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Boundary Integral Method for Thermoelastic Screen Scattering Problem in $\mathbb{R}^3$

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We investigate a three-dimensional mathematical thermoelastic scattering problem from an open surface which will be referred to as a screen. Under the assumption of the local finite energy of the unified thermoelastic scattered field, we give a weak model on the appropriate Sobolev spaces and derive equivalent integral equations of the first kind for the jump of some trace operators on the open surface. Uniqueness and existence theorems are proved, the regularity and the singular behaviour of the solution near the edge are established with the help of the Wiener–Hopf method in the halfspace, the calculus of pseudodifferential operators on the basis of the strong ellipticity property and Gårding's inequality. An improved Galerkin scheme is provided by simulating the singular behaviour of the exact solution at the edge of the screen. Copyright © 2000 John Wiley & Sons, Ltd.

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1. Introduction

Problems connected with the scattering of waves by a very thin obstacle have became very important, finding application especially in non-destructive tests of structure reconstruction. This work presents a complete analysis of three-dimensional thermoelastic scattering problems from an open surface 'screen'. A time-harmonic incident wave propagating in an isotropic, homogeneous medium is scattered by a screen, which can be a rigid scatterer or cavity either under constant temperature or in a thermally insulating condition. The scattered field satisfies the governing equation of coupled Biot system for thermoelastic oscillations together with boundary conditions on the screen, which depend on the physics of the screen and the incident

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wave. Moreover, in all cases the Kupradze asymptotic radiation conditions are assumed to hold at infinity.

In Section 2 we introduce the coupled thermoelastic differential and boundary operators, fundamental tensors and a unified formulation of the scattering problem in terms of the four-dimensional scattered field in which the first three components correspond to the displacement field and the last one to the temperature field. The mathematical theory of thermoelasticity is systematically given by Kupradze [14], Leis [15] and Nowacki [16]. Dassios and co-authors in [2, 3, 7] have considered thermoelastic scattering by a closed 'inclusion' in an isotropic medium. The potential methods for boundary value problems in closed domains are recently developed for anisotropic media by Jentsch and Natroshvili in [12, 13]. In the present work we consider the scattering from an open surface, which presents difficulties for both the mathematical analysis and the numerical approximation due to the fact that the solutions have a singularity at the edge of the screen. Similar problems have been investigated by Costabel, Duduchava, Kress, Stephan, Wendland [6, 17–20, 22, 23, 4, 8] in acoustic wave propagation, elasticity, and Stokes flows such as crack and screen problems.

Assuming that the unified thermoelastic scattered field has local finite energy near the screen, we present a weak model in appropriate Sobolev spaces, prove its unique solvability and derive equivalent integral equations of the first kind for the jump of some trace operators on the open surface. This is accomplished by using thermoelastic potential operators and Green's type integral formulae, which are developed in Appendix A. Although the integral operators possess different kernel singularities, either weak or hypersingular according to the considered boundary conditions, we show that they all belong to the class of strong elliptic pseudodifferential operators. In Section 4 we prove existence and consider the singular behaviour of the solution near the edge with the help of the Wiener–Hopf method in the halfspace developed by Eskin [9]. Duduchava and Wendland [8] avoided the explicit factorization of the matrix symbol, proving a new implicit version of the factorization theorem. However, here we can provide the explicit factorization for 4×4 matrix-valued two-dimensional symbols of the pseudodifferential operators, and decompose the solution of the integral equations into a regular and singular part.

We finally present a Galerkin scheme for the solution of the integral equations on the screen. In order to improve the asymptotic convergence we augment the boundary elements with special singular functions according to the singular behaviour of the exact solution at the edge of the screen. Our boundary element models for the four thermoelastic screen problems are similar to the boundary element model of the elastostatic crack problem and of the Stokes problem for an open pipe. For this reason we refer to the convergence and error analysis previously demonstrated in [6, 23].

2. Thermoelastic scattering by a screen

Let the open obstacle Γ be a bounded, simply connected, orientable smooth surface in $\mathbb{R}^3$ with a smooth non-self-intersecting boundary γ. The complement of Γ is occupied by the medium of propagation, a linear isotropic and homogeneous thermoelastic medium of Biot type. The Biot medium is characterized by the Lamé constants λ, μ, the constant mass density ρ, the coefficient of thermal diffusity κ and the coupling constants γ and η. Suppressing the time-harmonic dependence $e^{-i\omega t}$, the Biot system assumes the following spectral form:

$$\mu\Delta\mathbf{u}(\mathbf{r}) + (\lambda + \mu)\nabla(\nabla\cdot\mathbf{u}(\mathbf{r})) + \rho\omega^2\mathbf{u}(\mathbf{r}) = \gamma\nabla\theta(\mathbf{r}) \tag{2.1}$$

$$\Delta\theta(\mathbf{r}) + \frac{i\omega}{\kappa}\theta(\mathbf{r}) = i\omega\eta\nabla\cdot\mathbf{u}(\mathbf{r}) \tag{2.2}$$

where $\mathbf{u}$ is the elastic displacement field, θ denotes the temperature variation field, $\mathbf{r} = (x_1, x_2, x_3) \in \mathbb{R}^3$, and $q = i\omega/\kappa$ is a spectral thermal constant. A unified formulation of the thermoelastic problem can be obtained by introducing four-dimensional vector fields

$$\mathbf{U}(\mathbf{r}) = (\mathbf{u}(\mathbf{r}), \theta(\mathbf{r})) = (u_1(\mathbf{r}), u_2(\mathbf{r}), u_3(\mathbf{r}), \theta(\mathbf{r})) \tag{2.3}$$

The system formed by (2.1) and (2.2) is written as

$$\mathbf{L}(\partial_r)\mathbf{U}(\mathbf{r}) = \begin{bmatrix} (\mu\Delta + \rho\omega^2)\mathbf{I}_3 + (\lambda+\mu)\nabla\nabla\cdot & -\gamma\nabla \\ q\kappa\eta\nabla\cdot & \Delta + q \end{bmatrix}\mathbf{U}(\mathbf{r}) = \mathbf{0} \tag{2.4}$$

where the time-independent thermoelastic operator $\mathbf{L}(\partial_r)$ is a 4×4 matrix elliptic differential operator with the determinant of the principal symbol, the positive constant $\mu^2(\lambda + 2\mu)$. Note that the thermoelastic operator $\mathbf{L}(\partial_r)$ is not self-adjoint and the adjoint operator $\mathbf{L}^*(\partial_r)$ may be obtained from $\mathbf{L}(\partial_r)$ by replacing γ with $i\omega\eta$ and *vice versa*. The lack of the symmetry of the Biot system is reflected in the lack of self-adjointness of the operator $\mathbf{L}(\partial_r)$.

It is known from Kupradze [14] that the solution $\mathbf{U} \in \mathbf{C}^2(\mathbb{R}^3)$ of equation (2.4) admits the following representation:

$$\mathbf{U}(\mathbf{r}) = \mathbf{U}^1(\mathbf{r}) + \mathbf{U}^2(\mathbf{r}) + \mathbf{U}^s(\mathbf{r}) \tag{2.5}$$

with

$$\mathbf{U}^1(\mathbf{r}) = (\mathbf{u}^1(\mathbf{r}), \theta^1(\mathbf{r})),\ \mathbf{U}^2(\mathbf{r}) = (\mathbf{u}^2(\mathbf{r}), \theta^2(\mathbf{r})),\ \mathbf{U}^s(\mathbf{r}) = (\mathbf{u}^s(\mathbf{r}), 0),$$

such that the three-dimensional displacement fields $\mathbf{u}^1(\mathbf{r})$, $\mathbf{u}^2(\mathbf{r})$, $\mathbf{u}^s(\mathbf{r})$, satisfy the following vectorial Helmholtz equations:

$$\begin{aligned} &(\Delta + k_1^2)\mathbf{u}^1(\mathbf{r}) = \mathbf{0} \quad &&(\Delta + k_2^2)\mathbf{u}^2(\mathbf{r}) = \mathbf{0} \quad &&(\Delta + k_s^2)\mathbf{u}^s(\mathbf{r}) = \mathbf{0} \\ &\operatorname{curl}\mathbf{u}^1 = 0, &&\operatorname{curl}\mathbf{u}^2 = 0, &&\operatorname{div}\mathbf{u}^s = 0, \end{aligned} \tag{2.6}$$

and the scalar temperature fields satisfy the following scalar Helmholtz equations:

$$(\Delta + k_1^2)\theta^1(\mathbf{r}) = 0, \quad (\Delta + k_2^2)\theta^2(\mathbf{r}) = 0. \tag{2.7}$$

The dispersion relations characterizing (2.4) are given by

$$k_1^2 + k_2^2 = q(1+\varepsilon) + k_p^2, \quad k_1^2 k_2^2 = q k_p^2, \quad \mu k_s^2 = \rho\omega^2 \tag{2.8}$$

where k_1, k_2, such that $k_j = \omega/v_j + \mathrm{i}d_j$, $v_j > 0$, $d_j > 0$, $j = 1, 2$, are the complex wave numbers of the elastothermal and thermoelastic waves respectively; $k_s = \omega\sqrt{\rho/\mu}$ is the wavenumber of the uncoupled transverse wave; $k_p = \omega\sqrt{\rho/(\lambda + 2k)}$ is the wavenumber of the longitudinal wave in the absence of thermal coupling and $\varepsilon = \gamma\eta\kappa/(\lambda + 2\mu)$ is the dimensionless thermoelastic coupling constant. From (2.8) we see that the transverse elastic wave is not affected by the existence of the temperature field, and it behaves exactly in the same way as it does in the classical theory of elasticity.

The fundamental unified thermoelastic tensor $\mathbf{E}(\mathbf{r}, \mathbf{r}')$: $\mathbb{R}^3/\{0\} \to \mathbb{C}^{4\times 4}$ is written in a compact form

$$\mathbf{E}(\mathbf{r}, \mathbf{r}') = \mathbf{E}^1(\mathbf{r}, \mathbf{r}') + \mathbf{E}^2(\mathbf{r}, \mathbf{r}') + \mathbf{E}^s(\mathbf{r}, \mathbf{r}'), \tag{2.9}$$

where

$$\mathbf{E}^1(\mathbf{r}, \mathbf{r}') = \frac{-1}{\rho\omega^2(k_1^2 - k_1^2)} \begin{bmatrix} (k_p^2 - k_2^2)\nabla_r\nabla_r & \gamma k_p^2 \nabla_r \\ -\kappa\eta q k_p^2 \nabla_r & \rho(k_p^2 - k_1^2)\omega^2 \end{bmatrix} \frac{\mathrm{e}^{\mathrm{i}k_1|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}, \tag{2.10}$$

$$\mathbf{E}^2(\mathbf{r}, \mathbf{r}') = \frac{-1}{\rho\omega^2(k_1^2 - k_2^2)} \begin{bmatrix} (k_p^2 - k_1^2)\nabla_r\nabla_r & \gamma k_p^2 \nabla_r \\ -\kappa\eta q k_p^2 \nabla_r & \rho(k_p^2 - k_2^2)\omega^2 \end{bmatrix} \frac{\mathrm{e}^{\mathrm{i}k_2|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}, \tag{2.11}$$

$$\mathbf{E}^s(\mathbf{r}, \mathbf{r}') = \frac{1}{\rho\omega^2} \begin{bmatrix} (k_s^2 \mathbf{I}_3) + \nabla_r\nabla_r & 0 \\ 0 & 0 \end{bmatrix} \frac{\mathrm{e}^{\mathrm{i}k_s|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \tag{2.12}$$

and it behaves in any neighbourhood $R = |\mathbf{r} - \mathbf{r}'| < \varepsilon$ according to

$$|\mathbf{E}_{kj}(\mathbf{r}, \mathbf{r}')| \leqslant \frac{C}{R}, \quad \left|\frac{\partial}{\partial x_{l_1}} \mathbf{E}_{kj}(\mathbf{r}, \mathbf{r}')\right| \leqslant \frac{C}{R^2}, \quad \left|\frac{\partial^2}{\partial x_{l_1}\partial x_{l_2}} \mathbf{E}_{kj}(\mathbf{r}, \mathbf{r}')\right| \leqslant \frac{C}{R^3} \tag{2.13}$$

for $j, k = 1, 2, 3, 4$, $l_1, l_2, = 1, 2, 3$ and C a positive constant.

Let us suppose now that the screen is excited by a given thermoelastic incident wave $\Phi(\mathbf{r})$ which is an entire solution of the thermoelastic equation (2.4) in $\mathbb{R}^3$. The *direct scattering problem* asks for the total field $\Psi(\mathbf{r}) = \Phi(\mathbf{r}) + \mathbf{U}(\mathbf{r})$ such that the scattered field $\mathbf{U}(\mathbf{r})$ satisfies the thermoelastic equation (2.4) in the exterior of the screen $\mathbb{R}^3/\bar{\Gamma}$ and the boundary condition $\mathbf{B}(\partial_r, \hat{\mathbf{n}})\mathbf{U}(\mathbf{r}) = -\mathbf{B}(\partial_r, \hat{\mathbf{n}})\Phi(\mathbf{r})$ on the screen Γ. We also assume that the scattered field satisfies the Kupradze radiation conditions as $r \to \infty$ for $i = 1, 2, 3$ and $j = 1, 2$

$$\mathbf{u}^j(\mathbf{r}) = o\left(\frac{1}{r}\right), \quad \partial_{x_i}\mathbf{u}^j(\mathbf{r}) = O\left(\frac{1}{r^2}\right), \quad \theta^j(\mathbf{r}) = o\left(\frac{1}{r}\right), \quad \partial_{x_i}\theta^j(\mathbf{r}) = O\left(\frac{1}{r^2}\right),$$

$$\mathbf{u}^s(\mathbf{r}) = O\left(\frac{1}{r}\right), \quad r(\partial_{x_i}\mathbf{u}^s(\mathbf{r}) - ik_s\mathbf{u}^s(\mathbf{r})) = O\left(\frac{1}{r}\right). \tag{2.14}$$

For $\mathbf{B}(\partial_r, \hat{\mathbf{n}})$ we consider one of the following four thermoelastic boundary operators:

$$\mathbf{B}_1(\partial_r, \hat{\mathbf{n}}) = \mathbf{I}_4 = \begin{bmatrix} \mathbf{I}_3 & 0 \\ 0 & 1 \end{bmatrix}, \tag{2.15}$$

$$\mathbf{B}_2(\partial_r, \hat{\mathbf{n}}) = \mathbf{R}(\partial_r, \hat{\mathbf{n}}) = \begin{bmatrix} \mathbf{T}(\partial_r, \hat{\mathbf{n}}) & -\gamma\hat{\mathbf{n}} \\ 0 & \partial_n \end{bmatrix}, \tag{2.16}$$

$$\mathbf{B}_3(\partial_r, \hat{\mathbf{n}}) = \begin{bmatrix} \mathbf{I}_3 & 0 \\ 0 & \partial_n \end{bmatrix}, \tag{2.17}$$

$$\mathbf{B}_4(\partial_r, \hat{\mathbf{n}}) = \begin{bmatrix} \mathbf{T}(\partial_r, \hat{\mathbf{n}}) & -\gamma\hat{\mathbf{n}} \\ 0 & 1 \end{bmatrix}, \tag{2.18}$$

where $\hat{\mathbf{n}}$ is the unit normal vector according to the chosen orientation on Γ and the operator $\mathbf{T}(\partial_r, \hat{\mathbf{n}})$ is the surface traction operator of elasticity given by

$$\mathbf{T}(\partial_r, \hat{\mathbf{n}})\mathbf{u}(\mathbf{r}) = 2\mu\hat{\mathbf{n}}\cdot\nabla\mathbf{u}(\mathbf{r}) + \lambda\hat{\mathbf{n}}\nabla\cdot\mathbf{u}(\mathbf{r}) + \mu\hat{\mathbf{n}}\times(\nabla\times\mathbf{u}(\mathbf{r})).$$

From physical point of view, the first condition corresponds to a rigid surface at constant temperature and is the Dirichlet-type thermoelastic condition. The second one corresponds to a cavity in thermal insulation and is a Neumann-type condition. Finally, the third and the fourth correspond to a rigid surface in thermal insulation and to a cavity at constant temperature, respectively. Both are mixed-type Dirichlet and Neumann on the displacement and temperature fields.

