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Open problems in the qualitative approach to inverse electromagnetic scattering theory

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We formulate a number of open problems for time-harmonic inverse electromagnetic scattering theory focusing on uniqueness theorems, the determination of the support of a scattering object and the determination of material parameters

1 Introduction

Radar is one of the most important inventions of the twentieth century. However, the case can easily be made that radar has not yet realized its full potential. In particular, ever since its invention scientists and engineers have strived not only to detect but also to identify unknown objects through the use of electromagnetic waves. Indeed, as pointed out in [28], "Target identification is the great unsolved problem. We detect almost everything; we identify nothing". A significant step forward in the resolution of this problem occurred in the 1960's with the invention of synthetic aperture radar (SAR) and since that time numerous striking successes have been recorded in imaging by electromagnetic waves using SAR [4]. However, as the demands of radar imaging have increased, the limitations of SAR have become increasingly apparent. These limitations arise from the fact that SAR is based on the "weak scattering" approximation and ignores both multiple scattering and polarization effects. Indeed, such incorrect model assumptions have caused some scientists to ask "how (and if) the complications associated with radar based automatic target recognition can be surmounted" ([4], p.5).

In recent years, in an effort to avoid such incorrect model assumptions, considerable effort has been put into nonlinear optimization techniques (c.f. [3], [20], [22], [47]). Such an approach is in principle relatively straightforward and, where applicable, can produce striking results. However, it was quickly realized that for many, if not most, practical applications such an approach has severe limitations. For example, in the case of coated objects embedded in a piecewise constant background medium the implementation of nonlinear optimization techniques requires very precise a priori information which is in general not available. Hence, in recent years, alternative methods for imagining have been developed which still avoid incorrect model assumptions but, as opposed to nonlinear optimization techniques, only seek limited information about the scattering object. An example of such an approach is the linear sampling method [13], [38] which only seeks to determine an approximation to the shape of a scattering obstacle but in general can say little about the material properties of the scatterer (however, see [7], [9]). We refer to such methods as *qualitative* methods in inverse scattering theory and direct the reader to the Introduction in [42] for further discussion.

This is where we are at the moment. Given the increased activity and interest in the field of electromagnetic imaging, we thought it would be appropriate at this time to take stock of where we are and try and formulate a few basic mathematical problems which, if a solution could be found, would be of significant help in moving forward in the use of electromagnetic waves for target recognition, particularly in the case of the qualitative approach mentioned above. In this context, we firmly believe that reconstruction algorithms cannot be viewed in isolation but rather as part of an overall view of the inverse scattering problem for electromagnetic waves. More specifically, issues of uniqueness and continuous dependence are inseparable from the issue of reconstruction. To clarify these observations, we note that in many (if not most!) situations neither the shape nor material properties of the object being imaged are known. For example, it may not be known a priori if the scattering object is a perfect or imperfect conductor. Hence a uniqueness result for determining the support of a perfect or imperfect conductor should not depend on knowing the boundary condition a priori nor should the reconstruction algorithm. Furthermore, since the inverse scattering problem is ill-posed, in order to restore stability some type of a priori information is needed and such a priori information needs to be built into the reconstruction algorithm if successful imaging is to be achieved.

Keeping the above ideas in mind, the plan of our paper is as follows. We begin by formulating the direct scattering problem for three representative situations, the first being when the scatterer is a penetrable anisotropic inhomogeneous medium, the second when the scatterer is an imperfect conductor with variable impedance and the third being the case when the scatterer is a perfect conductor. In all cases we restrict our attention to the frequency domain, i.e. our focus is on time-harmonic (deterministic) scattering theory. We then consider the inverse scattering problem under the assumption that multi-static data is available and formulate (with motivation!) seven outstanding problems associated with the qualitative approach to inverse scattering theory which have perplexed us and others over the past ten years. We also include in the text a variety of directions and additional open problems that we feel are of importance. We have grouped these problems into three categories: (1) uniqueness problems, (2) determination of support of the scatterer, and (3) determination of material parameters. Our hope is that by publishing these problems fresh ideas will be found to enable others to succeed where we have failed!

2 Electromagnetic scattering problems

We begin by formulating the basic electromagnetic scattering problems that we will be considering in this paper. We first consider the general case of Maxwell's equations in an inhomogeneous anisotropic medium (which, of course, includes the isotropic medium as a special case). We assume that $D \subset \mathbb{R}^3$ is a bounded domain with connected complement such that the boundary Γ is in class C^2 with unit outward normal ν. We also introduce the 3×3 symmetric matrix $N = N(x)$ whose entries are piecewise smooth complex valued functions in $\mathbb{R}^3$ such that N is the identity matrix outside D. We will assume that there exists $\gamma > 0$ such that

$$\operatorname{Re}(N\xi, \xi) \geqslant \gamma|\xi|^2; \quad \xi \in \mathbb{C}^3, \quad x \in \mathbb{R}^3 \tag{2.1}$$

and

$$\operatorname{Im}(N\xi, \xi) > 0; \quad \xi \in \mathbb{C}^3 \setminus \{0\}, \quad x \in D. \tag{2.2}$$

