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An inverse electromagnetic scattering problem for cavity

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Abstract. We consider the inverse electromagnetic scattering problem of determining the shape of a perfectly conducting cavity from measurement of scattered electric field due to electric dipole sources on a surface inside the cavity. We prove a reciprocity relation for the scattered electric field and a uniqueness theorem for the inverse problem. Then the near field linear sampling method is employed to reconstruct the shape of the cavity. Preliminary numerical examples are provided to show the viability of the method.

1. Introduction

In this paper we consider the inverse electromagnetic scattering problem of determining the shape of a perfectly conducting cavity. In contrast to the typical exterior problems, such as radar or sonar imaging, the problem we are interested in can be called the interior inverse scattering problem due to the fact that the sources and measurements are in the interior of the cavity. This is desirable in some applications of non-destructive testing such as monitoring the structural integrity of the fusion reactor by electromagnetic waves [12]. To be precise, we consider a bounded domain $D \subset \mathbb{R}^3$ such that ∂D is perfectly conducting. The dipole sources and measurements are on a surface Λ inside D (see Fig. 1). The inverse problem considered in this paper is to determine ∂D from the measured scattered electric field on Λ due to dipole sources on the same surface. In particular, we apply the near field linear sampling method to reconstruct the cavity D (see [3]).

To the authors' knowledge, there are only a few papers dealing with the qualitative methods for this type of interior inverse scattering problems. In [12] Jakubik and Potthast used the solutions of the Cauchy problem by potential methods and the range test to test the integrity of the boudary of some cavity by acoustic waves. In [18], Qin and Colton applied the linear sampling method to exactly the same problem discussed in this paper but in 2D case. They further extended their method to reconstruct both the shape of the cavity and surface impedance in [19]. Nonlinear integral equations have also been used to reconstruct the cavity [17].

Note that in some ways the interior inverse scattering problem is physically more complicated since the scattered waves are "trapped" inside the cavity. Similar to [18], our numerical reconstructions by the linear sampling method are less satisfactory than the results for the exterior inverse scattering problem [3]. However, whether this phenomenon is due to the reconstruction method or the physics of the interior scattering problem remains unclear at this point.

The rest of our paper is organized as the following. In Section 2, we formulate the interior scattering problem mathematically and introduce some functional spaces. In addition, we prove a reciprocity property of the scattered electric field which is useful for the uniqueness proof and numerical scheme. We show that under suitable conditions the cavity is uniquely determined from a knowledge of incident dipole sources and measurements on a surface inside the cavity. In Section 3, we describe how to employ the near field linear sampling method to reconstruct the cavity. We provide some preliminary numerical examples to show the viability of the method in Section 4. Finally in Section 5, we make conclusions and discuss some future works.

2. The scattering problem for a cavity

Let $D \subset \mathbb{R}^3$ be a simply connected bounded Lipschitz domain in $\mathbb{R}^3$, and Λ be a surface contained in D (see Fig. 1). Let ν be the unit outward normal defined almost everywhere on ∂D. More generally, in the following ν denotes the unit normal to the

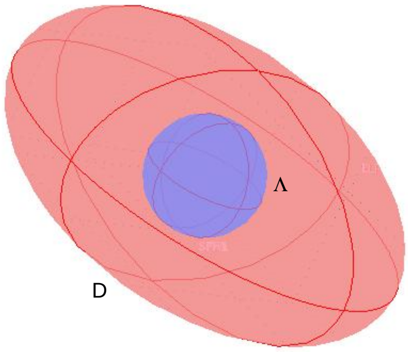

Figure 1. Explicative picture. The cavity is denoted by D. The surface Λ is inside D. Dipole sources and measurement locations are distributed on Λ.

indicated surface directed outward to the region bounded by the same surface. We consider the interior scattering problem of time-harmonic Maxwell's equations for the cavity D in terms of the electric field

$$\nabla \times \nabla \times \mathbf{E}^s - k^2 \mathbf{E}^s = 0, \quad \text{in} \quad \mathrm{D}, \tag{1}$$

$$\nu \times \mathbf{E}^s = \mathbf{h}, \quad \text{on} \quad \partial\mathrm{D}, \tag{2}$$

where k is the wave number, $\mathbf{h} = -\nu \times \mathbf{E}^i$, $\mathbf{E}^i$ is the incident wave in the form of the electric dipole given by

$$\mathbf{E}^i = \mathbb{G}(x,z)\mathbf{p} := \frac{i}{k}\nabla_x \times \nabla_x \times \Phi(x,z)\mathbf{p}$$

where $\mathbb{G}(x,z)$ is the Green's tensor, $\mathbf{p}$ is the polarization and $\Phi(x,z)$ is the fundamental solution of the Helmholtz equation given by

$$\Phi(x,z) = \frac{1}{4\pi}\frac{e^{ik|x-z|}}{|x-z|}.$$

Thus the scattered electric field depends on $x, z, \mathbf{p}$ which is indicated by writing $\mathbf{E}^s := \mathbf{E}^s(x,z,\mathbf{p})$.

For the following discussion, we need to introduce some functional spaces. Let $\Gamma = \partial D$ and we define (see, e.g., [15, 2])

$$H(\mathrm{curl}, \mathrm{D}) := \left\{\mathbf{u} \in (\mathrm{L}^2(\mathrm{D}))^3 : \nabla \times \mathbf{u} \in (\mathrm{L}^2(\mathrm{D}))^3\right\},$$

$$L^2_t(\Gamma) := \left\{\mathbf{u} \in (L^2(\Gamma))^3 : \nu \cdot \mathbf{u} = 0 \quad \text{on} \quad \Gamma\right\},$$

$$H^s_t(\Gamma) := \left\{\mathbf{u} \in (H^s(\Gamma))^3 : \nu \cdot \mathbf{u} = 0 \quad \text{on} \quad \Gamma\right\}, \quad s \in [-1,1],$$

$$H^{-1/2}_{\mathrm{div}}(\Gamma) := \left\{\mathbf{u} \in H^{-1/2}_t(\Gamma), \nabla_\Gamma \cdot \mathbf{u} \in H^{-1/2}(\Gamma)\right\},$$

$$H^{-1/2}_{\mathrm{curl}}(\Gamma) := \left\{\mathbf{u} \in H^{-1/2}_t(\Gamma), \nabla_\Gamma \times \mathbf{u} \in H^{-1/2}(\Gamma)\right\}.$$

For precise definitions of above spaces and surface divergence $\nabla_\Gamma \cdot$ and surface curl $\nabla_\Gamma \times$, we refer the readers to [1, 15]. It is well known that for $u \in H(\mathrm{curl}, \mathrm{D})$ the traces $\nu \times u$

and $(\nu \times u) \times \nu$ are well defined and belong to $H_{\mathrm{div}}^{-1/2}(\Gamma)$ and $H_{\mathrm{curl}}^{-1/2}(\Gamma)$, respectively. Furthermore, $H_{\mathrm{div}}^{-1/2}(\Gamma)$ and $H_{\mathrm{curl}}^{-1/2}(\Gamma)$ are dual of each other with $L_t^2(\Gamma)$ as the pivot space. In the following we use $\langle \cdot, \cdot \rangle_{H_{\mathrm{div}}^{-1/2}, H_{\mathrm{curl}}^{-1/2}}$ to denote the duality pairing between $H_{\mathrm{div}}^{-1/2}(\Gamma)$ and $H_{\mathrm{curl}}^{-1/2}(\Gamma)$. For any $u, v \in H(\mathrm{curl}, \mathrm{D})$ we have that

$$\langle \gamma_t u, \gamma_T v \rangle_{H_{\mathrm{div}}^{-1/2}, H_{\mathrm{curl}}^{-1/2}} := \int_\Gamma (\nu \times u) \cdot v \, \mathrm{d}s = \int_D (\nabla \times u \cdot v - u \cdot \nabla \times v) \, \mathrm{d}x$$

where

$$\gamma_t u = \nu \times u \quad \text{and} \quad \gamma_T v = (\nu \times v) \times v.$$

If k is not a Maxwell's eigenvalue and $\mathbf{h} \in H_{\mathrm{div}}^{-1/2}(\Gamma)$, the well-posedness of the cavity problem (1) is well-known [3]. We first prove a reciprocity relation for the scattered electric field.

