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Mathematics and Computers in Simulation 66 (2004) 325–335

MATHEMATICS AND COMPUTERS IN SIMULATION

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A target signature for distinguishing perfect conductors from anisotropic media of finite conductivity

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Dedicated to Professor Dr. Rainer Kress on the occasion of his 60th birthday and the many years of friendship and mathematical collaboration between RK and DC!

Available online 21 March 2004

Abstract

We exhibit a target signature obtainable from the electromagnetic far field pattern at fixed frequency that allows one to distinguish a perfect conductor from an anisotropic medium of finite conductivity.

Keywords: Electromagnetic inverse scattering; Target signature

1. Introduction

This paper can be viewed as a continuation of work initiated by Colton and Kress [9]. In that paper, the authors suggested "an alternative direction in inverse scattering theory than that pursued to date, where one only tries to obtain lower and upper bounds for a few relevant features of the scattering object rather than attempting a complete reconstruction". In particular, in [9] a lower bound for the arclength of the boundary of a coated scattering object in $\mathbb{R}^2$ was obtained from a knowledge of the location of the eigenvalues of the far field operator corresponding to the scattering of fixed frequency TM-polarized electromagnetic plane waves. The result was subsequently extended to the case of elastic waves by Alves and Kress [1]. However, the methods of [1,9] suffered from the fact that in order to obtain the desired estimate it was necessary to know the far field pattern of the scattered field for all angles of incidence and observation. In [12], this problem was overcome by using the far field equation associated with the linear sampling method instead of the eigenvalues of the far field operator. Unfortunately, the lower bounds obtained in [9,12] were in general rather crude and required an a priori knowledge of the surface impedance of the scattering object.

In some areas of practical importance in inverse scattering theory even upper and lower bounds are more than is needed. Indeed, in problems associated with the detection of hostile objects, it is often

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doi:10.1016/j.matcom.2004.02.002

only important to know if the scattering object is coated or not. For example, in the use of electromagnetic waves to distinguish between a perfectly conducting tank and a wooden decoy of the same shape coated with metallic paint, it is only desired to determine a "target signature" that distinguishes between an actual tank and a coated decoy. The purpose of this paper is to exhibit an appropriate target signature for problems of this type, i.e. a number obtainable from a knowledge of the far field pattern of the scattered electromagnetic wave at fixed frequency that will enable one to distinguish a perfect conductor from a coated anisotropic object of the same shape. Our approach to this problem is based on extending the ideas of [12] to the case of electromagnetic waves in an anisotropic medium with the key ingredient in doing this being the recent result of Haddar establishing the well-posedness of the interior transmission problem for Maxwell's equations in an anisotropic medium [13].

The plan of our paper is as follows. In the next section, we will formulate the mathematical problems associated with the scattering of electromagnetic waves by a perfect conductor and an anisotropic medium of finite conductivity. We then use the relationship between the far field operator and the interior transmission problem to exhibit a target signature that distinguishes between these two cases. We conclude by commenting on the physically important cases of limited aperture far field data, piecewise homogeneous background media and the limiting case when the width of coating and the resistivity in the coating both tend to zero.

2. Formulation of the problem

We consider the scattering of a time-harmonic electromagnetic plane wave by a bounded obstacle D (the scatterer) in $\mathbb{R}^3$. In particular let $D \subset \mathbb{R}^3$ be a bounded domain having a C^2-smooth boundary ∂D such that the exterior domain $D_e := \mathbb{R}^3 \setminus \bar{D}$ is connected. The unit normal vector to ∂D directed into the exterior of D is denoted by ν. Throughout the paper we denote by Ω the unit sphere in $\mathbb{R}^3$.

2.1. Scattering by a perfect conductor

We first assume that D is a perfectly conducting obstacle. After factoring out a term of the form $e^{-i\omega t}$ where ω is the frequency, we are led to the following boundary value problem for the electric field E and magnetic field H:

$$\operatorname{curl} E - ikH = 0, \qquad \operatorname{curl} H + ikE = 0 \quad \text{in} \quad D_e \tag{1}$$

and

$$\nu \times E = 0 \text{ on } \partial D. \tag{2}$$

The total field E and H is given by

$$E = E^i + E^s, \qquad H = H^i + H^s, \tag{3}$$

E^s and H^s are the scattered fields satisfying the Silver–Müller radiation condition:

$$\lim_{r\to\infty} (H^s \times x - rE^s) = 0 \tag{4}$$

uniformly in $\hat{x} = x/|x|$, $r = |x|$, and the incident fields $E^{\rm i}$ and $H^{\rm i}$ are given by

$$E^{\rm i}(x; d, p) = \frac{\rm i}{k} \operatorname{curl}\operatorname{curl} p\, {\rm e}^{{\rm i}kdx} = {\rm i}k(d \times p) \times d\, {\rm e}^{{\rm i}kdx},$$
$$H^{\rm i}(x; d, p) = \operatorname{curl} p\, {\rm e}^{{\rm i}kdx} = {\rm i}kd \times p\, {\rm e}^{{\rm i}kdx}, \tag{5}$$

where $k > 0$ is the wave number, $d \in \Omega$ a unit vector giving the direction of the incident plane wave and $p \in \mathbb{R}^3$ is the polarization. From now on the problems (1)–(5) will be referred to as problem (PC).