We further assume that $N - I$ is invertible in D and $\operatorname{Re}(N - I)^{-1}$ is uniformly positive definite in D. Let $H(\text{curl}, D)$ be the Hilbert space

$$H(\text{curl}, D) := \left\{ u \in (L^2(D))^3 : \ \text{curl}\, u \in (L^2(D))^3 \right\}$$

equipped with the inner product

$$(u, v)_{H(curl,D)} := (u, v)_{L^2(D)} + (\text{curl}\, u, \text{curl}\, v)_{L^2(D)}\,.$$

We denote by $H_{loc}(\text{curl}, \mathbb{R}^3)$ the Fréchet space of functions in $(L^2_{loc}(\mathbb{R}^3))^3$ such that $u \in H(\text{curl}, K)$ for all compact sets $K \subset \mathbb{R}^3$.

Now we consider the incident electromagnetic plane wave (with the term $e^{-i\omega t}$ being factored out where ω is the frequency)

$$\begin{aligned} E^i(x) &= \frac{i}{k}\text{curl}\,\text{curl}\, p\, e^{ikx\cdot d} \\ H^i(x) &= \text{curl}\, p\, e^{ikx\cdot d} \end{aligned} \tag{2.3}$$

where E^i and H^i represent the incident electric and magnetic fields respectively, p is the (constant) polarization vector and d is the direction of the plane wave with $|d| = 1$. Then the direct scattering problem corresponding to the scattering of the electromagnetic plane wave (2.3) by an anisotropic inhomogeneous medium D with constant permeability and refractive index N is to determine an electric field $E \in H(\text{curl}, D)$ such that

$$\text{curl}\,\text{curl}\, E - k^2\, N(x)\, E = 0, \qquad x \in \mathbb{R}^3 \tag{2.4}$$

$$E = E^s + E^i \tag{2.5}$$

$$\lim_{r\to\infty} (\text{curl}\, E^s \times x - ikr\, E^s) = 0 \tag{2.6}$$

uniformlly in $\hat{x} = x/|x|$ where $r = |x|$. The proof of the following theorem can be found in [33] (see also [38]).

Theorem 1 *There exists a unique solution* $E \in H_{loc}(\mathbb{R}^3)$ *to* (2.4)–(2.6). *For every ball* B_R *of radius* R *centered at the origin such that* $D \subset B_R$ *there exists a constant* $C = C(R)$ *such that*

$$\|E\|_{H(curl,B_R)} \leqslant C\|E^i\|_{H(curl,B_R)}.$$

Now assume that the scattering object D is an imperfect conductor, i.e. $\|\operatorname{Im} N(x)\|$ is very large for x inside D except for a thin region in the neighborhood of the boundary Γ. Then to successfully solve the direct problem (2.4)–(2.6) or the inverse problem using optimization techniques, the scattering problem (2.4)–(2.6) must be replaced by a simpler model which reflects the above properties of the refractive index N. Such a model is given by the *impedance boundary value problem*. In particular, let $X(D)$ be the space

$$X(D) := \left\{u \in H(\text{curl}, D):\ \ \nu \times u|_\Gamma \in L_t^2(\Gamma)\right\}$$

equipped with the inner product

$$(u, v)_{X(D)} := (u, v)_{H(curl,D)} + (u, v)_{L_t^2(\Gamma)}$$

where $L_t^2(\Gamma)$ denotes the space of square integrable tangential vector fields defined on Γ and let $\lambda \in L^\infty(\Gamma)$, $\lambda(x) \geqslant \lambda_0 > 0$, be the *surface impedance*. Let $D_e := \mathbb{R}^3 \setminus \overline{D}$ and define $X_{loc}(D_e)$ in the obvious way. Then the impedance boundary value problem is to find an electric field $E^s \in X_{loc}(D_e)$ such that

$$\text{curl}\,\text{curl}\, E^s - k^2 E^s = 0, \qquad x \in D_e \tag{2.7}$$

$$\nu \times \text{curl}\,(E^s + E^i) - i\lambda(x)(\nu \times (E^s + E^i)) = 0 \qquad x \in \Gamma \tag{2.8}$$

$$\lim_{r\to\infty} (\text{curl}\, E^s \times x - ikr\, E^s) = 0. \tag{2.9}$$

The proof of the following theorem can be found in [8] (see also [38]).

