On the existence of transmission eigenvalues in an inhomogeneous medium

Fioralba Cakoni* and Houssem Haddar†

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Abstract

We prove the existence of transmission eigenvalues corresponding to the inverse scattering problem for isotropic and anisotropic media for both the scalar problem and Maxwell's equations. Considering a generalized abstract eigenvalue problem, we are able to extend the ideas of Päivärinta and Sylvester [15] to prove the existence of transmission eigenvalues for a larger class of interior transmission problems. Our analysis includes both the case of a medium with positive contrast and of a medium with negative contrast provided that the contrasts are large enough.

Keywords: Interior transmission problem, transmission eigenvalues, inhomogeneous medium, inverse scattering

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1 Introduction

The interior transmission problem is a boundary value problem in a bounded domain which arises in inverse scattering theory for inhomogeneous media. Although simply stated, this problem is not covered by the standard theory of elliptic partial differential equation since as it stands it is neither elliptic nor self-adjoint. Of particular interest is the spectrum associated with this boundary value problem, more specifically the existence of eigenvalues which are called transmission eigenvalues. We note that the case of a zero contrast leads to a continuous spectrum for the corresponding eigenvalue problem. On the other hand, for non-zero contrast, the occurrence of a transmission eigenvalue corresponds to the scattering matrix having an eigenvalue equal to one. Besides the theoretical importance of transmission eigenvalues in connection with uniqueness and reconstruction results in

*Department of Mathematical Sciences, University of Delaware, Newark, Delaware 19716-2553, USA

†INRIA Saclay Ile de France / CMAP Ecole Polytechnique, Route de Saclay, 91128 Palaiseau Cedex FRANCE

inverse scattering theory, recently they have been used to obtain information about the index of refraction from measured data [1], [4]. For information on the interior transmission problem, we refer the reader to [2], [7] and in particular to the survey paper by Colton, Päivärinta and Sylvester [9].

Up to now, most of the known results on the interior transmission problem are concerned with when the problem is well-posed. Roughly speaking, two main approaches are available in this direction, namely integral equation methods [8], [11], and variational methods typically applied to a fourth order equivalent boundary value problem [3], [5], [6], [10], [17]. On the other hand, except for the case of spherically stratified medium [7], [9], until recently little was known about the existence and properties of transmission eigenvalues. Applying the analytic Fredholm theory it was possible to show that the transmission eigenvalues form at most a discrete set with infinity as the only possible accumulation point. However, nothing was known in general about the existence of transmission eigenvalues untill the recent important result of Päivärinta and Sylvester [15] who were the first to show that, in the case of (scalar) isotropic media a finite number of transmission eigenvalues exist provided the index of refraction is bounded away from one. Kirsch [12], has extended this existence result to the case of anisotropic media for both the scalar case and Maxwell's equations. However his approach works only if the index of refraction of the scattering medium is less then the index of refraction of the background medium.

In this paper, inspired by the ideas of [15], we present a general proof for the existence of transmission for a wide class of scattering problems. The main idea of our approach makes use of a generalized eigenvalue problem for a family of positive definite and self-adjoint operators with respect to a non negative compact operator. The plan of the paper is as follows. In the next section we develop the abstract analytical framework. Then in Section 3 we give examples of interior transmission problems where we can apply our theory to prove that transmission eigenvalues exist. In particular, we first recover the results of [15] and then apply our approach to anisotropic media for both the scalar case and Maxwell's equations. We show that if the anisotropic index of refraction is greater than or less than one everywhere in the scattering medium then finitely many transmission eigenvalues exist provided that the contrast is big enough. The number of recovered eigenvalues depends on how large the contrast is and we give explicit estimates for this number in terms of the support of scattering medium.

We conclude by noting that many questions related to the spectrum of interior transmission problems still remain open. In particular, we mention the analysis of the interior transmission problem for scattering media with contrast that changes sign or is zero on a set of finite measure.

2 Abstract analytical framework

Let U be a separable Hilbert space with scalar product $(\cdot,\cdot)$ and associated norm $\|\cdot\|$, and A be a bounded, positive definite and self-adjoint operator on U. We recall that the operators $A^{\pm 1/2}$ are defined by $A^{\pm 1/2} = \int_0^\infty \lambda^{\pm 1/2} dE_\lambda$ where dE_λ is the spectral measure associated with A. In particular, $A^{\pm 1/2}$ are also bounded, positive definite and self-adjoint operators on U, $A^{-1/2}A^{1/2} = I$ and $A^{1/2}A^{1/2} = A$. We shall consider the spectral decomposition of the operator A with respect to self-adjoint non negative compact operators and the next two theorems indicate the main properties of such decomposition.

Theorem 2.1 *Let A be a positive definite and self-adjoint bounded linear operator on U and let B be a non negative, self-adjoint and compact bounded linear operator on U. There exists an increasing sequence of positive real numbers $(\lambda_k)_{k\geq 1}$ and a sequence $(u_k)_{k\geq 1}$ of elements of U such that $Au_k = \lambda_k Bu_k$. The sequence $(u_k)_{k\geq 1}$ form a basis of $(A\ker(B))^{\perp}$ and can be chosen so that $(Bu_k, u_l) = \delta_{k,l}$. If $\ker(B)^{\perp}$ has infinite dimension then $\lambda_k \to +\infty$ as $k \to \infty$.*

Proof. This theorem is a direct consequence of the spectral decomposition of the non negative self-adjoint compact operator $\tilde{B} = A^{-1/2}BA^{-1/2}$ (known as the Hilbert-Scmidt theorem see e.g. [16]). Let $(\mu_k, v_k)_{k\geq 1}$ be the sequence of positive eigenvalues and corresponding eigenfunctions associated with $\tilde{B}$ such that $\{v_k,\ k = 1, 2, \ldots\}$ form an orthonormal basis for $\ker(\tilde{B})^{\perp}$. Note that 0 is the only possible accumulation point for the sequence μ_k. Then, one can easily check that $\lambda_k = 1/\mu_k$ and $u_k = \sqrt{\lambda_k}\, A^{-1/2}v_k$ for $k = 1, 2, \ldots$ satisfiy $Au_k = \lambda_k Bu_k$. Obviously, if $w \in A\ker(B)$ then $w = Az$ for some $z \in \ker B$ and hence $(u_k, w) = \lambda_k(A^{-1}Bu_k, w) = \lambda_k(A^{-1}Bu_k, Az) = \lambda_k(Bu_k, z) = 0$ which means that $u_k \in (A\ker(B))^{\perp}$. Furthermore, any $v \in (A\ker(B))^{\perp}$ can be written as $v = \sum_k \gamma_k u_k = \sum_k \gamma_k \sqrt{\lambda_k} A^{-1/2} v_k$ because it is easy to check that $A^{1/2}v \in \left(\ker(A^{-1/2}BA^{-1/2})\right)^{\perp}$. This ends the proof of the theorem. □

Theorem 2.2 *Let A, B and $(\lambda_k)_{k\geq 1}$ be as in Theorem 2.1 and define the Rayleigh quotient as*

$$R(u) = \frac{(Au, u)}{(Bu, u)}$$

for $u \notin \ker(B)$, where $(\cdot\,,\,\cdot)$ is the inner product in U. Then the following min-max principles (known as Courant-Fischer formulae) hold

$$\lambda_k = \min_{W\in \mathcal{U}_k^A}\left(\max_{u\in W\setminus\{0\}} R(u)\right) = \max_{W\in\mathcal{U}_{k-1}^A}\left(\min_{u\in(A(W+\ker(B)))^{\perp}\setminus\{0\}} R(u)\right)$$

where $\mathcal{U}_k^A$ denotes the set of all k-dimensional subspaces of $(A\ker(B))^{\perp}$.

Proof. The proof uses classical arguments and is given here for the reader's convenience. It is based on the fact that if $u \in (A\ker(B))^{\perp}$ then from Theorem 2.1 v we can write $u = \sum_k \gamma_k u_k$ for some coefficients γ_k, where u_k are defined in Theorem 2.1 (note that the u_k are orthogonal with respect to the inner-product induced by self-adjoint invertible operator A). Then using the facts that $(Bu_k, u_l) = \delta_{k,l}$ and $Au_k = \lambda_k Bu_k$ it is easy to see that

$$R(u) = \frac{1}{\sum_k |\gamma_k|^2}\sum_k \lambda_k|\gamma_k|^2.$$

Therefore, if $W_k \in \mathcal{U}_k^A$ denotes the space generated by $\{u_1, \dots, u_k\}$ we have that

$$\lambda_k = \max_{u \in W_k \setminus \{0\}} R(u) = \min_{u \in (A(W_{k-1} + \ker(B)))^{\perp} \setminus \{0\}} R(u).$$

Next, let W be any element of $\mathcal{U}_k^A$. Since W has dimension k and $W \subset (A \ker(B))^{\perp}$, then $W \cap (AW_{k-1} + A\ker(B))^{\perp} \neq \{0\}$. Therefore

$$\max_{u \in W \setminus \{0\}} R(u) \geq \min_{u \in W \cap (A(W_{k-1} + \ker(B)))^{\perp} \setminus \{0\}} R(u) \geq \min_{u \in (A(W_{k-1} + \ker(B)))^{\perp} \setminus \{0\}} R(u) = \lambda_k$$

which proves the first equality of the theorem. Similarly, if W has dimension $k-1$ and $W \subset (A\ker(B))^{\perp}$, then $W_k \cap (AW)^{\perp} \neq \{0\}$. Therefore

$$\min_{u \in (A(W + \ker(B)))^{\perp} \setminus \{0\}} R(u) \leq \max_{u \in W_k \cap (AW)^{\perp} \setminus \{0\}} R(u) \leq \max_{u \in W_k \setminus \{0\}} R(u) = \lambda_k$$

which proves the second equality of the theorem. □

The following corollary shows that it is possible to remove the dependence on A in the choice of the subspaces in the min-max principle for the eigenvalues λ_k.

