say $\underline{s_1 \geq s_2 \geq s_3}$ <u>then</u> the worst case is~~

~~if~~

~~$s_1+s_2+s_3 \geq (s_1+s_2-s_3) + |s_2+s_3-s_1|$~~ (Not true)

~~$= s_1+s_2-s_3 + s_2+s_3-s_1 = \underline{2s_2}$~~ ~~$s_2+s_3 \geq s_1$~~

Let us assume $s_1 \geq s_2 \geq s_3$ so the worst case

$$s_1+s_2+s_3 \geq |s_2+s_3-s_1| + |s_1+s_3-s_2| \ ?$$

$a_{ii} \geq 0 \ \forall i$

<u>2x2 principle</u> ≥ 0

Also

it is $\det \geq 0$

$$\left[\begin{array}{cc} s_1 \geq s_2 \geq s_3 \geq 0 & \Rightarrow ? \\ s_1+s_2+s_3 & \end{array}\right]$$

Why is the determinant positive?

the determinant $A = s_1+s_2+s_3 \Rightarrow$

$$\begin{bmatrix} A & A-2s_3 & A-2s_2 \\ A-2s_3 & A & A-2s_1 \\ A-2s_2 & A-2s_1 & A \end{bmatrix}$$

First the determinant $\neq 0$ since $\left[\frac{\partial y_i}{\partial t_j}\right]$ is invertible

We can solve t_j's from y_1, y_2, y_3.

Next, For the set of all possible t's is connected space.

Thus, ~~it so~~ the $\det\left[\frac{\partial y_i}{\partial t_j}\right]$ does not change signs

Now for the very specific one: $s_1 = s_2 = s_3 = 1$ it is $\det > 0$

$\square$

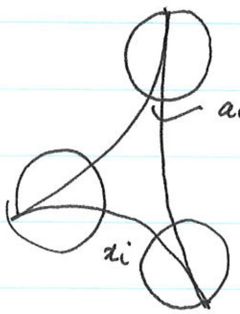
Corollary: $\exists$ a strictly convex function $F(x)$ s.t. $x \in \mathbb{R}^E$

$$\nabla F = \varphi_0(x)$$

$\Rightarrow$ the result.

(Guo-L)

A <u>Variational principle</u> consider decorated ideal triangle of lengths $x_1, x_2, x_3 \in \mathbb{R}$ and angles a_1, a_2, a_3



Let $y_i = -(a_j + a_k) + a_i$ $\quad \{i,j,k\} = \{1,2,3\}$

$= y_i(x)$

<u>Then</u> $A = \left[\dfrac{\partial y_i}{\partial x_j} \right]_{3\times 3}$ is symmetric and positive definite

HW direct <u>check</u>: $a_i = e^{\frac{1}{2}(x_i - x_j - x_k)}$.

<u>Corollary</u>. $\exists$ a strictly $\cancel{\text{concave}}$ convex function $F : \mathbb{R}^3 \to \mathbb{R}$ s.t

$$\nabla F(x) = (y_1, y_2, y_3)$$

$$F(x) = \int_0^x \left(\sum_{i=1}^{3} y_i \, dx_i \right)$$

A symmetric $\iff d\left(\sum y_i dx_i\right) = 0$

A negative def $\iff$ Hessian of $F = A < 0$

$\Rightarrow F$ strictly concave.

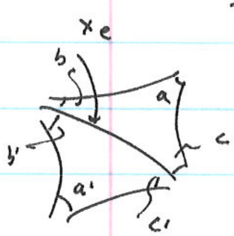
<u>Now</u> $\forall \; x \in \mathbb{R}^E$, define $W(x) : \mathbb{R}^E \to \mathbb{R}$ by

(strictly concave) $$W(x) = \sum_{x_i \triangle_{x_j} x_k \in F(T)} G(x_i, x_j, x_k)$$

$\forall\, e \in E$

$$\frac{\partial W(x)}{\partial x_e} = a + a' - b - b' - c - c' = \Phi_0(x)$$

$\uparrow$ Φ_0-<u>curvature</u> dis



$$\nabla W = \Phi_0$$

<u>But</u> <u>Lemma</u> $\Omega \subset \mathbb{R}^N$ open $\cancel{\text{convex}}$ convex $W : \Omega \to \mathbb{R}$

s.t concave $\Rightarrow \nabla W : \Omega \to \mathbb{R}^N$ 1-1

$\Rightarrow \nabla W(x) : \mathbb{R}^E \to \mathbb{R}^E$ is <u>1-1</u>

$x \mapsto \Phi_0(x)$