Theorem 2.1. *Let $\mathbf{E}^s(\cdot, z, \mathbf{p})$ be the scattered field satisfying (1) due to the incident field given by an electric dipole at z with polarization $\mathbf{p}$. Then we have*

$$\mathbf{q} \cdot \mathbf{E}^s(x_q, x_p, \mathbf{p}) = \mathbf{p} \cdot \mathbf{E}^s(x_p, x_q, \mathbf{q})$$

for $x_p, x_q \in \Lambda$ and $\mathbf{p}, \mathbf{q} \in \mathbb{R}^3$.

Proof. By the Stratton-Chu formula (see Thm 9.2 of [15]), we have

$$\begin{aligned} \mathbf{E}(x) = & -\nabla \times \int_{\partial D} (\nu \times \mathbf{E})(y) \Phi(x, y) \, \mathrm{d}s(y) \\ & + \frac{1}{ik} \nabla \times \nabla \times \int_{\partial D} (\nu \times \mathbf{H})(y) \Phi(x, y) \, \mathrm{d}s(y). \end{aligned}$$

Using the fact that Φ is the fundamental solution of the Helmholtz equation and $x \neq y$ and $\mathbf{H} = (1/ik) \nabla \times \mathbf{E}$, we have

$$\begin{aligned} & \frac{1}{ik} \nabla \times \nabla \times \int_{\partial D} (\nu \times \mathbf{H})(y) \Phi(x, y) \, \mathrm{d}s(y) \\ & \quad = -\frac{1}{ik} (\Delta - \nabla \nabla \cdot) \int_{\partial D} (\nu \times \mathbf{H})(y) \Phi(x, y) \, \mathrm{d}s(y) \\ & \quad = \frac{1}{ik} \int_{\partial D} \left\{ k^2 (\nu \times \mathbf{H})(y) \Phi(x, y) + \nabla_x [(\nu \times \mathbf{H})(y) \cdot \nabla_x \Phi(x, y)] \right\} \mathrm{d}s(y) \\ & \quad = -\int_{\partial D} \mathbb{G}^T(x, y) (\nu \times (\nabla \times \mathbf{E}))(y) \, \mathrm{d}s(y) \end{aligned}$$

since lth entry of the gradient term is

$$\begin{aligned} (\nabla_x [(\nu \times \mathbf{H})(y) \cdot \nabla_x \Phi(x, y)])_l &= \frac{\partial}{\partial x_l} \sum_{m=1}^{3} (\nu \times \mathbf{H})_m(y) \frac{\partial \Phi}{\partial x_m}(x, y) \\ &= \sum_{m=1}^{3} (\nu \times \mathbf{H})_m(y) \frac{\partial^2 \Phi}{\partial x_l \partial x_m}(x, y). \end{aligned}$$

Using the fact that $\nabla_x \Phi = -\nabla_y \Phi$, we have

$$\begin{aligned}\nabla \times \int_{\partial D} (\nu \times \mathbf{E})(y)\Phi(x,y)\,\mathrm{d}s(y) &= \int_{\partial D} (\nu \times \mathbf{E})(y) \times \nabla_y \Phi(x,y)\,\mathrm{d}s(y) \\ &= \int_{\partial D} (\nabla_y \times (\Phi \mathbb{I}))^T(x,y)(\nu \times \mathbf{E})(y)\,\mathrm{d}s(y) \\ &= \int_{\partial D} (\nabla_y \times \mathbb{G})^T(x,y)(\nu \times \mathbf{E})(y)\,\mathrm{d}s(y).\end{aligned}$$

Thus we have shown that

$$\begin{aligned}\mathbf{E}(x) = & -\int_{\partial D} (\nabla_y \times \mathbb{G})^T(x,y)(\nu \times \mathbf{E})(y) \\ & \qquad + \mathbb{G}^T(x,y)\,(\nu \times (\nabla \times \mathbf{E}))\,(y)\,\mathrm{d}s(y).\end{aligned}$$

Similar to the proof of Theorem 2.1 in [19], we first employ the above result to the scattered fields to obtain

$$\begin{aligned}\mathbf{E}^s(x_q, x_p, \mathbf{p}) = & -\int_{\partial D} (\nabla_y \times \mathbb{G})^T(x_q,y)(\nu \times \mathbf{E}^s)(y, x_p, \mathbf{p}) \\ & \qquad + \mathbb{G}^T(x_q,y)\,(\nu \times (\nabla \times \mathbf{E}^s(y,x_p,\mathbf{p}))\,(y)\,\mathrm{d}s(y)\end{aligned} \tag{3}$$

and

$$\begin{aligned}\mathbf{E}^s(x_p, x_q, \mathbf{q}) = & -\int_{\partial D} (\nabla_y \times \mathbb{G})^T(x_p,y)(\nu \times \mathbf{E}^s)(y, x_q, \mathbf{q}) \\ & \qquad + \mathbb{G}^T(x_p,y)\,(\nu \times (\nabla \times \mathbf{E}^s(y,x_q,\mathbf{q}))\,(y)\,\mathrm{d}s(y).\end{aligned} \tag{4}$$

Applying Green's second theorem (see Section 6.2 of [7]) to $\mathbf{E}^s$ and $\mathbb{G}$ and using the fact that $\mathbb{G}^T(x,y)\mathbf{p}$ is a radiating field, we obtain

$$\begin{aligned}\int_{\partial D} & \nu \times \mathbf{E}^s(y, x_q, \mathbf{q}) \cdot \nabla \times \mathbf{E}^s(y, x_p, \mathbf{p}) \\ & - \nu \times \mathbf{E}^s(y, x_p, \mathbf{p}) \cdot \nabla \times \mathbf{E}^s(y, x_q, \mathbf{q})\,\mathrm{d}s(y) = 0\end{aligned} \tag{5}$$

and

$$\begin{aligned}\int_{\partial D} & \nu \times \mathbb{G}^T(x_q,y)\mathbf{q} \cdot \nabla \times \mathbb{G}^T(x_p,y)\mathbf{p} \\ & - \nu \times \mathbb{G}^T(x_p,y)\mathbf{p} \cdot \nabla \times \mathbb{G}^T(x_q,y)\mathbf{q}\,\mathrm{d}s(y) = 0.\end{aligned} \tag{6}$$

Multiplying (3) by $\mathbf{q}$ and (4) by $\mathbf{p}$ yields

$$\begin{aligned}\mathbf{q} \cdot \mathbf{E}^s(x_q, x_p, \mathbf{p}) = \int_{\partial D} & \nu \times \mathbb{G}^T(x_q,y)\mathbf{q} \cdot \nabla \times \mathbf{E}^s(y, x_p, \mathbf{p})(y) \\ & - \nu \times \mathbf{E}^s(y, x_p, \mathbf{p}) \cdot \nabla \times \mathbb{G}^T(x_q,y)\mathbf{q}\,\mathrm{d}s(y)\end{aligned}$$

and

$$\begin{aligned}\mathbf{p} \cdot \mathbf{E}^s(x_p, x_q, \mathbf{q}) = \int_{\partial D} & \nu \times \mathbb{G}^T(x_p,y)\mathbf{p} \cdot \nabla \times \mathbf{E}^s(y, x_q, \mathbf{q})(y) \\ & - \nu \times \mathbf{E}^s(y, x_q, \mathbf{q}) \cdot \nabla \times \mathbb{G}^T(x_p,y)\mathbf{p}\,\mathrm{d}s(y).\end{aligned}$$

Taking the difference, and using (5) and (6), we obtain

$$\mathbf{p}\cdot\mathbf{E}^s(x_p, x_q, \mathbf{q}) - \mathbf{q}\cdot\mathbf{E}^s(x_q, x_p, \mathbf{p}) =$$
$$\int_{\partial D} \nu\times\mathbf{E}(y, x_q, \mathbf{q})\cdot\nabla\times\mathbf{E}(y, x_p, \mathbf{p}) - \nu\times\mathbf{E}(y, x_p, \mathbf{p})\cdot\nabla\times\mathbf{E}(y, x_q, \mathbf{q})\,\mathrm{d}s(y)$$

where $\mathbf{E}$ is the total electric field. Using the PEC boundary condition (2), we have

$$\mathbf{p}\cdot\mathbf{E}^s(x_p, x_q, \mathbf{q}) - \mathbf{q}\cdot\mathbf{E}^s(x_q, x_p, \mathbf{p}) = 0$$

which completes the proof.

□

The inverse scattering problem we are interested in is to determine D from a knowledge of tangential components $\nu\times\mathbf{E}^s$ of the scattered electric field $\mathbf{E}^s = \mathbf{E}^s(\cdot, z, \mathbf{p})$ measured on Λ for all incident field due to point sources $z\in\Lambda$ and all polarizations $\mathbf{p}\in\mathbb{R}^3$. To this end, we have the following uniqueness theorem.