2.2. Scattering by an anisotropic medium of finite conductivity

We now assume that the scatterer is a bounded anisotropic medium of finite conductivity. The domain D (the scatterer) is the support of the anisotropic material. If the scatterer is illuminated by a time-harmonic electromagnetic plane wave with frequency ω than the total electric and magnetic fields $\tilde{E}$ and $\tilde{H}$ satisfy

$$\operatorname{curl} \tilde{E} - {\rm i}\omega\mu_0 \tilde{H} = 0, \qquad \operatorname{curl} \tilde{H} + ({\rm i}\omega\epsilon(x) - \sigma(x))\tilde{E} = 0 \quad \text{in} \quad \mathbb{R}^3, \tag{6}$$

where the total fields $\tilde{E}$ and $\tilde{H}$ are given by

$$\tilde{E} = E^{\rm i} + \tilde{E}^{\rm s}, \qquad \tilde{H} = H^{\rm i} + \tilde{H}^{\rm s}, \tag{7}$$

where $\tilde{E}^{\rm s}$ and $\tilde{H}^{\rm s}$ are the radiating scattered fields; $E^{\rm i}$ and $H^{\rm i}$ the incident plane waves. The electric permittivity ϵ_0 and magnetic permeability μ_0 of the exterior domain $D_{\rm e} := \mathbb{R}^3 \setminus \bar{D}$ are positive constants whereas the scatterer has the same magnetic permeability μ_0 as the exterior medium but the electric permittivity ϵ is real 3×3 matrix-valued function such that $\epsilon_0 = \epsilon_0 I$ in $D_{\rm e}$ and the conductivity σ is a real 3×3 matrix-valued function such that $\sigma = 0$ in $D_{\rm e}$. If we now define $\tilde{E} = (1/\sqrt{\epsilon_0})E$, $\tilde{H} = (1/\sqrt{\mu_0})H$, $N(x) = (1/\epsilon_0)(\epsilon(x) + {\rm i}(\sigma(x)/\omega))$ and $k^2 = \epsilon_0\mu_0\omega^2$, and define $H := (1/{\rm i}k) \operatorname{curl} E$ we obtain the following problem for the electric field E:

$$\text{(i)}\ \operatorname{curl}\operatorname{curl} E - k^2 N(x) E = 0, \qquad \text{(ii)}\ E = E^{\rm s} + E^{\rm i},$$
$$\text{(iii)}\ \lim_{r\to\infty} (\operatorname{curl} E^{\rm s} \times x - {\rm i}krE^{\rm s}) = 0 \quad \text{in} \quad \mathbb{R}^3, \tag{8}$$

where $E^{\rm s}$ and $H^{\rm s}$ are the scattered fields; $E^{\rm i}$ and $H^{\rm i}$ the incident fields given by (5) and the Silver–Müller radiation condition (8(iii)) is satisfied uniformly in $\hat{x} = x/r$.

We assume that N is a symmetric matrix-valued function whose entries are bounded measurable complex-valued functions in $\mathbb{R}^3$ such that $N = I$ in $D_{\rm e}$.

Furthermore, we assume that there exists a constant $\gamma > 1$ such that, for all $\xi \in \mathbb{C}^3$, $\bar{\xi} \cdot \operatorname{Re}(N)\xi \geq \gamma|\xi|^2$ almost everywhere in $\mathbb{R}^3$ and $\bar{\xi} \cdot \operatorname{Im}(N)\xi > 0$ in D. The last two assumption are made in order to guarantee the existence of solution to the interior transmission problem (c.f. [13]). From the point of view of the application mentioned in the Introduction, we would require $\operatorname{Im}(N)$ to be very small except for a highly conducting portion near ∂D. In the sequel the problem (8) will be referred to as problem (AM).

For the mathematical framework of our problem we need to introduce the following Sobolev spaces. Letting $L^2(D)$ and $L^2(\partial D)$ denote the space of square integrable functions define on D and ∂D, respectively, we define

$$H(\mathrm{curl}, D) := \{U \in (L^2(D))^3 : \mathrm{curl}\, U \in (L^2(D))^3\},$$
$$\mathcal{U}(D) := \{U \in H(\mathrm{curl}, D) : \mathrm{curl}\, U \in H(\mathrm{curl}, D)\},$$
$$L^2_t(\partial D) := \{U \in (L^2(\partial D))^3 : \nu \cdot U = 0 \text{ on } \partial D\},$$
$$H^{-1/2}_{\mathrm{div}}(\partial D) := \{U \in H^{-1/2}(\partial D) : \nu \cdot U = 0, \mathrm{div}_{\partial D}\, U \in H^{-1/2}(\partial D)\},$$

where $H^{-1/2}(\partial D)$ is the trace space of the Sobolev space $H^1(D)$ (the space of functions that are in $L^2(D)$ together with their first-order derivatives). Moreover, we denote by $H_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3)$ and $\mathcal{U}_{\mathrm{loc}}(\mathbb{R}^3)$ the space of functions that are in $H(\mathrm{curl}, K)$ and $\mathcal{U}(K)$, respectively, for every compact set $K \subset \mathbb{R}^3$.