Theorem 2 *There exists a unique solution $E^s \in X_{loc}(D_e)$ to* (2.7)–(2.9). *For every ball B_R of radius R centered at the origin such that $D \subset B_R$ there exists a constant $C = C(R)$ such that*

$$\|E^s\|_{X(D_e\cap B_R)} \leqslant C\|E^i\|_{X(D_e\cap B_R)}.$$

Finally, we consider the case when the scattering object D is a perfect conductor, i.e. $\|\text{Im}\, N(x)\|$ is very large for x everywhere inside D. Let $H_{div}^{-1/2}(\Gamma)$ denote the trace space of $H(\text{curl}, B_R)$, i.e. for a $u \in H(\text{curl}, B_R)$, $\nu \times u$ is in

$$H_{div}^{-1/2}(\Gamma) := \left\{u \in H^{-1/2}(\Gamma):\ \ \nu \cdot u = 0,\ \ \text{div}_\Gamma\, u \in H^{-1/2}(\Gamma)\right\}.$$

Then the scattering problem (2.4)–(2.6) can be modeled by the *scattering problem for a perfect conductor*, i.e. defining $H_{loc}(\text{curl}, D_e)$ in an obvious way, to find an electric field $E^s \in H_{loc}(\text{curl}, D_e)$ such that

$$\text{curl}\,\text{curl}\, E^s - k^2 E^s = 0, \qquad x \in D_e \tag{2.10}$$

$$\nu \times (E^s + E^i) = 0 \qquad x \in \Gamma \tag{2.11}$$

$$\lim_{r\to\infty} (\text{curl}\, E^s \times x - ikr\, E^s) = 0. \tag{2.12}$$

The proof of the following theorem can be found in [38].

Theorem 3 *There exists a unique solution $E^s \in H_{loc}(curl, D_e)$ to* (2.10)–(2.12). *For every ball B_R of radius R centered at the origin such that $D \subset B_R$ there exists a constant $C = C(R)$ such that*

$$\|E^s\|_{H(curl,D_e\cap B_R)} \leqslant C\|E^i\|_{H_{div}^{-1/2}(\Gamma)}.$$

Having formulated the above electromagnetic scattering problems, we will now turn our attention to the corresponding inverse problems. We begin with the problem of determining uniqueness theorems for inverse scattering problems.

3 Uniqueness problems

There are many inverse problems in scattering theory. However, the inverse problems we are concerned with in this survey paper are to determine the support D of a scattering obstacle from far field data and, in the case of a penetrable inhomogeneous medium, information about the material properties of the scatterer.

We begin by considering the uniqueness of a solution to the inverse problem. For each of the three problems considered in the previous section it can be shown [15] that the scattered field E^s has the asymptotic behavior

$$E^s(x) = \frac{e^{ikr}}{r}\left(E_\infty(\hat{x}, d, p) + O\left(\frac{1}{r}\right)\right) \tag{3.1}$$

as $r \to \infty$ where E_∞ is known as the *electric far field pattern* and is an infinitely differentiable tangential vector field for $\hat{x}$ on the unit sphere Ω.

The first inverse scattering problems we are interested in are (2.7)–(2.9) and (2.10)–(2.12) where we want to determine D from a knowledge of $E_\infty(\hat{x}, d, p)$ for $\hat{x}, d \in \Omega$ and $p \in \mathsf{R}^3$ without knowing *a priori* which of these two scattering problems E_∞ is associated with. The proof of the following theorem is based on the ideas of Kirsch & Kress [32] (see [35] and Theorem 7.1 in [15]). We note that these ideas are also closely related to the linear sampling method for reconstructing D from noisy far field data which we will discuss in the next section of this paper.

Theorem 4 *Let E_∞ be the electric far field pattern corresponding to* (2.7)–(2.9) *or* (2.10)–(2.12). *Then D is uniquely determined by $E_\infty(\hat{x}, d, p)$ for $\hat{x}, d \in \Omega$, $p \in \mathsf{R}^3$. Furthermore, in the case of* (2.7)–(2.9), *$\lambda = \lambda(x)$ is uniquely determined.*

Remark Since $E_\infty(\hat{x}, d, p)$ in an analytic function of $\hat{x}$ and d on Ω, it suffices to know E_∞ for $\hat{x}$ and d on an open subset of Ω. Furthermore, since $E_\infty(\hat{x}, d, p)$ is linear in p, it suffices to know E_∞ for three linearly independent vectors p_1, p_2, p_3.

Proof of Theorem 4 We only prove that D is uniquely determined and refer the reader to [35] for the unique determination of λ. Assume that there are two different domains D_1 and D_2 giving rise to the same far field pattern E_∞. By Rellich's lemma, and the fact that an entire solution to Maxwell's equations satisfying the Silver-Müller radiation condition must be identically zero, we can restrict our attention to the case when $\overline{D}_1 \cap \overline{D}_2 \neq 0$. Finally, by using the mixed reciprocity relation of Potthast [42], we can assume that the scattered fields $E^s_{1,e}(x, z, p)$ and $E^s_{1,e}(x, z, p)$ corresponding to the scattering of electric dipoles with source point z and polarization p by D_1 and D_2 coincide for all x, z in the unbounded component G of $\overline{D}_1 \cup \overline{D}_2$ and all polarizations p.

