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ASYMPTOTIC EXPANSIONS OF TRANSMISSION EIGENVALUES FOR SMALL PERTURBATIONS OF MEDIA WITH GENERALLY SIGNED CONTRAST

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Abstract. In this paper we revisit the transmission eigenvalue problem for an inhomogeneous media of compact support perturbed by small penetrable homogeneous inclusions. Assuming that the inhomogeneous background media is known and smooth, we investigate how these small volume inclusions affect the transmission eigenvalues. Our perturbation analysis makes use of the formulation of the transmission eigenvalue problem introduced Kirsch in [8], which requires that the contrast of the inhomogeneity is of one-sign *only* near the boundary. Thus, our approach can handle small perturbations with positive, negative or zero (voids) contrasts. In addition to proving the convergence rate for the eigenvalues corresponding to the perturbed media as inclusions' volume goes to zero, we also provide the explicit first correction term in the asymptotic expansion for simple eigenvalues. The correction term involves computable information about the known inhomogeneity as well as the location, size and refractive index of small perturbations. Our asymptotic formula has the potential to be used to recover information about small inclusions from knowledge of the real transmission eigenvalues, which can be determined from scattering data.

1. Introduction.

The transmission eigenvalue problem is intrinsic to the scattering theory for inhomogeneous media [2]. Real transmission eigenvalues are related to non-scattering frequencies and can be determined from scattering data [1] and [9], hence they can be used to obtain information about the inhomogeneity (see e.g. [3] for monotonicity results on real transmission eigenvalues in terms of the refractive index in the media). One possible application is to identify small volume perturbations of a known inhomogeneity using measured transmission eigenvalues. In this case, asymptotic analysis is needed to quantify the effect of small perturbations on transmission eigenvalues. This task is complicated due to the fact that the transmission eigenvalue problem is non-selfadjoint and most of its mathematical formulations lead to nonlinear eigenvalue problems. The celebrated paper by Osborn [12], generalized to nonlinear problems in [11], provides a mathematical approach to obtain asymptotic formulas with first order correction term for perturbation of eigenvalues of a non-selfadjoint eigenvalue problem. This perturbation

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approach has been used in [5] and [6] to obtain asymptotic expressions for transmission eigenvalues for the isotropic case and in [4] for the anisotropic case, where in the latter preliminary results on the use of these asymptotic formulas to solve the inverse problem have been presented. Unfortunately, bounded by the mathematical formulation of the transmission eigenvalue problem, all the aforementioned work required that the contrast in the known inhomogeneous media as well as in the small perturbations does not change sign in the support of the inhomogeneity. This assumption significantly restrict the class of problems where our method can apply. More recent papers on the transmission eigenvalue problem [16], [13], [14] have obtained spectral results under the assumption that the contrast keep the same sign only in a neighborhood of the boundary. Under this assumption, the formulation introduced by Kirsch [8], which is a variational writing of the transmission eigenvalue problem formulation first introduced by Sylvester [16], provides a conducive framework to apply perturbation analysis in [12], [11], and the goal of this paper is to do exactly this. More specifically, the main result of our paper is obtaining convergence and asymptotic formulas with correction term for the transmission eigenvalues corresponding to isotropic inhomogeneous media of compact support perturbed by small penetrable homogeneous inclusions. The only assumption on the media, besides physical ones, is that the contrast in the refractive index of the known inhomogeneity is one sign only in a neighborhood of the boundary of its support. This allows for the known inhomogeneity to have more general contrast inside the support, as well as for the small volume perturbations to have positive, negative or zero (voids) constant contrasts simultaneously. Although the calculations are presented here for real transmission eigenvalues and real valued refractive index (motivated by the practical fact that only real transmission eigenvalues are determined from scattering data) our analysis can be carried through for complex eigenvalues and complex-valued refractive index with obvious modifications when computing the adjoint operators. We remark that, although the convergence analysis can be done for multiple eigenvalues, our asymptotic formula is obtained only for simple eigenvalues due to technical difficulties stemming from the non-linearity of the eigenvalue problem as pointed out in [11].