3. Boundary integral equations

In order to obtain the total thermoelastic field from the knowledge of the incident field, we can consider a boundary value problem with the data produced by the trace of the incident field on the screen. In the slightly more general mathematical model we choose arbitrary boundary data with a non-zero jump across Γ. Henceforth, the orientation of Γ defines the normal vector $\hat{\mathbf{n}}$ pointing to the side Γ_2. The opposite side of Γ will be denoted by Γ_1. Thus, the weak formulation of the original scattering screen problems reads:

For *given* $\mathbf{G}_1 = (\mathbf{g}_1, t_1)$ and $\mathbf{G}_2 = (\mathbf{g}_2, t_2)$ two four-dimensional vector-valued functions defined on Γ satisfying

(i) *in the case of the first boundary value problem* (*Dirichlet type*)

$$\mathbf{g}_1, \mathbf{g}_2 \in (\mathbf{H}^{1/2}(\Gamma))^3;\ t_1, t_2 \in \mathbf{H}^{1/2}(\Gamma) \text{ and } [\mathbf{g}] \in (\tilde{\mathbf{H}}^{1/2}(\Gamma))^3;\ [t] \in \tilde{\mathbf{H}}^{1/2}(\Gamma);$$

(ii) *in the case of the second boundary value problem* (*Neumann type*)

$$\mathbf{g}_1, \mathbf{g}_2 \in (\mathbf{H}^{-1/2}(\Gamma))^3;\ t_1, t_2 \in \mathbf{H}^{-1/2}(\Gamma) \text{ and } [\mathbf{g}] \in (\tilde{\mathbf{H}}^{-1/2}(\Gamma))^3;\ [t] \in \tilde{\mathbf{H}}^{-1/2}(\Gamma);$$

(iii) *in the case of the third boundary value problem (mixed type)*

$$\mathbf{g}_1, \mathbf{g}_2 \in (\mathbf{H}^{1/2}(\Gamma))^3;\ t_1, t_2 \in \mathbf{H}^{-1/2}(\Gamma) \text{ and } [\mathbf{g}] \in (\tilde{\mathbf{H}}^{1/2}(\Gamma))^3;\ [t] \in \tilde{\mathbf{H}}^{-1/2}(\Gamma);$$

(iv) *in the case of the fourth boundary value problem (mixed type)*

$$\mathbf{g}_1, \mathbf{g}_2 \in (\mathbf{H}^{-1/2}(\Gamma))^3;\ t_1, t_2 \in \mathbf{H}^{1/2}(\Gamma) \text{ and } [\mathbf{g}] \in (\tilde{\mathbf{H}}^{-1/2}(\Gamma))^3;\ [t] \in \tilde{\mathbf{H}}^{1/2}(\Gamma);$$

find $\mathbf{U} \in (\mathbf{H}^1_{\mathrm{loc}}(\mathbb{R}^3/\bar{\Gamma}))^4$ which solves the thermoelastic equation (2.4) in $\mathbb{R}^3/\bar{\Gamma}$, satisfies the boundary conditions $(\mathbf{B}_k\mathbf{U})|_{\Gamma_1} = \mathbf{G}_1$ and $(\mathbf{B}_k\mathbf{U})|_{\Gamma_2} = \mathbf{G}_2$ for a given $k = 1, 2, 3, 4$ and Kupradze radiation conditions (2.14) as $r \to \infty$.

We denote $[\mathbf{g}] = \mathbf{g}_1 - \mathbf{g}_2$, $[t] = t_1 - t_2$ and define for the closed surface $\partial\Omega$ with a piece $\Gamma \subset \partial\Omega$ the spaces $\mathbf{H}^s(\Gamma) = \{u|_\Gamma\colon u \in \mathbf{H}^s(\partial\Omega)\}$ and

$$\tilde{\mathbf{H}}^s(\Gamma) = \{u \in \mathbf{H}^s(\partial\Omega)\},\ \operatorname{supp} u \subset \bar{\Gamma}, \text{ equipped with the } \mathbf{H}^s(\partial\Omega)\text{-norm}\}.$$

Theorem 3.1. (*Uniqueness*). *The four thermoelastic boundary value problems for the open set* Γ *have at most one weak solution.*

Proof. For the proof and further analysis, we extend Γ to an arbitrary smooth, simply connected, closed, orientable surface $\partial\Omega$ enclosing the bounded domain Ω_1. Furthermore, let $B(o, R)$ be a sufficiently large ball with radius R including $\bar{\Omega}_1$, and let $\Omega_2 := B(o, R) \cap (\mathbb{R}^3/\bar{\Omega}_1)$ and ∂B denotes the boundary of $B(o, R)$. The jump across $\partial\Omega$ is defined as $[f] = f_1 - f_2$ where the subscripts (1) and (2) indicate the limit respectively from Ω_1 and $\mathbb{R}^3/\bar{\Omega}_1$ to $\partial\Omega$. For the sake of notational consistency, we assume that the normal vector according to the orientation of the screen is pointed to Ω_2. With $\partial/\partial n$ we denote the exterior normal derivative to the closed surface $\partial\Omega$.

To concretize we consider the second homogeneous boundary value problem; $\mathbf{U} = (\mathbf{u}, \theta) \in (\mathbf{H}^1_{\mathrm{loc}}(\mathbb{R}^3/\bar{\Gamma}))^4$ solves the thermoelastic equation (2.4), Kupradze radiation conditions (2.14) and the zero boundary conditions $(\mathbf{R}\mathbf{U})|_{\Gamma_1} = (\mathbf{R}\mathbf{U})|_{\Gamma_2} = 0$. Then $\mathbf{U} = (\mathbf{u}, \theta) \in (\mathbf{H}^1(\Omega_1))^4 \cup (\mathbf{H}^1_{\mathrm{loc}}(\mathbb{R}^3/\bar{\Omega}_1))^4$ solves the transmission problem

$$\mathbf{U} = \mathbf{U}_1 \quad \text{with } \mathbf{L}\mathbf{U}_1 = \mathbf{0} \quad \text{in } \Omega_1,$$
$$\mathbf{U} = \mathbf{U}_2 \quad \text{with } \mathbf{L}\mathbf{U}_2 = \mathbf{0} \quad \text{in } \mathbb{R}^3/\bar{\Omega}_1$$

satisfying radiation conditions (2.14) and the boundary transmission conditions

$$(\mathbf{T}\mathbf{u}_1 - \gamma\hat{\mathbf{n}}\theta_1, \partial\theta_1/\partial n)_1 = (\mathbf{T}\mathbf{u}_2 - \gamma\hat{\mathbf{n}}\theta_2, \partial\theta_2/\partial n)_2 \quad \text{on } \partial\Omega$$
$$(\mathbf{u}_1, \theta_1)_1 = (\mathbf{u}_2, \theta_2)_2 \quad \text{on } \partial\Omega/\bar{\Gamma}. \tag{3.1}$$

Applying the thermoelastic Green's integral formulae (see Appendix A.1) in Ω_1 and Ω_2, adding both equalities and considering the transmission conditions to eliminate the integral over $\partial\Omega$ we obtain

$$\frac{2\gamma}{\mathrm{i}\omega\eta}\left(\int_{\Omega_1} |\nabla\theta_1|^2\,\mathrm{d}\omega + \int_{\Omega_2} |\nabla\theta_2|^2\,\mathrm{d}\omega\right) \tag{3.2}$$
$$= \oint_{\partial B}\left(\bar{\mathbf{u}}_2\cdot(\mathbf{T}\mathbf{u}_2 - \gamma\hat{\mathbf{n}}\theta_2) + \frac{\gamma}{\mathrm{i}\omega\eta}\bar{\theta}_2\frac{\partial\theta_2}{\partial n} - \mathbf{u}_2\cdot(\mathbf{T}\bar{\mathbf{u}}_2 - \gamma\hat{\mathbf{n}}\bar{\theta}_2) + \frac{\gamma}{\mathrm{i}\omega\eta}\theta_2\frac{\partial\bar{\theta}_2}{\partial n}\right)\mathrm{d}s.$$

The radiation conditions (2.14) on the parts $\mathbf{u}_2^1$, $\mathbf{u}_2^2$, θ_2^1, θ_2^2 and some easy modifications yield, as $R \to \infty$,

$$\frac{2\gamma}{\mathrm{i}\omega\eta}\left(\int_{\Omega_1} |\nabla\theta_1|^2\,\mathrm{d}\omega + \int_{\Omega_2} |\nabla\theta_2|^2\,\mathrm{d}\omega\right) = 2\mathrm{i}\omega\sqrt{\rho\mu}\oint_{\partial B} |\mathbf{u}_2^s|^2\,\mathrm{d}s$$

$$+ \oint_{\partial B} (\bar{\mathbf{u}}_2^s\cdot(\mathbf{T}\mathbf{u}_2^s - \mathrm{i}\omega\sqrt{\rho\mu}\mathbf{u}_2^s) - \mathbf{u}_2^s\cdot(\mathbf{T}\bar{\mathbf{u}}_2^s + \mathrm{i}\omega\sqrt{\rho\mu}\bar{\mathbf{u}}_2^s)\,\mathrm{d}s + o(1). \tag{3.3}$$

It is easy to show that the radiation conditions for the transverse part $\mathbf{u}_s$ may be rewritten as $\mathbf{T}\mathbf{u}^s - \mathrm{i}\omega\sqrt{\rho\mu}\mathbf{u}^s = \mathrm{o}(r^{-1})$ and hence $\mathbf{T}\bar{\mathbf{u}}^s + \mathrm{i}\omega\sqrt{\rho\mu}\bar{\mathbf{u}}^s = \mathrm{o}(r^{-1})$. After passing to the limit in (3.3) as $R \to \infty$, we obtain

$$\lim_{R\to\infty}\oint_{\partial B} |\mathbf{u}_2^s|^2\,\mathrm{d}s = 0 \text{ and } \int_{\Omega_1} |\nabla\theta_1|^2\,\mathrm{d}\omega = \int_{\Omega_1} |\nabla\theta_2|^2\,\mathrm{d}\omega = 0.$$

From the first relation, using Rellich's lemma [5] we get that $\mathbf{u}_2^s = 0$ and by the transmission formulation $\mathbf{u}_1^s = 0$. Consequently $\mathbf{u}^s = 0$ in $\mathbb{R}^3/\bar{\Gamma}$. The second relation yields $\theta_2 = c$ and by radiation conditions it satisfies $\theta_2 = 0$. Moreover, $\theta_1 = c$ and hence $\theta_1 = 0$ by the transmission conditions. Both mean that $\theta = 0$ in $\mathbb{R}^3/\bar{\Gamma}$. Finally, if we substitute the thermoelastic equation (2.4) for the divergence-free part of the displacement field $\mathbf{u}^s = 0$ and the temperature vibration field $\theta = 0$ we get for the rotation-free part of the displacement field $\mathbf{u}^1 + \mathbf{u}^2 = 0$ in $\mathbb{R}^3/\bar{\Gamma}$. Thus, we conclude the uniqueness for the second boundary problem.

The uniqueness for the other three boundary problems follows exactly the same arguments if one considers the associated transmission problem as in (3.1) with the appropriate boundary transmission conditions, namely:

(i) for the first homogeneous boundary value problem

$$(\mathbf{u}_1, \theta_1)_1 = (\mathbf{u}_1, \theta_2)_2 \quad \text{on} \quad \partial\Omega,$$

$$(\mathbf{T}\mathbf{u}_1 - \gamma\hat{\mathbf{n}}\theta_1, \partial\theta_1/\partial n)_1 = (\mathbf{T}\mathbf{u}_2 - \gamma\hat{\mathbf{n}}\theta_2, \partial\theta_2/\partial n)_2 \quad \text{on} \quad \partial\Omega/\bar{\Gamma}; \tag{3.4}$$

(ii) for the third homogeneous boundary value problem

$$(\mathbf{u}_1, \partial\theta_1/\partial n)_1 = (\mathbf{u}_2, \partial\theta_2/\partial n)_2 \quad \text{on} \quad \partial\Omega,$$

$$(\mathbf{T}\mathbf{u}_1 - \gamma\hat{\mathbf{n}}\theta_1, \theta_1)_1 = (\mathbf{T}\mathbf{u}_2 - \gamma\hat{\mathbf{n}}\theta_2, \theta_2)_2 \quad \text{on} \quad \partial\Omega/\bar{\Gamma}; \tag{3.5}$$

(iii) for the fourth homogeneous boundary value problem

$$(\mathbf{T}\mathbf{u}_1 - \gamma\hat{\mathbf{n}}\theta_1, \theta_1)_1 = (\mathbf{T}\mathbf{u}_2 - \gamma\hat{\mathbf{n}}\theta_2, \theta_2)_2 \quad \text{on} \quad \partial\Omega,$$

$$(\mathbf{u}_1, \partial\theta_1/\partial n)_1 = (\mathbf{u}_2, \partial\theta_2/\partial n)_2 \quad \text{on} \quad \partial\Omega/\bar{\Gamma}. \tag{3.6}$$

For the formulation and analysis of properties of the boundary integral equations on Γ we shall need the following property of the traces of the weak thermoelastic solutions.