It is already known that both scattering problems are well posed. In particular in [8,15] it is shown that the problem (PC) has a unique solution $E, H \in H_{\mathrm{loc}}(\mathrm{curl}, D_{\mathrm{e}})$ and in [14] it is shown that the transmission problem (AM) has a solution $E, H \in H_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3)$. Moreover, the Silver–Müller radiation condition implies that the corresponding scattered fields E^{s} and H^{s} has the asymptotic behavior [8]:

$$E^{\mathrm{s}}(x) = \frac{\mathrm{e}^{\mathrm{i}k|x|}}{|x|}\left\{E_\infty(\hat{x}; d, p) + \mathrm{O}\left(\frac{1}{|x|}\right)\right\},$$
$$H^{\mathrm{s}}(x) = \frac{\mathrm{e}^{\mathrm{i}k|x|}}{|x|}\left\{H_\infty(\hat{x}; d, p) + \mathrm{O}\left(\frac{1}{|x|}\right)\right\}, \quad |x| \to \infty,$$

where $E_\infty(\cdot; d, p)$ and $H_\infty(\cdot; d, p)$ defined on the unit sphere Ω are the electric far field pattern and the magnetic far field pattern, respectively, corresponding to the incident direction d and polarization p. These far field patterns satisfy the relations:

$$H_\infty(\hat{x}; d, p) = \hat{x} \times E_\infty(\hat{x}; d, p), \qquad \hat{x} \cdot H_\infty(\hat{x}; d, p) = \hat{x} \cdot E_\infty(\hat{x}; d, p) = 0.$$

Assuming that the support D of the scatterer is already determined by using methods such as synthetic aperture radar or the linear sampling method, we want to determine if the scatterer D is a perfect conductor or an anisotropic media of finite conductivity from a knowledge of the (measured) electric far field pattern $E_\infty(\hat{x}; d, p)$ for $\hat{x}, d \in \Omega_0 \subset \Omega$ and $p \in \mathbb{R}^3$. In particular our aim is to give a "target signature" determined from the electric far field pattern that will distinguish between these two cases.

3. A target signature

Let us first assume that the electric far field pattern $E_\infty(\hat{x}; d, p)$ is known for all $\hat{x}$, d on the unit sphere Ω and $p \in \mathbb{R}^3$ (the case of limited aperture data will be discussed in the next section). We can then define the far field operator $F : L^2_t(\Omega) \to L^2_t(\Omega)$ by

$$(Fg)(\hat{x}) := \int_\Omega E_\infty(\hat{x}; d, g(d))\, \mathrm{d}s(d), \quad g \in L_t(\Omega). \tag{9}$$

An electromagnetic Herglotz pair is defined to be a pair of vector fields of the form:

$$E_g(x) = \int_\Omega \mathrm{e}^{\mathrm{i}kxd} g(d)\, \mathrm{d}s(d), \qquad H_g(x) = \frac{1}{\mathrm{i}k}\nabla \times E_g(x) \tag{10}$$

with kernel $g \in L^2_t(\Omega)$. Note that an electromagnetic Herglotz pair is an entire solution of Maxwell's equations. One can easily see by superposition that Fg is the electric far field pattern corresponding to the incident field being an electromagnetic Herglotz pair with kernel $\mathrm{i}kg(d)$. Next we consider the first kind integral equation known as the far field equation (the equation associated with the linear sampling method for finding the support of the scattering object from the knowledge of the far field pattern—c.f. [5,6,13]):

$$(Fg)(\hat{x}) = E_{\mathrm{e},\infty}(\hat{x}; z, q) \tag{11}$$

for an arbitrary point $z \in \mathbb{R}^3$ and polarization $q \in \mathbb{R}^3$ where $E_{\mathrm{e},\infty}$ is the electric far field pattern of the electric dipole:

$$E_{\mathrm{e}}(x; z, q) := \frac{\mathrm{i}}{k} \operatorname{curl}_x \operatorname{curl}_x q\Phi(x, z), \qquad H_{\mathrm{e}}(x; z, q) := \operatorname{curl}_x q\Phi(x, z) \tag{12}$$

with $\Phi(x, z) := (1/4\pi)(\mathrm{e}^{\mathrm{i}k|x-z|}/|x-z|)$ and $q \in \mathbb{R}^3$. In particular,

$$E_{\mathrm{e},\infty}(\hat{x}; z, q) = \frac{\mathrm{i}k}{4\pi}(\hat{x} \times q) \times \hat{x}\, \mathrm{e}^{-\mathrm{i}k\hat{x}\cdot z}. \tag{13}$$

If $z \in D$ than for every $\epsilon_n > 0$ there exists a $g_n \in L^2_t(\Omega)$ such that

$$||(Fg_n)(\hat{x}) - E_{\mathrm{e},\infty}(\hat{x}; z, q)||_{L^2_t(\Omega)} < \epsilon_n. \tag{14}$$

This result is proved in [5,6] for the case when k is not a Maxwell eigenvalue for D and F corresponds to (PC) and in [13] in for case when F corresponds to (AM).