Theorem 2.2. *If k^2 is not a Maxwell's eigenvalue for the interior of Λ, then D is uniquely determined from $\nu\times\mathbf{E}^s(x, z, \mathbf{p})$ for $x, z\in\Lambda$ and all polarizations $\mathbf{p}\in\mathbb{R}^3$.*

Proof. The proof is based on the approach used by Kirsch and Kress for the exterior scattering problem [13, 14]. Assume $D_1\neq D_2$ are two bounded domain and $\mathbf{E}_i^s, i = 1, 2$, satisfy equations (1), respectively. Suppose that $\nu\times\mathbf{E}_1^s(x, z, \mathbf{p}) = \nu\times\mathbf{E}_2^s(x, z, \mathbf{p})$ on Λ for all $z\in\Lambda$ and let $\mathbf{V} = \mathbf{E}_1^s - \mathbf{E}_2^s$. Then

$$\nabla\times\nabla\times\mathbf{V} - k^2\mathbf{V} = 0, \quad \text{in} \quad \dot{\Lambda},$$
$$\nu\times\mathbf{V} = 0, \quad \text{on} \quad \Lambda,$$

where $\dot{\Lambda}$ is the interior of Λ. Since k^2 is not a Maxwell's eigenvalue for $\dot{\Lambda}$, we have that $\mathbf{V} = 0$ in $\dot{\Lambda}\cup\Lambda$.

Let D_0 be the connected component of $D_1\cap D_2$ containing $\dot{\Lambda}$. Then by analyticity, $\mathbf{V} = 0$ in $\overline{D}_0$, i.e.,

$$\mathbf{E}_1^s(x, z, \mathbf{p}) = \mathbf{E}_2^s(x, z, \mathbf{p})$$

for all $x\in\overline{D}_0$, $z\in\Lambda$ and all polarization $\mathbf{p}$. By the reciprocity relation, we have, for $i = 1, 2, 3$,

$$\mathbf{e}_i\cdot\mathbf{E}_s^1(z, x, \mathbf{p}) = \mathbf{p}\cdot\mathbf{E}_s^1(x, z, \mathbf{e}_i) = \mathbf{p}\cdot\mathbf{E}_s^2(x, z, \mathbf{e}_i) = \mathbf{e}_i\cdot\mathbf{E}_s^2(z, x, \mathbf{p}).$$

Thus we obtain

$$\mathbf{E}_1^s(z, x, \mathbf{p}) = \mathbf{E}_2^s(z, x, \mathbf{p})$$

for all $x\in\overline{D}_0$, $z\in\Lambda$ and all polarization $\mathbf{p}$. Using the same argument as above, we have that

$$\mathbf{E}_1^s(x, z, \mathbf{p}) = \mathbf{E}_2^s(x, z, \mathbf{p})$$

for all $x, z\in\overline{D}_0$ and all polarization $\mathbf{p}$.

Without loss of generality, there exists $x^* \in \partial D_0$ such that $x^* \in \partial D_1$ and $x^* \notin \partial D_2$. In particular, we have that

$$z_n := x^* - \frac{1}{n}\nu(x^*) \in D_0$$

for sufficiently large n. Then, in view of the well-posedness of the cavity problem for scatterer D_2, on one hand we obtain that

$$\lim_{n\to\infty} \nu \times \mathbf{E}_2^s(x^*, z_n, \mathbf{p}) = \nu \times \mathbf{E}_2^s(x^*, x^*, \mathbf{p}).$$

On the other hand we find that

$$\lim_{n\to\infty} \nu \times \mathbf{E}_1^s(x^*, z_n, \mathbf{p}) = \infty \quad \text{for} \quad \mathbf{p} \perp \nu(x^*),$$

because of the boundary condition for $\mathbf{E}_1^s$ in terms of the electric dipole located at $z_n \to x^*$ for $n \to \infty$. This is a contradiction and thus $D_1 = D_2$. □

3. The Linear Sampling Method

In this section, we employ the linear sampling method [3] to the inverse problem stated in the previous section. For sake of simplicity, without loss of generality, from now on we assume that $\Lambda \subset D$ is a sphere centered at the origin, i.e., $\Lambda = \{x \in \mathbb{R}^3, |x| = r_c > 0\}$. We define the near field operator: $\mathcal{F} : L_t^2(\Lambda) \to L_t^2(\Lambda)$

$$(\mathcal{F}\phi)(x) := \int_\Lambda \nu(x) \times \mathbf{E}^s(x, y, \phi(y))\,\mathrm{d}s(y). \tag{7}$$

Since $\mathbf{E}^s$ is analytic, the operator $\mathcal{F}$ is compact. We also define the electric single layer potential $\mathcal{S} : H_{\mathrm{div}}^{-1/2}(\Lambda) \to (H_{loc}^1(\mathrm{curl}, \mathbb{R}^3 \setminus \Lambda)$ given by [9]:

$$(\mathcal{S}\phi)(x) := \int_\Lambda \phi(y)\mathbb{G}(x, y)\,\mathrm{d}s(y) \tag{8}$$

with density ϕ, i.e.,

$$(\mathcal{S}\phi)(x) := \frac{i}{k}\nabla_x \times \nabla_x \times \int_\Lambda \phi(y)\Phi(x, y)\,\mathrm{d}s(y). \tag{9}$$

By superposition, $\mathcal{F}\phi$ is the rotated tangential component on Λ of the scattered field due to $\mathcal{S}\phi$. In the following we prove an important property for $\mathcal{F}$.

Theorem 3.1. *The operator $\mathcal{F}$ is injective and has dense range if k^2 is not a Maxwell's eigenvalue in the interior of Λ.*

Proof. Let $\mathcal{F}\phi = 0$ and we need to show that $\phi = 0$. Define

$$\mathbf{W} = \int_\Lambda \mathbf{E}^s(x, y, \phi(y))\,\mathrm{d}s(y), \quad x \in D.$$

It is obvious that $\mathbf{W}$ satisfies

$$\nabla \times \nabla \times \mathbf{W} - k^2\mathbf{W} = 0, \quad \text{in} \quad \dot{\Lambda},$$

$$\nu \times \mathbf{W} = 0, \quad \text{on} \quad \Lambda.$$

Since k^2 is not a Maxwell's eigenvalue in the interior of Λ, we have $\mathbf{W} = 0$ in $\dot{\Lambda}$. Note that $\nabla \times \nabla \times \mathbf{W} - k^2\mathbf{W} = 0$ in D. Then $\mathbf{W} = 0$ in $\overline{D}$ by analyticity. In particular,

$$\mathbf{W} = -\int_\Lambda \phi(y)\mathbb{G}(x,y)\,\mathrm{d}s(y) = 0, \quad x \in \partial D.$$

Define

$$\mathbf{V}(x) = -\int_\Lambda \phi(y)\mathbb{G}(x,y)\,\mathrm{d}s(y), \quad x \in \mathbb{R}^3 \setminus \Lambda.$$

Then $\mathbf{V}$ satisfies

$$\nabla \times \nabla \times \mathbf{V} - k^2\mathbf{V} = 0, \quad \text{in} \quad \mathbb{R}^3 \setminus \overline{\mathrm{D}},$$

$$\nu \times \mathbf{V} = 0, \quad \text{on} \quad \partial\mathrm{D},$$

and the Silver-Müller condition

$$\lim_{r\to\infty} r\left(\nabla \times \mathbf{V} \times \hat{x} - ik\mathbf{V}\right) = 0$$

where $r = |x|$ and $\hat{x} = x/|x|$. Since the solution of the exterior Dirichlet problem is unique, $\mathbf{V} = 0$ in $\mathbb{R}^3 \setminus \overline{D}$. The unique continuation principle implies that $\mathbf{V} = 0$ in the exterior of Λ. By the jump relation for the single layer potential, we have

$$\nu \times \mathbf{V}_- - \nu \times \mathbf{V}_+ = 0, \quad \nu \times \nabla \times \mathbf{V}_- - \nu \times \nabla \times \mathbf{V}_+ = \phi,$$

where ν is the unit outward normal to Λ and $+$ and $-$ denote the limit as $x \to \Lambda$ from outside and inside of Λ, respectively. Since $\nu \times \mathbf{V}_+ = \nu \times \nabla \times \mathbf{V}_+ = 0$, we have that $\nu \times \mathbf{V}_- = 0$ and $\nu \times \nabla \times \mathbf{V}_- = \phi$. Since k^2 is not a Maxwell's eigenvalue in the interior of Λ, we have $\mathbf{V} = 0$ in the interior of Λ. Hence $\phi = 0$, i.e., $\mathcal{F}$ is injective.