Since $D_1 \neq D_2$, without loss of generality, there exists $x^* \in \partial G$ such that $x^* \in \Gamma_1$ and $x^* \notin \Gamma_2$ where Γ_j is the boundary of D_j, $j = 1, 2$. In particular, we have that $z_n := x^* + n^{-1}v(x^*) \in G$ for n sufficiently large. Then, by using either Theorem 2 or Theorem 3, we have that $E^s_{2,e}(x^*, z_n, p)$ remains bounded in $X(D_{2,e} \cap B_R)$ or $H(\mathrm{curl}, D_{2,e} \cap B_R)$ as $n \to \infty$ but on the other hand $E^s_{1,e}(x^*, z_n, p)$ cannot remain bounded in $X(D_{1,e} \cap B_R)$ or

$H(\mathrm{curl}, D_{i,e} \cap B_R)$ as $n \to \infty$ due to the dipole at $x = x^*$. But this contradicts $E^s_{1,e} = E^s_{2,e}$ in G and therefore $D_1 = D_2$. □

In the case of acoustic scattering by a sound-soft obstacle it has been shown by Colton & Sleeman [19] and Rondi [44] that only a finite number of incident plane waves is sufficient to determine D (see also [1] and [11]. The exact number of waves needed depends on a priori information on the size of the obstacle). This brings us to our first open problem.

Open Problem *Assume that D is a perfect conductor. Is D uniquely determined by the electric far field pattern $E_\infty(\hat{x}, d, p)$ for $\hat{x} \in \Omega$, $p \in \mathbb{R}^3$ and a finite number of directions d. Under what conditions is D determined by a single incident plane wave?*

We now return our attention to the uniqueness questions for inverse problems associated with the scattering problem (2.4)–(2.6). We first assume that the medium is isotropic, i.e. the refractive index $N(x)$ is of the form $n(x)I$ where $n(x)$ is a scalar. The proof of the following theorem is quite technical and hence we only give the key steps in the proof. The result is due Colton & Päivärinta [17] with a subsequent simplification being given by Hähner [26]. Generalizations to the case of variable permeability have been given by Ola, Päivärinta & Somersalo [40], Ola & Somersalo [41] and Sun & Uhlmann [45].

Theorem 5 *Let E_∞ be the electric far field pattern corresponding to* (2.4)–(2.6) *for $N(x) = n(x)I$ where n is a scalar complex valued function such that $n \in C^{1,\alpha}(\mathbb{R}^3)$ for $0 < \alpha < 1$, $Re\,(n) > 0$, $Im\,(n) \geqslant 0$ and $n(x) = 1$ for $x \in D_e$. Then n is uniquely determined by $E_\infty(\hat{x}, d, p)$ for $\hat{x}, d \in \Omega$, $p \in \mathbb{R}^3$.*

Remark The remark after the statement of Theorem 4 also holds in this case.

Proof of Theorem 5 The key steps in the proof are as follows. Full details can be found in [17].

(1) It is shown that the set of all solutions to (2.4)–(2.6) for $N(x) = n(x)I$, $d \in \Omega$, $p \in \mathbb{R}^3$ is complete in the closure in $L^2(B)$ of all solution to

$$\mathrm{curl}\,\mathrm{curl}\,E - k^2 n(x)\,E = 0 \tag{3.2}$$

in B where B is a ball containing D.

(2) If there exist two refractive indices n_1 and n_2 having the same electric far field pattern, then it is shown using step 1 that

$$\int_{\mathbb{R}^3} E_1(x)\,(n_1(x) - n_2(x))\,E_2(x)\,dx = 0 \tag{3.3}$$

where E_j is any solution of (3.2) in B with $n = n_j$, $j = 1, 2$.

(3) A solution E of (3.2) is constructed such that E has the form

$$E(x) = e^{i\zeta\cdot x}\left[\eta + R_\zeta(x)\right] \tag{3.4}$$

where $\zeta, \eta \in \mathsf{C}^3$, $\eta\cdot\zeta = 0$ and $\zeta\cdot\zeta = k^2$.

(4) Choose E_j to be of the form (3.4) where $\zeta = \zeta_j$ with $\zeta_1 + \zeta_2 = \xi \in \mathsf{R}^3$. By choosing $\eta_j = \eta(\zeta_j)$ appropriately and substituting E_j into (3.3) we have, letting $|\zeta_j| \to \infty$, that

$$\int\limits_{\mathsf{R}^3} e^{i\xi\cdot x}\,(n_1(x) - n_2(x))\,dx = 0.$$

Hence, by the Fourier integral theorem, $n_1(x) = n_2(x)$ for all $x \in \mathsf{R}^3$.

□

The proof of the above theorem breaks down if n is not continuously differentiable in R^3, in particular if n has a jump discontinuity across Γ. Unfortunately, in almost all practical applications n *does* have a jump across Γ. A first step in treating the case when n is discontinuous across Γ was taken by Hähner [25], who assumed that n was constant in a neighborhood of Γ. This brings us to our next open problem.