Lemma 3.1. *Let* $\mathbf{U}=(\mathbf{u},\theta)\in(\mathbf{H}^1_{\text{loc}}(\mathbb{R}^3/\bar{\Gamma}))^4$ *be the weak solution of*:

(a) *The first boundary value problem*; *then*

$$[\mathbf{Tu}-\gamma\hat{\mathbf{n}}\tilde{\theta}]|_\Gamma\in(\tilde{\mathbf{H}}^{-1/2}(\Gamma))^3 \quad \textit{and} \quad \left[\frac{\partial\theta}{\partial n}\right]\Big|_\Gamma\in\tilde{\mathbf{H}}^{-1/2}(\Gamma). \tag{3.7}$$

(b) *The second boundary value problem*; *then*

$$[\mathbf{u}]|_\Gamma\in(\tilde{\mathbf{H}}^{1/2}(\Gamma))^3 \quad \textit{and} \quad [\theta]|_\Gamma\in\tilde{\mathbf{H}}^{1/2}(\Gamma). \tag{3.8}$$

(c) *The third boundary value problem*; *then*

$$[\mathbf{Tu}-\gamma\hat{\mathbf{n}}\tilde{\theta}]|_\Gamma\in(\tilde{\mathbf{H}}^{-1/2}(\Gamma))^3 \quad \textit{and} \quad [\theta]|_\Gamma\in\tilde{\mathbf{H}}^{1/2}(\Gamma). \tag{3.9}$$

(d) *The fourth boundary value problem*, *then*

$$[\mathbf{u}]|_\Gamma\in(\tilde{\mathbf{H}}^{1/2}(\Gamma))^3 \quad \textit{and} \quad \left[\frac{\partial\theta}{\partial n}\right]\Big|_\Gamma\in\tilde{\mathbf{H}}^{-1/2}(\Gamma). \tag{3.10}$$

Note that θ and $\tilde{\theta}$ are isomorph elements from the Riesz Representation Theorem in the dual spaces $\mathbf{H}^{1/2}$ and $\mathbf{H}^{-1/2}$, respectively.

Proof. Let the first case of Dirichlet condition be considered. For a weak solution $\mathbf{U}=(\mathbf{u},\theta)\in(\mathbf{H}^1_{\text{loc}}(\mathbb{R}^3/\bar{\Gamma}))^4$, we apply a test vector-valued function $\mathbf{V}=(\mathbf{v},\vartheta)\in(\mathbf{H}^1_{\text{loc}}(\mathbb{R}^3/\bar{\Gamma}))^4$ such that supp $\mathbf{V}\subset\subset B(o,R)$. The thermoelastic variational formula (see Appendix A.1) for each Ω_j, $j=1,2$, reads

$$\int_{\Omega_j}\left(\mathscr{E}(\mathbf{u},\mathbf{v})+\nabla\mathbf{u}\cdot\nabla\mathbf{v}-\mathrm{i}\omega\eta\nabla\cdot\mathbf{u}\vartheta-\gamma\theta\nabla\cdot\mathbf{v}-\rho\omega^2\mathbf{u}\cdot\mathbf{v}-\frac{\mathrm{i}\omega}{\kappa}\theta\vartheta\right)\mathrm{d}\omega$$
$$=\oint_{\partial\Omega_j}\left((\mathbf{Tu}-\gamma\hat{\mathbf{n}}\tilde{\theta})\cdot\mathbf{v}+\frac{\partial\theta}{\partial n}\vartheta\right)\mathrm{d}s. \tag{3.11}$$

From the trace theorem, we have that $\mathbf{V}|_{\partial\Omega_j}\in(\mathbf{H}^{1/2}(\partial\Omega_j))^4$, and by the duality in (3.11) we conclude that

$$(\mathbf{Tu}-\gamma\hat{\mathbf{n}}\tilde{\theta})|_{\partial\Omega_j}\in(\mathbf{H}^{-1/2}(\partial\Omega_j))^3 \quad \text{and} \quad \frac{\partial\theta}{\partial n}\Big|_{\partial\Omega_j}\in\mathbf{H}^{-1/2}(\partial\Omega_j).$$

Hence, the jumps satisfy

$$[(\mathbf{Tu}-\gamma\hat{\mathbf{n}}\tilde{\theta})]|_{\partial\Omega_1}\in(\mathbf{H}^{-1/2}(\partial\Omega_1))^3 \quad \text{and} \quad \left[\frac{\partial\theta}{\partial n}\right]\Big|_{\partial\Omega_1}\in\mathbf{H}^{-1/2}(\partial\Omega_1).$$

In addition, we see from the transmission formulation (3.4) that

$$\operatorname{supp}[\mathbf{Tu}-\gamma\hat{\mathbf{n}}\tilde{\theta}]|_{\partial\Omega_1}\subset\bar{\Gamma} \quad \text{and} \quad \operatorname{supp}\left[\frac{\partial\theta}{\partial n}\right]\Big|_{\partial\Omega_1}\subset\bar{\Gamma}.$$

This proves the assertion in (3.7).

The other assertions in (3.8)–(3.10) follow from the variational formula (3.11) by similar arguments.

Remark 3.1. A weak solution of the four boundary screen problems may be written in the following variational form:

$$\int_{\mathbb{R}^3/\Gamma}\left(\mathscr{E}(\mathbf{u},\mathbf{v})+\nabla\mathbf{u}\cdot\nabla\mathbf{v}-\mathrm{i}\omega\eta\nabla\cdot\mathbf{u}\vartheta-\gamma\theta\nabla\cdot\mathbf{v}-\rho\omega^2\mathbf{u}\cdot\mathbf{v}-\frac{\mathrm{i}\omega}{\kappa}\theta\vartheta\right)\mathrm{d}r$$

$$=\int_{\Gamma}\left((\mathbf{Tu}-\gamma\hat{\mathbf{n}}\widetilde{\theta})_1\cdot\mathbf{v}_1+\left(\frac{\partial\theta}{\partial n}\right)_1\vartheta_1-(\mathbf{Tu}-\gamma\hat{\mathbf{n}}\widetilde{\theta})_2\cdot\mathbf{v}_2-\left(\frac{\partial\theta}{\partial n}\right)_2\vartheta_2\right)\mathrm{d}s$$

for all test vector-valued functions $\mathbf{V}=(\mathbf{v},\vartheta)\in(\mathbf{H}^1_{\mathrm{loc}}(\mathbb{R}^3/\bar{\Gamma}))^4$ having support compact in $\mathbb{R}^3$. The boundary conditions related to the considered problem are substituted on the right-hand side of the above integral equality. Here the subindex (2) denotes the limit from the side to which the normal vector is pointed and (1) from the other side.

The properties of the boundary data, as stated in Lemma 3.1, enable us to derive boundary integral equations of the first kind with singular kernel equivalent to the boundary problems.

Green's formula (Appendix A.1) provides a representation of the weak thermoelastic solution $\mathbf{U}$ by single- and double-layer potentials. For any fixed $\mathbf{r}\in\boldsymbol{\Omega}_1$ we have

$$\mathbf{U}(\mathbf{r})=-\frac{1}{4\pi}\oint_{\partial\Omega}[\mathbf{U}_1(\mathbf{r}')\cdot\mathbf{R}^*(\partial_{r'},\hat{\mathbf{n}})\mathbf{E}^{\mathrm{T}}(\mathbf{r},\mathbf{r}')-\mathbf{E}(\mathbf{r},\mathbf{r}')\cdot\mathbf{R}(\partial_{r'},\hat{\mathbf{n}})\mathbf{U}_1(\mathbf{r}')]\,\mathrm{d}s(r'),\tag{3.12}$$

$$0=-\frac{1}{4\pi}\oint_{\partial\Omega}[\mathbf{U}_2(\mathbf{r}')\cdot\mathbf{R}^*(\partial_{r'},\hat{\mathbf{n}})\mathbf{E}^{\mathrm{T}}(\mathbf{r},\mathbf{r}')-\mathbf{E}(\mathbf{r},\mathbf{r}')\cdot\mathbf{R}(\partial_{r'},\hat{\mathbf{n}})\mathbf{U}_2(\mathbf{r}')]\,\mathrm{d}s(r').\tag{3.13}$$

This representation formula is known for smooth boundary and smooth layers $\mathbf{U}_1$, $\mathbf{U}_2$. Since the potentials have $\mathbf{C}^\infty$ kernels for $\mathbf{r}\in\boldsymbol{\Omega}_1$, they remain valid for a weak solution as well.

Let us first consider the *thermoelastic Dirichlet problem*. Summing the relations (3.12), (3.13) using the jump relations of the boundary potentials (Lemma A.1) and the result of Lemma 3.2 for the jumps $[\mathbf{Tu}-\gamma\hat{\mathbf{n}}\widetilde{\theta})$, $[\partial\theta/\partial n]$ together with the assumption on the boundary data, we obtain

$$\frac{1}{2}(\mathbf{U}_1(\mathbf{r})+\mathbf{U}_2(\mathbf{r}))|_{\partial\Omega_1}=-\frac{1}{4\pi}\int_{\Gamma}[\mathbf{G}](\mathbf{r}')\cdot\mathbf{R}^*(\partial_{r'},\hat{\mathbf{n}}')\mathbf{E}^{\mathrm{T}}(\mathbf{r},\mathbf{r}')\,\mathrm{d}s(r')$$

$$+\frac{1}{4\pi}\int_{\Gamma}\mathbf{E}(\mathbf{r},\mathbf{r}')[\mathbf{Tu}-\gamma\hat{\mathbf{n}}\widetilde{\theta},\partial\theta/\partial n]^{\mathrm{T}}(\mathbf{r}')\,\mathrm{d}s(r')$$

$$+\frac{1}{4\pi}\oint_{\partial B}[\mathbf{U}_2(\mathbf{r}')\cdot\mathbf{R}^*(\partial_{r'},\hat{\mathbf{n}})\mathbf{E}^{\mathrm{T}}(\mathbf{r},\mathbf{r}')-\mathbf{E}(\mathbf{r},\mathbf{r}')\cdot\mathbf{R}(\partial_{r'},\hat{\mathbf{n}})\mathbf{U}_2(\mathbf{r}')]\,\mathrm{d}s(r').\tag{3.14}$$

Thus, restricting this equation to Γ, letting $R \to \infty$, and using Kupradze radiation conditions (2.14) which holds for the solution $\mathbf{U}_2$ and as well as for the thermoelastic fundamental solution $\mathbf{E}$, we obtain for the jump of the thermoelastic surface traction

$$\mathbf{S}[\mathbf{Tu} - \gamma\hat{\mathbf{n}}\tilde{\theta}, \partial\theta/\partial n],(\mathbf{r}) := \frac{1}{4\pi}\int_\Gamma \mathbf{E}(\mathbf{r}, \mathbf{r}')[\mathbf{Tu} - \gamma\hat{\mathbf{n}}\tilde{\theta}, \partial\theta/\partial n]^{\mathrm{T}}(\mathbf{r}')\,\mathrm{d}s(r')$$

$$= \frac{1}{2}(\mathbf{G}^+ + \mathbf{G}^-) + \frac{1}{4\pi}\int_\Gamma [\mathbf{G}](\mathbf{r}')\cdot\mathbf{R}^*(\partial_{r'}, \hat{\mathbf{n}}')\mathbf{E}^{\mathrm{T}}(\mathbf{r}, \mathbf{r}')\,\mathrm{d}s(r') \text{ for } \mathbf{r} \in \Gamma. \quad (3.15)$$

The integral equation (3.15) is a boundary integral equation of the first kind on Γ with weakly singular kernel which is evident from the asymptotic relations (2.13). Conversely, let $[\mathbf{Tu} - \gamma\hat{\mathbf{n}}\tilde{\theta}, \partial\theta/\partial n]|_\Gamma \in (\tilde{\mathbf{H}}^{-1/2}(\Gamma))^4$ be a solution of equation (3.15). Since, for $\mathbf{r} \notin \Gamma$, $\mathbf{E}_{i,j}(\mathbf{r}, \mathbf{r}')|_\Gamma \in \mathbf{H}^{1/2}(\Gamma)$ and $(\mathbf{R}^*(\partial_{r'}, \hat{\mathbf{n}}')\mathbf{E}^{\mathrm{T}}(\mathbf{r}, \mathbf{r}'))_{i,j}|_\Gamma \in \mathbf{H}^{-1/2}(\Gamma)$, together with their derivatives, it is obvious that the weakly singular potential

$$\mathbf{K}_1[\mathbf{Tu} - \gamma\hat{\mathbf{n}}\tilde{\theta}, \partial\theta/\partial n](\mathbf{r}) = \frac{1}{4\pi}\int_\Gamma \mathbf{E}(\mathbf{r}, \mathbf{r}')[\mathbf{Tu} - \gamma\hat{\mathbf{n}}\tilde{\theta}, \partial\theta/\partial n]^{\mathrm{T}}(\mathbf{r}')\,\mathrm{d}s(r')$$

and Cauchy singular potential

$$(\mathbf{K}_2[\mathbf{G}])(\mathbf{r}) = \frac{1}{4\pi}\int_\Gamma [\mathbf{G}](\mathbf{r}')\cdot\mathbf{R}^*(\partial_{r'}, \hat{\mathbf{n}}')\mathbf{E}^{\mathrm{T}}(\mathbf{r}, \mathbf{r}')\,\mathrm{d}s(r')$$

are well-defined for $\mathbf{r} \notin \Gamma$. They are respectively, pseudodifferential operators of order $-3/2$, $-1/2$ as a mapping from functions on the boundary $\partial\Omega$ into functions on Ω_1 and $\mathbb{R}^3/\bar{\Omega}_1$. Thus, for $[\mathbf{Tu} - \gamma\hat{\mathbf{n}}\tilde{\theta}, \partial\theta/\partial n]^* \in (\tilde{\mathbf{H}}^{-1/2}(\partial\Omega))^4$ and $[\mathbf{G}]^* \in (\tilde{\mathbf{H}}^{1/2}(\partial\Omega))^4$, we have that both $\mathbf{K}_1[\mathbf{Tu} - \gamma\hat{\mathbf{n}}\tilde{\theta}, \partial\theta/\partial n]^*$ and $\mathbf{K}_2[\mathbf{G}]^*$ belong to $(\mathbf{H}^1(\Omega_1))^4 \cup (\mathbf{H}^1_{\mathrm{loc}}(\Omega_2))^4$, where f^* is zero extension of f on $\partial\Omega/\bar{\Gamma}$. Moreover, their difference satisfies the thermoelastic equation, radiation conditions and the given thermoelastic Dirichlet boundary conditions on both sides of the screen Γ_1 and Γ_2.