The L^2 adjoint $\mathcal{F}^* : L^2_t(\Lambda) \to L^2_t(\Lambda)$ is given by

$$(\mathcal{F}^*\psi)(x) = \overline{\int_\Lambda \nu(x) \times \mathbf{E}^s(x, y, \overline{\psi(y)})\,\mathrm{d}s(y)},$$

for $\psi \in L^2_t(\Lambda)$ and $x \in \Lambda$. Then we have $(\mathcal{F}^*\psi)(x) = \overline{(\mathcal{F}\phi)(x)}$ if $\phi(z) = \overline{\psi(z)}$. Since $\mathcal{F}$ is injective, then $\mathcal{F}^*$ is injective. Moreover, since $N(\mathcal{F}^*) = (R(\mathcal{F}))^\perp$, $\mathcal{F}$ has dense range in $L^2_t(\Lambda)$. □

We remark that the assumption that k^2 is not a Maxwell's eigenvalue in the interior of Λ is not a restriction since Λ can be chosen such that this assumption is satisfied.

Now we define a linear operator $\mathcal{B} : H^{-1/2}_{\mathrm{div}}(\partial D) \to L^2_t(\Lambda)$ mapping the boundary value $\mathbf{h}$ to $\nu \times \mathbf{E}^s$ on Λ where $\mathbf{E}^s$ is the corresponding scattered field satisfying (1). Then we have

$$\mathcal{F} = -\mathcal{B}\left[\nu \times \mathcal{S}|_{\partial D}\right].$$

Theorem 3.2. *Assume that k^2 is not a Maxwell's eigenvalue either in D or in the interior of Λ. Then the operator $\mathcal{B}$ is injective, compact and has dense range in $L^2_t(\Lambda)$.*

Proof. Since k^2 is not a Maxwell's eigenvalue in D, $\mathcal{B}$ is well defined.

Let $\mathcal{B}\mathbf{h} = 0$, i.e., $\nu \times \mathbf{E}^s = 0$ on Λ. Since k^2 is not a Maxwell's eigenvalue in the interior of Λ, $\mathbf{E}^s = 0$ in the interior of Λ. By analyticity $\mathbf{E}^s = 0$ in $\overline{D}$ and $\mathbf{h} = 0$, i.e., the operator $\mathcal{B}$ is injective.

Next we choose a ball $\Omega = \{x \in \mathbb{R}^3, |x| \le r > r_c\}$ such that $\Lambda \subset \Omega \subset D$. Then we have

$$\mathbf{E}^s(x) = -\int_{\partial D} (\nabla_y \times \mathbb{G})^T(x,y)(\nu \times \mathbf{E}^s)(y) + \mathbb{G}^T(x,y)\left(\nu \times (\nabla \times \mathbf{E}^s)\right)(y)\mathrm{d}s(y)$$

for $x \in D$. We decompose the operator $\mathcal{B} = \mathcal{B}_1\mathcal{B}_2$, where

$$\mathcal{B}_2 : H^{-1/2}_{\mathrm{div}}(\partial D) \to H^{-1/2}_{\mathrm{div}}(\partial\Omega) \times H^{-1/2}_{\mathrm{div}}(\partial\Omega)$$

is defined by

$$(\mathcal{B}_2\mathbf{h})(x) = (\nu \times (\nabla \times \mathbf{E}^s)|_{\partial\Omega}, \nu \times \mathbf{E}^s|_{\partial\Omega}) =: (\mathbf{h}_1, \mathbf{h}_2)$$

and

$$\mathcal{B}_1 : H^{-1/2}_{\mathrm{div}}(\partial\Omega) \times H^{-1/2}_{\mathrm{div}}(\partial\Omega) \to L^2_t(\Lambda)$$

is defined by

$$\mathcal{B}_1(\mathbf{h}_1, \mathbf{h}_2)(x) = -\int_{\partial\Omega} \left\{(\nabla_y \times \mathbb{G})^T(x,y)\mathbf{h}_2(y) + \mathbb{G}^T(x,y)\mathbf{h}_1(y)\right\}\mathrm{d}s(y).$$

Then $\mathcal{B}_2$ is bounded and $\mathcal{B}_1$ is compact. Hence, the operator $\mathcal{B}$ is compact.

We now show that the operator $\mathcal{B}$ has a dense range in $L^2_t(\Lambda)$. Following [7, 15], let Y_n^m be the spherical harmonic and define

$$\tilde{\mathbf{M}}_n^m = \nabla \times \{x j_n(k|x|)Y_n^m(\hat{x})\}, \quad \tilde{\mathbf{N}}_n^m = \frac{1}{ik} \times \tilde{\mathbf{M}}_n^m$$

where $\hat{x} = x/|x|$ and j_n is the spherical Bessel's function. Let

$$\mathbf{E}_n = \sum_{m=-n}^{n} a_{n,m}\tilde{\mathbf{M}}_n^m + b_{n,m}\tilde{\mathbf{N}}_n^m.$$

Then $\mathbf{E}_n$ satisfies the interior cavity problem with $\mathbf{h} = \mathbf{E}_n|_{\partial D}$. Since the spherical harmonics are complete in $L^2_t(\Lambda)$ and k^2 is not a Maxwell's eigenvalue in the interior of Λ, $\mathcal{B}$ has dense range. □

Theorem 3.3. *Let $\mathbb{G}$ be the Green's tensor. Then $\nu \times \mathbb{G}(\tilde{x}, z)\mathbf{p}, \tilde{x} \in \Lambda$ is in the range of $\mathcal{B}$ if and only if $z \in \mathbb{R}^3 \setminus \overline{D}$.*

Proof. If $z \in \mathbb{R}^3 \setminus \overline{D}$, then $\mathbb{G}(\cdot, z)\mathbf{p}$ is the solution of (1) with $\mathbf{h} = \nu \times \mathbb{G}(\cdot, z)\mathbf{p}|_{\partial D}$ and $\mathcal{B}\mathbf{h} = \nu \times \mathbb{G}(\tilde{x}, z)\mathbf{p}$ for $\tilde{x} \in \Lambda$.

Now let $z \in \overline{D} \setminus \Lambda$ and assume on the contrary that $\nu \times \mathbb{G}(\tilde{x}, z)\mathbf{p}$ is in the range of $\mathcal{B}$. Then $\mathbb{G}(\cdot, z)\mathbf{p}$ is a solution of (1) with $\mathbf{h} = \nu \times \mathbb{G}(\cdot, z)\mathbf{p}|_{\partial D}$. However, $\mathbb{G}(\cdot, z)\mathbf{p}$ is not in the $H(\mathrm{curl}, \mathrm{D})$. This leads to a contradiction which completes the proof. □

Theorem 3.4. *Assume that k^2 is not a Maxwell's eigenvalue either in D or in the interior of Λ. Then the operator $\mathcal{S}_{\partial D} = \nu \times \mathcal{S}|_{\partial D} : H^{-1/2}_{\mathrm{div}}(\Lambda) \to H^{-1/2}_{\mathrm{div}}(\partial D)$ is injective, compact and has dense range in $H^{-1/2}_{\mathrm{div}}(\partial D)$.*

Proof. It is obvious that $\mathcal{S}_{\partial D}$ is compact since its kernel is analytic.

To show that $\mathcal{S}_{\partial D}$ is injective, we first assume that

$$(\mathcal{S}_{\partial D}\mathbf{g})(x) = \nu(x) \times \int_\Lambda \mathbf{g}(y)\mathbb{G}(x,y)\,\mathrm{d}s(y) = 0, \quad x \in \partial D.$$

Define

$$\mathbf{E}(x) = \int_\Lambda \mathbf{g}(y)\mathbb{G}(x,y)\,\mathrm{d}s(y), \quad x \in \mathbb{R}^3 \setminus \Lambda.$$

Then $\nu \times \mathbf{E}(x) = 0$ on ∂D. By the same arguments as in the proof that $\mathcal{F}$ is injective, we have that $\mathbf{g} = 0$, i.e. the operator $\mathcal{S}_{\partial D}$ is injective.