2. **Formulation of the transmission eigenvalue problem.**

In this section we state a formulation of the transmission eigenvalue problem by Kirsch [8], which gave a variational formulation for the transmission eigenvalue problem formulation first introduced by Sylvester [16]. This formulation has the advantage of allowing for a much more general class of contrasts in the media. Here we translate Kirsch's variational version back to an operator form, which we find convenient to work with.

Let $D \subset \mathbb{R}^d$ ($d = 2, 3$) be a given C^2 domain with coefficient (e.g. squared index of refraction) given by $(1 + q) \in L^\infty(D)$. Consider the transmission eigenvalue problem of finding k^2 and nontrivial V, W such that

$$\Delta V + k^2(1 + q)V = 0 \qquad \text{in } D \tag{1}$$

$$\Delta W + k^2 W = 0 \qquad \text{in } D \tag{2}$$

$$W = V \qquad \text{on } \partial D \tag{3}$$

$$\frac{\partial W}{\partial \nu} = \frac{\partial V}{\partial \nu} \qquad \text{on } \partial D. \tag{4}$$

If one subtracts the second equation from the first and sets $\lambda := -k^2$, $v := V - W$ and $w := -k^2 W$, the transmission eigenvalue problem may be written: find $\lambda \in \mathbb{C}$

such that there exists nontrivial $w \in L^2(D)$ and $v \in H_0^2(D)$ satisfying

$$\Delta v - \lambda(1+q)v = qw \qquad \text{in } D \tag{5}$$

$$\Delta w - \lambda w = 0 \qquad \text{in } D. \tag{6}$$

We define the Hilbert space $X := L^2(D) \times H_0^2(D)$ equipped with the inner product

$$(w, v; \phi, \psi)_X := (w, \phi)_{L^2(D)} + (\Delta v, \Delta\psi)_{L^2(D)} \text{ for } (w, v), (\phi, \psi) \in X. \tag{7}$$

One may check that a weak formulation of (5)-(6) on X is to find $\lambda \in \mathbb{C}$ and nontrivial $(w, v) \in X$ such that (see e.g. [8])

$$\int_D (\Delta\overline{\psi} - \lambda\overline{\psi})w \,\mathrm{d}x + \int_D (\Delta v - \lambda(1+q)v)\overline{\phi} - qw\overline{\phi} \,\mathrm{d}x = 0 \tag{8}$$

for all $(\phi, \psi) \in X$. Now we define the linear operator on X

$$A_\lambda = \begin{pmatrix} -q & \Delta - \lambda(1+q) \\ (\Delta\Delta)^{-1}(\Delta - \lambda) & 0 \end{pmatrix}. \tag{9}$$

Here $(\Delta\Delta)^{-1} : H^{-2}(D) \to H^2(D)$ is defined by the weak solution of the equation

$$\Delta\Delta u = f, \quad u \in H_0^2(D) \text{ and } f \in H^{-2}(D), \tag{10}$$

that is, u solves

$$(u, \phi)_{H_0^2(D)} = (f, u)_{L^2(D)}, \quad \text{for all } \phi \in H_0^2(D). \tag{11}$$

Similarly, $\Delta : L^2(D) \to H^{-2}(D)$ is defined by

$$(u, \Delta\phi)_{L^2(D)} = \langle \Delta u, \phi \rangle_{H^{-2}(D)}, \quad \phi \in H_0^2(D). \tag{12}$$

We abuse notation slightly and also use Δ to denote the usual mapping $\Delta : H^2(D) \to L^2(D)$. We will also use the sesquilinear form corresponding to A_λ (introduced in [8]) defined by

$$a_\lambda(w, v; \phi, \psi) = (A_\lambda(w, v); (\phi, \psi))_X \text{ for all } (w, v), (\phi, \psi) \in X. \tag{13}$$