Corollary 2.1 *Let A, B, $(\lambda_k)_{k \geq 1}$ and R be as in Theorem 2.2. Then*

$$\lambda_k = \min_{W \subset \mathcal{U}_k} \left(\max_{u \in W \setminus \{0\}} R(u) \right) \tag{1}$$

where $\mathcal{U}_k$ denotes the set of all k-dimensional subspaces W of U such that $W \cap \ker(B) = \{0\}$.

Proof. From Theorem 2.2, since $\mathcal{U}_k^A \subset \mathcal{U}$ it suffices to prove that

$$\lambda_k \leq \min_{W \subset \mathcal{U}_k} \left(\max_{u \in W \setminus \{0\}} R(u) \right).$$

Let $W \in \mathcal{U}_k$ and let $v_1, v_2, \dots, v_k$ be a basis for W Each vector v_k can be decomposed into a sum $v_k^0 + \tilde{v}_k$ where $\tilde{v}_k \in (A\ker(B))^{\perp}$ and $v_k^0 \in \ker(B)$ (which is the orthogonal decomposition with respect to the scalar product induced by A). Since $W \cap \ker(B) = \{0\}$, the space $\tilde{W}$ generated by $\tilde{v}_1, \tilde{v}_2, \dots, \tilde{v}_k$ has dimension k. Moreover, $\tilde{W} \subset (A\ker(B))^{\perp}$. Now let $\tilde{u} \in \tilde{W}$. Obviously $\tilde{u} = u - u^0$ for some $u \in W$ and $u^0 \in \ker(B)$. Since $Bu^0 = 0$ and $(Au_0, \tilde{u}) = 0$ we have that

$$R(u) = \frac{(A\tilde{u}, \tilde{u}) + (Au^0, u^0)}{(B\tilde{u}, \tilde{u})} = R(\tilde{u}) + \frac{(Au^0, u^0)}{(B\tilde{u}, \tilde{u})}.$$

Consequently, since A is positive definite and B is non negative, we obtain

$$R(\tilde{u}) \leq R(u) \leq \max_{u \in W \setminus \{0\}} R(u).$$

Finally, taking the maximum with respect to $\tilde{u} \in \tilde{W} \subset (A \ker(B))^{\perp}$ in the above inequality, we obtain from Theorem 2.2 that

$$\lambda_k \leq \max_{u \in W \setminus \{0\}} R(u),$$

which completes the proof after taking the minimum over all $W \subset \mathcal{U}_k$.

□

In the following we formulate the main result of this section which provides the theoretical basis of our analysis on the existence of transmission eigenvalues. This theorem is a simple consequence of Theorem 2.2 and Corollary 2.1.

Theorem 2.3 *Let $\tau \longmapsto A_\tau$ be a continuous mapping from $]0,\infty[$ to the set of self-adjoint and positive definite bounded linear operators on U and let B be a self-adjoint and non negative compact bounded linear operator on U. We assume that there exists two positive constant $\tau_0 > 0$ and $\tau_1 > 0$ such that*

1. *$A_{\tau_0} - \tau_0 B$ is positive on U,*

2. *$A_{\tau_1} - \tau_1 B$ is non positive on a k-dimensional subspace W_k of U.*

Then each of the equations $\lambda_j(\tau) = \tau$ for $j = 1, \ldots, k$, has at least one solution in $[\tau_0, \tau_1]$ where $\lambda_j(\tau)$ is the j^{th} eigenvalue (counting multiplicity) of A_τ with respect to B, i.e. $\ker(A_\tau - \lambda_j(\tau)B) \neq \{0\}$.

Proof. First we can deduce from (1) that for all $j \geq 1$, $\lambda_j(\tau)$ is a continuous function of τ. Assumption *1.* shows that $\lambda_j(\tau_0) > \tau_0$ for all $j \geq 1$. Assumption *2.* implies in particular that $W_k \cap \ker(B) = \{0\}$. Hence, another application of (1) implies that $\lambda_j(\tau_1) \leq \tau_1$ for $1 \leq j \leq k$. The desired result is then obtained by applying the intermediate value theorem.
□

3 The existence of transmission eigenvalues

Ou goal is to apply Theorem 2.3 to show the existence of one or more transmission eigenvalues corresponding to different scattering problems for inhomogeneous media. In all the examples presented here, the corresponding interior transmission eigenvalue problem is formulated as $A_\tau - \tau B = 0$ where $\{A_\tau\}$ is a family of positive definite self adjoint bounded linear operators and B is a non negative compact bounded linear operator, both defined on appropriate Hilbert spaces. Then a transmission eigenvalue is the solution of $\lambda(\tau) - \tau = 0$ where $\lambda(\tau)$ is an eigenvalue of the generalized eigenvalue problem $A_\tau - \lambda(\tau)B = 0$.

Our first application concerns with the existence of transmission eigenvalues corresponding to the scattering problem for an isotropic inhomogenouse medium in $\mathbb{R}^2$. This is the simplest scattering problem for inhomogeneous media where we can present our basic ideas with the least technicality. Using the analytical framework developed in Section 2, we recover the results obtained in [15]. Next, we carry our approach to prove the existence of transmission eigenvalues for the cases of electromagnetic scattering for anisotropic media in $\mathbb{R}^3$ (the corresponding interior transmission problem is considered in [10] and [5]) and for the scattering problem for anisotropic media in $\mathbb{R}^2$ (the corresponding interior transmission problem is considered in [6]). We remark that the abstract analytical framework presented here, although in the same spirit as [15], is applicable to a larger class of problems then the analysis of [15]. In particular, our approach is based on a generalized eigenvalue problem with respect to a non negative compact operator B which allows us, as opposed to the approach in [15], to consider cases when the identity operator is no longer a compact operator. Note that for all the problems considered here it is already known that the transmission eigenvalues form at most a discrete set with infinity as the only possible accumulation point [2], [5], [6], [9]-[11], [15] [17].

3.1 Scalar isotropic media

The *interior transmission eigenvalue problem* corresponding to the scattering problem for the isotropic inhomogenous medium in $\mathbb{R}^2$ reads:

$$\Delta w + k^2 n(x) w = 0 \qquad \text{in} \qquad D \tag{2}$$

$$\Delta v + k^2 v = 0 \qquad \text{in} \qquad D \tag{3}$$

$$w = v \qquad \text{on} \qquad \partial D \tag{4}$$

$$\frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} \qquad \text{on} \qquad \partial D \tag{5}$$

for $w \in L^2(D)$ and $v \in L^2(D)$ such that $w - v \in H_0^2(D)$ where

$$H_0^2(D) = \left\{ u \in H^2(D) : \; u = 0 \quad \text{and} \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \partial D \right\}.$$

Here we assume that $n(x)$ and and $1/|n(x) - 1| > 0$ are bounded positive real valued functions defined in D. Furthermore, we assume that $D \subset \mathbb{R}^2$ is a bounded simply connected region with piece-wise smooth boundary ∂D and denote by ν the outward normal vector to ∂D. (Everything in this section holds true for the same equations in $\mathbb{R}^3$.) Transmission eigenvalues are the values of $k > 0$ for which the above homogeneous interior transmission has non zero solutions. It is possible to write (2)-(5) as an equivalent eigenvalue problem for $u = w - v \in H_0^2(D)$ for the following forth order equation

$$\left(\Delta + k^2 n\right) \frac{1}{n-1} \left(\Delta + k^2\right) u = 0 \tag{6}$$

which in variational form is formulated as finding a function $u \in H_0^2(D)$ such that

$$\int_D \frac{1}{n-1} (\Delta u + k^2 u)(\Delta \overline{v} + k^2 n \overline{v}) \, dx = 0 \qquad \text{for all } v \in H_0^2(D). \tag{7}$$

We now define the following bounded sesquilinear forms on $H_0^2(D) \times H_0^2(D)$:

$$\mathcal{A}_\tau(u,v) = \left(\frac{1}{n-1}(\Delta u + \tau u),\, (\Delta v + \tau v)\right)_D + \tau^2\,(u,\, v)_D \tag{8}$$

$$\begin{aligned}\tilde{\mathcal{A}}_\tau(u,v) &= \left(\frac{1}{1-n}(\Delta u + \tau n u),\, (\Delta v + \tau n v)\right)_D + \tau^2\,(nu,\, v)_D \\ &= \left(\frac{n}{1-n}(\Delta u + \tau u),\, (\Delta v + \tau v)\right)_D + (\Delta u,\, \Delta v)_D\end{aligned} \tag{9}$$

and

$$\mathcal{B}(u,v) = (\nabla u,\, \nabla v)_D \tag{10}$$

where $\tau := k^2$ and $(\cdot\,,\,\cdot)_D$ denotes the $L^2(D)$ inner product. Then (7) can be written as either

$$\mathcal{A}_\tau(u,v) - \tau\mathcal{B}(u,v) = 0 \qquad \text{for all } v \in H_0^2(D), \tag{11}$$

or

$$\tilde{\mathcal{A}}_\tau(u,v) - \tau\mathcal{B}(u,v) = 0 \qquad \text{for all } v \in H_0^2(D). \tag{12}$$