We now show that $\mathcal{S}_{\partial D}$ has dense range. Let $\psi \in H^{-1/2}_{\mathrm{curl}}(\partial D)$ be such that

$$\langle \mathcal{S}_{\partial D}\mathbf{g}, \psi \rangle_{H^{-1/2}_{\mathrm{div}}(\partial D), H^{-1/2}_{\mathrm{curl}}(\partial D)} = 0$$

for all $\mathbf{g}$, i.e.

$$\int_{\partial D} \nu(x) \times \int_\Lambda \mathbf{g}(y)\mathbb{G}(x,y)\,\mathrm{d}s(y)\psi(x)\,\mathrm{d}s(x) = 0.$$

We need to show that $\psi = 0$. By interchanging the order of integration we have

$$\int_\Lambda \int_{\partial D} \nu(x) \times \mathbb{G}(x,y)\psi(x)\,\mathrm{d}s(x)\mathbf{g}(y)\,\mathrm{d}s(y) = 0$$

for all $\mathbf{g}$. Then

$$\mathbf{V}(z) = \int_{\partial D} \nu(x) \times \mathbb{G}(x,z)\psi(x)\,\mathrm{d}s(x) = 0$$

for all $z \in \Lambda$. Define

$$\mathbf{W}(z) = \int_{\partial D} \nu(x) \times \mathbb{G}(x,z)\psi(x)\,\mathrm{d}s(x), \quad z \in \mathbb{R}^3 \setminus \partial D.$$

Then $\nu \times \mathbf{W}(z) = 0$ on Λ and $\mathbf{W}$ satisfies

$$\nabla \times \nabla \times \mathbf{W} - k^2\mathbf{W} = 0 \quad \text{in} \quad \dot{\Lambda}.$$

Since k^2 is not a Maxwell's eigenvalue in the interior of Λ, we have $\mathbf{W} = 0$ in the interior of Λ. By analyticity $\mathbf{W} = 0$ in D. The jump condition for the single layer potential gives

$$\nu \times \mathbf{W}_- - \nu \times \mathbf{W}_+ = 0, \quad \nu \times \nabla \times \mathbf{W}_- - \nu \times \nabla \times \mathbf{W}_+ = \psi.$$

Since the solution for the exterior Dirichlet problem is unique we have $\mathbf{W} = 0$ in $\mathbb{R}^3 \setminus D$. Hence $\psi = 0$ and the operator $\mathcal{S}_{\partial D}$ has a dense range. □

Now we employ the linear sampling method for the inverse problem. Define the near field equation:

$$\int_{\Lambda} \nu(x) \times \mathbf{E}^s(x, y, \phi_z(y))\, \mathrm{d}s(y) = \nu(x) \times \mathbb{G}(x, z)\mathbf{p}, \quad x \in \Lambda \tag{10}$$

where $\mathbf{p}$ is an artificial polarization and ν is the unit outward normal to Λ. The linear sampling method is based on solving the above linear near field equation for the indicator function $\phi_z \in L^2_t(\Lambda)$. If $z \in \mathbb{R}^3 \setminus \overline{D}$, we can see that if ϕ_z is a solution for the near field equation, the tangential trace of $\mathbf{E}^s$ and the electric dipole $\mathbb{G}(x, z)\mathbf{p}$ coincide on ∂D. As $z \to \partial D$, we have that $\|\nu \times \mathbb{G}(x, z)\mathbf{p}\|_{H^{-1/2}_{\mathrm{div}}(\partial D)} \to \infty$, and hence $\|\nu \times \mathbf{E}^{\mathbf{s}}\|_{H^{-1/2}_{\mathrm{div}}(\partial D)} \to \infty$. Thus $\|\phi_z\|_{L^2_t(\Lambda)} \to \infty$ and this behavior determines ∂D (see [3] for the case of the exterior inverse scattering problem). The above argument is only heuristic because of the ill-posed nature of (10). However, we can always solve for an approximate solution of the near field equation and expect the similar behavior of the solution as $z \to \partial D$. The following theorem is of fundamental importance to the linear sampling method for the interior inverse scattering problem.

Theorem 3.5. *Assume that k^2 is not a Maxwell's eigenvalue either in D or in the interior of Λ. Then*

- *If $z \in \mathbb{R}^3 \setminus \overline{D}$, then for every $\epsilon > 0$, there exist a solution $\phi^\epsilon_z \in L^2_t(\Lambda)$ satisfying*

  $$\|\mathcal{F}\phi^\epsilon_z(x) - \nu(x) \times \mathbb{G}(x, z)\mathbf{p}\|_{L^2_t(\Lambda)} < \epsilon$$

  such that $\mathcal{S}_{\partial D}\phi^z_\epsilon$ converges to the solution of the problem (1) with $\mathbf{h} = -\nu \times \mathbb{G}\mathbf{p}$ as $\epsilon \to 0$. Furthermore, for a fixed ϵ,

  $$\lim_{z \to \partial D} \|\mathcal{S}_{\partial D}\phi^\epsilon_z\|_{H^{-1/2}_{\mathrm{div}}(\partial D)} = \infty \tag{11}$$

  and

  $$\lim_{z \to \partial D} \|\phi^\epsilon_z\|_{L^2_t(\Lambda)} = \infty. \tag{12}$$

- *If $z \in D \setminus \Lambda$, then for every $\epsilon > 0$, there exists a solution $\phi^\epsilon_z \in L^2_t(\Lambda)$ such that*

  $$\|\mathcal{F}\phi^\epsilon_z(x) - \nu(x) \times \mathbb{G}(x, z)\mathbf{p}\|_{L^2_t(\Lambda)} < \epsilon$$

  such that

  $$\lim_{\epsilon \to 0} \|\mathcal{S}_{\partial D}\phi^\epsilon_z\|_{H^{-1/2}_{\mathrm{div}}(\partial D)} = \infty$$

  and

  $$\lim_{\epsilon \to 0} \|\phi^\epsilon_z\|_{L^2_t(\Lambda)} = \infty.$$

Proof. We first assume that $z \in \mathbb{R}^3 \setminus \overline{D}$. Then there exists $\mathbf{h}_z \in H^{-1/2}_{\mathrm{div}}(\partial D)$ such that $\mathcal{B}\mathbf{h}_z = \nu \times \mathbb{G}(x, z)\mathbf{p}$ for $x \in \Lambda$. For every $\epsilon_0 > 0$, there exists a function $\phi^{\epsilon_0}_z \in L^2_t(\Lambda)$ such that

$$\|\mathcal{S}_{\partial D}\phi^{\epsilon_0}_z - \mathbf{h}_z\|_{L^2_t(\partial D)} < \epsilon_0. \tag{13}$$

Since the operator $\mathcal{B}$ is bounded, we have

$$\|\mathcal{B}\mathcal{S}_{\partial D}\phi_z^{\epsilon_0} - \mathcal{B}\mathbf{h}_z\|_{L_t^2(\Lambda)} < c\epsilon_0$$

where $c > 0$ is a constant. Letting $\epsilon = c\epsilon_0$, we have

$$\|\mathcal{F}\phi_z^{\epsilon}(\cdot) + \nu \times \mathbb{G}(\cdot, z)\mathbf{p}\|_{L_t^2(\Lambda)} < \epsilon. \tag{14}$$

Obviously $\mathbf{h}_z$ is $\nu \times \mathbb{G}\mathbf{p}$ on ∂D by definition. As $z \to \partial D$, $\|\mathbf{h}_z\|_{H_{\rm div}^{-1/2}(\partial D)}$ blows up. For a fixed ϵ, from (13), we have that (11) holds, and thus (12) also holds.