Clearly the operator A_λ given by (9) is bounded on X. Furthermore, a straightforward calculation shows that $(w, v) \in Ker(A_\lambda)$ if and only if (w, v) is a solution to the weak formulation of the transmission eigenvalue problem (8). So the transmission eigenvalue problem can be written as: Find $\lambda \in \mathbb{C}$ and $U = (w, v) \in X = L^2(D) \times H_0^2(D)$ such that

$$A_\lambda U = 0.$$

3. Decomposition into invertible plus compact.

In [16] and afterwards in [8] it was shown with some restrictions on q (near the boundary only) that the set of transmission eigenvalues is discrete. We redo the result from [8] here both for completeness and to show uniformity for families of q. Additionally, the operator form (9) simplifies some of the proofs.

Note that A_λ is not self adjoint even for real λ; however, it would be if we were to shift it slightly. To this end define the bounded operator on X

$$\hat{A}_\lambda = \begin{pmatrix} -q & \Delta - \lambda \\ (\Delta\Delta)^{-1}(\Delta - \lambda) & 0 \end{pmatrix}, \tag{14}$$

and its associated sesquilinear form

$$\hat{a}_\lambda(w, v; \phi, \psi) = \left(\hat{A}_\lambda(w, v); (\phi, \psi)\right)_X. \tag{15}$$

Note that now for $\lambda \in \mathbb{R}$, $\hat{A}_\lambda$ is self-adjoint with respect to the inner product on X. We also define the operator

$$K_{\mu,\lambda} = A_\mu - \hat{A}_\lambda = \begin{pmatrix} 0 & \lambda - \mu(1+q) \\ (\Delta\Delta)^{-1}(\lambda - \mu) & 0 \end{pmatrix}, \tag{16}$$

noting that

$$A_\mu = \hat{A}_\lambda + K_{\mu,\lambda}.$$

We would like to find λ such that $\hat{A}_\lambda$ is invertible and $K_{\mu,\lambda}$ is compact. The second is the easier of the two.

Proposition 3.1. *For any $\lambda, \mu \in \mathbb{C}$ and $q \in L^\infty(D)$, $K_{\mu,\lambda}$ is compact.*

Proof. This is obvious; the top right component of $K_{\mu,\lambda}$ is multiplication by $(\lambda - \mu(1+q))$ as a mapping from $H_0^2(D)$ to $L^2(D)$, which is clearly compact by Sobolev embedding. The bottom left component is the mapping $(\Delta\Delta)^{-1}$ from $L^2(D)$ to $H_0^2(D)$. By standard elliptic regularity theory, $(\Delta\Delta)^{-1}$ takes $L^2(D)$ functions into $H^4(D)$, which embeds compactly into $H_0^2(D)$. □

We first prove a lemma that is essential for the next proposition.

Lemma 3.2. *Let $\lambda > 0$ and w_j be a sequence in $L^2(D)$ that weakly converges to 0. Then, there exists a sequence $z_j \in H^2(D)$ defined by*

$$\begin{cases} -\Delta z_j + \lambda z_j = w_j \text{ in } D \\ z_j = 0 \text{ on } \partial D \end{cases}$$

that converges to 0 weakly in $H^2(D)$.

Proof. The bilinear form on $H_0^1(D) \times H_0^1(D)$

$$a(z, \phi) := (\nabla z, \nabla \phi)_{L^2(D)} + \lambda (z, \phi)_{L^2(D)} \tag{17}$$

is coercive, so by Lax Milgram, the partial differential equation

$$\begin{cases} -\Delta z + \lambda z = g \text{ in } D \\ z = 0 \text{ on } \partial D \end{cases} \tag{18}$$

has a unique weak solution $z \in H^2(D) \cap H_0^1(D)$ for any $g \in L^2(D)$. By standard elliptic regularity, z satisfies the inequality

$$\|z\|_{H^2(D)} \leq C \|g\|_{L^2(D)}. \tag{19}$$

For each j, let $g = w_j$ and define z_j as the solution to (18). Since w_j converges weakly, it is norm bounded in $L^2(D)$, hence there exists an $M > 0$ such that