Open Problem *In the scattering problem* (2.4)–(2.6) *assume that N is of the form $N(x) = n(x)I$ where $n \in C^1(\overline{D})$ but $n(x)$ is not necessarily equal to one for $x \in \Gamma$. Show that n is uniquely determined by the electric far field pattern $E_\infty(\hat{x}, d, p)$ for $\hat{x}, d \in \Omega$, $p \in \mathsf{R}^3$.*

Standard examples show that for anisotropic media (i.e. when N is a matrix) Theorem 5 is not valid, i.e. N is not uniquely determined from E_∞ [46]. However it can be shown that the support D of $I - N$ is uniquely determined [6]. The proof of the following theorem is based on the idea used in Theorem 4 together with a detailed analysis of a modified version of the *interior transmission problem* of finding a solution E, $E_0 \in (L^2(D))^3$ such that $E - E_0 \in H(\mathrm{curl}, D)$ and $\mathrm{curl}(E - E_0) \in H(\mathrm{curl}, D)$ of

$$\begin{matrix} \mathrm{curl}\,\mathrm{curl}\,E - k^2 N(x)\,E = 0 \\ \mathrm{curl}\,\mathrm{curl}\,E_0 - k^2 E_0 = 0 \end{matrix} \qquad \text{in} \quad D \tag{3.5}$$

$$\begin{matrix} (E - E_0)\times\nu = \mathscr{E}\times\nu \\ \mathrm{curl}\,(E - E_0)\times\nu = \mathrm{curl}\,\mathscr{E}\times\nu \end{matrix} \qquad \text{on} \quad \Gamma \tag{3.6}$$

where $\mathscr{E}$ is an electric dipole with source $z \in D$ (For more information on the interior transmission problem we refer the reader to [23] and Chapters 9 and 10 of [15]). We will meet the interior transmission problem again in the next section of this paper.

Theorem 6 *Let E_∞ be the electric far field pattern corresponding to* (2.4)–(2.6)*, and assume that either*

(1) $\bar{\xi}\,Re(N)\,\xi \geqslant \gamma|\xi|^2$*, or*

(2) $\bar{\xi}\,Re(N^{-1})\,\xi \geqslant \gamma|\xi|^2$*,*

for all $\xi \in \mathsf{C}^3$, $x \in \overline{D}$, *where* $\gamma > 1$ *is a constant. Then the support* D *of* $I - N$ *is uniquely determined by* $E_\infty(\hat{x}, d, p)$ *for* $\hat{x}, d \in \Omega$, $p \in \mathsf{R}^3$.

The remark after the statement of Theorem 4 again holds in the case of Theorem 6. Noting that in the isotropic case

$$n(x) := \frac{1}{\epsilon_0}\left(\epsilon(x) + i\frac{\sigma(x)}{\omega}\right)$$

where ϵ_0 is the permittivity of the background medium, $\epsilon = \epsilon(x)$ the permittivity of D and $\sigma = \sigma(x)$ the conductivity of D we see that in this case conditions (1) and (2) of the above theorem imply that $\epsilon(x) > \epsilon_0$ for all $x \in \overline{D}$ or $\epsilon(x) < \epsilon_0$ for all $x \in \overline{D}$. To us, this restriction seems artificial. Hence we have the following open problem.

Open Problem *Show that if* E_∞ *is the electric far field pattern for the scattering problem* (2.4)–(2.6) *then the support of* D *of* $I - N$ *is uniquely determined by* $E_\infty(\hat{x}, d, p)$ *for* $\hat{x}, d \in \Omega$ *and* $p \in \mathsf{R}^3$ *(without making the assumptions (1) and (2) of Theorem 6).*

4 Determination of the support of a scattering object

As already mentioned in the introduction, a considerable effort has been made in recent years to develop algorithms that determine only certain relevant physical properties of the scattering object rather than trying to reconstruct the complete model of the scattering problem. The aim of this program is of course the hope that such algorithms will be faster and more robust than those methods that try to reconstruct complete information of the scatterer. Probably the most important physical property of a scattering object is its support D and hence in this section we will address ourselves to the problem of determining (an approximation to) D from a knowledge of the electric far field pattern. In particular, we want an algorithm that determines D without requiring the *a priori* knowledge that the electric far field pattern is associated specifically with either of the scattering problems (2.4)–(2.6), (2.7)–(2.9) and (2.10)–(2.12). A method for doing this is the *linear sampling method* which was first introduced for acoustic waves by Colton & Kirsch [14] and Colton *et al.* [18], and for electromagnetic waves by Kress [34], Colton *et al.* [12] and Haddar & Monk [24]. Here we present a heuristic introduction to the method, refering the reader to [13] and [38] for its mathematical justification and numerical implementation.