In the rest of our paper we want to develop a simple mathematical criteria based on an approximate (regularized) solution of the far field equation (11) that distinguishes between a perfect conductor and an anisotropic medium of finite conductivity.

To this end we need the following technical lemma.

Lemma 3.1. *Let $E^{\mathrm{s}}_1, H^{\mathrm{s}}_1 \in H_{\mathrm{loc}}(\mathrm{curl}, D_{\mathrm{e}})$ and $E^{\mathrm{s}}_2, H^{\mathrm{s}}_2 \in H_{\mathrm{loc}}(\mathrm{curl}, D_{\mathrm{e}})$ be radiating solutions of the Maxwell equations with electric far field patterns $E_{1,\infty}, E_{2,\infty}$, respectively. Then*

$$\int_{\partial D} (\nu \times E^{\mathrm{s}}_1 \cdot \operatorname{curl} \bar{E}^{\mathrm{s}}_2 - \nu \times \bar{E}^{\mathrm{s}}_2 \cdot \operatorname{curl} E^{\mathrm{s}}_1)\, \mathrm{d}s = -2\mathrm{i}k \int_{\Omega} E_{1,\infty,} \cdot \bar{E}_{2,\infty}\, \mathrm{d}s. \tag{15}$$

The proof of this lemma for smooth radiating solutions can be found in [10] (Lemma 4.1). It is easy to see that the proof also holds for $H_{\mathrm{loc}}(\mathrm{curl}, D_{\mathrm{e}})$ radiating solutions.

3.1. An equality for perfect conductors

Let D be the support of a perfectly conducting obstacle and F the far field operator corresponding to the problem (PC). We consider a sequence ϵ_n tending to zero as n tends to infinity and a fixed point $z \in D$. Assume that k is not a Maxwell eigenvalue for D and let $g_n \in L^2_t(\Omega)$ be such that for every n (14) holds. In particular it is shown in [5] (if one sets $\mu = 0$) and in [6] (if one sets $\Gamma_I = \emptyset$) that g_n is the kernel of the electromagnetic Herglotz pair (E_{g_n}, H_{g_n}) such that $\mathrm{i}kE_{g_n}$ approximates with accuracy of order ϵ_n the unique solution E_0 of the interior boundary value problem:

$$\operatorname{curl}\operatorname{curl} E_0 - k^2 E_0 = 0 \quad \text{in} \quad D, \qquad \nu \times E_0 = -\nu \times E_{\mathrm{e}} \quad \text{on} \quad \partial D, \tag{16}$$

where $E_{\mathrm{e}} := E_{\mathrm{e}}(\cdot; q, z)$ is the electric field of the electric dipole given by (12).

Now let us denote by E_n^{s}, H_n^{s} the unique solution of (PC) corresponding to the incident wave being the electromagnetic Herglotz pair $(\mathrm{i}kE_{g_n}, \mathrm{i}kH_{g_n})$, and let $E_n = E_n^{\mathrm{s}} + \mathrm{i}kE_{g_n}$ be the total electric field. By applying the second vector Green formula and using the boundary condition $\nu \times E_n = 0$ on ∂D we obtain

$$\begin{aligned} 0 &= \int_{\partial D} (\nu \times E_n \cdot \operatorname{curl} \bar{E}_n - \nu \times \bar{E}_n \cdot \operatorname{curl} E_n)\,\mathrm{d}s \\ &= \int_{\partial D} (\nu \times E_n^{\mathrm{s}} \cdot \operatorname{curl} \bar{E}_n^{\mathrm{s}} - \nu \times \bar{E}_n^{\mathrm{s}} \cdot \operatorname{curl} E_n^{\mathrm{s}})\,\mathrm{d}s - \mathrm{i}k\int_{\partial D} (\nu \times E_n^{\mathrm{s}} \cdot \operatorname{curl} \bar{E}_{g_n} \\ &\quad - \nu \times \bar{E}_{g_n} \cdot \operatorname{curl} E_n^{\mathrm{s}})\,\mathrm{d}s + \mathrm{i}k\int_{\partial D} (\nu \times E_{g_n} \cdot \operatorname{curl} \bar{E}_n^{\mathrm{s}} - \nu \times \bar{E}_n^{\mathrm{s}} \cdot \operatorname{curl} E_{g_n})\,\mathrm{d}s \\ &= \int_{\partial D} (\nu \times E_n^{\mathrm{s}} \cdot \operatorname{curl} \bar{E}_n^{\mathrm{s}} - \nu \times \bar{E}_n^{\mathrm{s}} \cdot \operatorname{curl} E_n^{\mathrm{s}})\,\mathrm{d}s \\ &\quad - 2\mathrm{i}k \operatorname{Re}\left(\int_{\partial D} (\nu \times E_n^{\mathrm{s}} \cdot \operatorname{curl} \bar{E}_{g_n} - \nu \times \bar{E}_{g_n} \cdot \operatorname{curl} E_n^{\mathrm{s}})\,\mathrm{d}s\right). \end{aligned} \tag{17}$$