In order to determine the order of the mentioned pseudodifferential operators we have to calculate their principal homogeneous symbol. We use the known argument of the local identification of Ω_2 with $\mathbb{R}^3_+$, Ω_1 with $\mathbb{R}^3_-$ and the boundary $\partial\Omega$ with $\mathbb{R}^2$. After introducing a basis of orthonormal co-ordinates in the cotangential bundle T_0^* (see Eskin [9, pp. 255–256]), we obtain a general representation of the principal symbol of the operators on the manifold Γ that locally coincides with the principal symbol in the halfspace with boundary $\mathbb{R}^2$. Thus, we can first take the three-dimensional Fourier transform of the kernels $\mathbf{E}$ and $\mathbf{R}^*\mathbf{E}^{\mathrm{T}}$ of the integral potentials $\mathbf{K}_1$ and $\mathbf{K}_2$ in $\mathbb{R}^3_+$ with boundary $\mathbb{R}^2$ and then patch the local results together. These calculations yield the transformed kernel $\hat{\mathbf{E}}$

$$\hat{\mathbf{E}}^1(\xi) = -\frac{c(|\xi|^2 - k_1^2)^{-1}}{\rho\omega^2(k_1^2 - k_2^2)}\begin{bmatrix} -(k_p^2 - k_2^2)\xi\cdot\xi^{\mathrm{T}} & \gamma k_p^2 \mathrm{i}\xi^{\mathrm{T}} \\ -\kappa\eta q k_p^2 \mathrm{i}\xi & \rho(k_p^2 - k_1^2)\omega^2 \end{bmatrix}, \quad (3.16)$$

$$\hat{\mathbf{E}}^2(\xi) = \frac{c(|\xi|^2 - k_2^2)^{-1}}{\rho\omega^2(k_1^2 - k_2^2)} \begin{bmatrix} -(k_p^2 - k_1^2)\xi\cdot\xi^{\mathrm{T}} & \gamma k_p^2 \mathrm{i}\xi^{\mathrm{T}} \\ -\kappa\eta q k_p^2 \mathrm{i}\xi & \rho(k_p^2 - k_2^2)\omega^2 \end{bmatrix}, \tag{3.17}$$

$$\hat{\mathbf{E}}^s(\rho) = \frac{c(|\xi|^2 - k_s^2)^{-1}}{\rho\omega^2} \begin{bmatrix} k_s^2 \mathbf{I}_3 - \xi\cdot\xi^{\mathrm{T}} & 0 \\ 0 & 0 \end{bmatrix}, \tag{3.18}$$

where $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^2$, $|\xi| = \sqrt{\xi_1^2 + \xi_2^2 + \xi_3}$ and c is a positive constant. For the calculation of the Fourier image $\widehat{\mathbf{RE}}$, it is enough to multiply each matrix $\hat{\mathbf{E}}^i(\xi)$, $i = 1, 2, s$, with the matrix $\hat{\mathbf{R}}$ by using the derivative property of the Fourier transform. Now, from the asymptotic expression

$$(|\xi|^2 - k^2)^{-1} \sim |\xi|^{-2}(1 + k^2|\xi|^{-2} + k^4|\xi|^{-4} + \cdots) \quad \text{for } |\xi| > |k|,$$

we see that the principal homogeneous symbol $\sigma_0(\mathbf{S})(\xi)$ and $\sigma_0(\mathbf{K}_2)(\xi)$ are homogeneous matrix-valued functions of order -2 and -1, respectively. Therefore $\mathbf{K}_1$ maps $(\mathbf{H}^{s+1/2}(\partial\Omega_1))^4$ into $(\mathbf{H}^{s+2}(\Omega_1))^4 \cup (\mathbf{H}^{s+2}_{\mathrm{loc}}(\Omega_2))^4$ and $\mathbf{K}_2$ maps $(\mathbf{H}^{s+1/2}(\partial\Omega_1))^4$ into $(\mathbf{H}^{s+1}(\Omega_1))^4 \cup (\mathbf{H}^{s+1}_{\mathrm{loc}}(\Omega_2))^4$ ((see Eskin [9, 8.2]).

The above analysis proves the following equivalence theorem:

Theorem 3.2. *Let* $\mathbf{G}_1 = (\mathbf{g}_1, t_1)$, $\mathbf{G}_2 = (\mathbf{g}_2, t_2) \in (\mathbf{H}^{1/2}(\Gamma))^4$ *be given with* $\mathbf{G}_1 - \mathbf{G}_2 \in (\tilde{\mathbf{H}}^{1/2}(\Gamma))^4$. *A function* $\mathbf{U} = (\mathbf{u}, \theta) \in (\mathbf{H}^1_{\mathrm{loc}}(\mathbb{R}^3/\bar{\Gamma}))^4$ *is a weak solution of the first Dirichlet thermoelastic screen problem if and only if the jump* $[\mathbf{Tu} - \gamma\hat{\mathbf{n}}\tilde{\theta}, \partial\theta/\partial n]_\Gamma \in (\tilde{\mathbf{H}}^{-1/2}(\Gamma))^4$ *satisfies the integral equation* (3.15).

Let us continue with the *second thermoelastic Neumann problem.* We follow almost the same procedure as in the previous case, but before summing, we apply the operator $\mathbf{R}(\partial_r, \hat{\mathbf{n}})$ to both the integral relations (3.12) and (3.13). Finally, we conclude that the jump of the weak solution satisfies the following integral equation of the first kind on the screen Γ:

$$\begin{aligned} \mathbf{D}[\mathbf{u}, \theta](\mathbf{r}) &:= \frac{1}{4\pi} \mathbf{R}(\partial_r, \hat{\mathbf{n}}) \int_\Gamma [\mathbf{u}, \theta](\mathbf{r}')\cdot \mathbf{R}^*(\partial_{r'}, \hat{\mathbf{n}}')\mathbf{E}^{\mathrm{T}}(\mathbf{r}, \mathbf{r}')\,\mathrm{d}s(r') \\ &= -\frac{1}{2}(\mathbf{G}^+ + \mathbf{G}^-) + \frac{\mathbf{1}}{\mathbf{4\pi}} \int_\Gamma \mathbf{R}(\partial_r, \hat{\mathbf{n}})\cdot \mathbf{E}(\mathbf{r}, \mathbf{r}')[\mathbf{G}]^{\mathrm{T}}(\mathbf{r}')\,\mathrm{d}s(r') \\ &\quad \text{for } \mathbf{r} \in \Gamma. \end{aligned} \tag{3.19}$$

The kernel of the integral operator $\mathbf{D}$ is $\mathbf{R}(\partial_r, \hat{\mathbf{n}})\mathbf{E}(\mathbf{r}, \mathbf{r}')\mathbf{R}^{*\mathrm{T}}(\partial_{r'}, \hat{\mathbf{n}}')$ with hyper-singular point $\mathbf{r}' = \mathbf{r}$.

Conversely, let $[\mathbf{u}, \theta]|_\Gamma \in (\tilde{\mathbf{H}}^{1/2}(\Gamma))^4$ be a solution of equation (3.19). Since for $\mathbf{r} \notin \Gamma$ $\mathbf{E}_{i,j}(\mathbf{r}, \mathbf{r}')|_\Gamma \in \mathbf{H}^{1/2}(\Gamma)$ and $(\mathbf{R}^*(\partial_{r'}, \hat{\mathbf{n}}')\mathbf{E}^{\mathrm{T}}(\mathbf{r}, \mathbf{r}'))_{i,j}|_\Gamma \in \mathbf{H}^{-1/2}(\Gamma)$ together with their derivates, it is obvious that the potential

$$\mathbf{K}_2[\mathbf{u}, \theta](\mathbf{r}) = \frac{1}{4\pi} \int_\Gamma [\mathbf{u}, \theta](\mathbf{r}')\cdot \mathbf{R}^*(\partial_{r'}, \hat{\mathbf{n}}')\mathbf{E}^{\mathrm{T}}(\mathbf{r}, \mathbf{r}')\,\mathrm{d}s(r')$$

and

$$\mathbf{K}_1[\mathbf{G}](\mathbf{r}) = \frac{1}{4\pi}\int_\Gamma \mathbf{E}(\mathbf{r}, \mathbf{r}')[\mathbf{G}]^{\mathrm{T}}(\mathbf{r}')\,\mathrm{d}s(r')$$

are well defined for $\mathbf{r}\notin\Gamma$. The halfspace techniques and the same analysis as in Dirichlet problem show that $\mathbf{K}_2$ maps continuously $(\mathbf{H}^{s+1/2}(\partial\Omega_1))^4$ into $(\mathbf{H}^{s+1}(\Omega_1))^4 \cup (\mathbf{H}^{s+1}_{\mathrm{loc}}(\Omega_2))^4$ and $\mathbf{K}_1$ maps $(\mathbf{H}^{s+1/2}(\partial\Omega_1))^4$ into $(\mathbf{H}^{s+2}(\Omega_1))^4 \cup (\mathbf{H}^{s+2}_{\mathrm{loc}}(\Omega_2))^4$. Hence both $\mathbf{K}_2[\mathbf{u}, \theta]$ and $\mathbf{K}_1[\mathbf{G}]$ belong to $(\mathbf{H}^1(\Omega_1))^4 \cup (\mathbf{H}^1_{\mathrm{loc}}(\Omega_2))^4$ which completes the proof of the following equivalence theorem:

Theorem 3.3. *Let* $\mathbf{G}_1 = (\mathbf{g}_1, t_1)$, $\mathbf{G}_2 = (\mathbf{g}_2, t_2) \in (\mathbf{H}^{-1/2}(\Gamma))^4$ *be given with* $\mathbf{G}_1 - \mathbf{G}_2 \in (\widetilde{\mathbf{H}}^{-1/2}(\Gamma))^4$. *A function* $\mathbf{U} = (\mathbf{u}, \theta) \in (\mathbf{H}^1_{\mathrm{loc}}(\mathbb{R}^3/\bar{\Gamma}))^4$ *is a weak solution of the second Neuman thermoelastic screen problem if and only if the jump* $[\mathbf{u}, \theta]_\Gamma \in (\widetilde{\mathbf{H}}^{1/2}(\Gamma))^4$ *satisfies the integral equation* (3.19).

To conclude the same for the third and fourth mixed type thermoelastic screen problems, we do the previous analysis on the vector-valued operators of potential type separately for the first three components and for the fourth component. We obtain for the third screen problem the following weakly singular boundary integral equation of the first kind on the screen Γ

$$\begin{aligned}
&\mathbf{V}[\mathbf{Tu} - \gamma\hat{\mathbf{n}}\tilde{\theta}, \theta](\mathbf{r}) \\
&:= \frac{1}{4\pi}\mathbf{B}_3(\partial_r, \hat{\mathbf{n}})\int_\Gamma [\mathbf{Tu} - \gamma\hat{\mathbf{n}}\tilde{\theta}, \theta](\mathbf{r}')\cdot\mathbf{B}_3^*(\partial_{r'}, \hat{\mathbf{n}}')\mathbf{E}^{\mathrm{T}}(\mathbf{r}, \mathbf{r}')\,\mathrm{d}s(r') \\
&= -\frac{1}{2}(\mathbf{G}^+ + \mathbf{G}^-) + \frac{1}{4\pi}\int_\Gamma \mathbf{B}_3(\partial_r, \hat{\mathbf{n}})[\mathbf{G}](\mathbf{r}')\cdot\mathbf{B}_4^*(\partial_{r'}, \hat{\mathbf{n}}')\mathbf{E}^{\mathrm{T}}(\mathbf{r}, \mathbf{r}')\,\mathrm{d}s(r')
\end{aligned} \tag{3.20}$$

and for the fourth screen problem the following hypersingular boundary integral equation of the first kind on the screen:

$$\begin{aligned}
&\mathbf{W}[\mathbf{u}, \partial\theta/\partial n](\mathbf{r}) \\
&:= \frac{1}{4\pi}\mathbf{B}_4(\partial_r, \hat{\mathbf{n}})\int_\Gamma [\mathbf{u}, \partial\theta/\partial n](\mathbf{r}')\cdot\mathbf{B}_4^*(\partial_{r'}, \hat{\mathbf{n}}')\mathbf{E}^{\mathrm{T}}(\mathbf{r}, \mathbf{r}')\,\mathrm{d}s(r') \\
&= -\frac{1}{2}(\mathbf{G}^+ + \mathbf{G}^-) + \frac{1}{4\pi}\int_\Gamma \mathbf{B}_4(\partial_r, \hat{\mathbf{n}})[\mathbf{G}](\mathbf{r}')\cdot\mathbf{B}_3^*(\partial_{r'}, \hat{\mathbf{n}}')\mathbf{E}^{\mathrm{T}}(\mathbf{r}, \mathbf{r}')\,\mathrm{d}s(r').
\end{aligned} \tag{3.21}$$

Thus the solution of the third and fourth screen boundary value problem can be reduced to the resolution of the integral equations (3.20), (3.21), respectively according to:

Theorem 3.4. *Let* $\mathbf{G}_1 = (\mathbf{g}_1, t_1)$, $\mathbf{G}_2 = (\mathbf{g}_2, t_2) \in (\mathbf{H}^{1/2}(\Gamma))^3 \times \mathbf{H}^{-1/2}(\Gamma)$ *be given with* $\mathbf{G}_1 - \mathbf{G}_2 \in (\tilde{\mathbf{H}}^{1/2}(\Gamma))^3 \times \tilde{\mathbf{H}}^{-1/2}(\Gamma)$. *A function* $\mathbf{U} = (\mathbf{u}, \theta) \in (\mathbf{H}^1_{\text{loc}}(\mathbb{R}^3/\bar{\Gamma}))^4$ *is a weak solution of the third thermoelastic screen problem if and only if the jump* $[\mathbf{Tu} - \gamma\hat{\mathbf{n}}\tilde{\theta}, \theta]_\Gamma \in (\tilde{\mathbf{H}}^{-1/2}(\Gamma))^3 \times \tilde{\mathbf{H}}^{1/2}(\Gamma)$ *satisfies the integral equation* (3.20).

Theorem 3.5. *Let* $\mathbf{G}_1 = (\mathbf{g}_1, t_1)$, $\mathbf{G}_2 = (\mathbf{g}_2, t_2) \in (\mathbf{H}^{-1/2}(\Gamma))^3 \times \mathbf{H}^{1/2}(\Gamma)$ *be given with* $\mathbf{G}_1 - \mathbf{G}_2 \in (\tilde{\mathbf{H}}^{-1/2}(\Gamma))^3 \times \tilde{\mathbf{H}}^{1/2}(\Gamma)$. *A function* $\mathbf{U} = (\mathbf{u}, \theta) \in (\mathbf{H}^1_{\text{loc}}(\mathbb{R}^3/\bar{\Gamma}))^4$ *is a weak solution of the fourth thermoelastic screen problem if and only if the jump* $[\mathbf{u}, \partial\theta/\partial n]_\Gamma \in (\tilde{\mathbf{H}}^{1/2}(\Gamma))^3 \times \tilde{\mathbf{H}}^{-1/2}(\Gamma)$ *satisfies the integral equation* (3.21).