$$\|z_j\|_{H^2(D)} \leq M. \tag{20}$$

Since the sequence $\{z_j\}$ is bounded in $H^2(D)$, there exists a subsequence (we will again denote it by z_j) that converges to some $z \in H^2(D)$ weakly. From this weak convergence and the equation for z_j, for any $\phi \in H_0^1(D)$,

$$\begin{aligned} \int_D (-\Delta z + \lambda z)\overline{\phi}\, \mathrm{d}x &= \lim_{k\to\infty} \int_D (-\Delta z_j + \lambda z_j)\overline{\phi}\, \mathrm{d}x \\ &= \lim_{k\to\infty} \int_D w_j \overline{\phi}\, \mathrm{d}x \\ &= 0, \end{aligned}$$

which implies that z solves

$$-\Delta z + \lambda z = 0 \text{ in } D.$$

By taking another subsequence, we must have the convergence strong in H_0^1, which implies that $z = 0$ on ∂D, and hence we have $z = 0$. The result follows. $\square$

We now introduce a family of operators so we may state our final theorems of the section, and we do this by first defining the parameter family $\{q_\epsilon\} \subset L^\infty(D)$.

Assumption 3.3. *We assume the family $\{q_\epsilon\} \subset L^\infty(D)$ satisfies the following properties:*

1. *$\epsilon \in \mathcal{I}$ where $\mathcal{I}$ is a compact subset of $\mathbb{R}$.*
2. *The family $\{q_\epsilon\}$ is uniformly bounded in $L^\infty(D)$.*
3. *Let R be a neighborhood of the boundary ∂D; that is, R is an open set contained in D with $\partial D \subset \overline{R}$. We assume q_ϵ is of one sign on R with either $q_\epsilon > \alpha > 0$ or $-q_\epsilon > \alpha > 0$ holding in R.*
4. *The family has the property that for each $\epsilon, \epsilon' \in \mathbb{R}$, $q_\epsilon - q_{\epsilon'} = 0$ except on some measurable subset $D(\epsilon, \epsilon') \subset D \setminus \overline{R}$ satisfying $m(D(\epsilon, \epsilon')) \to 0$ as $\epsilon \to \epsilon'$ where m is the Lebesgue measure on $\mathbb{R}^d$.*

The following Lemma and Proposition are generalizations of Kirsch's results in [8]. The arguments are nearly identical to those of Kirsch, with the exception of a few modifications to allow for a family $\{q_\epsilon\}$.

Lemma 3.4 (Kirsch, [8], Lemma 2.3)**.** *Let $\{q_\epsilon\}$ be a family of functions satisfying Assumption 3.3. Then, there exists a $\lambda_0 > 0$ such that*

$$\int_{D\setminus R} |q_\epsilon||w|^2 \, \mathrm{d}x \leq \frac{1}{2} \int_R |q_\epsilon||w|^2 \mathrm{d}x$$

for all $\epsilon > 0$ and all $w \in L^2(D)$ solving $\Delta w - \lambda_0 w = 0$ in D.

Proof. Let $\lambda > 0$ and let R be the neighborhood of the boundary defined in Assumption 3.3. Let R' be another neighborhood of ∂D such that $\hat{d} := \mathrm{dist}(D \setminus R, R')$ is positive. This implies that $R' \subset R$. Define $\rho \in C_c^\infty(D)$ to satisfy $\rho = 1$ on $D \setminus R'$. For $w \in L^2(D)$ solving $\Delta w - \lambda w = 0$, from standard elliptic regularity we know that w is in $C_{loc}^\infty(D)$, so $\rho w \in C^\infty(D)$ and we can apply Green's representation theorem for $\Delta - \lambda$:

$$(\rho w)(x) = -\int_D [(\Delta(\rho w)(y) - \lambda(\rho w)(y)] \frac{e^{-\sqrt{\lambda}|x-y|}}{4\pi|x-y|} \mathrm{d}y. \tag{21}$$