Obviously, if $\frac{1}{n(x)-1} > \gamma > 0$ almost everywhere in D then $\mathcal{A}_\tau$ is a coersive sesquilinear form on $H_0^2(D) \times H_0^2(D)$ whereas if $\dfrac{n(x)}{1-n(x)} > \gamma > 0$ almost everywhere in D then $\tilde{\mathcal{A}}_\tau$ is a coersive sesquilinear form on $H_0^2(D) \times H_0^2(D)$. Indeed we have

$$\begin{aligned}\mathcal{A}_\tau(u,u) &\geq \gamma\|\Delta u + \tau u\|_{L^2}^2 + \tau^2\|u\|_{L^2}^2 \geq \gamma X^2 - 2\gamma XY + (\gamma+1)Y^2 \\ &= \epsilon\left(Y - \frac{\gamma}{\epsilon}X\right)^2 + \left(\gamma - \frac{\gamma^2}{\epsilon}\right)X^2 + (1+\gamma-\epsilon)Y^2 \\ &\geq \left(\gamma - \frac{\gamma^2}{\epsilon}\right)X^2 + (1+\gamma-\epsilon)Y^2\end{aligned} \tag{13}$$

for $\gamma < \epsilon < \gamma + 1$, where $X = \|\Delta u\|_{L^2(D)}$ and $Y = \tau\|u\|_{L^2(D)}$. Furthermore, since $\nabla u \in H_0^1(D)^2$, using the Poincaré inequality we have that

$$\|\nabla u\|_{L^2(D)}^2 \leq \frac{1}{\lambda_0(D)}\|\Delta u\|_{L^2(D)}^2 \tag{14}$$

where $\lambda_0(D)$ is the first Dirichlet eigenvalue of $-\Delta$ on D. Hence we can conclude that

$$\mathcal{A}_\tau(u,u) \geq C_\tau\|u\|_{H^2(D)}^2$$

for some positive constant C_τ. Similarly if $\frac{n(x)}{1-n(x)} > \gamma > 0$ then

$$\begin{aligned}\tilde{\mathcal{A}}_\tau(u,u) &\geq \gamma\|\Delta u + \tau u\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 \geq (1+\gamma)X^2 - 2\gamma XY + \gamma Y^2 \\ &= \epsilon\left(X - \frac{\gamma}{\epsilon}Y\right)^2 + \left(\gamma - \frac{\gamma^2}{\epsilon}\right)Y^2 + (1+\gamma-\epsilon)X^2 \end{aligned} \tag{15}$$

$$\geq (1+\gamma-\epsilon)X^2 + \left(\gamma - \frac{\gamma^2}{\epsilon}\right)Y^2 \tag{16}$$

for $\gamma < \epsilon < \gamma + 1$, whence as in the above, by using the Poincaré inequality,

$$\mathcal{A}_\tau(u,u) \geq C_\tau \|u\|^2_{H^2(D)}$$

for some positive constant C_τ.

Using the Riesz representation theorem we now define the bounded linear operators $A_\tau : H^2_0(D) \to H^2_0(D)$, $\tilde{A}_\tau : H^2_0(D) \to H^2_0(D)$ and $B : H^2_0(D) \to H^2_0(D)$ by

$$(A_\tau u, v)_{H^2(D)} = \mathcal{A}_\tau(u,v), \quad \left(\tilde{A}_\tau u, v\right)_{H^2(D)} = \tilde{\mathcal{A}}_\tau(u,v) \quad \text{and} \quad (Bu,v)_{H^2(D)} = \mathcal{B}(u,v).$$

Since n is real the sesquilinear forms $\mathcal{A}_\tau$, $\tilde{\mathcal{A}}_\tau$ and $\mathcal{B}$ are hermitian and therefore the operators A_τ, $\tilde{A}_\tau$ and B are self-adjoint. Furthermore, by definition, B is a non negative operator and if $\frac{1}{n(x)-1} > \gamma > 0$ then A_τ is a positive definite operator, whereas if $\frac{n(x)}{1-n(x)} > \gamma > 0$ then $\tilde{A}_\tau$ is a positive definite operator. Finally, noting that for $u \in H^2_0(D)$ we have that $\nabla u \in H^1_0(D)^2$, since $H^1_0(D)^2$ is compactly embedded in $L^2(D)^2$ we can conclude that $B : H^2_0(D) \to H^2_0(D)$ is a compact operator. Also A_τ and $\tilde{A}_\tau$ depend continuously on $\tau \in (0, +\infty)$. Hence, depending on the assumptions on n, we have that A_τ and B or $\tilde{A}_\tau$ and B , $\tau > 0$ satisfy the conditions of Theorem 2.3 with $U = H^2_0(D)$.

To prove existence of eigenvalues we shall prove that under some further assumptions on n there exist τ_0 and τ_1 satisfying assumption 1 and assumption 2, respectively, of Theorem 2.3. This result is proven in Theorem 17 of [15] but we present here a slightly modified proof which can be generalized in a straight forward manner to other applications such as for anisotropic media and Maxwell's equations which will be discussed next.

To this end let us set $n_* = \inf_D(n)$ and $n^* = \sup_D(n)$, denote by $\mu_p(D) > 0$ the $(p+1)-st$ clamped plate eigenvalue (counting the multiplicity) in D and then set

$$\theta_p(D) := 4\frac{\mu_p(D)^{1/2}}{\lambda_0(D)} + 4\frac{\mu_p(D)}{\lambda_0(D)^2}.$$

Theorem 3.1 *Let $n \in L^\infty(D)$ satisfying either one of the following assumptions*

1) $1 + \theta_p(D) \leq n_* \leq n(x) \leq n^* < \infty$,

2) $0 < n_* \leq n(x) \leq n^* < \frac{1}{1+\theta_p(D)}$.

Then, there exist $p+1$ transmission eigenvalues (counting multiplicity).

Proof. First assume that the assumption 1) holds. This assumption also implies that

$$0 < \frac{1}{n^*-1} \leq \frac{1}{n(x)-1} \leq \frac{1}{n_*-1} < \infty$$

and according to the above, A_τ and B, $\tau > 0$ satisfy the assumptions of Theorem 2.3 with $U = H^2_0(D)$. Fom (13) and (14) we have

$$\begin{aligned}(A_\tau u - \tau Bu,\, u)_{H^2_0} &= \mathcal{A}_\tau(u,u) - \tau\|\nabla u\|^2_{L^2} \\ &\geq \left(\gamma - \frac{\gamma^2}{\epsilon} - \frac{\tau}{\lambda_0(D)}\right)\|\Delta u\|^2_{L^2} + \tau(1+\gamma-\epsilon)\|u\|^2_{L^2}\end{aligned} \tag{17}$$

with $\gamma = \frac{1}{n^*-1}$ and $\gamma < \epsilon < \gamma+1$. Hence $A_\tau - \tau B$ is positive as long as $\tau < \left(\gamma - \frac{\gamma^2}{\epsilon}\right)\lambda_0(D)$. In particular, taking ϵ arbitrary closed to $\gamma+1$, the latter becomes $\tau < \frac{\gamma}{1+\gamma}\lambda_0(D) = \frac{\lambda_0(D)}{\sup_D(n)}$. Then any positive number τ_0 smaller then $\frac{\lambda_0(D)}{sup_D(n)}$ satisfies assumption 1 of Theorem 2.3.

Set $M = \sup_D(\frac{1}{n-1}) = \frac{1}{n_*-1}$. Then, restricting ourselves to functions in $H_0^2(D)$ such that $\|u\|_{L^2} = 1$, and using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}(A_\tau u - \tau Bu,\, u)_{H_0^2} &= \int_D \frac{1}{n-1}|\Delta u|^2\,dx + \tau^2 \int_D \frac{n}{n-1}|u|^2\,dx \\ &+ \tau \int_D \frac{1}{n-1}(\overline{u}\Delta u + u\Delta\overline{u})\,dx - \tau \int_D |\nabla u|^2\,dx \\ &\leq M\|\Delta u\|_{L^2}^2 + \tau^2(1+M) + 2\tau M\|\Delta u\|_{L^2} - \tau\|\nabla u\|_{L^2}^2.\end{aligned}$$

Applying the Poincaré inequality to $u \in H_0^1(D)$ one has

$$\|\nabla u\|_{L^2}^2 \geq \lambda_0(D).$$

Now let us denote by V_p the $p+1$ dimensional eigenspace associated with the lowest $p+1$ clamped plate eigenvalues. In particular, if $u \in V_p$ such that $\|u\|_{L^2} = 1$ then $\|\Delta u\|_{L^2}^2 \leq \mu_p$. Note that the kernel of B contains only constant functions which are not in V_p. Hence for $u \in V_p$ we have

$$(A_\tau u - \tau Bu,\, u)_{H_0^2} \leq \tau^2(1+M) - \tau\left(\lambda_0(D) - 2M\mu_p(D)^{1/2}\right) + M\mu_p(D)$$

for any $\tau > 0$. In particular, the value of $\tau_1 = \frac{\lambda_0(D) - 2M\mu_p(D)^{1/2}}{2+2M}$ minimizes the right hand side, whence we obtain

$$(A_\tau u - \tau Bu,\, u)_{H_0^2} \leq -\frac{\left(\lambda_0(D) - 2M\mu_p(D)^{1/2}\right)^2}{4+4M} + M\mu_p(D)$$

which becomes non positive if $M \leq \frac{\lambda_0(D)^2}{4\mu_p(D)^{1/2}\left(\lambda_0(D)+\mu_p(D)^{1/2}\right)}$ which means that

$$\inf_D(n) \geq 1 + 4\frac{\mu_p(D)^{1/2}}{\lambda_0(D)} + 4\frac{\mu_p(D)}{\lambda_0(D)^2} = 1 + \theta_p(D).$$