Hence, making use of Lemma 3.1, we have

$$0 = ||F_{g_n}||^2_{L_t^2(\Omega)} + \operatorname{Re}(\mathbb{I}_n), \tag{18}$$

where

$$\mathbb{I}_n := \int_{\partial D} (\nu \times E_n^{\mathrm{s}} \cdot \operatorname{curl} \bar{E}_{g_n} - \nu \times \bar{E}_{g_n} \cdot \operatorname{curl} E_n^{\mathrm{s}})\,\mathrm{d}s.$$

Since $\lim_{n\to\infty} \mathrm{i}kE_{g_n} = E_0$ in $H(D, \operatorname{curl})$, we have that

$$\lim_{n\to\infty} (\nu \times E_n^{\mathrm{s}}) = -\lim_{n\to\infty} (\mathrm{i}k\nu \times E_{g_n}) = -\nu \times E_0 = \nu \times E_{\mathrm{e}}$$

in $H_{\mathrm{div}}^{-1/2}(\partial D)$. Hence, from the continuous dependence of the solution of the problem (PC) on the boundary condition (c.f. [15]), we have that

$$\lim_{n\to\infty} E_n^{\mathrm{s}} = E_{\mathrm{e}}(\cdot; z, q)$$

in $H_{\mathrm{loc}}(\operatorname{curl}, D_{\mathrm{e}})$ (note that g_n as well as E_n^{s} depend on z and the polarization q!).

Therefore

$$\begin{aligned} \lim_{n\to\infty} \mathbb{I}_n &= \lim_{n\to\infty} \int_{\partial D} (\nu \times E_n^{\mathrm{s}} \cdot \operatorname{curl} \bar{E}_{g_n} - \nu \times \bar{E}_{g_n} \cdot \operatorname{curl} E_n^{\mathrm{s}})\,\mathrm{d}s \\ &= -\frac{1}{\mathrm{i}k}\int_{\partial D} (\nu \times E_{\mathrm{e}}(y; z, q) \cdot \operatorname{curl}_y \bar{E}_0(y) - \nu \times \bar{E}_0(y) \cdot \operatorname{curl}_y E_{\mathrm{e}}(y; z, q))\,\mathrm{d}s. \end{aligned} \tag{19}$$

One can easily show that

$$\begin{aligned} \nu \times E_{\mathrm{e}}(y; z, q) \cdot \operatorname{curl}_y \bar{E}_0(y) &= -\frac{\mathrm{i}}{k}(-\mathrm{i}k) \operatorname{curl}_z \operatorname{curl}_z q\Phi(y, z) \cdot (\nu \times \bar{H}_0(y)) \\ &= -q \cdot \operatorname{curl}_z \operatorname{curl}_z \Phi(y, z)(\nu \times \bar{H}_0(y)) \end{aligned} \tag{20}$$

and

$$\nu \times \bar{E}_0(y) \cdot \operatorname{curl}_y E_{\mathrm{e}}(y; z, q) = \mathrm{i}k\nu \times \bar{E}_0(y) \cdot H_{\mathrm{e}}(y; z, q) = \mathrm{i}kq \cdot \operatorname{curl}_z \Phi(y, z)(\nu \times \bar{E}_0(y)). \tag{21}$$

Hence substituting (20) and (21) into (19) and using Stratton–Chu formula yields

$$\lim_{n\to\infty} \mathbb{I}_n = -q\cdot \bar{E}_0(z). \tag{22}$$

But from (14) we have that

$$\lim_{n\to\infty} ||Fg_n||_{L_t^2(\Omega)} = ||E_{e,\infty}||^2_{L_t^2(\Omega)} = \frac{k^2}{(4\pi)^2}\int_\Omega ((\hat{x}\times q)\times\hat{x})\cdot((\hat{x}\times q)\times\hat{x})\,\mathrm{d}\hat{x}$$

and, since $((\hat{x}\times q)\times\hat{x})\cdot((\hat{x}\times q)\times\hat{x}) = ||q||^2 - (\hat{x}\cdot q)^2$, by simple computation we obtain

$$\lim_{n\to\infty} ||Fg_n||_{L_t^2(\Omega)} = ||E_{e,\infty}||^2_{L_t^2(\Omega)} = \frac{k^2}{6\pi}||q||^2. \tag{23}$$

Finally, by taking the limit as $n\to\infty$ in (18) and using (22) and (23) we have the following theorem.

Theorem 3.2. *Let D be a perfectly conducting obstacle and assume that k is not a Maxwell eigenvalue for D. Then*

$$\frac{k^2}{6\pi}||q||^2 - \mathrm{Re}(q\cdot E_0(z)) = 0, \tag{24}$$

where E_0 is the unique solution of (16) *and z an arbitrary fixed point in D.*

Note that for different points $z\in D$ we have different solutions E_0 of (16) since the boundary condition itself depends on z.