The product rule yields

$$\Delta(\rho w) = w\Delta\rho + 2\nabla\rho \cdot \nabla w + \rho\Delta w, \tag{22}$$

which gives

$$(\rho w)(x) = -\int_D [w\Delta\rho + 2\nabla\rho \cdot \nabla w + \rho\Delta w - \lambda(\rho w)(y)] \frac{e^{-\sqrt{\lambda}|x-y|}}{4\pi|x-y|} \, \mathrm{d}y. \tag{23}$$

Note that $\rho\Delta w - \lambda(\rho w) = 0$ from the equation for w, and so we have

$$(\rho w)(x) = -\int_D [w\Delta\rho + 2\nabla\rho \cdot \nabla w] \frac{e^{-\sqrt{\lambda}|x-y|}}{4\pi|x-y|} \, \mathrm{d}y. \tag{24}$$

Since $\rho = 1$ on $D \setminus R'$, $\Delta\rho = \nabla\rho = 0$ on $D \setminus R'$ and so

$$(\rho w)(x) = -\int_{R'} [w\Delta\rho + 2\nabla\rho \cdot \nabla w] \frac{e^{-\sqrt{\lambda}|x-y|}}{4\pi|x-y|} \, \mathrm{d}y. \tag{25}$$

We integrate the second term by parts to find

$$(\rho w)(x) = \int_{R'} \left[-\Delta\rho \frac{e^{-\sqrt{\lambda}|x-y|}}{4\pi|x-y|} + 2\mathrm{div}_y \left(\nabla\rho \frac{e^{-\sqrt{\lambda}|x-y|}}{4\pi|x-y|} \right) \right] w \,\mathrm{d}y, \tag{26}$$

where the boundary term disappears since $\rho \in C_c^\infty(D)$ and $\nabla\rho = 0$ on $D\backslash R'$ implies that $\partial_{\nu^+}\rho = 0$ on $\partial R'$.

Now, letting $x \in D \setminus R'$, we have

$$|w(x)| \leq Ce^{-d\sqrt{\lambda}} \int_{R'} |w| \,\mathrm{d}y \tag{27}$$

where C depends only on D, R, R', and ρ. By Cauchy-Schwartz and $R' \subset R$, we deduce

$$\begin{aligned} |w(x)|^2 \leq \left(Ce^{-d\sqrt{\lambda}} \int_{R'} |w| \,\mathrm{d}y \right)^2 \leq & C^2 e^{-2d\sqrt{\lambda}} \left(\|\chi_{R'}\|_{L^2(R)} \|w\|_{L^2(R)} \right)^2 \\ \leq & C^2 e^{-2d\sqrt{\lambda}} |R'| \int_R |w(y)|^2 \,\mathrm{d}y. \end{aligned} \tag{28}$$

Since $q_\epsilon > \alpha$ on R, we have

$$|w(x)|^2 \leq C^2 e^{-2d\sqrt{\lambda}} |R'| \int_R \frac{|q_\epsilon(y)|}{\alpha} |w(y)|^2 \,\mathrm{d}y. \tag{29}$$

As $R' \subset R$, the above inequality holds for $x \in D \setminus R$. We multiply by $|q_\epsilon(x)|$ and integrating with respect to x over $D \setminus R$:

$$\int_{D\setminus R} |q_\epsilon(y)||w(y)|^2 \,\mathrm{d}y \leq C^2 e^{-2d\sqrt{\lambda}} |R'||D \setminus R| \|q_\epsilon\|^2_{L^\infty(D)} \int_R |w(y)|^2 \,\mathrm{d}y. \tag{30}$$

Since q_ϵ is uniformly bounded, we can choose a λ_0 large enough so that the result holds. □

This next proposition states that the family of operators $\{\hat{A}^\epsilon_{\lambda_0}\}$ is uniformly weakly coercive. Again, our arguments follow [8] with the only difference that here we make sure that all estimates hold uniformly in $\epsilon \in \mathcal{I}$.