We begin by defining the *far field operator* $F : L^2_t(\Omega) \to L^2_t(\Omega)$ by

$$(Fg)(\hat{x}) := \int_\Omega E_\infty(\hat{x}, d, g(d))ds(d) \tag{4.1}$$

where E_∞ is the electric far field pattern of (2.4)–(2.6), (2.7)–(2.9) or (2.10)–(2.12). We further define a *Herglotz wave function with kernel* $g \in L^2_t(\Omega)$ by

$$E_g(x) := \int_\Omega e^{ikx\cdot d} g(d)ds(d). \tag{4.2}$$

We note that by superposition we have that Fg is the electric far field pattern for (2.4)–(2.6), (2.7)–(2.9) and (2.10)–(2.12), respectively, corresponding to $E_h(x)$ and $H_h(x) := \frac{1}{ik}\operatorname{curl} E_h(x)$, $h = ikg$, as the incident electromagnetic field. Finally, we define the *far field equation* by

$$Fg(\hat{x}) = \mathscr{E}_\infty(\hat{x}, z, q) \tag{4.3}$$

where

$$\mathscr{E}_\infty(\hat{x}, z, q) = \frac{ik}{4\pi}(\hat{x} \times q) \times \hat{x}\, e^{ik\hat{x}\cdot z}$$

is the electric far field pattern of the electric dipole

$$\mathscr{E}(x, z, q) = \frac{i}{k}\operatorname{curl}_x \operatorname{curl}_x q\, \Phi(x, z)$$

$$\mathscr{H}(x, z, q) = \operatorname{curl}_x q\, \Phi(x, z)$$

with

$$\Phi(x, z) = \frac{1}{4\pi}\frac{e^{ik|x-z|}}{|x-z|}, \qquad x \neq z.$$

We are now in a position to introduce the basic idea of the linear sampling method. Suppose that for each $z \in D$ there exists $g(\cdot, z) \in L_t^2(\Omega)$ such that the far field equation (4.3) is satisfied. Then by Rellich's lemma

$$\int_\Omega E^s(x, d, g(d))\, ds(d) = \mathscr{E}(x, z, q) \tag{4.4}$$

for $x \in \overline{D}_e$ and in particular for $x \in \Gamma$. As $z \to x \in \Gamma$ we have that $\mathscr{E}(x, z, q) \to \infty$ and hence from (4.4) and the regularity of E^s we must have that

$$\lim_{\substack{z \to x \in \Gamma \\ z \in D}} \|g(\cdot, z)\|_{L_t^2(\Omega)} = \infty.$$

Under the above assumptions it is also possible to conclude that the Herglotz wave function with kernel g becomes infinite as $z \to \Gamma$. Hence Γ is characterized by points where the solution of the far field equation becomes unbounded as a function of z.

The above argument is purely heuristic since in general there is no solution $g \in L_t^2(\Omega)$ of the far field equation! Indeed, it can be shown that for $z \in D$ a solution to the far field equation exists if and only if, in the cases of (2.7)–(2.9) and (2.10)–(2.12), the solution of the corresponding interior problems (2.7)–(2.9) and (2.10)-(2.12) with D_e replaced by D and E^i replaced by the electric dipole E_e is a Herglotz wave function and, in the case of (2.4)–(2.6), the function E_0 in the definition of the interior transmission problem (3.5)–(3.6) is a Herglotz wave function [15], [23]. In general this is not true for any of the above problems and hence a solution to the far field equation does not exist. A second problem is that even the above heuristic argument breaks down if $z \in D_e$. However, it is possible to prove the following theorem (see [13] for references). Recall that a *Maxwell eigenvalue* is a value of k such that there exists a nontrivial solution to the homogeneous interior problem corresponding to (2.10)–(2.11) (i.e. $D_e = D$ and $E^i = 0$) and a *transmission eigenvalue* is a value of k such that there exists a nontrivial solution to the homogeneous interior transmission problem (i.e. $\mathscr{E} = 0$ in (3.5)–(3.6)) [15, 23].

Theorem 7 *Assume that k is neither a Maxwell eigenvalue nor a transmission eigenvalue. Then if F is the far field operator corresponding to* (2.4)–(2.6), (2.7)–(2.9) *or* (2.10)–(2.12), *respectively, the following is true:*

(1) *If $z \in D$ then for every $\epsilon > 0$ there exists a solution $g_\epsilon(\cdot, z) = g_\epsilon(\cdot, z, q) \in L_t^2(\Omega)$ of the inequality*

$$\|Fg_\epsilon(\cdot, z) - E_{e,\infty}(\cdot, z, q)\|_{L_t^2(\Omega)} < \epsilon$$

such that

$$\lim_{z \to \Gamma} \|E_{g_\epsilon}(\cdot, z)\|_Y = \infty \qquad \text{and} \qquad \lim_{z \to \Gamma} \|g_\epsilon(\cdot, z)\|_{L_t^2(\Omega)} = \infty$$

where $Y = L^2(D)$, $X(D)$ or $H(curl, D)$, respectively.