4. Existence and regularity results

Theorem 4.1. (Existence). *There exists exactly one solution of the integral equations* (3.15), (3.19), (3.20) *and* (3.21) *for the given boundary data on* Γ *in the appropriate Sobolev spaces according to the considered boundary value problem.*

The proof of Theorem 4.1 is based on the following lemma:

Lemma 4.1. *The following mappings are continuous for any real number s*:

$$\mathbf{S}\colon (\tilde{\mathbf{H}}^s(\Gamma))^4 \to (\mathbf{H}^{s+1}(\Gamma))^4, \tag{4.1}$$

$$\mathbf{D}\colon (\tilde{\mathbf{H}}^s(\Gamma))^4 \to (\mathbf{H}^{s-1}(\Gamma))^4, \tag{4.2}$$

$$\mathbf{V}\colon (\tilde{\mathbf{H}}^s(\Gamma))^3 \times \tilde{\mathbf{H}}^s(\Gamma) \to (\mathbf{H}^{s+1}(\Gamma))^3 \times \mathbf{H}^{s-1}(\Gamma), \tag{4.3}$$

$$\mathbf{W}\colon (\tilde{\mathbf{H}}^s(\Gamma))^3 \times \tilde{\mathbf{H}}^s(\Gamma) \to (\mathbf{H}^{s-1}(\Gamma))^3 \times \mathbf{H}^{s+1}(\Gamma). \tag{4.4}$$

The operators **S**, **D**, **V**, **W** *satisfy a Gårding's inequality, i.e. there exist constants* $c_i > 0$, $i = 1, 2, 3, 4$, *and compact operators*

$$\mathbf{O}_1\colon (\tilde{\mathbf{H}}^{-1/2}(\Gamma))^4 \to (\mathbf{H}^{1/2+\varepsilon}(\Gamma))^4,$$

$$\mathbf{O}_2\colon (\tilde{\mathbf{H}}^{1/2}(\Gamma))^4 \to (\mathbf{H}^{-1/2+\varepsilon}(\Gamma))^4,$$

$$\mathbf{O}_3\colon (\tilde{\mathbf{H}}^{-1/2}(\Gamma))^3 \times \tilde{\mathbf{H}}^{1/2}(\Gamma) \to (\mathbf{H}^{1/2+\varepsilon}(\Gamma))^3 \times \mathbf{H}^{-1/2+\varepsilon}(\Gamma),$$

$$\mathbf{O}_4\colon (\tilde{\mathbf{H}}^{-1/2}(\Gamma))^3 \times \tilde{\mathbf{H}}^{-1/2}(\Gamma) \to (\mathbf{H}^{1/2+\varepsilon}(\Gamma))^3 \times \mathbf{H}^{-1/2+\varepsilon}(\Gamma)$$

such that

1. $\langle(\mathbf{S} + \mathbf{O}_1)\mathbf{v}, \mathbf{v}\rangle_{\mathbf{L}^2(\Gamma)} \geqslant c_1\|\mathbf{v}\|^2_{\tilde{\mathbf{H}}^{-1/2}(\Gamma)}$ *for all* $\mathbf{v} \in (\tilde{\mathbf{H}}^{-1/2}(\Gamma))^4$.
2. $\langle(\mathbf{D} + \mathbf{O}_2)\mathbf{v}, \mathbf{v}\rangle_{\mathbf{L}^2(\Gamma)} \geqslant c_2\|\mathbf{v}\|^2_{\tilde{\mathbf{H}}^{1/2}(\Gamma)}$ *for all* $\mathbf{v} \in (\tilde{\mathbf{H}}^{1/2}(\Gamma))^4$.
3. $\langle(\mathbf{V} + \mathbf{O}_3)\mathbf{v}, \mathbf{v}\rangle_{\mathbf{L}^2(\Gamma)} \geqslant c_3(\|\mathbf{w}\|^2_{\tilde{\mathbf{H}}^{-1/2}(\Gamma)} + \|t\|^2_{\tilde{\mathbf{H}}^{1/2}(\Gamma)})$

   for all $\mathbf{v} = (\mathbf{w}, t) \in (\tilde{\mathbf{H}}^{-1/2}(\Gamma))^3 \times \tilde{\mathbf{H}}^{1/2}(\Gamma)$.

4. $\langle (\mathbf{W} + \mathbf{O}_4)\mathbf{v}, \mathbf{v}\rangle_{\mathbf{L}^2(\Gamma)} \geqslant c_4(\|\mathbf{w}\|^2_{\tilde{\mathbf{H}}^{1/2}(\Gamma)} + \|t\|^2_{\tilde{\mathbf{H}}^{-1/2}(\Gamma)})$
for all $\mathbf{v} = (\mathbf{w}, t) \in (\tilde{\mathbf{H}}^{1/2}(\Gamma))^3 \times \tilde{\mathbf{H}}^{-1/2}(\Gamma)$.

Proof. Let $\mathbf{v}$ be a vector-valued function in $(\tilde{\mathbf{H}}^s(\Gamma))^4$. The extension $\mathbf{v}^*$ by zero on $\partial\Omega_1/\Gamma$ belongs to $(\mathbf{H}^s(\partial\Omega))^4$. We consider the vectorial operators $\mathbf{S}$, $\mathbf{D}$, $\mathbf{V}$, $\mathbf{W}$ as mappings from functions on $\partial\Omega$ into functions on $\partial\Omega$. Thus, the assertions of the first part of Lemma 4.1 follow from the fact that the operators $\mathbf{S}$, $\mathbf{D}$ are ψdos of order -1 and $+1$, respectively, while the operators $\mathbf{V}$, $\mathbf{W}$ are ψdos of order -1 and $+1$, respectively, considering the first three components, and of order $+1$ and -1, respectively, considering the last component. We have calculated the principal homogeneous symbol of the ψdos in $\mathbb{R}^2$ (see [9, 17, 6]) to determine their order.

The principal symbol of the operator $\mathbf{S}$ is easily obtained by applying the $\mathbb{R}^2$-Fourier transform to the kernel $\mathbf{E}$. The transformed kernel has the same form as (3.16)–(3.18) but in the considered case $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$, $|\xi| = \sqrt{\xi_1^2 + \xi_2^2}$ and $(|\xi|^2 - k_1^2)^{-1/2}$ substitutes $(|\xi|^2 - k_1^2)^{-1}$. Now, using the asymptotic expression

$$(|\xi|^2 - k^2)^{-1/2} \sim |\xi|^{-1}\left(1 + \frac{1}{2}k^2|\xi|^{-2} + \frac{3}{8}k^4|\xi|^{-4} + \cdots\right)$$

for $|\xi| > |k|$, we obtain the two-dimensional principal homogeneous symbol

$$\sigma_0(\mathbf{S})(\xi) = \frac{c}{\rho\omega^2}\frac{1}{|\xi|^3}$$
$$\times \begin{bmatrix} k_s^2|\xi|^2 + \left(\frac{k_p^2}{2} - k_s^2\right)\xi_1^2 & \left(\frac{k_p^2}{2} - k_s^2\right)\xi_1\xi_2 & 0 & 0 \\ \left(\frac{k_p^2}{2} - k_s^2\right)\xi_1\xi_2 & k_s^2|\xi|^2 + \left(\frac{k_p^2}{2} - k_s^2\right)\xi_2^2 & 0 & 0 \\ 0 & 0 & k_s^2|\xi|^2 & 0 \\ 0 & 0 & 0 & \rho\omega^2|\xi|^2 \end{bmatrix}.$$

Further, we use the relations (Appendix A.3)

$$\mathbf{SD} = \mathbf{K}_2^2 - \tfrac{1}{4}\mathbf{I}, \qquad \mathbf{DS} = \mathbf{K}_2^{*2} - \tfrac{1}{4}\mathbf{I} \tag{4.5}$$

which show that $\mathbf{S}$ is like a regularizer to $\mathbf{D}$ since $\mathbf{K}_2$ and $\mathbf{K}_2^*$ are ψdo of zero order mapping a function on $\partial\Omega$ to a function on $\partial\Omega$ and therefore compact pertubations. Note that $\mathbf{K}_2^*$ is the adjoint operator of $\mathbf{K}_2$. Now, taking the $\mathbb{R}^2$-Fourier transformation of the both sides in relations (4.5) and collecting the lower order contributions on the right-hand side, we get the principal symbol of the ψdo $\mathbf{D}$:

$$\sigma_0(\mathbf{D})(\xi) = \rho\omega^2\frac{c}{|\xi|}$$

$$\times \begin{bmatrix} \frac{1}{k_s^2}|\xi|^2 + \left(\frac{2}{k_p^2} - \frac{1}{k_s^2}\right)\xi_1^2 & \left(\frac{2}{k_p^2} - \frac{1}{k_s^2}\right)\xi_1\xi_2 & 0 & 0 \\ \left(\frac{2}{k_p^2} - \frac{1}{k_s^2}\right)\xi_1\xi_2 & \frac{1}{k_s^2}|\xi|^2 + \left(\frac{2}{k_p^2} - \frac{1}{k_s^2}\right)\xi_2^2 & 0 & 0 \\ 0 & 0 & \frac{1}{k_s^2}|\xi|^2 & 0 \\ 0 & 0 & 0 & \frac{1}{\rho\omega^2}|\xi|^2 \end{bmatrix}.$$

By the combination of the previous calculations componentwise for the operators **V** and **W**, we obtain their $\mathbb{R}^2$ homogeneous principal symbol, namely

$$\sigma_0(\mathbf{V})(\xi) = \frac{c}{\rho\omega^2}\frac{1}{|\xi|^3}$$

$$\times \begin{bmatrix} k_s^2|\xi|^2 + \left(\frac{k_p^2}{2} - k_s^2\right)\xi_1^2 & \left(\frac{k_p^2}{2} - k_s^2\right)\xi_1\xi_2 & 0 & 0 \\ \left(\frac{k_p^2}{2} - k_s^2\right)\xi_1\xi_2 & k_s^2|\xi|^2 + \left(\frac{k_p^2}{2} - k_s^2\right)\xi_2^2 & 0 & 0 \\ 0 & 0 & k_s^2|\xi|^2 & 0 \\ 0 & 0 & 0 & |\xi|^4 \end{bmatrix},$$

$$\sigma_0(\mathbf{W})(\xi) = \rho\omega^2\frac{c}{|\xi|}$$

$$\times \begin{bmatrix} \frac{1}{k_s^2}|\xi|^2 + \left(\frac{2}{k_p^2} - \frac{1}{k_s^2}\right)\xi_1^2 & \left(\frac{2}{k_p^2} - \frac{1}{k_s^2}\right)\xi_1\xi_2 & 0 & 0 \\ \left(\frac{2}{k_p^2} - \frac{1}{k_s^2}\right)\xi_1\xi_2 & \frac{1}{k_s^2}|\xi|^2 + \left(\frac{2}{k_p^2} - \frac{1}{k_s^2}\right)\xi_2^2 & 0 & 0 \\ 0 & 0 & \frac{1}{k_s^2}|\xi|^2 & 0 \\ 0 & 0 & 0 & \frac{1}{\rho\omega^2} \end{bmatrix}.$$

The principal symbols $\sigma_0(\mathbf{S})(\xi)$, $\sigma_0(\mathbf{D})(\xi)$, $\sigma_0(\mathbf{V})(\xi)$, $\sigma_0(\mathbf{W})(\xi)$ are positive definite 4×4 matrix, because all their principal minors have strictly positive determinants. Thus, the Ψdos **S**, **D**, **V**, **W** are strongly elliptic operators. The strong ellipticity property implies the validity of a Gårding's inequality [9]

$$\langle \mathbf{A}\mathbf{v}, \mathbf{v}\rangle_{\mathbf{L}^2(\Gamma)} \geqslant C\|\mathbf{v}\|^2_{\tilde{\mathbf{H}}^p(\Gamma)} - c\|\mathbf{v}\|^2_{\tilde{\mathbf{H}}^{p-\eta}(\Gamma)}, \tag{4.6}$$

where $\mathbf{A}$ stands for a strongly elliptic pseudodifferential operator of order $2p$ and C, η, c are constants such that $C, \eta > 0$, $c \geqslant 0$. The claimed Gårding's inequalities for the operators $\mathbf{S}$, $\mathbf{D}$, $\mathbf{V}$, $\mathbf{W}$ in the second part of Lemma 4.1 are a modification of 4.6 by means of the Rellich's imbedding theorem. Therefore, the second term on the right-hand side of the inequality (4.6) is represented by a compact bilinear form in each considered case.

Now, we are in the position to prove the existence Theorem 4.1. The inequalities (a)–(d) in the second part of Lemma 4.1 imply that the ψdos $\mathbf{S}$, $\mathbf{D}$, $\mathbf{V}$, $\mathbf{W}$ are Fredholm continuous operators of index zero. Moreover, we have proved that the integral equations are equivalent to the weak formulation of the corresponding screen problem that has at most one solution. As a result, the operators $\mathbf{S}$, $\mathbf{D}$, $\mathbf{V}$, $\mathbf{W}$ are injective in the determined spaces and therefore bijective, which ends the proof of Theorem 4.1.

The solution of the four thermoelastic screen problems, even for $\mathbf{C}^\infty$ data, have singularity at the edge γ of the screen Γ. In the following we obtain the local behaviour and a decomposition of the solution of the integral equations into a regular part and a singular part. Our analysis uses Eskin's procedure [9] applying near the edge the Wiener–Hopf technique based on the factorization of the homogeneous elliptic symbols in halfspace. This technique is modified for the 2×2 matrix case by Costabel and Stephan [6].

We denote by s the parameter of arc length of the smooth closed curve γ and $\rho(\mathbf{r})$ the distance from $\mathbf{r} \in \Gamma$ to γ. Let $\chi(\rho)$ be a $\mathbf{C}^\infty$ cut-off function with $\chi \equiv 1$ for small ρ and $\chi \equiv 0$ for $|\rho| > 1$. Then the following theorems hold.