3.2. An equality for anisotropic media of finite conductivity

Now we consider the scattering problem for an anisotropic medium. Let F be the far field operator corresponding to the problem (AM) and z, ϵ_n, and g_n be as in Section 3.1. For each n we again construct the total field $E_n = E_n^{\mathrm{s}} + \mathrm{i}kE_{g_n}$ that is the solution of (8) corresponding to the electromagnetic Herglotz pair with kernel $\mathrm{i}kg_n$ as incident field. In particular the scattered field $E_n^{\mathrm{s}} \in U_{\mathrm{loc}}(\mathbb{R}^3)$ is the unique solution of

$$\text{(i)}\ \mathrm{curl\,curl}\, E_n^{\mathrm{s}} - k^2NE_n^{\mathrm{s}} = (\mathrm{i}k)k^2(N-I)E_{g_n}, \qquad \text{(ii)}\ \lim_{n\to\infty}(\mathrm{curl}\, E_n^{\mathrm{s}}\times x - \mathrm{i}krE_n^{\mathrm{s}}) = 0 \quad \text{in} \quad \mathbb{R}^3. \tag{25}$$

Our analysis is based on the interior transmission problem associated with (AM). In particular in [13] it is proved that there exists a unique solution $E, E_0 \in L^2(D)$ of the interior transmission problem:

$$\begin{aligned} &\mathrm{curl\,curl}\, E_0 - k^2E_0 = 0, \qquad \mathrm{curl\,curl}\, E - k^2NE = 0 \quad \text{in} \quad D;\\ &\nu\times(E-E_0) = \nu\times E_{\mathrm{e}}, \qquad \nu\times(\mathrm{curl}\, E - \mathrm{curl}\, E_0) = \nu\times\mathrm{curl}\, E_{\mathrm{e}} \quad \text{on} \quad \partial D, \end{aligned} \tag{26}$$

where $E_{\mathrm{e}} = E_{\mathrm{e}}(\cdot; z, q)$ is the electric field of the electric dipole given by (12).

Indeed, for every n, $g_n \in L_t^2(\Omega)$ which satisfies (14) is the kernel of an electromagnetic Herglotz pair (E_{g_n}, H_{g_n}) such that $\lim_{n\to\infty}\mathrm{i}kE_{g_n} = E_0$ in $L^2(D)$. In [13] it is also shown that $\lim_{n\to\infty} E_n^{\mathrm{s}} =$

E^s in $U_{loc}(\mathbb{R}^3)$, where $E^s = E_e$ for $x \in D_e$ and $E^s = E - E_0$ for $x \in D$. Obviously $\lim_{n\to\infty} E_n = E$ in $L^2(D)$. In particular E^s is the unique solution of

$$\text{(i) } \operatorname{curl}\operatorname{curl} E^s - k^2 N E^s = k^2(N-I)E_0, \qquad \text{(ii) } \lim_{n\to\infty} (\operatorname{curl} E^s \times x - \mathrm{i}krE^s) = 0 \quad \text{in} \quad \mathbb{R}^3. \tag{27}$$

To deduce an equality similar to the one in Section 3.1, we again consider E_{g_n}, E_n^s and $E_n = E_n^s + \mathrm{i}kE_{g_n}$ and write

$$k^2\int_D (N-I)E_n\cdot\bar{E}_n\,\mathrm{d}x = k^2\int_D (N-I)(E_n^s+\mathrm{i}kE_{g_n})\cdot\bar{E}_n^s\,\mathrm{d}x + k^2(-\mathrm{i}k)\int_D (N-I)(E_n^s+\mathrm{i}kE_{g_n})\cdot\bar{E}_{g_n}\,\mathrm{d}x. \tag{28}$$

We first evaluate the imaginary part of the first integral on the right-hand side of (28). To this end we use (25) and the vector Green formula to obtain

$$\begin{aligned}
&\operatorname{Im}\left(k^2\int_D (N-I)(E_n^s+\mathrm{i}kE_{g_n})\cdot\bar{E}_n^s\,\mathrm{d}x\right)\\
&= \operatorname{Im}\left(\int_D (\operatorname{curl}\operatorname{curl}E_n^s - k^2E_n^s)\cdot\bar{E}_n^s\,\mathrm{d}x\right) = \operatorname{Im}\left(\int_D \operatorname{curl}\operatorname{curl}E_n^s\cdot\bar{E}_n^s\,\mathrm{d}x\right)\\
&= \operatorname{Im}\left(\int_D \operatorname{curl}E_n^s\cdot\operatorname{curl}\bar{E}_n^s\,\mathrm{d}x - \int_{\partial D}\nu\times\bar{E}_n^s\cdot\operatorname{curl}E_n^s\,\mathrm{d}s\right) = -\operatorname{Im}\left(\int_{\partial D}\nu\times\bar{E}_n^s\cdot\operatorname{curl}E_n^s\,\mathrm{d}s\right)\\
&= -\frac{1}{2}\operatorname{Im}\left(\int_{\partial D}(\nu\times\bar{E}_n^s\cdot\operatorname{curl}E_n^s - \nu\times E_n^s\cdot\operatorname{curl}\bar{E}_n^s)\,\mathrm{d}s\right).
\end{aligned} \tag{29}$$