Now if $z \in D\backslash\Lambda$, $\nu\times\mathbb{G}(\cdot, z)\mathbf{p}$ is not in the range of $\mathcal{B}$. But $\mathcal{B}$ has dense range. Hence, using Tikhonov regularization, for every $\epsilon > 0$ we can construct a unique regularized solution $\mathbf{h}_\alpha^z$ of equation $(\mathcal{B}\mathbf{h})(\cdot) = -\nu \times \mathbb{G}(\cdot, z)\mathbf{p}$ given by

$$\mathbf{h}_\alpha^z = -\sum_{j=1}^{\infty} \frac{\mu_j}{\alpha + \mu_j^2}(\nu \times \mathbb{G}(\cdot, z)\mathbf{p}, y_j)x_j$$

where (μ_j, x_j, y_j) is a singular system for the compact operator $\mathcal{B}$ such that

$$\|(\mathcal{B}\mathbf{h}_\alpha^z)(\cdot) + \nu \times \mathbb{G}(\cdot, z)\mathbf{p}\|_{L_t^2(\Lambda)} < \frac{\epsilon}{2}$$

and

$$\lim_{\alpha\to 0}\|\mathbf{h}_\alpha^z\|_{L_t^2(\partial D)} = \infty.$$

Since $\mathcal{S}_{\partial D}$ has a dense range, there exists ϕ_z^α such that

$$\|\mathcal{S}_{\partial D}\phi_z^\alpha - \mathbf{h}_\alpha^z\|_{H_{\rm div}^{-1/2}(\partial D)} < \frac{\epsilon}{2c}$$

where c is a constant such that $\|\mathcal{B}\| < c$. Hence we have

$$\begin{aligned} &\|\mathcal{F}\phi_z^\alpha(\cdot) - \nu(x) \times \mathbb{G}(\cdot, z)\mathbf{p}\|_{L_t^2(\Lambda)} \\ &\qquad \le \|\mathcal{B}\mathcal{S}_{\partial D}\phi_z^\alpha - \mathcal{B}\mathbf{h}_\alpha^z\|_{L_t^2(\Lambda)} + \|\mathcal{B}\mathbf{h}_\alpha^z + \nu(x) \times \mathbb{G}(\cdot, z)\mathbf{p}\|_{L_t^2(\Lambda)} < \epsilon. \end{aligned}$$

Since $\lim_{\epsilon\to 0}\alpha(\epsilon) = 0$ we have that

$$\lim_{\epsilon\to 0}\|\mathbf{h}_{\alpha(\epsilon)}^z\|_{H_{\rm div}^{-1/2}(\partial D)} = \infty.$$

Since $\mathcal{S}_{\partial D}$ is bounded, we obtain

$$\lim_{\epsilon\to 0}\|\phi_z^\epsilon\|_{L_t^2(\Lambda)} = \infty.$$

□

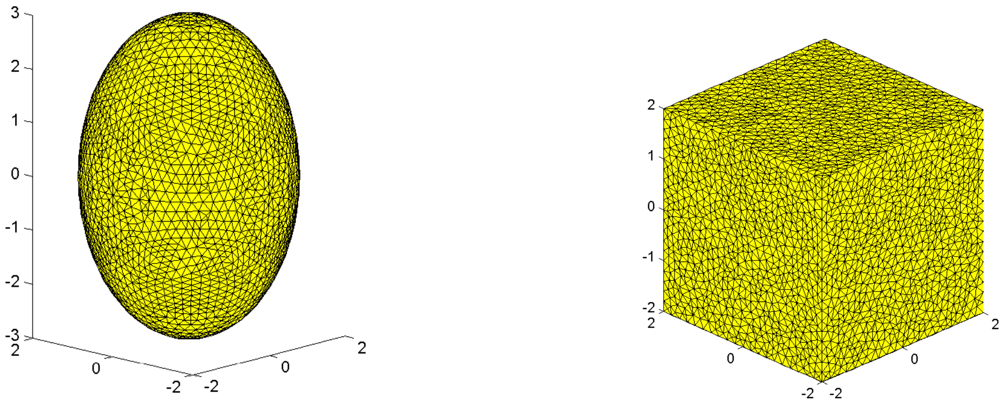

Figure 2. The geometries for two targets, an ellipsoid and a cube, and the meshes used for the edge element method for the interior scattering problem. The mesh size $h \approx 0.2$. Left: The ellipsoid. Right: The cube.

4. Numerical examples

In this section, we show some preliminary examples using synthetic data. We use finite element method to solve the interior scattering problems. We first choose the cavity D and a sphere Λ inside D. Then we generate a tetrahedra mesh $\mathcal{T}$ for D (see Fig. 2). We also generate a tetrahedra mesh for $\dot{\Lambda}$ which induces a triangulation $\mathbb{T}$ for Λ whose vertices restricted to Λ, denoted by $x_i, i = 1, \dots N$, are chosen to be positions for the dipole sources and measurement locations (see Fig. 3). For each dipole source at $x_i \in \Lambda$ and polarization $\mathbf{p}$, we use the linear edge element (see [15]) to solve (1) for the scattered field $\mathbf{E}^s(\cdot, x_i, \mathbf{p})$ and record the values at all x_i's. All the numerical examples are done using Matlab on a desktop with 12G memory. Due to the memory restriction and the 3D nature of the scattering problem, the mesh size is restricted to 0.2 which leads to roughly 10% error (maximum norm) of the scattered field. This is one of the reasons of the poor reconstruction of the cavity D. However, on the other hand, even with the inaccurate synthetic data, the position, the size and a rough shape can still be obtained.

After we have recorded the synthetic data on Λ, we turn to the problem of solving the linear ill-posed integral equations (10). Note that the left hand side of (10) obscures the dependence of the far field operator on ϕ_z. Similar to [15], we will derive an equivalent form. Let $(\mathbf{e}_1(x), \mathbf{e}_2(x), \nu(x))$ be an orthonormal basis on Λ. Then (10) is equivalent to the two scalar equations,

$$\int_\Lambda \mathbf{e}_l(x) \cdot \mathbf{E}^s(x, y, \phi_z(y))\, \mathrm{d}s(y) = \mathbf{e}_l(x) \cdot \mathbb{G}(x, z)\mathbf{p}, \quad x, z \in \Lambda$$

for $l = 1, 2$. Since the scattered field satisfies the reciprocity relation, we may rewrite the above equation as

$$\int_\Lambda \phi_z(y) \cdot \mathbf{E}^s(y, x, \mathbf{e}_l(x))\, \mathrm{d}s(y) = \mathbf{e}_l(x) \cdot \mathbb{G}(x, z)\mathbf{p}$$

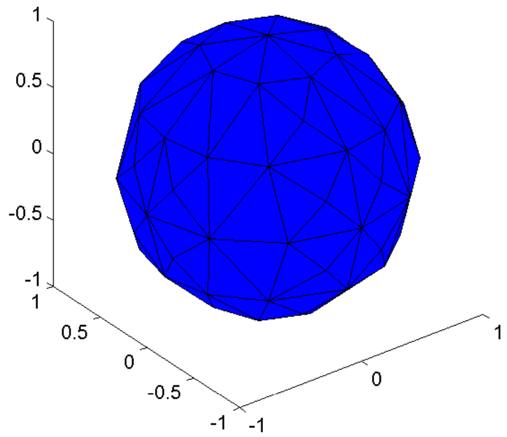

Figure 3. Triangulation $\mathbb{T}$ of the unit sphere Λ with 152 triangles and 78 vertices.

for $x, y \in \Lambda$, $z \in \mathbb{R}^3$ and $l = 1, 2$.

The integral in (10) will be approximated by a quadrature. The vertices of each triangle $T \in \mathbb{T}$ is denoted by $a_j^T, j = 1, 2, 3$. For any smooth function f on Λ, we have

$$\int_T f \, \mathrm{d}A \approx \frac{1}{3}\mathrm{area}(T) \sum_{j=1}^{3} f(a_j^T).$$

Suppose the triangulation $\mathbb{T}$ has N vertices given by $x_m, m = 1, \ldots, N$, we obtain

$$\int_\Lambda f \, \mathrm{d}A \approx \sum_{m=1}^{N} \omega_m f(x_m)$$

where the quadrature weights are $\omega_m, m = 1, \ldots, N$. We may write $\phi_j = \phi_{1,j}\mathbf{e}_1 + \phi_{2,j}\mathbf{e}_2$ where $\phi_{1,j}, \phi_{2,j} \in \mathbb{C}$. The fully discrete problem corresponding to (10) is to find $\phi_{1,j}, \phi_{2,j}, j = 1, \ldots, N$ such that

$$\sum_{j=1}^{N} \sum_{n=1}^{2} \omega_j \mathbf{e}_n(x_j) \cdot \mathbf{E}^s(x_j, x_m, \mathbf{e}_l(x_m))\phi_{n,j} = \mathbf{e}_l(x_m) \cdot \Phi(x_m, z)\mathbf{p}$$

for $l = 1, 2$, $m = 1, \ldots, N$. We can rewrite this linear system as

$$A\vec{\phi} = \vec{F} \tag{15}$$

where A is a $2N \times 2N$ matrix, $\vec{\phi}$ is is the vector of unknowns, and $\vec{F}$ is the right-hand side depending on the sampling point z. Since A comes from a compact operator, we choose to employ the Tikhonov regularization method for (15). The regularization parameter is chosen by Morozov's discrepancy principle as discussed in [5].