Theorem 4.2. *Let* $|\delta| < 1/2$ *and* $\mathbf{G}_1 = (\mathbf{g}_1, t_1)$, $\mathbf{G}_2 = (\mathbf{g}_2, t_2) \in (\mathbf{H}^{3/2+\delta}(\Gamma))^4$ *with* $\mathbf{G}_1 - \mathbf{G}_2 \in (\tilde{\mathbf{H}}^{3/2+\delta}(\Gamma))^4$. *Then, the solution* $\Psi :\equiv [\mathbf{Tu} - \gamma\hat{\mathbf{n}}\tilde{\theta}, \partial\theta/\partial n]_\Gamma$ *of the integral equation* (3.15) *has the form*

$$\Psi = \beta(s)\rho^{-1/2}\chi(\rho) + \Psi_0 \tag{4.7}$$

with

$$\beta(s) \in (\mathbf{H}^{1/2+\delta}(\Gamma))^4, \quad \Psi_0 \in (\tilde{\mathbf{H}}^{1/2+\delta'}(\Gamma))^4 \text{ for any } \delta' < \delta.$$

Theorem 4.3. *Let* $|\delta| < 1/2$ *and* $\mathbf{G}_1 = (\mathbf{g}_1, t_1)$, $\mathbf{G}_2 = (\mathbf{g}_2, t_2) \in (\mathbf{H}^{3/2+\delta}(\Gamma))^4$ *with* $\mathbf{G}_1 - \mathbf{G}_2 \in (\tilde{\mathbf{H}}^{3/2+\delta}(\Gamma))^4$. *Then, the solution* $\Psi :\equiv [\mathbf{u}, \theta]_\Gamma$ *of the integral equation* (3.19) *has the form*

$$\Psi = \beta(s)\rho^{1/2}\chi(\rho) + \Psi_0 \tag{4.8}$$

with

$$\beta(s) \in (\mathbf{H}^{3/2+\delta}(\gamma))^4, \quad \Psi_0 \in (\tilde{\mathbf{H}}^{3/2+\delta}(\Gamma))^4.$$

Theorem 4.4. *Let* $|\delta| < 1/2$ *and* $\mathbf{G}_1 = (\mathbf{g}_1, t_1)$, $\mathbf{G}_2 = (\mathbf{g}_2, t_2) \in (\mathbf{H}^{3/2+\delta}(\Gamma))^4$ *with* $\mathbf{G}_1 - \mathbf{G}_2 \in (\tilde{\mathbf{H}}^{3/2+\delta}(\Gamma))^4$. *Then, the solution* $\pi = (\Psi, \theta) :\equiv [\mathbf{Tu} - \gamma\hat{\mathbf{n}}\tilde{\theta}, \theta]_\Gamma$ *of the integral equation* (3.20) *has the form*

$$\Psi = \beta(s)\rho^{-1/2}\chi(\rho) + \Psi_0, \qquad \theta = \alpha(s)\rho^{1/2}\chi(\rho) + \psi_0 \tag{4.9}$$

and for any $\delta' < \delta$

$$\beta(s) \in (\mathbf{H}^{1/2+\delta}(\gamma))^3,\ \alpha(s) \in \mathbf{H}^{3/2+\delta}(\gamma),\ \Psi_0 \in (\tilde{\mathbf{H}}^{1/2+\delta'}(\Gamma))^3,\ \psi_0 \in \tilde{\mathbf{H}}^{3/2+\delta}(\Gamma).$$

Theorem 4.5. *Let* $|\delta| < 1/2$ *and* $\mathbf{G}_1 = (\mathbf{g}_1, t_1)$, $\mathbf{G}_2 = (\mathbf{g}_2, t_2) \in (\mathbf{H}^{3/2+\delta}(\Gamma))^4$ *with* $\mathbf{G}_1 - \mathbf{G}_2 \in (\tilde{\mathbf{H}}^{3/2+\delta}(\Gamma))^4$. *Then, the solution* $\pi = (\Psi, \theta) :\equiv [\mathbf{u}, \partial\theta/\partial n]_\Gamma$ *of the integral equation* (3.21) *has the form*

$$\Psi = \beta(s)\rho^{1/2}\chi(\rho) + \Psi_0, \qquad \theta = \alpha(s)\rho^{-1/2}\chi(\rho) + \psi_0 \tag{4.10}$$

and for any $\delta' < \delta$

$$\beta(s) \in (\mathbf{H}^{3/2+\delta}(\gamma))^3,\ \alpha(s) \in \mathbf{H}^{1/2+\delta}(\gamma),\ \Psi_0 \in (\tilde{\mathbf{H}}^{3/2+\delta}(\Gamma))^3,\ \psi_0 \in \tilde{\mathbf{H}}^{1/2+\delta'}(\Gamma).$$

Using the known technique of localization and a partition of unity, the smooth surface $\partial\Omega$ is mapped by the $\mathbf{C}^\infty$-isomorphism in every chart into the plane $x_3 = 0$, while its piece Γ is mapped into $\mathbb{R}^2_+$ for $x_3 = 0$ and $x_2 > 0$. Then, the edge γ of Γ is locally mapped into the line $\mathbb{R}$. The integral equations on Γ can be transformed into a finite sum of integral equations, each of them being defined on a local chart, and the principal part can be represented with the local mapping by collecting compact perturbations on the right. The main idea of such a consideration is to deal locally with the Fourier dual algebraic equations in $\mathbb{R}^2_+$ which can be solved by the Wiener–Hopf techniques. Our proof of the above formulated theorems provides the factorization of 4×4 matrix principal symbols in a suitable form to perform the well-developed Wiener–Hopf techniques.

Proof of Theorem 4.2. Equation (3.15) can be written as $\mathbf{P} + \mathbf{S}_0\mathbf{U} = \mathbf{G}$ on $\mathbb{R}^2_+$, where $\mathbf{P}_+$ denotes the projection operator of restriction to $\mathbb{R}^2_+$ and $\mathbf{S}_0$ is the Ψdo with symbol given by $\sigma_0(\mathbf{S})$. Following the idea of Costabel and Stephan [6], we factorize the 4×4 matrix $\sigma_0(\mathbf{S})(\xi)$ as follows:

$$\sigma_0(\mathbf{S})(\xi) = \frac{ck_p^2}{2\rho\omega^2}\mathbf{S}_-(\xi)\mathbf{S}_+(\xi),\ \text{for } \xi = (\xi_1, \xi_2, 0)$$

where

$$\mathbf{S}_-(\xi) = (\xi_2 - \mathrm{i}|\xi_1|)^{-3/2}$$

$$\times \begin{bmatrix} \xi_2 - 2\mathrm{i}\dfrac{k_p^2}{2k_s^2 + k_p^2}|\xi_1| & -\mathrm{i}\xi_2 - |\xi_1| & 0 & 0 \\ \dfrac{k_p^2 - 2k_s^2}{2k_s + k_p^2}\xi_1 & (\xi_2 - \mathrm{i}|\xi_1|)\operatorname{sign}\xi_1 & 0 & 0 \\ 0 & 0 & \dfrac{\sqrt{2}k_s}{k_p}(\xi_2 - \mathrm{i}|\xi_1|) & 0 \\ 0 & 0 & 0 & \dfrac{\sqrt{2}\rho\omega}{k_p}(\xi_2 - \mathrm{i}|\xi_1|) \end{bmatrix},$$

$$\mathbf{S}_+(\xi) = (\xi_2 + \mathrm{i}|\xi_1|)^{-3/2}$$

$$\times \begin{bmatrix} \frac{2k_s^2}{k_p^2}\xi_2 + 2\mathrm{i}|\xi_1| & \left(\mathrm{i}\xi_2 - \frac{2k_s^2}{k_p^2}|\xi_1|\right)\operatorname{sign}\xi_1 & 0 & 0 \\ \frac{k_p^2 - 2k_s^2}{2k_s + k_p^2}|\xi_1| & \left(\xi_2 + 2\mathrm{i}\frac{k_p^2}{2k_s + k_p^2}|\xi_1|\right)\operatorname{sign}\xi_1 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{2}k_s}{k_p}(\xi_2 + \mathrm{i}|\xi_1|) & 0 \\ 0 & 0 & 0 & \frac{\sqrt{2\rho}\omega}{k_p}(\xi_2 + \mathrm{i}|\xi_1|) \end{bmatrix},$$

Hence, the determinants are

$$\det \mathbf{S}_-(\xi) = (\xi_2 - i|\xi_1|)^{-1}, \quad \det \mathbf{S}_+(\xi) = \frac{2k_s^2}{k_p^2}(\xi_2 + \mathrm{i}|\xi_1|)^{-1}\operatorname{sign}\xi_1$$

we obtain the inverses

$$\mathbf{S}_-(\xi)^{-1} = (\xi_2 - \mathrm{i}|\xi_1|)^{-1/2}$$

$$\times \begin{bmatrix} \frac{2k_s^2 + k_p^2}{k_p^2}(\xi_2 - \mathrm{i}|\xi_1|)\operatorname{sign}\xi_1 & \frac{2k_s^2 + k_p^2}{k_p^2}(\mathrm{i}\xi_2 + |\xi_1|) & 0 & 0 \\ \frac{2k_s^2 - k_p^2}{k_p^2}\xi_1 & \frac{2k_s^2 + k_p^2}{k_p^2}\xi_2 - 2\mathrm{i}|\xi_1| & 0 & 0 \\ 0 & 0 & \frac{k_p}{\sqrt{2}k_s}(\xi_2 - \mathrm{i}|\xi_1|) & 0 \\ 0 & 0 & 0 & \frac{k_p}{\sqrt{2\rho}\omega}(\xi_2 - \mathrm{i}|\xi_1|) \end{bmatrix},$$

$$\mathbf{S}_+(\xi)^{-1} = (\xi_2 - \mathrm{i}|\xi_1|)^{-1/2}$$

$$\times \begin{bmatrix} \frac{2k_s^2 + k_p^2}{2k_s^2}\xi_2 + 2\mathrm{i}|\xi_1| & -\frac{k_p^2}{2k_s^2}\mathrm{i}\xi_2 + |\xi_1| & 0 & 0 \\ \frac{4k_s^4 - k_p^4}{2k_s^2 k_p^2}\xi_1 & \left(\frac{2k_s^2 + k_p^2}{k_p^2}\xi_2 + \frac{2k_s^2 + k_p^2}{2k_s^2}\mathrm{i}|\xi_1|\right)\operatorname{sign}\xi_1 & 0 & 0 \\ 0 & 0 & \frac{k_p}{\sqrt{2}k_s}(\xi_2 + \mathrm{i}|\xi_1|) & 0 \\ 0 & 0 & 0 & \frac{k_p}{\sqrt{2\rho}\omega}(\xi_2 + \mathrm{i}|\xi_1|) \end{bmatrix},$$

Now the theorem is proved by exactly the same procedure as in the 2×2 matrix case of Theorem 3.1 in [6] applied to the first two components of the vectorial operator and as in the scalar case of Theorem 2.9(i) in [19] separately applied to the third and fourth components.

Proof of Theorem 4.3. Again, in order to apply the Wiener–Hopf technique we consider the equation $\mathbf{P}_+\mathbf{D}_0\mathbf{U} = \mathbf{G}$ on $\mathbb{R}^2_+$, where $\mathbf{P}_+$ denotes the projection operator of restriction to $\mathbb{R}^2_+$ and $\mathbf{D}_0$ is the Ψdo with symbol given by $\sigma_0(\mathbf{D})$. We observe that $\sigma_0(\mathbf{D})(\xi) = \rho^2\omega^4\sigma_0(\mathbf{S}')$, where $\sigma_0(\mathbf{S}')$ denotes $\sigma_0(\mathbf{S})$ with k'_p, k'_s instead of k_p, k_s and $k'_p = 2/k_p$, $k'_s = 1/k_s$.

So, we have immediately a factorization for $\sigma_0(\mathbf{D})$ as follows:

$$\sigma_0(\mathbf{D}) = c\rho\omega^2 \frac{2}{k_p}\mathbf{S}'_-(\xi)\mathbf{S}'_+(\xi),$$

where $\mathbf{S}'_-(\xi)$, $\mathbf{S}'_+(\xi)$ are the same as $\mathbf{S}_-(\xi)$, $\mathbf{S}_+(\xi)$ with k'_p, k'_s instead of k_p, k_s. The resulting regularities follow from Theorem 3.2 of Costabel and Stephan [6] for the first two components of the vectorial operator, working with the 2×2 first minor, and Theorem 2.9(ii) of Stephan [19] for the scalar case applied separately to the third and fourth components.

Proof of Theorems 4.4 *and* 4.5. The proof of both theorems is a right combination of the regularity results of Theorems 4.2 and 4.3 considering separately the first three components and the last component.

Remark 4.1. If we relax the assumptions of Theorem 4.3, asking for $\mathbf{G}_1$ and $\mathbf{G}_2$ to be in $(\mathbf{H}^{1/2+\delta}(\Gamma))^4$ with $\mathbf{G}_1 - \mathbf{G}_2 \in (\tilde{\mathbf{H}}^{1/2+\delta}(\Gamma))^4$, we obtain the decomposition $\Phi = \beta(s)\rho^{1/2}\chi(\rho) + \Psi_r + \Psi_1$, where $\beta(s) \in (\mathbf{H}^{1/2+\delta}(\gamma))^4$, $\Phi_r \in (\tilde{\mathbf{H}}^{3/2+\delta}(\Gamma))^4$ and $\Psi_1 \in \mathbf{L}^2(I; (\mathbf{H}^{1/2+\delta}(\gamma))^4) \cap (\tilde{\mathbf{H}}^{3/2+\delta'}(I; (\mathbf{L}^2(\gamma))^4)$ for $0 < \delta' < \delta$ and $\Gamma \equiv I \times \gamma$, $I = [0, 1]$ (for details see [6, p. 476]).

5. Boundary elements approximation

In this section we give a Galerkin boundary element scheme to approximate the exact solution of the thermoelastic scattering from the open set. In the following, we assume that the jump of the boundary data satisfies $[\mathbf{B}_i\Psi]|_\Gamma = 0$. This is not an essential restriction considering that the incident field, which excites the scatterer, is an entire solution of thermoelastic equation. Recalling the previous regularity analysis of the solution near the edge γ, it is appropriate to adapt for our model the improved Galerkin approach [18, 6, 21]. This consists of incorporating the special behaviour of the exact solution into the Galerkin scheme by augmenting the finite element space via singular elements.