We therefore have shown that if assumption 1 holds then $A_{\tau_1} - \tau_1 B$ is non positive on a $p+1$ dimentional subspace of $H_0^2(D)$. The theorem is now proven in this case by an application of Theorem 2.3

Next we assume that assumption 2) holds. The proof for this case uses similar arguments as in the previous case after replacing A_τ with $\tilde{A}_\tau$. In this case we have that

$$0 < \frac{n_*}{1-n_*} \leq \frac{n(x)}{1-n(x)} \leq \frac{n^*}{1-n^*} < \infty,$$

and therefore according to the above, $\tilde{A}_\tau$ and B, $\tau > 0$ satisfy the assumptions of Theorem 2.3 with $U = H_0^2(D)$. From (15) and (14) we have

$$\begin{aligned}\left(\tilde{A}_\tau u - \tau B u,\, u\right)_{H_0^2} &= \tilde{\mathcal{A}}_\tau(u,u) - \tau\|\nabla u\|^2_{L^2} \\ &\geq \left(1+\gamma-\epsilon-\frac{\tau}{\lambda_0(D)}\right)\|\Delta u\|_{L^2} + \tau\left(\gamma - \frac{\gamma^2}{\epsilon}\right)\|u\|^2_{L^2}\end{aligned} \tag{18}$$

with $\gamma = \frac{n_*}{1-n_*}$ and $\gamma < \epsilon < \gamma+1$. Hence $\tilde{A}_\tau - \tau B$ is positive as long as $\tau < (1+\gamma-\epsilon)\,\lambda_0(D)$. In particular letting ϵ be arbitrarily close to γ shows in this case that any $\tau_0 < \lambda_0(D)$ satisfies the assumption 1 in Theorem 2.3.

Now set $M = \sup_D \frac{n}{1-n} = \frac{n^*}{1-n^*}$ and observe that $\frac{1}{1-n} \leq M+1$. Then doing the same type of calculations as above assuming that $u \in V_p$ and $\|u\|^2_{L^2} = 1$, we obtain

$$\begin{aligned}\left(\tilde{A}_\tau u - \tau B u,\, u\right)_{H_0^2} &= \int_D \frac{1}{1-n}|\Delta u|^2\,dx + \tau^2 \int_D \frac{n}{1-n}|u|^2\,dx \\ &+ \tau \int_D \frac{n}{1-n}(\overline{u}\Delta u + u\Delta\overline{u})\,dx - \tau\int_D |\nabla u|^2\,dx \\ &\leq (M+1)\|\Delta u\|^2_{L^2} + \tau^2 M + 2\tau M\|\Delta u\|_{L^2} - \tau\|\nabla u\|^2_{L^2} \\ &\leq \tau^2 M - \tau\left(\lambda_0(D) - 2M\mu_p(D)^{1/2}\right) + (M+1)\mu_p(D).\end{aligned}$$

The minimizing value of τ of the right hand side is now $\tau_1 = \frac{\lambda_0(D)-2M\mu_p(D)^{1/2}}{2M}$ which gives

$$(A_\tau u - \tau B u,\, u)_{H_0^2} \leq -\frac{\left(\lambda_0(D) - 2M\mu_p(D)^{1/2}\right)^2}{4M} + (M+1)\mu_p(D).$$

Hence the latter becomes non positive if $M \leq \frac{\lambda_0(D)^2}{4\mu_p(D)^{1/2}\left(\lambda_0(D)+\mu_p(D)^{1/2}\right)}$ which means that $\sup_D(n) \leq 1/(1+\theta_p(D))$. Consequently if assumption 2 holds then $\tilde{A}_{\tau_1} - \tau_1 B$ is non positive on a $p+1$ dimentional subspace of $H_0^2(D)$ and the result is proven in this case again by an application of Theorem 2.3. □

3.2 The anisotropic Maxwell's equations

Now we turn our attention to proving the existence of transmission eigenvalues corresponding to the electromagnetic scattering problem for an anisotropic medium. Let $D \subset \mathbb{R}^3$ now be a bounded simply connected region of $\mathbb{R}^3$ with piece-wise smooth boundary ∂D and denote by ν the outward normal vector to ∂D. Let $(\cdot,\,\cdot)_D$ denote the $L^2(D)^3$ scalar product and consider the Hilbert spaces

$$\begin{aligned}&H(\mathrm{curl}\,, D) := \{\mathbf{u} \in L^2(D)^3 :\ \mathrm{curl}\,\mathbf{u} \in L^2(D)^3\}, \\ &H_0(\mathrm{curl}\,, D) := \{\mathbf{u} \in H(\mathrm{curl}\,, D) :\ \mathbf{u}\times\nu = 0 \text{ on } \partial D\},\end{aligned}$$

equipped with the scalar product $(\mathbf{u}, \mathbf{v})_{\text{curl}} = (\mathbf{u}, \mathbf{v})_D + (\text{curl}\,\mathbf{u}, \text{curl}\,\mathbf{v})_D$ and the corresponding norm $\|\cdot\|_{\text{curl}}$. Next we define

$$\mathcal{U}(D) := \{\mathbf{u} \in H(\text{curl}\,, D) :\ \text{curl}\,\mathbf{u} \in H(\text{curl}\,, D)\},$$
$$\mathcal{U}_0(D) := \{\mathbf{u} \in H_0(\text{curl}\,, D) :\ \text{curl}\,\mathbf{u} \in H_0(\text{curl}\,, D)\},$$

equipped with the scalar product $(\mathbf{u}, \mathbf{v})_{\mathcal{U}} = (\mathbf{u}, \mathbf{v})_{\text{curl}} + (\text{curl}\,\mathbf{u}, \text{curl}\,\mathbf{v})_{\text{curl}}$ and the corresponding norm $\|\cdot\|_{\mathcal{U}}$. Let N be a 3×3 matrix valued function defined on D with $L^\infty(D)$ real valued entries, i.e. $N \in L^\infty(D, \mathbb{R}^{3\times 3})$. We first need to make precise the definition of a bounded positive definite real matrix field.

Definition 3.1 *A real matrix field K is said to be bounded positive definite on D if $K \in L^\infty(D, \mathbb{R}^{3\times 3})$ and if there exists a constant $\gamma > 0$ such that*

$$\overline{\boldsymbol{\xi}} \cdot K\boldsymbol{\xi} \geq \gamma\,|\boldsymbol{\xi}|^2, \ \ \forall \boldsymbol{\xi} \in \mathbb{C}^3 \quad \text{and a.e. in } D. \tag{19}$$

We will assume that N, N^{-1} and either $(N-I)^{-1}$ or $(I-N)^{-1}$ are bounded positive definite real matrix fields on D. The interior transmission eigenvalue problem for the anisotropic Maxwell's equations in terms of electric fields is formulated as the problem of finding two vector valued functions $\mathbf{E} \in L^2(D)^3$ and $\mathbf{E}_0 \in L^2(D)^3$ such that $\mathbf{E} - \mathbf{E}_0 \in \mathcal{U}_0(D)$ satisfies

$$\begin{aligned}
\text{curl}\,\text{curl}\,\mathbf{E} - k^2\, N\,\mathbf{E} = 0 &\qquad \text{in} \qquad D\\
\text{curl}\,\text{curl}\,\mathbf{E}_0 - k^2\,\mathbf{E}_0 = 0 &\qquad \text{in} \qquad D\\
\mathbf{E} \times \nu = \mathbf{E}_0 \times \nu &\qquad \text{on} \qquad \partial D\\
\text{curl}\,\mathbf{E} \times \nu = \text{curl}\,\mathbf{E}_0 \times \nu &\qquad \text{on} \qquad \partial D.
\end{aligned}$$

As it is shown in [10] and [5] the transmission eigenvalue problem is equivalent to finding $\mathbf{u} = \mathbf{E} - \mathbf{E}_0 \in \mathcal{U}_0(D)$ such that

$$(\text{curl}\,\text{curl}\ - k^2 N)(N - I)^{-1}(\text{curl}\,\text{curl}\,\mathbf{u} - k^2\mathbf{u}) = 0. \tag{20}$$