Proposition 3.5 (Kirsch [8], Theorem 2.4)**.** *Let $\{q_\epsilon\}$ be a family of functions satisfying Assumption 3.3, and for each ϵ, let $\{\hat{A}^\epsilon_{\lambda_0}\}$ be the operator (14) corresponding to $q = q_\epsilon$ and $\lambda = \lambda_0$, and let $\hat{a}^\epsilon_{\lambda_0}$ denote their associated sesquilinear forms. Then there exists a $\lambda_0 > 0$ and a $c > 0$ such that for all $\epsilon \in \mathcal{I}$,*

$$\sup_{(\phi,\psi)\neq 0} \frac{\hat{a}^\epsilon_{\lambda_0}(w, v; \phi, \psi)}{\|(\phi,\psi)\|_X} \geq c\|(w,v)\|_X \quad \textit{for all } (w,v) \in X \tag{31}$$

where c is independent of ϵ.

Proof. Assume that the estimate (31) does not hold, in which case we have a sequence $(\epsilon_j, w_j, v_j) \in \mathbb{R} \times X$ where (w_j, v_j) has norm 1 in X and

$$\lim_{j\to\infty} \sup_{(\phi,\psi)\neq 0} \frac{\hat{a}^{\epsilon_j}_{\lambda_0}(w_j, v_j; \phi, \psi)}{\|(\phi,\psi)\|_X} \to 0. \tag{32}$$

Now, since U_j is of norm 1 and X is a Hilbert Space, it has a subsequence (denoted U_j for simplicity) that converges weakly to some $U \in X$. We will first prove that $U = 0$.

Since $\epsilon_j \in \mathcal{I}$, there is a subsequence (which we abuse notation and denote $\{\epsilon_j\}$) such that $\epsilon_j \to \epsilon^*$ as $j \to \infty$. From Lemma 3.4 there exists a λ_0 such that

$$\int_{D\setminus R} |q_\epsilon||w|^2 \,\mathrm{d}x \le \frac{1}{2}\int_R |q_\epsilon||w|^2 \mathrm{d}x \tag{33}$$

for all $\epsilon \in \mathcal{I}$ and all w solving $\Delta w - \lambda_0 w = 0$. Define $U_j = (w_j, v_j)$ and $\Phi = (\phi, \psi)$. Then,

$$\begin{aligned}\frac{((\hat{A}^{\epsilon_j}_{\lambda_0} - \hat{A}^{\epsilon^*}_{\lambda_0})U_j, \Phi)_X}{\|\Phi\|_X} =& \frac{(U_j, (\hat{A}^{\epsilon_j}_{\lambda_0} - \hat{A}^{\epsilon^*}_{\lambda_0})\Phi)_X}{\|\Phi\|_X} \\ \le& \frac{\|U_j\|_X}{\|\Phi\|_X}\|(\hat{A}^{\epsilon_j}_{\lambda_0} - \hat{A}^{\epsilon^*}_{\lambda_0})\Phi\|_X \end{aligned}\tag{34}$$

since the operators are self adjoint. We claim

$$\|(\hat{A}^{\epsilon_j}_{\lambda_0} - \hat{A}^{\epsilon^*}_{\lambda_0})\Phi\|_X \to 0. \tag{35}$$

Indeed, for $\Psi = (\phi_1, \psi_1) \in X$,

$$((\hat{A}^{\epsilon_j}_{\lambda_0} - \hat{A}^{\epsilon^*}_{\lambda_0})\Phi, \Psi)_X = \int_D (q_{\epsilon_j} - q_{\epsilon^*})\phi\overline{\phi_1}\,\mathrm{d}x \tag{36}$$

$$= \int_{D(\epsilon_j,\epsilon^*)} (q_1 - q_0)\phi\overline{\phi_1}\,\mathrm{d}x \tag{37}$$

where $D(\epsilon_j, \epsilon^*)$ is the set defined in Assumption 3.3. Thus,

$$((\hat{A}^{\epsilon_j}_{\lambda_0} - \hat{A}^{\epsilon^*}_{\lambda_0})\Phi, \Psi)_X \le \|q_1 - q_0\|_{L^\infty D}\|\phi\|_{L^2(D(\epsilon_j,\epsilon^*))}\|\Psi\|_X \to 0 \tag{38}$$