Let us assume that the smooth surface Γ is given by a smooth parameter representation $\mathbf{r} = \phi(u, v)$, $(u, v) \in D \subset \mathbb{R}^2$. By the same smooth mapping ϕ, the boundary ∂D is mapped onto the edge γ of Γ. With the regular triangulation D_τ and the bijective mapping ϕ, we construct a family of finite element space on the surface Γ, with maximal mesh size h, regular according to Babǔska and Aziz (see [1, pp. 83–84]). We denote the family of finite element spaces with $\mathbf{S}_h^{d+1,k}(\Gamma)$, for $d+1 > k \geqslant 0$. The elements of $\mathbf{S}_h^{d+1,k}(\Gamma)$ are piecewise polynomials of a degree greater or equal to d belonging to $\tilde{\mathbf{H}}^k(\Gamma)$. In this way, two finite-dimensional subspaces $\mathbf{H}_h^1$ and $\mathbf{H}_h^2$, namely of $\tilde{\mathbf{H}}^{1/2}(\Gamma)$ and $\tilde{\mathbf{H}}^{-1/2}(\Gamma)$ are defined. Thus, for $d+1>k\geqslant 1$ we have

$$\mathbf{H}_h^1(\Gamma)=\mathbf{S}_h^{d,k-1}(\Gamma)\subset\mathbf{H}^{k-1}(\Gamma)\subset\tilde{\mathbf{H}}^{-1/2}(\Gamma),$$
$$\mathbf{H}_h^2(\Gamma)=\mathbf{S}_h^{d+1,k}(\Gamma)\subset\mathbf{H}^{k}(\Gamma)\cap\tilde{\mathbf{H}}^{1}(\Gamma)\subset\tilde{\mathbf{H}}^{1/2}(\Gamma).$$

Analogously, the finite-dimensional space $\mathbf{S}_h^{d+1,k}(\gamma)$ is defined using a regular grid in the parameter of one-dimensional domain γ.

The Galerkin scheme using standard finite elements gives a very low convergence rate of $O(h^{1/2-\varepsilon})$, which cannot be improved by only using higher order elements, due to the singular term with $\rho^{-1/2}\in\mathbf{H}^{-\varepsilon}(\Gamma)$. However, the asymptotic convergence can be improved by simultaneously approximating each part β, Ψ, respectively. To this end we define the augmented finite spaces:

$$\mathbf{Z}_h^{1/2}:=\{\psi_h=\beta_h\rho^{-1/2}\chi+\psi_{0h}\,|\,\beta_h\in\mathbf{S}_h^{d'+1,k'}(\gamma),\ \psi_{0h}\in\mathbf{S}_{0h}^{d,k-1}(\Gamma)\},$$
$$\mathbf{Z}_h^{3/2}:=\{\psi_h=\beta_h\rho^{1/2}\chi+\psi_{0h}\,|\,\beta_h\in\mathbf{S}_h^{d'+1,k'}(\gamma),\ \psi_{0h}\in\mathbf{S}_{0h}^{d+1,k}(\Gamma)\},$$

for $d'+1>k'\geqslant 1$ and $d+1>k\geqslant 2$. Here $\psi\in\mathbf{S}_{0h}^{d+1,k}(\Gamma)$ if $\psi\in\mathbf{S}_h^{d+1,k}(\Gamma)$ and $\psi=0$ on γ. It is clear that

$$\mathbf{Z}_h^{1/2}\subset\mathbf{Z}^{1/2+\delta}\subset\tilde{\mathbf{H}}^{-1/2}(\Gamma)\ \text{and}\ \mathbf{Z}_h^{3/2}\subset\mathbf{Z}^{3/2+\delta}\subset\tilde{\mathbf{H}}^{1/2}(\Gamma),$$

where $\mathbf{Z}^{1/2+\delta}$ and $\mathbf{Z}^{3/2+\delta}$ are the spaces of the scalar distribution having the form $\beta\rho^{-1/2}\chi+\psi_0$, for $\beta\in\mathbf{H}^{1/2+\delta}(\gamma)$, $\psi_0\in\tilde{\mathbf{H}}^{1/2+\delta'}(\Gamma)$, $\delta'<\delta$ and $\beta\rho^{1/2}\chi+\psi_0$, for $\beta\in\mathbf{H}^{3/2+\delta}(\gamma)$, $\psi_0\in\tilde{\mathbf{H}}^{3/2+\delta}(\Gamma)$, respectively.

The simplest elements belonging to the space $\mathbf{Z}_h^{1/2}$ are continuous piecewise linear one-dimensional elements β_h on γ and piecewise linear elements ψ_{0h} on Γ with $\psi_{0h}\in\mathbf{C}^0(\Gamma)$ and $\psi_{0h}=0$ on γ. The simplest elements belonging to the space $\mathbf{Z}_h^{3/2}$ are continuous piecewise linear one-dimensional elements β_h on γ and piecewise quadratic elements ψ_{0h} on Γ with $\psi_{0h}\in\mathbf{C}^1(\Gamma)$, $D\psi_{0h}=0$ and $\psi_{0h}=0$ on γ. The modified Galerkin schemes for each screen scattering problem reads as follows:

1. *The modified Galerkin scheme for the first (Dirichlet) thermoelastic screen scattering problem.* Find $\Psi_h=(\psi_h^i)_{i=1}^4$ with $\psi_h^i\in\mathbf{Z}_h^{1/2}(\Gamma)$ such that for all test functions $\mathbf{v}_h=(v_h^i)_{i=1}^4$ with $v_h^i\in\mathbf{Z}_h^{1/2}(\Gamma)$ we have

$$\frac{1}{4\pi}\int_\Gamma\int_\Gamma\mathbf{E}(\mathbf{r},\mathbf{r}')\Psi_h(\mathbf{r}')\cdot\mathbf{v}_h^{\mathrm{T}}(\mathbf{r})\,\mathrm{d}s(r)\,\mathrm{d}s(r')=-\int_\Gamma\Phi_{\mathrm{inc}}(\mathbf{r})\cdot\mathbf{v}_h^{\mathrm{T}}(\mathbf{r})\,\mathrm{d}s(r). \tag{5.1}$$

2. *The modified Galerkin scheme for the second (Neumann) thermoelastic screen scattering problem.* Find $\Psi_h=(\psi_h^i)_{i=1}^4$ with $\psi_h^i\in\mathbf{Z}_h^{3/2}(\Gamma)$ such that for all test functions $\mathbf{v}_h=(v_h^i)_{i=1}^4$ with $v_h^i\in\mathbf{Z}_h^{3/2}(\Gamma)$ we have

$$\frac{1}{4\pi}\int_\Gamma\int_\Gamma\mathbf{R}(\partial_r,\hat{\mathbf{n}})\mathbf{E}(\mathbf{r},\mathbf{r}')\mathbf{R}^{*\mathrm{T}}(\partial'_r,\hat{\mathbf{n}}')\Psi_h(\mathbf{r}'))^{\mathrm{T}}\cdot\mathbf{v}_h^{\mathrm{T}}(\mathbf{r})\,\mathrm{d}s(r)\,\mathrm{d}s(r')$$
$$=-\int_\Gamma\mathbf{R}(\partial_r,\hat{\mathbf{n}})\Phi_{\mathrm{inc}}(\mathbf{r})\cdot\mathbf{v}_h^{\mathrm{T}}(\mathbf{r})\,\mathrm{d}s(r). \tag{5.2}$$

3. *The modified Galerkin scheme for the third (mix-type) thermoelastic screen scattering problem.* Find $\Psi_h=(\psi_h^1,\psi_h^2,\psi_h^3,\theta_h)$ with $\psi_h^i\in\mathbf{Z}_h^{1/2}(\Gamma)$, $i=1,2,3$ and $\theta_h\in\mathbf{Z}_h^{3/2}(\Gamma)$

such that for all test functions $\mathbf{v}_h = (v_h^i)_{i=1}^4$ with $v_h^i \in \mathbf{Z}_h^{1/2}(\Gamma)$, $i = 1, 2, 3$, and $(v_h^4) \in \mathbf{Z}_h^{3/2}(\Gamma)$ we have

$$\frac{1}{4\pi}\int_\Gamma \int_\Gamma \mathbf{B}_3(\partial_r, \hat{\mathbf{n}})\mathbf{E}(\mathbf{r}, \mathbf{r}')\mathbf{B}_3^{*\mathrm{T}}(\partial_r', \hat{\mathbf{n}}')\Psi_h(\mathbf{r}'))^{\mathrm{T}} \cdot \mathbf{v}_h^{\mathrm{T}}(\mathbf{r})\,\mathrm{d}s(r)\,\mathrm{d}s(r')$$

$$= -\int_\Gamma \mathbf{B}_3(\partial_r, \hat{\mathbf{n}})\Phi_{\mathrm{inc}}(\mathbf{r}) \cdot \mathbf{v}_h^{\mathrm{T}}(\mathbf{r})\,\mathrm{d}s(r). \tag{5.3}$$

4. *The modified Galerkin scheme for the fourth* (*mix-type*) *thermoelastic screen scattering problem*. Find $\Psi_h = (\psi_h^1, \psi_h^2, \psi_h^3, \theta_h)$ with $\psi_h^i \in \mathbf{Z}_h^{3/2}(\Gamma)$, $i = 1, 2, 3$ and $\theta_h \in \mathbf{Z}_h^{1/2}(\Gamma)$ such that for all the test functions $\mathbf{v}_h = (v_h^i)_{i=1}^4$ with $v_h^i \in \mathbf{Z}_h^{3/2}(\Gamma)$, $i = 1, 2, 3$, and $(v_h^4) \in \mathbf{Z}_h^{1/2}(\Gamma)$ we have

$$\frac{1}{4\pi}\int_\Gamma \int_\Gamma \mathbf{B}_4(\partial_r, \hat{\mathbf{n}})\mathbf{E}(\mathbf{r}, \mathbf{r}')\mathbf{B}_3^{*\mathrm{T}}(\partial_r', \hat{\mathbf{n}}')\Psi_h(\mathbf{r}'))^{\mathrm{T}} \cdot \mathbf{v}_h^{\mathrm{T}}(\mathbf{r})\,\mathrm{d}s(r)\,\mathrm{d}s(r')$$

$$= -\int_\Gamma \mathbf{B}_4(\partial_r, \hat{\mathbf{n}})\Phi_{\mathrm{inc}}(\mathbf{r}) \cdot \mathbf{v}_h^{\mathrm{T}}(\mathbf{r})\,\mathrm{d}s(r). \tag{5.4}$$

The Gårding's inequalities in Lemma 4.1 imply directly that the approximated equations have a unique solution. For the convergence and error analysis of the above boundary element models we refer the readers to [6, 23].

Remark 5.1. (1) In the scattering problems the boundary data are infinitely smooth. In this case the solution of the integral equations has still singularity of the form as in Theorems 4.2–4.5, but with smooth functions $\alpha, \beta, \psi, \Psi$ for a smooth edge curve γ. The Galerkin method can be improved if more singular elements like $\rho^{3/2}, \rho^{5/2}, \ldots$ are used in the augmented spaces.

(2) Wendland and Zhu have constructed a mixed finite-boundary elements approximating approach in order to overcome the difficulty of the numerical implementation in the above augmented spaces on Γ of an arbitrary shape. They suggest firstly to approximate the surface Γ with the help of curved finite elements Γ_h and then to construct the boundary element spaces on Γ_h by using the method of Lagrangian multipliers. For details see [22].

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Appendix A

A.1. *Green's integral formulae in thermoelasticity*

Let $\Omega \in \mathbb{R}^3$ be a closed subset, bounded with Lipschitz boundary $\partial\Omega$. Moreover, let $\mathbf{U} = (\mathbf{u}, \theta)$, $\mathbf{V} = (\mathbf{v}, \vartheta)$ be two solutions of thermoelastic equation (2.4) such that $\mathbf{U}$,

$\mathbf{V} \in (\mathbf{C}^2(\Omega))^4 \cap (\mathbf{C}(\bar{\Omega}))^4$. Then the following identity holds:

$$\int_\Omega \left[\mathscr{E}(\mathbf{u}, \mathbf{v}) + \nabla\theta \cdot \nabla\vartheta - \mathrm{i}\omega\eta\nabla \cdot \mathbf{u}\vartheta - \gamma\theta\nabla \cdot \mathbf{v} - \rho\omega^2 \mathbf{u} \cdot \mathbf{v} - \frac{\mathrm{i}\omega}{\kappa}\theta\vartheta \right] \mathrm{d}\omega$$
$$= \oint_{\partial\Omega} \left[(\mathbf{Tu} - \gamma\hat{\mathbf{n}}\theta) \cdot \mathbf{v} + \frac{\partial\theta}{\partial n}\vartheta \right] \mathrm{d}s. \qquad \text{(A.1)}$$

The bilinear symmetric form $\mathscr{E}(\mathbf{u}, \mathbf{v})$ is given as

$$\mathscr{E}(\mathbf{u}, \mathbf{v}) = \lambda(\nabla \cdot \mathbf{u})(\nabla \cdot \mathbf{v}) + \mu \sum_{p,q=1}^{3} \frac{\partial \mathbf{v}_p}{\partial x_q} \left(\frac{\partial \mathbf{u}_p}{\partial x_q} + \frac{\partial \mathbf{u}_q}{\partial x_p} \right).$$

In the case of the two conjugated solutions, we have relation (6.1) as

$$\frac{2\gamma}{\mathrm{i}\omega\eta} \int_\Omega |\nabla\theta|^2 \, \mathrm{d}\omega$$
$$= \oint_{\partial\Omega} \left[\bar{\mathbf{u}} \cdot (\mathbf{Tu} - \gamma\hat{\mathbf{n}}\theta) + \frac{\gamma}{\mathrm{i}\omega\eta} \bar{\theta} \frac{\partial\theta}{\partial n} - \mathbf{u} \cdot (\mathbf{T}\bar{\mathbf{u}} - \gamma\hat{\mathbf{n}}\bar{\theta}) + \frac{\gamma}{\mathrm{i}\omega\eta} \theta \frac{\partial\bar{\theta}}{\partial n} \right] \mathrm{d}s. \qquad \text{(A.2)}$$

Let us assume that $\mathbf{LU}$, $\mathbf{LV} \in (\mathbf{C}(\bar{\Omega}))^4$. The following formula is true:

$$\int_\Omega [\mathbf{U} \cdot \mathbf{L}^*\mathbf{V} - \mathbf{V} \cdot \mathbf{LU}] \, \mathrm{d}r = \oint_{\partial\Omega} [\mathbf{U} \cdot \mathbf{R}^*\mathbf{V} - \mathbf{V} \cdot \mathbf{RU}] \, \mathrm{d}s(r) \qquad \text{(A.3)}$$

which when applied properly for $\mathbf{U}$ and $\mathbf{E}(\mathbf{r}, \mathbf{r}')$ implies the integral representation

$$\mathbf{U}(\mathbf{r}) = -(+)\frac{1}{4\pi} \oint_{\partial\Omega} [\mathbf{U}(\mathbf{r}') \cdot \mathbf{R}^*(\partial_{r'}, \hat{\mathbf{n}})\mathbf{E}^{\mathrm{T}}(\mathbf{r}, \mathbf{r}')$$
$$- \mathbf{E}(\mathbf{r}, \mathbf{r}') \cdot \mathbf{R}(\partial_{r'}, \hat{\mathbf{n}})\mathbf{U}(\mathbf{r}')] \, \mathrm{d}s(r') \ \text{ for } \mathbf{r} \in \Omega \ (\mathbf{r} \in \mathbb{R}^3/\bar{\Omega}).$$