Putting (20) into a variational framework and letting $\tau := k^2$ we obtain that (20) is equivalent to the problem of finding $\mathbf{u} \in \mathcal{U}_0(D)$ that satisfies

$$\mathcal{A}_\tau(\mathbf{u}, \mathbf{v}) - \tau\mathcal{B}(\mathbf{u}, \mathbf{v}) = 0 \ \text{ for all } \mathbf{v} \in \mathcal{U}_0(D), \tag{21}$$

or

$$\tilde{\mathcal{A}}_\tau(\mathbf{u}, \mathbf{v}) - \tau\mathcal{B}(\mathbf{u}, \mathbf{v}) = 0 \ \text{ for all } \mathbf{v} \in \mathcal{U}_0(D), \tag{22}$$

where here $\mathcal{A}_\tau$, $\tilde{\mathcal{A}}_\tau$ and $\mathcal{B}$ are the continuous sesquilinear forms on $\mathcal{U}(D) \times \mathcal{U}_0(D)$ defined by

$$\mathcal{A}_\tau(\mathbf{u}, \mathbf{v}) = \big((N - I)^{-1}(\text{curl}\,\text{curl}\,\mathbf{u} - \tau\mathbf{u}),\ (\text{curl}\,\text{curl}\,\mathbf{v} - \tau\mathbf{v})\big)_D + \tau^2\,(\mathbf{u},\, \mathbf{v})_D$$

$$\begin{aligned}
\tilde{\mathcal{A}}_\tau(\mathbf{u}, \mathbf{v}) &= \big((I - N)^{-1}(\text{curl}\,\text{curl}\,\mathbf{u} - \tau N\mathbf{u}),\ (\text{curl}\,\text{curl}\,\mathbf{v} - \tau N\mathbf{v})\big)_D + \tau^2\,(N\mathbf{u},\, \mathbf{v})_D\\
&= \big(N(I - N)^{-1}(\text{curl}\,\text{curl}\,\mathbf{u} - \tau\mathbf{u}),\ (\text{curl}\,\text{curl}\,\mathbf{v} - \tau\mathbf{v})\big)_D + (\text{curl}\,\text{curl}\,\mathbf{u},\, \text{curl}\,\text{curl}\,\mathbf{v})_D
\end{aligned}$$

and

$$\mathcal{B}(\mathbf{u},\mathbf{v}) = (\operatorname{curl}\mathbf{u},\, \operatorname{curl}\mathbf{v})_D$$

respectively, with $(\cdot\,,\,\cdot)_D$ denoting the $L^2(D)^3$ inner product. In a similar way as in Section 3.1, in [5] (Lemma 3.1) and [10] (Lemma 3.3), it is shown that if $(N-I)^{-1}$ is a bounded positive definite matrix field on D then $\mathcal{A}_k$ is a coercive hermitian sesquilinear form on $\mathcal{U}_0(D)\times\mathcal{U}_0(D)$, whereas if $N(I-N)^{-1}$ is a bounded positive definite matrix field on D then $\tilde{\mathcal{A}}_k$ is a coercive hermitian sesquilinear form on $\mathcal{U}_0(D)\times\mathcal{U}_0(D)$. Hence the bounded linear operators $\mathbb{A}_\tau : \mathcal{U}_0(D)\to\mathcal{U}_0(D)$ and $\tilde{\mathbb{A}}_\tau : \mathcal{U}_0(D)\to\mathcal{U}_0(D)$ defined using the Riesz representation theorem by

$$(\mathbb{A}_\tau\mathbf{u},\mathbf{v})_{\mathcal{U}_0} = \mathcal{A}_\tau(\mathbf{u},\mathbf{v}) \qquad \text{and} \qquad \left(\tilde{\mathbb{A}}_\tau\mathbf{u},\mathbf{v}\right)_{\mathcal{U}_0} = \tilde{\mathcal{A}}_\tau(\mathbf{u},\mathbf{v})$$

are positive definite self-adjoint operators if $(N-I)^{-1}$ and $N(I-N)^{-1}$, respectively, are bounded positive definite. It is obvious that the sesquilinear form $\mathcal{B}(\cdot\,,\,\cdot)$ is hermitian and non negative. In [10] (Lemma 3.4) it is shown that the non-negative self-adjoint bounded linear operator $\mathbb{B} : \mathcal{U}_0(D)\to\mathcal{U}_0(D)$ defined using the Riesz representation theorem by $(\mathbb{B}\mathbf{u},\mathbf{v})_{\mathcal{U}_0} = \mathcal{B}(\mathbf{u},\mathbf{v})$ is also compact. Finally, the families of operators $\mathbb{A}_\tau$ and $\tilde{\mathbb{A}}_\tau$ depend continuously on $\tau\in(0,+\infty)$. Hence, since the eigenvalue problems (21) and (22) are equivalent to

$$\mathbb{A}_\tau - \tau\mathbb{B} = 0 \qquad \text{and} \qquad \tilde{\mathbb{A}}_\tau - \tau\mathbb{B} = 0$$

respectively, we are at the position to apply Theorem 2.3 with $U := \mathcal{U}_0(D)$. In particular it remains to check whether the assumptions 1. and 2. of this theorem hold true for the above generalized eigenvalue problems.

To this end let $0<\eta_1(x)\le\eta_2(x)\le\eta_3(x)$ be the eigenvalues of the positive definite symmetric matrix N. Recall that the largest eigenvalue $\eta_3(x)$ which coincides with the Euclidean norm $\|N(x)\|_2$ is given by by $\eta_3(x) = \sup_{\|\xi\|=1}(\bar{\xi}\cdot N(x)\,\xi)$ and the smallest eigenvalue $\eta_1(x)$ is given by $\eta_1(x) = \inf_{\|\xi\|=1}(\bar{\xi}\cdot N(x)\,\xi)$. We denote by $N^* = \sup_D \eta_3(x)$ and $N_* = \inf_D \eta_1(x)$. Furthermore, let us consider the eigenvalue problem for the $(\operatorname{curl}\operatorname{curl})^2$ operator written in variational form as

$$\int_D (\operatorname{curl}\operatorname{curl}\mathbf{u}\,\operatorname{curl}\operatorname{curl}\overline{\mathbf{v}} - \kappa\mathbf{u}\overline{\mathbf{v}})\,dx = 0 \qquad \text{for all} \qquad \mathbf{v}\in\mathcal{W}_0(D) \tag{23}$$

where $\mathcal{W}_0(D) := \mathcal{U}_0(D)\cap H_0(\operatorname{div}0,D)$ with

$$H_0(\operatorname{div}0,D) := \left\{\mathbf{u}\in L^2(D)^3 :\ \operatorname{div}\mathbf{u}=0 \text{ and } \nu\cdot\mathbf{u}=0\right\}.$$

The following decomposition is orthogonal with respect to $L^2(D)^3$-inner product

$$\mathcal{U}_0(D) = \mathcal{W}_0(D)\oplus\left\{\mathbf{u} := \nabla\varphi,\ \varphi\in H^1(D)\right\}.$$

Note that

$$\text{the kernel of } \mathbb{B} = \left\{\mathbf{u}\in\mathcal{U}_0(D) \quad \text{such that } \mathbf{u} := \nabla\varphi,\ \varphi\in H^1(D)\right\}.$$

Moreover, $\mathcal{W}_0(D)$ is continuously embedded in $H^1(D)$ (see [14]). It is easily seen that the eigenvalues of this problem exist, are strictly positive and accumulate only at infinity, and the corresponding eigenspaces are finite dimensional. The eigenfunctions of this eigenvalue problem are divergent free functions with zero tangential and normal traces on ∂D and therefor are in $H^1_0(D)$. Since

$$\Delta \mathbf{u} = \nabla\nabla\cdot\mathbf{u} - \mathrm{curl}\,\mathrm{curl}\,\mathbf{u} = \mathbf{0}$$

the eigenfunctions of (23) coincide with the divergent free eigenfunctions of the vector bi-harmonic equation with clamped plate boundary conditions. We denote by $\kappa_p(D) > 0$ the $(p+1)$-th eigenvalue of (23) (eigenvalues are ordered in increasing order) and set

$$\Theta_p(D) := 4\frac{\kappa_p(D)^{1/2}}{\lambda_0(D)} + 4\frac{\kappa_p(D)}{\lambda_0(D)^2}.$$

Theorem 3.2 *Let $N \in L^\infty(D, \mathbb{R}^{3\times 3})$ be a positive definite symmetric real matrix field on D that satisfies either one of the following assumptions*

1) $1 + \Theta_p(D) \le N_* \le \left(\bar{\xi}\cdot N(x)\,\xi\right) \le N^* < \infty$,

2) $0 < N_* \le \left(\bar{\xi}\cdot N(x)\,\xi\right) \le N^* < \frac{1}{1+\Theta_p(D)}$.

for every $\xi \in \mathbb{C}^3$ such that $\|\xi\| = 1$ and for almost all $x \in D$. Then, there exist $p+1$ transmission eigenvalues (counting multiplicity).

Proof. First assume that the assumption 1) holds. This assumption also implies that

$$0 < \frac{1}{N^*-1}\|\xi\|^2 \le \left(\bar{\xi}\cdot (N-I)^{-1}\,\xi\right) \le \frac{1}{N_*-1}\|\xi\|^2 < \infty$$

and according to the above, $\mathbb{A}_\tau$ and $\mathbb{B}$, $\tau > 0$ satisfy the assumptions of Theorem 2.3 with $U = \mathcal{U}(D)$. Hence following [5] and [10] in a similar way as in Section 3.1 we obtain that

$$\begin{aligned}(\mathbb{A}_\tau\mathbf{u} - \tau\mathbb{B}\mathbf{u},\, \mathbf{u})_{\mathcal{U}_0} &= \mathcal{A}_\tau(\mathbf{u},\mathbf{u}) - \tau\|\mathrm{curl}\,\mathbf{u}\|^2_{L^2} \\ &\ge \left(\gamma - \frac{\gamma^2}{\epsilon}\right)\|\mathrm{curl}\,\mathrm{curl}\,\mathbf{u}\|^2_{L^2} + \tau(1+\gamma-\epsilon)\|\mathbf{u}\|^2_{L^2} - \tau\|\mathrm{curl}\,\mathbf{u}\|^2_{L^2}\end{aligned} \tag{24}$$

with $\gamma = \frac{1}{N^*-1}$ and $\gamma < \epsilon < \gamma + 1$. First we observe that since $\mathbf{u}\times\nu = 0$ on ∂D, then