(2) *If $z \in D_e$ then for every $\epsilon > 0$ and $\delta > 0$ there exists a solution $g_{\delta,\epsilon}(\cdot, z, q) \in L_t^2(\Omega)$ of the inequality*

$$\|Fg_{\epsilon,\delta}(\cdot, z) - E_{e,\infty}(\cdot, z, q)\|_{L_t^2(\Omega)} < \epsilon$$

such that

$$\lim_{\delta \to 0} \|E_{g_{\epsilon,\delta}}(\cdot, z)\|_Y = \infty \qquad \text{and} \qquad \lim_{\delta \to 0} \|g_{\epsilon,\delta}(\cdot, z)\|_{L_t^2(\Omega)} = \infty$$

where $Y = L^2(D)$, $X(D)$ or $H(curl, D)$, respectively.

It is also possible to consider limited aperture far field data, in which case in the definition of F the region of integration Ω is replaced by $\Omega_0 \subset \Omega$ and in the above theorem $L_t^2(\Omega)$ is replaced by $L_t^2(\Omega_1)$ where $\Omega_1 \subset \Omega$ [5].

For the above theorem to be meaningful, it is clearly important to know that the set of Maxwell eigenvalues and the set of transmission eigenvalues form (at most) discrete sets. This is well known for the case of Maxwell eigenvalues [39]. However, except for the case of a spherical symmetric scalar index of refraction [15], it is unknown if transmission eigenvalues even exist, much less form a discrete set! Hence we are led to our next open problem.

Open Problem *Show that transmission eigenvalues exist and form a discrete set.*

In the scalar case for a spherically stratified index of refraction it has been shown that the transmission eigenvalues uniquely determine the index of refraction [37]. If the above open problem has a positive solution, it would be of considerable interest to prove an analogous result for (3.5)–(3.6) in the case when $N(x)$ is a scalar. Finally, there exists the problem of numerically constructing the approximate solution g_ϵ and $g_{\epsilon,\delta}$ whose existence is given by Theorem 7. In practice this is typically done by using the method of Tikhonov regularization and Morozov's generalized discrepancy principle [13, 21]. However, the question remains as to whether or not the solution obtained by such regularization methods in fact constructs a function that behaves like the above approximate solution. In all numerical examples constructed to date this is indeed the case, but this connection has not been proved except for certain problems in the scalar case [2]. Hence we have the following open problem.

Open Problem *Show that the use of Tikhonov regularization and Morozov's generalized discrepancy principle applied to the far field equation yields a function that behaves like the function* g_ϵ *and* $g_{\epsilon,\delta}$ *whose existence is given by Theorem 7.*

In the scalar case, the problem of the numerical reconstruction of g_ϵ and $g_{\epsilon,\delta}$ led Andreas Kirsch to introduce a new method for reconstructing the support of a scattering object called the *factorization method* [29, 30, 31]. The main idea behind this approach is to replace the far field operator by another operator whose range can be explicitly characterized in terms of the support D. For example, in the case of the scattering of acoustic waves, Kirsch considered the far field equation

$$F_{\#}^{1/2} g = \Phi_\infty(\cdot, z)$$

where $\Phi_\infty(\hat{x}, z) = \exp(-ik\hat{x} \cdot z)$, $F_{\#} := |\mathrm{Re}\, F| + \mathrm{Im}\, F$ where F is the acoustic far field operator and $|\mathrm{Re}\, F|$, $\mathrm{Im}\, F$ and $F_{\#}^{1/2}$ are defined by the spectral decomposition of F. He was then able to show that $z \in D$ if and only if Φ_∞ is in the range of $F_{\#}^{1/2}$. Such a result leads to straightforward methods for reconstructing D without having to deal with delicate issues of regularization theory as is the case for the linear sampling method. Hence it would be highly desirable to extend the results of Kirsch to the case of Maxwell's equations.

Open Problem *Extend Kirsch's factorization method to the case of the scattering problems* (2.4)–(2.6), (2.7)–(2.9) *and* (2.10)–(2.12).

The factorization method requires a knowledge of the far field pattern for $\hat{x}$ and d on the unit sphere Ω whereas the linear sampling method requires a knowledge of the far field pattern for $\hat{x}$ and d on an open subset of Ω. Its would be highly desirable to reduce the amount of data needed to determine the support of the scattering object. However doing so is problematic since, as we have seen in the previous section, it is not known whether or not the electric far field pattern for a fixed incident direction d uniquely determines the support of a perfect conductor (nor, for that matter, the support of a penetrable inhomogeneous medium). In the scalar case a promising start to the problem of using reduced data has been made by Kusiak and Sylvester [36] and Potthast, Sylvester and Kusiak [43] who have shown that a knowledge of the far field pattern for a fixed incident direction d determines the *convex scattering support* which is a subset of the convex hull of the support of any scatterer that produces the given far field pattern. It would be of considerable interest to extend the results of Potthast, Sylvester and Kusiak to the case of electromagnetic waves.