by the dominated convergence theorem. This proves assertion (35) after taking the supremum over all $\Psi \in X$. Combining (34) and (35) yields

$$\frac{(\hat{A}^{\epsilon_j}_{\lambda_0} U_j, \Phi)_X}{\|\Phi\|_X} = \frac{(\hat{A}^{\epsilon^*}_{\lambda_0} U_j, \Phi)_X}{\|\Phi\|_X} + o(1). \tag{39}$$

Next, by (32), for any $\Phi \in X$, we have

$$\lim_{j\to\infty} \frac{\hat{a}^{\epsilon_j}_{\lambda_0}(U_j, \Phi)_X}{\|\Phi\|_X} \to 0, \tag{40}$$

and therefore, (39) becomes

$$\frac{(\hat{A}^{\epsilon^*}_{\lambda_0} U_j, \Phi)_X}{\|\Phi\|_X} \to 0. \tag{41}$$

Recall U_j converges weakly to some $U \in X$. Therefore, for any $\Phi \in X$,

$$\lim_{j\to\infty} (\hat{A}^{\epsilon^*}_{\lambda_0} U_j, \Phi)_X = (\hat{A}^{\epsilon^*}_{\lambda_0} U, \Phi)_X. \tag{42}$$

Then, (42) and (41) gives

$$(\hat{A}^{\epsilon^*}_{\lambda_0} U, \Phi)_X = 0 \tag{43}$$

for any $\Phi \in X$. We will show that this implies $U = 0$. Choosing $\Phi = (-w, v)$, (43) implies

$$\int_D -(\Delta\overline{v} - \lambda_0\overline{v})w + (\Delta v - \lambda_0 v)\overline{w} - q_{\epsilon^*}|w|^2\,\mathrm{d}x = 0. \tag{44}$$

By taking the real part, we have

$$\int_D q_{\epsilon^*}|w|^2\,\mathrm{d}x = 0. \tag{45}$$

Since q_{ϵ^*} is of one sign on R,

$$\int_R |q_{\epsilon^*}||w|^2\mathrm{d}x = \left|\int_R q_{\epsilon^*}|w|^2\,\mathrm{d}x\right|, \tag{46}$$

and from (45) and Lemma 3.4,

$$\left|\int_R q_{\epsilon^*}|w|^2\,\mathrm{d}x\right| = \left|\int_{D\setminus R} q_{\epsilon^*}|w|^2\,\mathrm{d}x\right| \le \left|\int_{D\setminus R} |q_{\epsilon^*}||w|^2\,\mathrm{d}x\right| \le \frac{1}{2}\int_R |q_{\epsilon^*}||w|^2\mathrm{d}x. \tag{47}$$

Thus, (46) and the above imply $w = 0$ on R because

$$\int_R |q_{\epsilon^*}||w|^2\mathrm{d}x \le \frac{1}{2}\int_R |q_{\epsilon^*}||w|^2\mathrm{d}x \tag{48}$$

and $q_{\epsilon^*} > \alpha$ on R. Since w solves $\Delta w - \lambda_0 w = 0$ in D, w is analytic by an extension of Weyl's theorem (see Corollary 11.4.13 [7]). We may then use analytic continuation to conclude $w = 0$ on D. This also implies $v = 0$, since by choosing $\Phi = (v, 0)$ and substituting Φ into (43),

$$\int_D (\Delta v - \lambda_0 v)\overline{v}\,\mathrm{d}x = -\int_D |\nabla v|^2 + \lambda|v|^2\,\mathrm{d}x = 0. \tag{49}$$

Assuming (31), we will show $\|U_j\|_X \to 0$, which contradicts that each U_j has norm 1. Recall that in our case the bilinear form is not fixed.