From Lemma 3.1 we can conclude that

$$\operatorname{Im}\left(k^2\int_D (N-I)(E_n^s+\mathrm{i}kE_{g_n})\cdot\bar{E}_n^s\,\mathrm{d}x\right) = -k\|Fg_n\|^2_{L_t^2(\Omega)}. \tag{30}$$

Since $Fg_n \to E_{e,\infty}$ in $L_t^2(\Omega)$ as $n\to\infty$, using (23) we have that

$$\lim_{n\to\infty}\operatorname{Im}\left(k^2\int_D (N-I)(E_n^s+\mathrm{i}kE_{g_n})\cdot\bar{E}_n^s\,\mathrm{d}x\right) = -k\|E_{e,\infty}\|^2_{L_t^2(\Omega)} = -\frac{k^3}{3\pi}\|q\|^2.$$

Next we evaluate the second integral on the right-hand side of (28). We have

$$\begin{aligned}
&(-\mathrm{i}k)k^2\int_D (N-I)(E_n^s+\mathrm{i}kE_{g_n})\cdot\bar{E}_{g_n}\,\mathrm{d}x\\
&= (-\mathrm{i}k)k^2\int_D (N-I)(E_n^s+E^s)\cdot\bar{E}_{g_n}\,\mathrm{d}x + (-\mathrm{i}k)k^2\int_D (N-I)(\mathrm{i}kE_{g_n}-E_0)\cdot\bar{E}_{g_n}\,\mathrm{d}x\\
&\quad +(-\mathrm{i}k)k^2\int_D (N-I)(E^s+E_0)\cdot\bar{E}_{g_n}\,\mathrm{d}x.
\end{aligned}$$

We now observe that

$$\lim_{n\to\infty} k^2\int_D (N-I)(E_n^s - E^s)\cdot\bar{E}_{g_n} = 0, \tag{31}$$

$$\lim_{n\to\infty} k^2 \int_D (N-I)(\mathrm{i}kE_{g_n} - E_0)\cdot \bar{E}_{g_n} = 0. \tag{32}$$

Furthermore, by using (27) and the vector Green formula, (20) and (21), and the Stratton–Chu formula together with the fact that $\nu \times E^{\mathrm{s}}(y) = \nu \times E_{\mathrm{e}}(y; z, q)$ and $\nu \times \mathrm{curl}\, E^{\mathrm{s}}(y) = \nu \times \mathrm{curl}\, E_{\mathrm{e}}(y; z, q)$ for $y \in \partial D$ we obtain

$$\begin{aligned}
&(-\mathrm{i}k)k^2 \int_D (N-I)(E^{\mathrm{s}} + E_0)\cdot \bar{E}_{g_n}\,\mathrm{d}y \\
&\quad = (-\mathrm{i}k)\int_D (\mathrm{curl}\,\mathrm{curl}\, E^{\mathrm{s}} - k^2 E^{\mathrm{s}})\cdot \bar{E}_{g_n}\,\mathrm{d}y = (-\mathrm{i}k)\int_D (\mathrm{curl}\,\mathrm{curl}\, E^{\mathrm{s}}\cdot \bar{E}_{g_n} - E^{\mathrm{s}}\cdot \mathrm{curl}\,\mathrm{curl}\,\bar{E}_{g_n})\,\mathrm{d}y \\
&\quad = (-\mathrm{i}k)\int_{\partial D} (\nu \times E^{s}(y)\cdot \mathrm{curl}_y\, \bar{E}_{g_n}(y) - \nu \times \bar{E}_{g_n}(y)\cdot \mathrm{curl}_y\, E^{\mathrm{s}}(y))\,\mathrm{d}s \\
&\quad = (-\mathrm{i}k)\int_{\partial D} (\nu \times E_{\mathrm{e}}(y; z, q)\cdot \mathrm{curl}_y\, \bar{E}_{g_n}(y) - \nu \times \bar{E}_{g_n}(y)\cdot \mathrm{curl}_y\, E_{\mathrm{e}}(y; z, q))\,\mathrm{d}s \\
&\quad = (-\mathrm{i}k)(\mathrm{i}k)q\cdot \bar{E}_{g_n}(z).
\end{aligned}$$

Therefore we have

$$\lim_{n\to\infty} (-\mathrm{i}k)k^2 \int_D (N-I)(E^{\mathrm{s}} + E_0)\cdot \bar{E}_{g_n}\,\mathrm{d}y = \mathrm{i}kq\cdot \bar{E}_0(z),$$

where $\lim_{n\to\infty} (-\mathrm{i}k)\bar{E}_{g_n}(z) = \bar{E}_0(z)$ is interpreted through the use of the mean value theorem as in [12].