$$\mathrm{curl}\,\mathbf{u}\cdot\nu = 0 \text{ on } \partial D.$$

This holds true for Lipshitz boundaries by interpreting the relationship $\mathrm{curl}\,\mathbf{u}\cdot\nu = \mathrm{div}_{\partial D}(\mathbf{u}\times\nu)$ in the weak sense [14]. On the other hand, the continuous embedding of

$$\{\mathbf{u} \in H_0(\mathrm{curl}\,, D) : \ \mathrm{div}\,\mathbf{u} = 0 \text{ in } D\}$$

into $H^1(D)^3$ implies that $\operatorname{curl}\mathbf{u} \in H^1_0(D)^3$. Then the Poincaré inequality implies that

$$\|\operatorname{curl}\mathbf{u}\|^2 \leq \frac{1}{\lambda_0(D)}\|\nabla \operatorname{curl}\mathbf{u}\|^2_{L^2(D)}$$

where $\lambda_0(D)$ is the first Dirichlet eigenvalue of $-\Delta$ in D. Let $\tilde{\mathbf{v}}$ be the extension of $\operatorname{curl}\mathbf{u}$ by 0 outside D. Then

$$\|\nabla \operatorname{curl}\mathbf{u}\|^2_{L^2(D)} = \|\nabla \tilde{\mathbf{v}}\|^2_{L^2(\mathbb{R}^3)} = \|\operatorname{curl}\tilde{\mathbf{v}}\|^2_{L^2(\mathbb{R}^3)} + \|\operatorname{div}\tilde{\mathbf{v}}\|^2_{L^2(\mathbb{R}^3)} = \|\operatorname{curl}\tilde{\mathbf{v}}\|^2_{L^2(D)} + \|\operatorname{div}\tilde{\mathbf{v}}\|^2_{L^2(D)}.$$

We therefore obtain that

$$\|\operatorname{curl}\mathbf{u}\|^2_{L^2(D)} \leq \frac{1}{\lambda_0(D)}\|\operatorname{curl}\operatorname{curl}\mathbf{u}\|^2_{L^2(D)}. \tag{25}$$

Now from (24) and (25) we obtain

$$(\mathbb{A}_\tau\mathbf{u} - \tau\mathbb{B}\mathbf{u},\, \mathbf{u})_{\mathcal{U}_0} \geq \left(\gamma - \frac{\gamma^2}{\epsilon} - \frac{\tau}{\lambda(D)}\right)\|\operatorname{curl}\operatorname{curl}\mathbf{u}\|^2_{L^2} + \tau(1+\gamma-\epsilon)\|\mathbf{u}\|^2_{L^2}.$$

Hence $\mathbb{A}_\tau - \tau\mathbb{B}$ is positive as long as $\tau < \left(\gamma - \frac{\gamma^2}{\epsilon}\right)\lambda_0(D)$. In particular taking ϵ arbitrary close to $\gamma+1$, the latter becomes $\tau < \frac{\gamma}{1+\gamma}\lambda_0(D) = \frac{\lambda_0(D)}{\sup_D \|N\|_2}$. Then any positive number τ_0 smaller then $\frac{\lambda_0(D)}{\sup_D \|N\|_2}$ satisfies assumption 1 of Theorem 2.3.

Next we set $M = \sup_D \sup_{\|\xi\|=1}\left(\bar{\xi}\cdot (N(x)-I)^{-1}\xi\right) = \frac{1}{N_*-1}$. Then, restricting ourselves to functions in $\mathcal{U}_0(D)$ such that $\|u\|_{L^2} = 1$, and using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
(\mathbb{A}_\tau\mathbf{u} - \tau\mathbb{B}\mathbf{u},\, u)_{\mathcal{U}_0} &\leq M\int_D \left[(\operatorname{curl}\operatorname{curl}\mathbf{u} - \tau\mathbf{u})\,(\operatorname{curl}\operatorname{curl}\overline{\mathbf{u}} - \tau\overline{\mathbf{u}}) + \tau^2|\mathbf{u}|^2 - \tau|\operatorname{curl}\,\mathbf{u}|^2\right] dx \\
&= M\|\operatorname{curl}\operatorname{curl}\mathbf{u}\|^2_{L^2} + \tau^2(1+M)\|\mathbf{u}\|^2_{L^2} - \tau\|\operatorname{curl}\,u\|^2_{L^2} \\
&- M\tau\int_D (\overline{\mathbf{u}}\operatorname{curl}\operatorname{curl}\mathbf{u} + \mathbf{u}\operatorname{curl}\operatorname{curl}\overline{\mathbf{u}})\,dx \\
&\leq M\|\operatorname{curl}\operatorname{curl}\mathbf{u}\|^2_{L^2} + \tau^2(1+M) + 2M\tau\|\operatorname{curl}\operatorname{curl}\mathbf{u}\|_{L^2} - \tau\|\operatorname{curl}\,\mathbf{u}\|^2_{L^2}.
\end{aligned}$$

Now let us denote by W_p the $p+1$ dimensional eigenspace associated with the lowest $p+1$ eigenvalues of (23). In particular, if $\mathbf{u} \in W_p$ such that $\|\mathbf{u}\|_{L^2} = 1$ then $\|\operatorname{curl}\operatorname{curl}\mathbf{u}\|^2_{L^2} \leq \kappa_p$. Furthermore for such $\mathbf{u} \in W_p$ we have that $\operatorname{div}\mathbf{u} = 0$ and $\mathbf{u} \in H^1_0(D)$, whence arguing as in the first part of the proof we have that $\|\operatorname{curl} u\|_{L^2} \geq \Lambda_0(D)$. Hence, restricted to $u \in W_p$, we have

$$(\mathbb{A}_\tau\mathbf{u} - \tau\mathbb{B}\mathbf{u},\, \mathbf{u})_{\mathcal{U}_0} \leq \tau^2(1+M) - \tau\left(\lambda_0(D) - 2M\kappa_p(D)^{1/2}\right) + M\kappa_p(D)$$

for any $\tau > 0$. In particular, the value of $\tau_1 = \frac{\lambda_0(D)-2M\kappa_p(D)^{1/2}}{2+2M}$ minimizes the right hand side, whence we obtain

$$(\mathbb{A}_\tau \mathbf{u} - \tau\mathbb{B}\mathbf{u},\, \mathbf{u})_{\mathcal{U}_0} \leq -\frac{\left(\lambda_0(D) - 2M\kappa_p(D)^{1/2}\right)^2}{4+4M} + M\kappa_p(D)$$

which becomes non positive if $M \leq \frac{\lambda_0(D)^2}{4\kappa_p(D)^{1/2}\left(\lambda_0(D)+\kappa_p(D)^{1/2}\right)}$ which means that

$$\inf_D \eta_1(x) \geq 1 + 4\frac{\kappa_p(D)^{1/2}}{\lambda_0(D)} + 4\frac{\kappa_p(D)}{\lambda_0(D)^2} = 1 + \Theta_p(D)$$

where $\eta_1(x)$ is the smallest eigenvalue of $N(x)$.
We therefore have shown that if assumption 1 holds then $\mathbb{A}_{\tau_1} - \tau_1\mathbb{B}$ is non positive on a $p+1$ dimentional subspace of $\mathcal{U}_0(D)$. The theorem is then proven in this case by an application of Theorem 2.3

Next we assume that assumption 2) holds. We proceed in the same way as in the previous case after replacing $\mathbb{A}_\tau$ with $\tilde{\mathbb{A}}_\tau$. Now we have that

$$0 < \frac{N_*}{1-N_*}\|\xi\|^2 \leq \left(\bar{\xi}\cdot N(I-N)^{-1}\,\xi\right) \leq \frac{N^*}{1-N^*}\|\xi\|^2 < \infty.$$

Moreover, we have shown that $\tilde{\mathbb{A}}_\tau$ and $\mathbb{B}$, $\tau > 0$ satisfy the assumptions of Theorem 2.3 with $U = \mathcal{U}_0(D)$. Hence following [5] and using (25) we obtain

$$\begin{aligned}\left(\tilde{\mathbb{A}}_\tau \mathbf{u} - \tau\mathbb{B}\mathbf{u},\, \mathbf{u}\right)_{\mathcal{U}_0} &= \tilde{\mathcal{A}}_\tau(\mathbf{u},\mathbf{u}) - \tau\|\mathrm{curl}\,\mathbf{u}\|^2_{L^2} &(26)\\ &\geq (1+\gamma-\epsilon)\|\mathrm{curl}\,\mathrm{curl}\,\mathbf{u}\|^2_{L^2} + \tau\left(\gamma - \frac{\gamma^2}{\epsilon}\right)\|\mathbf{u}\|^2_{L^2} - \tau\|\mathrm{curl}\,\mathbf{u}\|^2_{L^2}\\ &\geq \left(1+\gamma-\epsilon-\frac{\tau}{\lambda_0(D)}\right)\|\mathrm{curl}\,\mathrm{curl}\,\mathbf{u}\|^2_{L^2} + \tau\left(\gamma-\frac{\gamma^2}{\epsilon}\right)\|\mathbf{u}\|^2_{L^2} &(27)\end{aligned}$$

with $\gamma = \frac{N_*}{1-N_*}$ and $\gamma < \epsilon < \gamma + 1$. Hence $\tilde{\mathbb{A}}_\tau\mathbf{u} - \tau\mathbb{B}\mathbf{u}$ is positive as long as $\tau < (1+\gamma-\epsilon)\,\lambda_0(D)$. In particular letting ϵ be arbitrarily close to γ shows in this case that any $\tau_0 < \lambda_0(D)$ satisfies the assumption 1 of Theorem 2.3.