To reconstruct the cavity D, we choose a uniform grid of the sampling region, i.e. a region S contains D in the exterior of Λ. According to the results of previous section,

$\|\vec{\phi}_z\|$ (where $\|\cdot\|$ denotes $\|\cdot\|_{l^2}$) should become large as $z \to \partial D$ from outside of D and $z \in D \setminus \Lambda$. For simplicity, we just choose the sampling region S to be a cube containing D. After we solve (15) for all the sampling points in S, the reconstruction is done by plotting an iso-surface such that $\|\vec{\phi}_z\| = C$ for some constant C. Note that in practice it is difficult to choose C since $\vec{\phi}_z$ is computed from noisy data using Morozov's discrepancy and we have no a priori knowledge of the scatterer. In the following examples we choose C using the heuristic calibration approach of [8], where the value of C is determined from the reconstruction of a ball. In both of our examples, we choose

$$C = 0.4 \max_{z_i} \|\vec{\phi}_{z_i}\|$$

where z_i is the sampling point.

For the first example the cavity D is an ellipsoid with x and y axis 4 and z axis 6. We choose Λ to be the unit sphere. The left picture of Fig. 2 shows the exact shape of D and the tetrahedra mesh we use for the interior scattering problem. We first set the wave number k to be 2. In Fig. 4, for better visualization, we show the contour plot of $1/\|\vec{\phi}\|$ on different planes for the ellipsoid. The dashed lines are the exact boundary of the cavity on the corresponding planes.

Then we consider other wave numbers. In Fig. 5 we show the iso-surface reconstructions of the ellipsoid for $k = 0.5$, $k = 1$, $k = 2$ and $k = 3$. It can be seen that for the wave numbers we considered, the reconstructions are similar.

The second example is a cube, centered at the origin, whose sides are 4. The right picture of Fig. 2 shows the exact shape of D and the tetrahedra mesh we use for the interior scattering problem. We first set the wave number k to be 2 and Λ to be the unit sphere. In Fig. 6, we show the contour plot of $1/\|\vec{\phi}\|$ on different planes for the cube. The dashed lines are the exact boundary of the cavity on the corresponding planes. Fig. 7 is the iso-surface reconstructions of the cube for different wave numbers. Similarly, the reconstructions are not very different from each other.

Finally, we consider the cases for different locations of the point sources and measurement. We fix the wave number $k = 2$. In addition to the unit sphere, we also choose spheres with radii $r = 1.2, 0.8, 0.6$ for Λ. We show the reconstructions of the cube in Fig. 8. Different Λ's do not change the results significantly, indicating the linear sampling method is rather stable with respect to source and measurement locations.

5. Conclusions and future work

In this paper, an interior inverse electromagnetic scattering problem for cavities is considered. We prove a reciprocity relation for the scattered field and a uniqueness theorem of the inverse problem. Then we employ the linear sampling method to reconstruct the shape of the cavity. Numerical examples are provided to show the viability of the method.

Similar to [19], the method can be extended to cavities with impedance boundary condition which is currently under our consideration. Due to the near field setting of

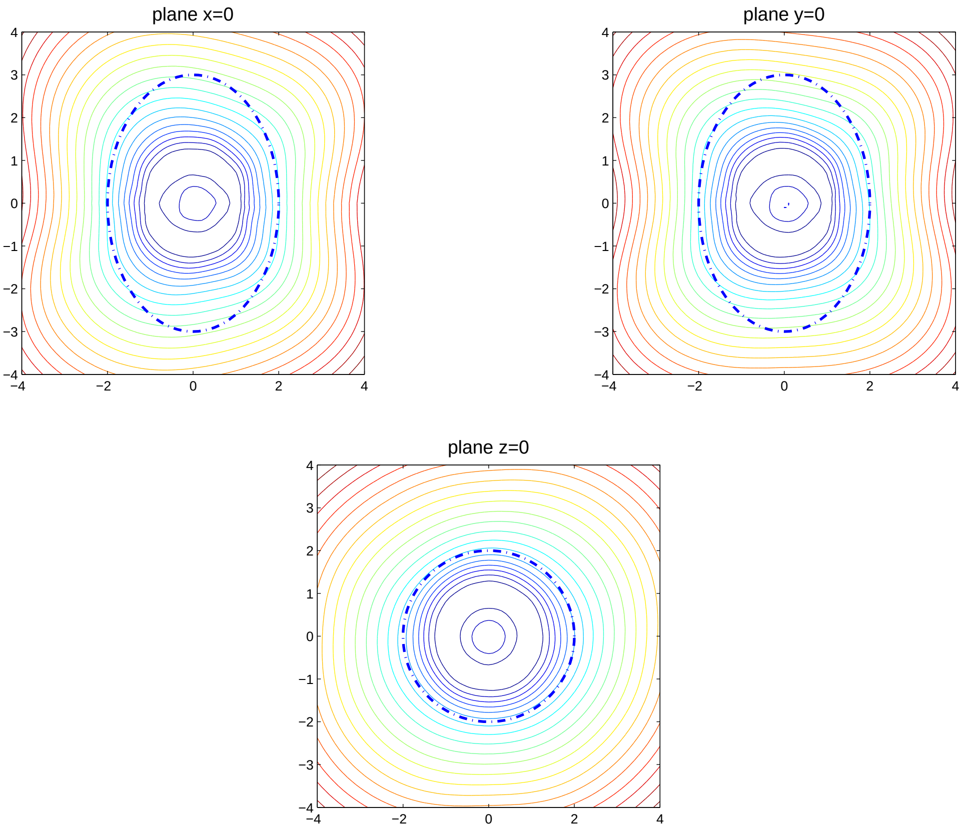

Figure 4. The contour plot of $1/\|\vec{\phi}\|$ on different planes for the ellipsoid when $k = 2$. The dashed lines are the exact boundary of the cavity on the corresponding planes.

the problem, another qualitative method, the reciprocity gap method should work as well. We refer the readers to [6, 10, 4, 11, 16] for the details and applications of the reciprocity gap method.

It can be seen that the reconstruction is not as satisfactory as the results of the linear sampling method for exterior inverse scattering problems. How to refine the method to obtain a better reconstruction is another interesting research topic worthy of effort.

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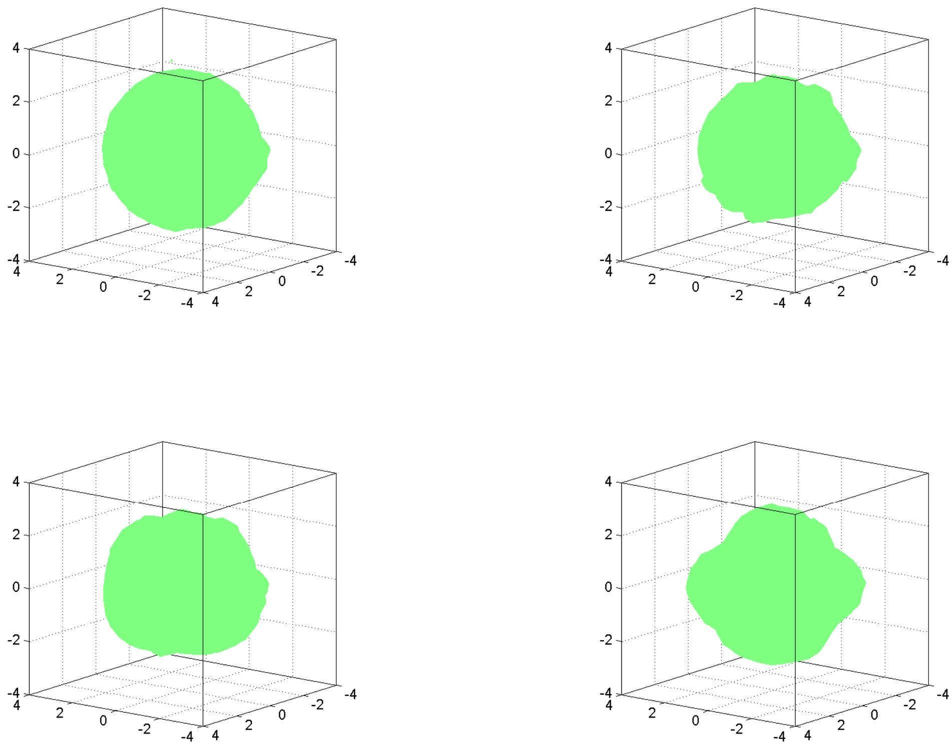

Figure 5. The reconstruction of the ellipsoid, i.e. the iso-surface of $\|\vec{\phi}\| = 0.4\max_{z_i}\|\vec{\phi}_{z_i}\|$ for different wave numbers. Upper left: $k = 0.5$, Upper right: $k = 1$, Lower left: $k = 2$, Lower right: $k = 3$.