Finally, we take the imaginary part of both sides of (28) and let $n \to \infty$ to obtain

$$\lim_{n\to\infty} \mathrm{Im}\left(k^2 \int_D (N-I)E_n\cdot \bar{E}_n\,\mathrm{d}x\right) = -k||E_{\mathrm{e},\infty}|| + \mathrm{Im}(\mathrm{i}kq\cdot \bar{E}_0(z)).$$

Hence

$$\mathrm{Im}\left(k^2 \int_D (N-I)E\cdot \bar{E}\,\mathrm{d}x\right) = -\frac{k^3}{6\pi}||q||^2 + k\,\mathrm{Re}(q\cdot E_0(z)).$$

We have proved the following theorem.

Theorem 3.3. *Let D be the support of a conducting anisotropic medium of finite conductivity. Then*

$$k\int_D \bar{E}\cdot \mathrm{Im}(N)E\,\mathrm{d}x = -\frac{k^2}{6\pi}||q||^2 + \mathrm{Re}(q\cdot E_0(z)), \tag{33}$$

where (E_0, E) *is the unique solution of* (26) *and z an arbitrary fixed point in D.*

Our analysis suggests that a target signature that distinguishes between a perfect conductor and an anisotropic medium of finite conductivity is the function E_0. In particular, for a fixed $q \in \mathbb{R}^3$ and $z \in D$, $\mathrm{Re}(q\cdot E_0(z))$ is equal to $(k^2/3\pi)|q|^2$ if D is the support of a perfect conductor while it is different form $(k^2/3\pi)|q|^2$ if D is the support of a medium of finite conductivity. Obviously the function E_0 cannot be computed as long as one does not know the nature of the material of the scatterer D. However, E_0 can be approximated arbitrarily closely by the electric field E_g of the electromagnetic Herglotz pair with the kernel $g \in L^2_t(\Omega)$ where g is an approximate solution of the far field equation:

$$(Fg)(\hat{x}) = E_{e,\infty}(\hat{x}; z, q). \tag{34}$$

This g is then sought for by applying regularization methods to solve the ill-posed integral equation (34) (c.f. [7,11]).

4. Physical considerations

We conclude by briefly addressing some questions that arise in connection with the target signature discussed in the previous section.

4.1. Limited aperture

In most cases of practical interest the far field data $E_\infty(\hat{x}, d)$ is restricted to the case when $\hat{x}$ and d are on a subset Ω_0 of the unit sphere Ω, i.e. we are concerned with limited aperture scattering data. This situation can be handled by using the results of [5]. In particular in [5] it is shown that the set of electromagnetic Herglotz pairs can be approximated uniformly on compact subsets of a ball B_R of radius R by a Herglotz pair with kernel supported on a subset of Ω. The electric field of this new Herglotz pair can now be used in place of E_g to approximate the target signature E_0.

4.2. Piecewise homogeneous background

Our model problem of the scattering from a perfect conductor or an anisotropic medium of finite conductivity assumes that the scatterer is immersed in a homogeneous background. Obviously in applications (such as the one mentioned in Introduction) it is important to consider the scatterer in a piecewise homogeneous background. There are two ways to treat this situation. Neglecting multiple scattering effects, one way is to simply subtract the scattering due to the known background when solving the far field equation. In particular, the far field data used in the far field equation (11) is the difference of the measured far field data and the far field due to the scattering from the known background. Examples using this approach for a related problem can be found in [4]. If multiple scattering effects cannot be ignored, then one can modify the far field equation by incorporating the background in the model as is done in [7].

4.3. Thin coating

In the limiting case where the conductivity become arbitrarily large in a thin layer near ∂D and $\mathrm{Im}(N)$ is arbitrarily small in the remainder of D one obtains the so-called conductive transmission condition on the coated part Γ_2 of the surface $\partial D = \bar{\Gamma}_1 \cup \bar{\Gamma}_2$ of the anisotropic medium D [2]. In particular we are led to the mixed transmission problem:

(i) $\nabla \times E^s - ikH^s = 0$ and $\nabla \times H^s + ikE^s = 0$ in D_e,
(ii) $\nabla \times E - ikH = 0$ and $\nabla \times H + ikN(x)E = 0$ in D,
(iii) $\nu \times E^s - \nu \times E = -\nu \times E^i$ on ∂D,

(iv) $\nu \times H^{\rm s} - \nu \times H = -\nu \times H^{\rm i}$ on Γ_1,
(v) $\nu \times H^{\rm s} - \nu \times H = -\nu \times H^{\rm i} + \eta(x)[\nu \times (E^{\rm s} + E^{\rm i}) \times \nu]$ on Γ_2,
(vi) $\lim_{r\to\infty} (H^{\rm s} \times x - rE^{\rm s}) = 0$,

where the positive real-valued function $\eta > 0$ defined on Γ_2 describes the conducting properties of the thin coating layer. This problem is formulated and studied in [3].

In this limiting case the equality corresponding to the one in Theorem 3.3 contains on the left-hand side

$$\int_{\Gamma_2} \eta ||\nu \times E||^2_{L^2_t} \,{\rm d}s > 0$$

instead of the integral over the whole domain D.

Acknowledgements

This research was supported in part by grants from the Air Force Office of Scientific Research.

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