Finally set $M = \sup_D \sup_{\|\xi\|=1}\left(\bar{\xi}\cdot N(x)(I-N(x))^{-1}\,\xi\right) = \frac{N^*}{1-N_*}$ and observe that $\bar{\xi}(I-N)^{-1}\xi \leq (M+1)\|\xi\|^2$ for any $\xi \in \mathbb{C}^3$. Then doing the same type of calculations as in the first case, assuming that $\mathbf{u} \in W_p$ and $\|\mathbf{u}\|^2_{L^2} = 1$, we obtain

$$\begin{aligned}(\mathbb{A}_\tau\mathbf{u} - \tau\mathbb{B}\mathbf{u},\, u)_{\mathcal{U}_0} &\leq M\int_D (\mathrm{curl}\,\mathrm{curl}\,\mathbf{u} - \tau\mathbf{u})\,(\mathrm{curl}\,\mathrm{curl}\,\overline{\mathbf{u}} - \tau\overline{\mathbf{u}})\;dx\\ &+ \int_D \left(|\mathrm{curl}\,\mathrm{curl}\,\mathbf{u}|^2 - \tau|\mathrm{curl}\;\mathbf{u}|^2\right)\;dx\end{aligned}$$

$$
\begin{aligned}
&= (M+1)\|\operatorname{curl}\operatorname{curl}\mathbf{u}\|^2_{L^2} + \tau^2 M\|\mathbf{u}\|^2 - \tau\|\operatorname{curl}\, u\|^2_{L^2} \\
&- M\tau \int_D (\overline{\mathbf{u}}\operatorname{curl}\operatorname{curl}\mathbf{u} + \mathbf{u}\operatorname{curl}\operatorname{curl}\overline{\mathbf{u}})\, dx \\
&\le (M+1)\|\operatorname{curl}\operatorname{curl}\mathbf{u}\|^2_{L^2} + \tau^2 M + 2\tau M\|\operatorname{curl}\operatorname{curl}\mathbf{u}\|_{L^2} - \tau\|\operatorname{curl}\,\mathbf{u}\|^2_{L^2}. \\
&\le \tau^2 M - \tau\left(\lambda_0(D) - 2M\kappa_p(D)^{1/2}\right) + (M+1)\kappa_p(D).
\end{aligned}
$$

The minimizing value of τ of the right hand side is now $\tau_1 = \frac{\lambda_0(D)-2M\mu_p(D)^{1/2}}{2M}$ which gives

$$
(\mathbb{A}_\tau \mathbf{u} - \tau\mathbb{B}\mathbf{u},\, u)_{\mathcal{U}_0} \le -\frac{\left(\lambda_0(D) - 2M\mu_p(D)^{1/2}\right)^2}{4M} + (M+1)\mu_p(D).
$$

Hence the latter becomes non positive if $M \le \frac{\lambda_0(D)^2}{4\kappa_p(D)^{1/2}\left(\lambda_0(D)+\kappa_p(D)^{1/2}\right)}$ which means that $\sup_D \|N\|_2 \le 1/(1+\theta_p(D))$. Consequently if assumption 2 holds then $\tilde{\mathbb{A}}_{\tau_1} - \tau_1\mathbb{B}$ is non positive on a $p+1$ dimentional subspace of $\mathcal{U}_0(D)$ and the result is then proven in this case again by application of Theorem 2.3.

Remark 3.1 Exactly the same analysis can be applied to prove the existence of transmission eigenvalues for the anisotropic Maxwell's equations with conducting transmission conditions, i.e. for the problem considered in [5] with the surface conductivity η being a bounded and purely complex valued function.

3.3 Scalar anisotropic media

The last example we consider is the interior transmission eigenvalue problem corresponding to the scattering problem for an anisotropic scalar medium (for a physical model and more on this interior transmission problem see [6]). Let D again be a bounded simply connected region in $\mathbb{R}^2$ with piecewise smooth boundary ∂D. We consider a real 2×2 matrix-valued function A whose entries are bounded functions defined on D, i.e. $A \in L^\infty(D, \mathbb{R}^{2\times 2})$. We assume that A, A^{-1} and either $(A^{-1}-I)^{-1}$ or $(I-A^{-1})^{-1}$ are bounded positive definite matrices according to Definition 3.1 where we replace $\mathbb{C}^3$ by $\mathbb{C}^2$. Then the interior transmission eigenvalue problem is formulated as

$$\nabla\cdot A\nabla w + k^2\, w = 0 \qquad \text{in} \quad D \tag{28}$$

$$\Delta v + k^2\, v = 0 \qquad \text{in} \quad D \tag{29}$$

$$w = v \qquad \text{on} \quad \partial D \tag{30}$$

$$\frac{\partial w}{\partial \nu_A} = \frac{\partial v}{\partial \nu} \qquad \text{on} \quad \partial D \tag{31}$$

where

$$\frac{\partial w}{\partial \nu_A}(x) := \nu(x)\cdot A(x)\nabla v(x), \qquad x \in \partial D.$$

We say that $k > 0$ is a transmission eigenvalue if (28)-(31) has a nontrivial solution $w, v \in H^1(D)$. The main idea to study (28)-(31) is based on making an appropriate substitution and rewriting (28)-(31) as an eigenvalue problem for a fourth order differential equation for which we can apply the machinery developed above. To this end we make the substitution

$$\mathbf{w} = A\nabla w \in L^2(D)^2, \quad \text{and} \quad \mathbf{v} = \nabla v \in L^2(D)^2$$

and hence $\nabla w = A^{-1}\mathbf{w}$. Taking the gradient of (28) and (29), we obtain that $\mathbf{w}$ and $\mathbf{v}$ satisfy

$$\nabla(\nabla \cdot \mathbf{w}) + k^2 A^{-1}\mathbf{w} = 0 \qquad \text{and} \qquad \nabla(\nabla \cdot \mathbf{v}) + k^2\mathbf{v} = 0, \qquad \text{in } D.$$

Obviously (31) implies that $\nu \cdot \mathbf{w} = \nu \cdot \mathbf{v}$ on ∂D. Furthermore, from (28) and (29) we have that

$$-k^2 w = \nabla \cdot \mathbf{w} \qquad \text{and} \qquad -k^2 v = \nabla \cdot \mathbf{v}$$

and the transmission condition (30) yields $\nabla \cdot \mathbf{w} = \nabla \cdot \mathbf{v}$ on ∂D. We now formulate the interior transmission eigenvalue problem in terms of $\mathbf{w}$ and $\mathbf{v}$. To this end we introduce the Sobolev spaces

$$\begin{aligned} H(\mathrm{div}\,, D) : &= \left\{\mathbf{u} \in L^2(D)^3 : \ \nabla \cdot \mathbf{u} \in L^2(D)\right\} \\ H_0(\mathrm{div}\,, D) : &= \{\mathbf{u} \in H(\mathrm{div}\,, D) : \ \nu \cdot \mathbf{u} = 0 \text{ on } \partial D\} \end{aligned}$$

and

$$\begin{aligned} \mathcal{H}(D) : &= \left\{\mathbf{u} \in H(\mathrm{div}\,, D) : \ \nabla \cdot \mathbf{u} \in H^1(D)\right\} \\ \mathcal{H}_0(D) : &= \left\{\mathbf{u} \in H_0(\mathrm{div}\,, D) : \ \nabla \cdot \mathbf{u} \in H_0^1(D)\right\} \end{aligned}$$

equipped with the scalar product $(\mathbf{u}, \mathbf{v})_{\mathcal{H}(D)} := (\mathbf{u}, \mathbf{v})_{L^2(D)} + (\nabla \cdot \mathbf{u}, \nabla \cdot \mathbf{v})_{H^1(D)}$.

The interior transmission eigenvalue problem in terms of $\mathbf{w}$ and $\mathbf{v}$ now reads: Find $\mathbf{w} \in L^2(D)$ and $\mathbf{v} \in L^2(D)$ such that $\mathbf{w} - \mathbf{v} \in \mathcal{H}_0(D)$ satisfies

$$\nabla(\nabla \cdot \mathbf{w}) + k^2 A^{-1}\mathbf{w} = 0 \qquad \text{in} \quad D \tag{32}$$

$$\nabla(\nabla \cdot \mathbf{v}) + k^2\mathbf{v} = 0 \qquad \text{in} \quad D. \tag{33}$$

Note that the above boundary conditions for $\mathbf{w}$ and $\mathbf{v}$ are incorporated in the fact that $\mathbf{w} - \mathbf{v} \in \mathcal{H}_0(D)$. From the above analysis we have the following result:

Lemma 3.1 *If k is a transmission eigenvalue, i.e. if $w \in H^1(D)$ and $v \in H^1(D)$ satisfy (28)-(31), then $\mathbf{w} = A\nabla w \in L^2(D)^2$ and $\mathbf{v} = \nabla v \in L^2(D)^2$ satisfy $\mathbf{w} - \mathbf{v} \in \mathcal{H}_0(D)$ and (32)-(33).*

We now formulate (32)-(33) as an eigenvalue problem for a fourth order differential equation. Hence we have that $\mathbf{u} = \mathbf{w} - \mathbf{v} \in \mathcal{H}_0(D)$ satisfies

$$\nabla(\nabla \cdot \mathbf{u}) + k^2\mathbf{u} = k^2\left(I - A^{-1}\right)\mathbf{w} \qquad \text{in} \quad D. \tag{34}$$

and from (34) using (32) we obtain the fourth order differential equation

$$\left(\nabla\nabla \cdot + k^2 A^{-1}\right)\left(A^{-1} - I\right)^{-1}\left(\nabla\nabla \cdot \mathbf{u} + k^2\mathbf{u}\right) = 0 \qquad \text{in} \quad D. \tag{35}$$