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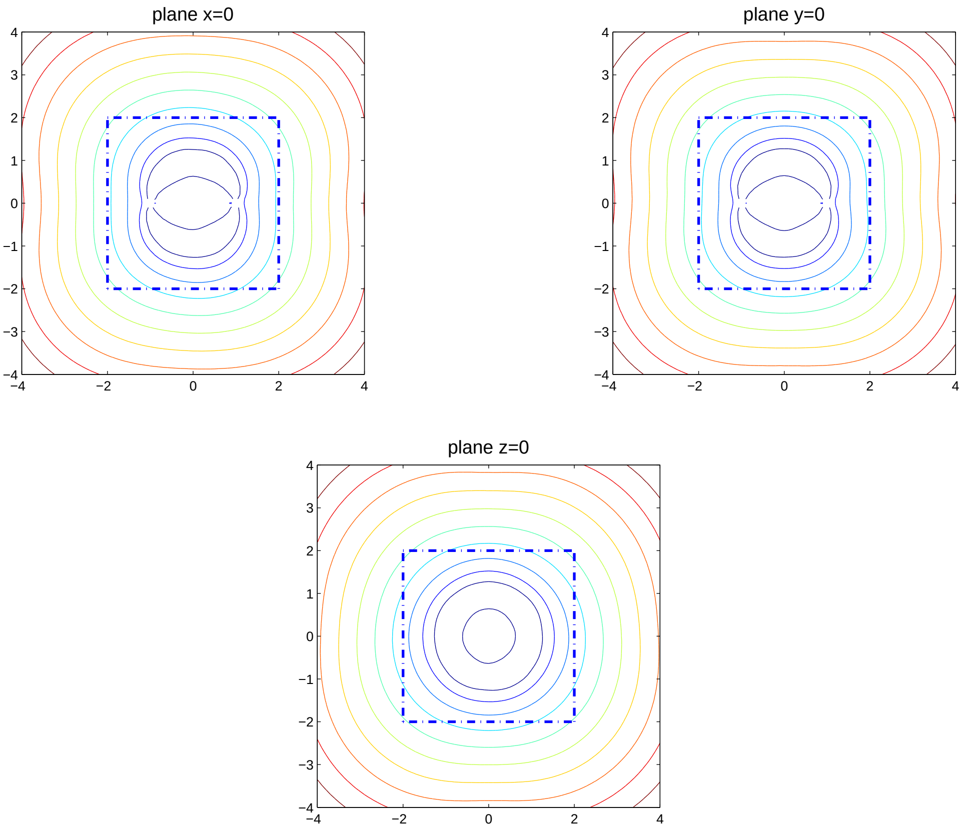

Figure 6. The contour plot of $1/\|\vec{\phi}\|$ on different planes for the cube. The dashed lines are the exact boundary of the cavity on the corresponding planes.

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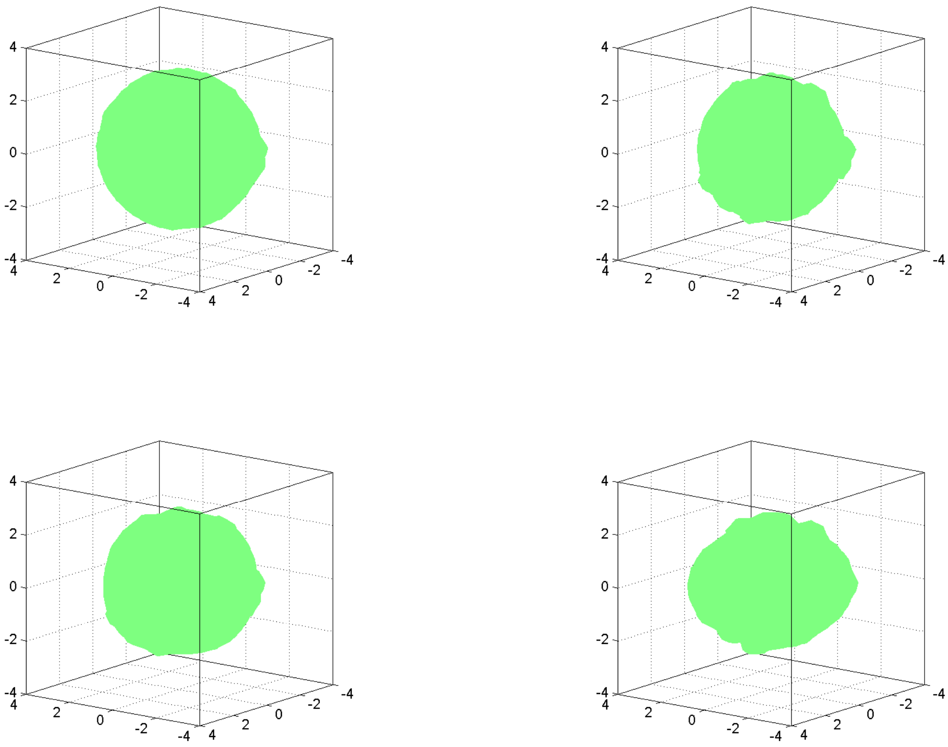

Figure 7. The reconstruction of the cube, i.e. the iso-surface of $\|\vec{\phi}\| = 0.4 \max_{z_i} \|\vec{\phi}_{z_i}\|$ for different wave numbers. Upper left: $k = 0.5$, Upper right: $k = 1$, Lower left: $k = 2$, Lower right: $k = 3$.

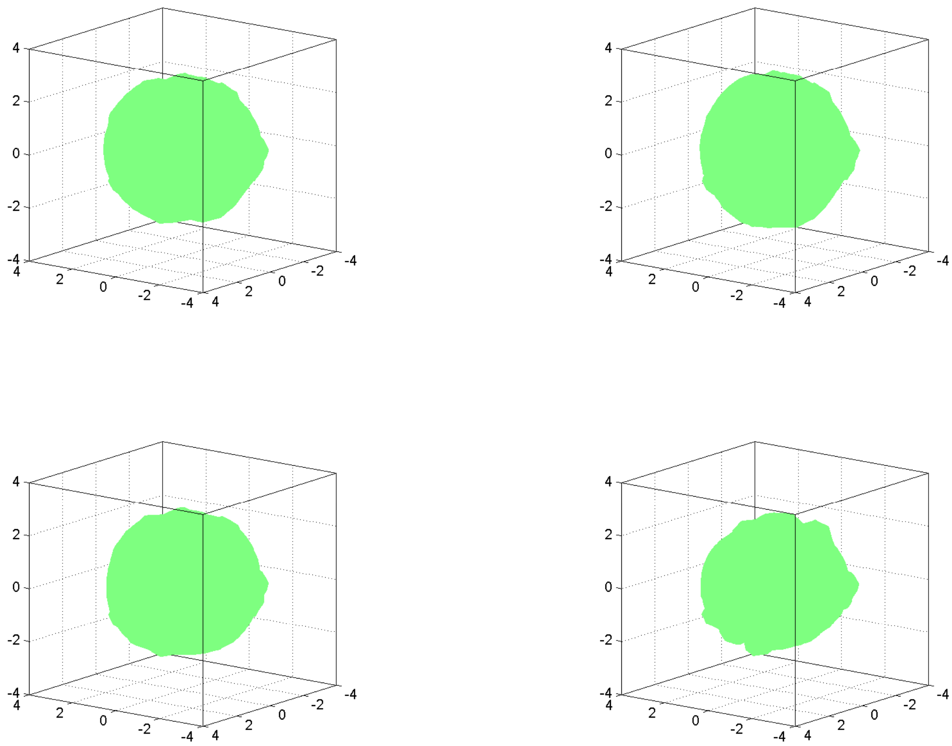

Figure 8. The reconstruction of the cube, i.e. the iso-surface of $\|\vec{\phi}\| = 0.4\max_{z_i}\|\vec{\phi}_{z_i}\|$ for different Λ's ($k = 2$). Upper left: $\Lambda := \{x, |x| = 0.6\}$. Upper right: $\Lambda := \{x, |x| = 0.8\}$. Lower left: $\Lambda := \{x, |x| = 1.0\}$. Lower right: $\Lambda := \{x, |x| = 1.2\}$.