5 Determination of material parameters

The support is of course not the only quantity of physical interest associated with a scattering object. Indeed, very recently efforts have been made to determine various coefficients in the boundary condition of a scatterer that has been coated by a thin layer of highly absorbing material [7, 9]. In particular, consider the impedance boundary value problem (2.7)–(2.9). It is not difficult to show that both the support D and surface impedance $\lambda = \lambda(x)$ are uniquely determined by the electric far field pattern $E_\infty(\hat{x}, d, p)$ for $\hat{x}, d \in \Omega$ and $p \in \mathbb{R}^3$ (Recall again the remark after Theorem 4) [35]. Under the assumption that the regularized solution of the far field equation approximates the function g_ϵ and $g_{\epsilon,\delta}$ of Theorem 7, D can be determined by the linear sampling method (without needing to know $\lambda = \lambda(x)$ a priori!). Furthermore, if g is the regularized solution of the far field equation and E_g is the Herglotz wave function with kernel g, then ikE_g approximates the solution $E_z \in X(D)$ of

$$\begin{aligned} &\operatorname{curl}\operatorname{curl} E_z - k^2 E_z = 0, && x \in D \\ &\nu \times \operatorname{curl}(E_z + \mathscr{E}) - i\lambda(x)\,[\nu \times (E_z + \mathscr{E})] = 0 && x \in \Gamma \end{aligned}$$

where $\mathscr{E} = \mathscr{E}(\cdot, z, q)$ is an electric dipole. A straightforward (but lengthy) use of Green's formula and the Stratton-Chu formula [7] now show that for $W_z := E_z + \mathscr{E}(\cdot, z, q)$ and every two points z_1 and z_2 in D and polarization $q \in \mathbb{R}^3$ we have

$$\begin{aligned} 2\int_\Gamma \left(W_{z_1}\right)_\top \cdot \lambda(x) \left(\overline{W}_{z_1}\right)_\top ds \\ = -|q|^2 A(z_1, z_2, k, q) + k\left(q \cdot E_{z_1}(z_2) + q \cdot \overline{E}_{z_2}(z_1)\right) \end{aligned} \tag{5.1}$$

where $u_\top := (\nu \times u) \times \nu$ and

$$A(z_1, z_2, k, q) = \frac{k^3}{6\pi}\left[2j_0(k|z_1 - z_2|) + j_2(k|z_1 - z_2|)(3\cos^2\phi - 1)\right]$$

with j_0 and j_2 being spherical Bessel functions of order 0 and 2 respectively and ϕ is the angle between $(z_1 - z_2)$ and q. By varying z_1 and z_2 over a ball B_r of radius r contained in D one can now use (5.1) to determine $\|\lambda\|_{L_\infty(\Gamma)}$ [7]. In particular, when λ is a positive constant and setting $z_1 = z_2 = z_0 \in B_r$ we obtain

$$\lambda = \frac{-\frac{k^2}{6\pi}|q|^2 + k\mathrm{Re}\left(q \cdot E_{z_0}\right)}{\|(W_{z_0})_\top\|^2_{L^2_t(\Gamma)}}. \tag{5.2}$$

It would be of considerable interest to derive analogous expressions for physical parameters of scattering objects that can be computed from a knowledge of the electric far field pattern. A step in this direction can be found in [9] where an expression similar to (5.2) is given for the surface conductivity of a coated dielectric. An interesting aspect of the results of [7] and [9] is that they remain valid for partially coated objects as well where it is not necessary to know the extent of the coating a priori! In the case of the scattering problem (2.4)–(2.6) it seems reasonable to expect that an expression analogous to (5.2) could be found for $\|N\|$ where $\|\cdot\|$ is the operator norm on $\left(L^2(D)\right)^3$. However, this has not yet been done.

Open Problem *Let $N(x)$ be the matrix index of refraction in the scattering problem* (2.4)–(2.6). *Derive an expression for $\|N\|$ that can be computed from a knowledge of the electric far field pattern $E_\infty(\hat{x}, d, q)$ for $\hat{x}, d \in \Omega$ and $p \in \mathbb{R}^3$.*

In closing we note that the spectral properties of the far field operator contain considerable information about both the material and geometric properties of the scattering object. This has been examined to some extent in the scalar case [16, 27] but much remains to be done. In our opinion, this is a particularly promising area for further research for both the scalar and vector case. Further work is also needed in making clever use of multi-frequency data. An example of promising work in this direction is the recent paper by Bao & Li [3]. Finally, in most practical problems the scattering object is situated in a (known) nonhomogeneous background medium rather than a homogeneous background. In many applications one can assume a piecewise-constant background medium, but such situations still present formidable problems in attempting to solve the inverse problem. In particular, new methods are needed which have the simplicity of the linear sampling method but at the same time avoid the need to know the (dyadic) Green's function for the background medium as is currently the case for the linear sampling method in a piecewise constant background medium (c.f. [13]). An example of progress in this direction is the recent paper by Cakoni *et al.* [10] but this is only a starting point and much remains to be done.

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