Note that $\mathbf{u} \in \mathcal{H}_0(D)$ implies that $\nu \cdot \mathbf{u} = 0$ and $\nabla \cdot \mathbf{u} = 0$ on ∂D. The eigenvalue problem for (35) can be written in variational form as the problem of finding $\mathbf{u} \in \mathcal{H}_0(D)$ that satisfies

$$\int_D (A^{-1} - I)^{-1} \left(\nabla\nabla \cdot \mathbf{u} + k^2\mathbf{u}\right) \cdot \left(\nabla\nabla \cdot \overline{\mathbf{v}} + k^2 A^{-1}\overline{\mathbf{v}}\right) \, dx = 0 \qquad \text{for all } \mathbf{v} \in \mathcal{H}_0(D) \quad (36)$$

which, setting $\tau := k^2$, can be put into the following concise forms

$$\mathcal{A}_\tau(\mathbf{u}, \mathbf{v}) - \tau \mathcal{B}(\mathbf{u}, \mathbf{v}) = 0 \qquad \text{for all } \mathbf{v} \in \mathcal{H}_0(D) \quad (37)$$

or

$$\tilde{\mathcal{A}}_\tau(\mathbf{u}, \mathbf{v}) - \tau \mathcal{B}(\mathbf{u}, \mathbf{v}) = 0 \qquad \text{for all } \mathbf{v} \in \mathcal{H}_0(D). \quad (38)$$

The sesquilinear forms $\mathcal{A}_\tau$, $\tilde{\mathcal{A}}$ and $\mathcal{B}$ here are defined by

$$\mathcal{A}_\tau(\mathbf{u}, \mathbf{v}) := \left((A^{-1} - I)^{-1} \left(\nabla\nabla \cdot \mathbf{u} + \tau\mathbf{u}\right), \left(\nabla\nabla \cdot \mathbf{v} + \tau\mathbf{v}\right)\right)_{L^2} + \tau^2 \left(\mathbf{u}, \mathbf{v}\right)_{L^2}$$

$$\begin{aligned} \tilde{\mathcal{A}}_\tau(\mathbf{u}, \mathbf{v}) : &= \left(A^{-1}(I - A^{-1})^{-1} \left(\nabla\nabla \cdot \mathbf{u} + \tau\mathbf{u}\right), \left(\nabla\nabla \cdot \mathbf{v} + \tau\mathbf{v}\right)\right)_{L^2} \\ &+ \left(\nabla\nabla \cdot \mathbf{u}, \nabla\nabla \cdot \mathbf{v}\right)_{L^2}. \end{aligned}$$

and

$$\mathcal{B}(\mathbf{u}, \mathbf{v}) := \left(\nabla \cdot \mathbf{u}, \nabla \cdot \mathbf{v}\right)_{L^2}.$$

Obviously, $\mathcal{A}_\tau$, $\tilde{\mathcal{A}}_\tau$ and $\mathcal{B}$ are continuous hermitian sesquilinear forms on $\mathcal{H}_0(D) \times \mathcal{H}_0(D)$. Let us denote by $\mathbf{A}_\tau$, $\tilde{\mathbf{A}}_\tau$ and $\mathbf{B}$ the bounded linear operators from $\mathcal{H}_0(D)$ to $\mathcal{H}_0(D)$ defined using the Riesz representation theorem by

$$(\mathbf{A}_\tau \mathbf{u}, \, \mathbf{v})_{\mathcal{H}_0} = \mathcal{A}_\tau(\mathbf{u}, \mathbf{v}), \; \left(\tilde{\mathbf{A}}_\tau \mathbf{u}, \, \mathbf{v}\right)_{\mathcal{H}_0} = \tilde{\mathcal{A}}_\tau(\mathbf{u}, \mathbf{v}), \text{ and } (\mathbf{B}\mathbf{u}, \, \mathbf{v})_{\mathcal{H}_0} = \mathcal{B}(\mathbf{u}, \mathbf{v})$$

for all $\mathbf{v} \in \mathcal{H}_0(D)$. In [6] The following result is proven

Lemma 3.2 *The bounded linear operators $\mathbf{A}_\tau : \mathcal{H}_0(D) \to \mathcal{H}_0(D)$ and $\tilde{\mathbf{A}}_\tau : \mathcal{H}_0(D) \to \mathcal{H}_0(D)$ are positive definite and self-adjoint, and depend continuously on $\tau \in (0, +\infty)$. The bounded linear operator $\mathbf{B}_\tau : \mathcal{H}_0(D) \to \mathcal{H}_0(D)$ is non negative, self-adjoint and compact.*

The sesquilinear froms $\mathcal{A}$, $\tilde{\mathcal{A}}$ and $\mathcal{B}$ for the current problem have exactly the same structure as the respective sesquilinear forms in Section 3.2 where the curl curl operator is replaced by grad div operator and the space $\mathcal{U}_0(D)$ is replaced by $\mathcal{H}_0(D)$. The operator $\mathbf{B}$ has a big kernel, namely all divergence free functions, and since D is simply connected

$$\text{the kernel of } \mathbf{B} = \{\mathbf{u} \in \mathcal{H}_0(D) \quad \text{such that } \mathbf{u} := \operatorname{curl} \boldsymbol{\psi}, \; \boldsymbol{\psi} \in H(\operatorname{curl}, D)\}.$$

Next, let $0 < \alpha_1(x) \leq \alpha_2(x)$ be the eigenvalues of the positive definite symmetric 2×2 matrix A^{-1}. The largest eigenvalue $\alpha_2(x)$, which coincides with the Euclidean norm

$\|A^{-1}(x)\|_2$, is given by $\alpha_2(x) = \sup_{\|\xi\|=1}(\bar{\xi} \cdot A^{-1}(x)\,\xi)$ and the smallest eigenvalue $\alpha_1(x)$ is given by $\alpha_1(x) = \inf_{\|\xi\|=1}(\bar{\xi} \cdot A^{-1}(x)\,\xi)$. We denote by $A^* = \sup_D \alpha(x)$ and $A_* = \inf_D \alpha_1(x)$. To state our result on the existence of transmission eigenvalues we need to consider the eigenvalue problem for $(\nabla\nabla\cdot)^2$ which can be written in the variational form as

$$\int_D (\nabla\nabla \cdot \mathbf{u}\ \nabla\nabla \cdot \overline{\mathbf{v}} - \rho \mathbf{u}\overline{\mathbf{v}})\ dx = 0 \qquad \text{for all} \qquad \mathbf{v} \in \mathcal{K}_0(D) \tag{39}$$

where $\mathcal{K}_0(D) := \mathcal{H}_0(D) \cap H_0(\mathrm{curl}\,0, D)$ with

$$H_0(\mathrm{curl}\,0, D) := \left\{\mathbf{u} \in L^2(D)^2 :\ \mathrm{curl}\,\mathbf{u} = \mathbf{0} \text{ and } \nu \times \mathbf{u} = 0\right\}.$$

The following decomposition is orthogonal with respect to $L^2(D)^2$-inner product

$$\mathcal{H}_0(D) = \mathcal{K}_0(D) \oplus \{\mathbf{u} := \mathrm{curl}\,\boldsymbol{\psi},\ \boldsymbol{\psi} \in H(\mathrm{curl}\,, D)\}.$$

Again, we can easily see that the eigenvalues of the problem (39) exist, are strictly positive and accumulate only at infinity and the corresponding eigenspaces are finite dimensional. The eigenfunctions of this eigenvalue problem coincide with the curl free eigenfunctions of the vector bi-harmonic operator with clamped plate boundary conditions. We denote by $\rho_p(D) > 0$ the $(p+1)$-th eigenvalue of (39) (eigenvalues are ordered in increasing order) and set

$$\Sigma_p(D) := 4\frac{\rho_p(D)^{1/2}}{\lambda_0(D)} + 4\frac{\rho_p(D)}{\lambda_0(D)^2}.$$

Exactly in the same way as in the proof of Theorem 3.2 by replacing the operators curl curl by grad div, and grad by div and the space $\mathcal{U}_0(D)$ by $\mathcal{H}_0(D)$ it is now possible to prove the following theorem.

Theorem 3.3 *Let $A \in L^\infty(D, \mathbb{R}^{2\times 2})$ be a positive definite symmetric real matrix field on D satisfying either one of the following assumptions*

1) $1 + \Sigma_p(D) \le A_* \le \left(\bar{\xi} \cdot A^{-1}(x)\,\xi\right) \le A^* < \infty$,

2) $0 < A_* \le \left(\bar{\xi} \cdot A^{-1}(x)\,\xi\right) \le A^* < \frac{1}{1+\Sigma_p(D)}$.

for every $\xi \in \mathbb{C}^2$ such that $\|\xi\| = 1$ and for almost all $x \in D$. Then there exist $p+1$ transmission eigenvalues (counting multiplicity).

4 Conclusions

We have developed an analytical frame work based on a generalized eigenvalue problem which enables to apply the idea of [15] to prove the existence of transmission eigenvalues for a much larger class of scattering problems for inhomogeneous medium than that considered in [15]. We have shown the existence of transmission eigenvalues corresponding to the scattering problem for the isotropic and anisotropic media for both the scalar case and for Maxwell's equations (in the latter case when the contrast is only on one of the constitutive parameters). Our method can also be adapted to the case of more complicated transmission conditions such as conducting boundary conditions.

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