A.2. *Thermoelastic single- and double-layer potentials*

We define the following thermoelastic potentials for $\mathbf{r} \notin \partial\Omega$:

$$(\mathbf{K}_1\mathbf{V})(\mathbf{r}) := \frac{1}{4\pi} \int_{\partial\Omega} \mathbf{E}(\mathbf{r}, \mathbf{r}')\mathbf{V}(\mathbf{r}') \, \mathrm{d}s(r'), \qquad \text{(A.4)}$$

$$(\mathbf{K}_2\mathbf{V})(\mathbf{r}) := \frac{1}{4\pi} \int_{\partial\Omega} \mathbf{V}(\mathbf{r}') \cdot \mathbf{R}^*(\partial_{r'}, \hat{\mathbf{n}}')\mathbf{E}^{\mathrm{T}}(\mathbf{r}, \mathbf{r}') \, \mathrm{d}s(r'), \qquad \text{(A.5)}$$

$$(\mathbf{K}_3\mathbf{V})(\mathbf{r}) := \frac{1}{4\pi} \int_{\partial\Omega} \mathbf{V}(\mathbf{r}') \cdot \mathbf{B}_3^*(\partial_{r'}, \hat{\mathbf{n}}')\mathbf{E}^{\mathrm{T}}(\mathbf{r}, \mathbf{r}') \, \mathrm{d}s(r'), \qquad \text{(A.6)}$$

$$(\mathbf{K}_4\mathbf{V})(\mathbf{r}) := \frac{1}{4\pi} \int_{\partial\Omega} \mathbf{V}(\mathbf{r}') \cdot \mathbf{B}_4^*(\partial_{r'}, \hat{\mathbf{n}}')\mathbf{E}^{\mathrm{T}}(\mathbf{r}, \mathbf{r}') \, \mathrm{d}s(r'), \qquad \text{(A.7)}$$

Obviously, $\mathbf{K}_1$ is a single-layer potential with a weak singular kernel, $\mathbf{K}_2$ is a double-layer potential with a Cauchy singular kernel, while $\mathbf{K}_3$, $\mathbf{K}_4$ are componentwise of a different type with a weak singular and Cauchy singular kernel, respectively.

Lemma A.1. *Let $k \geqslant 0$ be an integer, $\partial\Omega \in \mathbf{C}^{k+1,\alpha'}$ and $0 < \alpha < \alpha'$. The following operators are bounded*:

$$\mathbf{K}_1 : (\mathbf{C}^{k,\alpha}(\partial\Omega))^4 \to (\mathbf{C}^{k+1,\alpha}(\bar{\Omega}^{\pm}))^4, \tag{A.8}$$

$$\mathbf{K}_2 : (\mathbf{C}^{k,\alpha}(\partial\Omega))^4 \to (\mathbf{C}^{k,\alpha}(\bar{\Omega}^{\pm}))^4, \tag{A.9}$$

$$\mathbf{K}_3 : (\mathbf{C}^{k,\alpha}(\partial\Omega))^4 \to (\mathbf{C}^{k+1,\alpha}(\bar{\Omega}^{\pm}))^3 \times \mathbf{C}^{k,\alpha}(\bar{\Omega}^{\pm}), \tag{A.10}$$

$$\mathbf{K}_4 : (\mathbf{C}^{k,\alpha}(\partial\Omega))^4 \to (\mathbf{C}^{k,\alpha}(\bar{\Omega}^{\pm}))^3 \times \mathbf{C}^{k+1,\alpha}(\bar{\Omega}^{\pm}). \tag{A.11}$$

Their values for $\mathbf{r} \in \partial\Omega$ are computed via the following relations on the boundary:

$$[(\mathbf{K}_1\mathbf{V})(\mathbf{r})]^+ = [(\mathbf{K}_1\mathbf{V})(\mathbf{r})]^-,$$

$$[\mathbf{R}(\partial_r, \hat{\mathbf{n}})(\mathbf{K}_2\mathbf{V})(\mathbf{r})]^+ = [\mathbf{R}(\partial_r, \hat{\mathbf{n}})(\mathbf{K}_2\mathbf{V})(\mathbf{r})]^-,$$

$$[\mathbf{B}_3(\partial_r, \hat{\mathbf{n}})(\mathbf{K}_3\mathbf{V})(\mathbf{r})]^+ = [\mathbf{B}_3(\partial_r, \hat{\mathbf{n}})(\mathbf{K}_3\mathbf{V})(\mathbf{r})]^-,$$

$$[\mathbf{B}_4(\partial_r, \hat{\mathbf{n}})(\mathbf{K}_4\mathbf{V})(\mathbf{r})]^+ = [\mathbf{B}_4(\partial_r, \hat{\mathbf{n}})(\mathbf{K}_4\mathbf{V})(\mathbf{r})]^-,$$

$$[(\mathbf{K}_2\mathbf{V})(\mathbf{r})]^{\pm} = \pm\frac{1}{2}\mathbf{V}(\mathbf{r}) + \frac{1}{4\pi}\int_{\partial\Omega} \mathbf{V}(\mathbf{r}')\cdot\mathbf{R}^*(\partial_{r'}, \hat{\mathbf{n}}')\mathbf{E}(\mathbf{r}, \mathbf{r}')\,\mathrm{d}s(r'),$$

$$[\mathbf{R}(\partial_r, \hat{\mathbf{n}})(\mathbf{K}_1\mathbf{V})(\mathbf{r})]^{\pm} = \mp\frac{1}{2}\mathbf{V}(\mathbf{r}) + \frac{1}{4\pi}\int_{\partial\Omega} \mathbf{R}(\partial_r, \hat{\mathbf{n}})\cdot\mathbf{E}(\mathbf{r}, \mathbf{r}')\mathbf{V}(\mathbf{r}')\,\mathrm{d}s(r'),$$

$$[\mathbf{B}_3(\partial_r, \hat{\mathbf{n}})(\mathbf{K}_4\mathbf{V})(\mathbf{r})]^{\pm} = \pm\frac{1}{2}\mathbf{V}(\mathbf{r}) + \frac{1}{4\pi}\int_{\partial\Omega} \mathbf{B}_3(\partial_r, \hat{\mathbf{n}}) \times [\mathbf{V}(\mathbf{r}')\cdot\mathbf{B}_4^*(\partial_{r'}, \hat{\mathbf{n}}')\mathbf{E}(\mathbf{r}, \mathbf{r}')]\,\mathrm{d}s(r'),$$

$$[\mathbf{B}_4(\partial_r, \hat{\mathbf{n}})(\mathbf{K}_3\mathbf{V})(\mathbf{r})]^{\pm} = \mp\frac{1}{2}\mathbf{V}(\mathbf{r}) + \frac{1}{4\pi}\int_{\partial\Omega} \mathbf{B}_4(\partial_r, \hat{\mathbf{n}}) \times [\mathbf{V}(\mathbf{r}')\cdot\mathbf{B}_3^*(\partial_{r'}, \hat{\mathbf{n}}')\mathbf{E}(\mathbf{r}, \mathbf{r}')]\,\mathrm{d}s(r'),$$

where

$$[f(\mathbf{r})]^{\pm} = \lim_{h\to 0^+} f(\mathbf{r} \pm h\hat{\mathbf{n}}(\mathbf{r})), \quad \left[\frac{\partial f}{\partial n}(\mathbf{r})\right]^{\pm} = \lim_{h\to 0^+} \hat{\mathbf{n}}(\mathbf{r})\cdot\nabla f(\mathbf{r} \pm h\hat{\mathbf{n}}(\mathbf{r})).$$

The integrals for $\mathbf{r} \in \partial\Omega$ are defined as Cauchy principal value.

Proof. Let us consider the thermoelastostatic equation for the frequency $\omega = 0$. The elements of 4×4 fundamental matrix $\mathbf{E}^o(\mathbf{r}, \mathbf{r}')$ are

$$\mathbf{E}^o_{kj}(\mathbf{r}, \mathbf{r}') = (1 - \delta_{k4})(1 - \delta_{j4})\Gamma^o_{kj}(\mathbf{r}, \mathbf{r}') + \frac{\gamma\delta_{j4}(1 - \delta_{k4})(x_k - x'_k)}{(\lambda + 2\mu)|\mathbf{r} - \mathbf{r}'|} + \frac{\mathbf{2}\delta_{4k}\delta_{j4}}{|\mathbf{r} - \mathbf{r}'|}.$$

The following estimates due to Kupradze [14],

$$|\mathbf{E}_{kj}(\mathbf{r},\mathbf{r}')-\mathbf{E}^{o}_{kj}(\mathbf{r},\mathbf{r}')|\leqslant C_2,\quad \left|\frac{\partial}{\partial x_{l_1}}(\mathbf{E}_{kj}(\mathbf{r},\mathbf{r}')-\mathbf{E}^{o}_{kj}(\mathbf{r},\mathbf{r}'))\right|\leqslant\frac{C_2}{|\mathbf{r}-\mathbf{r}'|}$$

for $j,k=1,2,3,4$ and $l_1,l_2=1,2,3$

$$\left|\frac{\partial^2}{\partial x_{l_1}\partial x_{l_2}}(\mathbf{E}_{kj}(\mathbf{r},\mathbf{r}')-\mathbf{E}^{o}_{kj}(\mathbf{r},\mathbf{r}'))\right|\leqslant\frac{C_2}{|\mathbf{r}-\mathbf{r}'|}$$

for $j,k=1,2,3$ or $j=k=4$ and $l_1,l_2,=1,2,3$

$$\left|\frac{\partial^2}{\partial x_{l_1}\partial x_{l_2}}(\mathbf{E}_{k4}(\mathbf{r},\mathbf{r}')-\mathbf{E}^{o}_{k4}(\mathbf{r},\mathbf{r}'))\right|\leqslant C_2,$$

for $k=1,2,3$ and $l_1,l_2,=1,2,3$, show that the thermoelastostatic integral operators behave near the boundary $\partial\Omega$ partly as elastostatic single- and double-layer integral operators (this refers to the first three components) and partly as harmonic single- and double-layer integral operators (this refers to the last component). So, the assertions in Lemma A.1 follow from [14, 10].

Lemma A.2. *The boundary integral potential operators* (A.4)–(A.7) *can be extended continuously to the following bounded operators for every* $s\in\mathbb{R}$:

$$\mathbf{K}_1:(\mathbf{H}^s(\partial\Omega))^4\to(\mathbf{H}^{s+1}(\partial\Omega))^4,$$

$$\mathbf{K}_2:(\mathbf{H}^s(\partial\Omega))^4\to(\mathbf{H}^{s}(\partial\Omega))^4,$$

$$\mathbf{K}_3:(\mathbf{H}^s(\partial\Omega))^4\to(\mathbf{H}^{s+1}(\partial\Omega))^3\times\mathbf{H}^s(\partial\Omega),$$

$$\mathbf{K}_4:(\mathbf{H}^s(\partial\Omega))^4\to(\mathbf{H}^{s}(\partial\Omega))^3\times\mathbf{H}^{s+1}(\partial\Omega).$$

Proof. The operator $\mathbf{K}_1$ is Ψdo of order -1 as a mapping of $\partial\Omega$ into $\partial\Omega$. Furthermore, the operator $\mathbf{K}_2$ is Ψdo of order 0, while the operators $\mathbf{K}_3$ and $\mathbf{K}_4$ are Ψdo of order -1 and 0, respectively, referring to the first three components and of order 0 and -1, respectively, referring to the last component.

6.3. *The proof of the relations* (4.5)

Firstly, it is easy to see that the operators $\mathbf{K}_1,\mathbf{K}_2$ are not self-adjoint and the adjoint operators are namely

$$(\mathbf{K}_1^*\mathbf{V})(\mathbf{r}):=\frac{1}{4\pi}\int_{\partial\Omega}\mathbf{E}^*(\mathbf{r},\mathbf{r}')\mathbf{V}(\mathbf{r}')\,\mathrm{d}s(r'),\tag{A.12}$$

$$(\mathbf{K}_2^*\mathbf{V})(\mathbf{r}):=\frac{1}{4\pi}\int_{\partial\Omega}\mathbf{V}(\mathbf{r}')\cdot\mathbf{R}(\partial_{r'},\hat{\mathbf{n}}')\mathbf{E}^{*\mathrm{T}}(\mathbf{r},\mathbf{r}')\,\mathrm{d}s(r'),\tag{A.13}$$

where $\mathbf{E}^*(\mathbf{r}, \mathbf{r}')$ is the fundamental solution of the adjoint thermoelastic operator $\mathbf{L}^*$ and is obtained by interchanging γ with $i\omega\eta$ in $\mathbf{E}(\mathbf{r}, \mathbf{r}')$. In addition, it satisfies the relation $\mathbf{E}^*(\mathbf{r}, \mathbf{r}') = \mathbf{E}^{\mathrm{T}}(\mathbf{r}', \mathbf{r})$. Furthermore, we have for all $\phi \in (\mathbf{C}(\partial\Omega))^4$ and $\psi \in (\mathbf{C}^{1,\alpha}(\partial\Omega))^4$

$$\langle \phi, \mathbf{SD}\psi \rangle_{\mathbf{L}^2} = \langle \mathbf{S}^*\phi, \mathbf{D}\psi \rangle_{\mathbf{L}^2}$$

$$= \int_{\partial\Omega} \mathbf{K}_1^*\phi \cdot \mathbf{R}(\mathbf{K}_2\psi)\,\mathrm{d}s = \int_{\partial\Omega} \mathbf{K}_2\psi \cdot \mathbf{R}^*(\mathbf{K}_1^*\phi)\,\mathrm{d}s \tag{A.14}$$

$$= \int_{\partial\Omega} (\mathbf{K}_2 - \tfrac{1}{2}\mathbf{I})\psi \cdot (\mathbf{R}^*\mathbf{K}_1^* + \tfrac{1}{2}\mathbf{I})\phi\,\mathrm{d}s = \int_{\partial\Omega} (\mathbf{K}_2^2 - \tfrac{1}{4}\mathbf{I})\psi \cdot \phi\,\mathrm{d}s. \tag{A.15}$$

In equality (A.14) we have used the Green formula (A.3), while equality (A.15) is due to the jump relation for the single- and double-layer potentials. Obviously, it follows that $\mathbf{SD} = \mathbf{K}_2^2 - \frac{1}{4}\mathbf{I}$. In a similar way, it can be shown that the adjoint relation $\mathbf{DS} = \mathbf{K}_2^{*2} - \frac{1}{4}\mathbf{I}$ is also valid.

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