Let R' be a neighborhood of R so that $\overline{R'} \subset R \cup \partial D$. We take a non-negative $\rho_1 \in C^\infty(D)$ defined to be

$$\rho_1(x) = \begin{cases} 1 & x \in R' \\ 0 & x \in D \setminus R \end{cases} \tag{50}$$

and construct the sequence $\Phi_j = (-\rho_1 w_j, \rho_1 v_j) \in X$. Of course, due to (32) we have that

$$\frac{\hat{a}^{\epsilon_j}_{\lambda_0}(U_j, \Phi_j)}{\|\Phi_j\|_X} \to 0. \tag{51}$$

More explicitly,

$$\frac{1}{\|\Phi_j\|_X}\int_D (\Delta(\rho_1\overline{v_j}) - \lambda_0\rho_1\overline{v_j})w_j - (\Delta v_j - \lambda_0 v_j)\rho_1\overline{w_j} - q_\epsilon\rho_1|w_j|^2 dx \to 0. \tag{52}$$

Given the support of ρ_1, we have

$$\frac{1}{\|\Phi_j\|_X}\int_R (\Delta(\rho_1\overline{v_j}) - \lambda_0\rho_1\overline{v_j})w_j - (\Delta v_j - \lambda_0 v_j)\rho_1\overline{w_j} - q_\epsilon\rho_1|w_j|^2 dx \to 0. \tag{53}$$

From the product rule,

$$\Delta(\rho_1\overline{v_j}) = \overline{v_j}\Delta\rho_1 + 2\nabla\rho_1\cdot\nabla\overline{v_j} + \rho_1\Delta\overline{v_j}. \tag{54}$$

We substitute this into (53) and take the real part which yields

$$\mathrm{Re}\frac{1}{\|\Phi_j\|_X}\int_R \overline{v_j}w_j\Delta\rho_1 + 2w_j\nabla\rho_1\cdot\nabla\overline{v_j} - q_\epsilon\rho_1|w_j|^2 dx \to 0. \tag{55}$$

Since $\|\Phi_j\|_X \le \|U_j\|_X = 1$ by construction, we must have

$$\mathrm{Re}\int_R \overline{v_j}w_j\Delta\rho_1 + 2w_j\nabla\rho_1\cdot\nabla\overline{v_j} - q_\epsilon\rho_1|w_j|^2 dx \to 0 \tag{56}$$

since $U_j \rightharpoonup 0$, $v_j \rightharpoonup 0$ in $H^2(D)$. Since $H^2(D)$ is compactly embedded in $H^1(D)$, we have v_j converges strongly to 0 in $H^1(D)$, since the inclusion operator is compact and maps weakly convergent sequences to strongly convergent sequences. Therefore, the first two terms in (56) go to 0, implying

$$\int_R q_\epsilon \rho_1 |w_j|^2 dx \to 0. \tag{57}$$

Using the assumption that $q_\epsilon > \alpha > 0$ on R and $\rho_1 = 1$ on R', the previous line implies $w_j \to 0$ in $L^2(R')$.

We again define a neighborhood R'' of ∂D such that its closure is in $R' \cup \partial D$. For this neighborhood, we define a non-negative $\rho_2 \in C^\infty(D)$ such that

$$\rho_2(x) = \begin{cases} 0 & x \in R'' \\ 1 & x \in D \setminus R'. \end{cases} \tag{58}$$

Furthermore, we construct a sequence $\{z_j\} \subset H^2(D)$ which for each j solves

$$\begin{cases} \Delta z_j - \lambda_0 z_j & = w_j \text{ in } D \\ z_j & = 0 \text{ on } \partial D. \end{cases} \tag{59}$$

By choosing $\Phi'_j = (0, \rho_2 z_j)$, we may conclude from (32) that

$$\frac{1}{\|\Phi'_j\|_X} \int_D (\Delta(\rho_2 \overline{z_j}) - \lambda_0 \rho_2 \overline{z_j}) w_j \mathrm{d}x \to 0. \tag{60}$$

We use the product rule to obtain

$$\Delta(\rho_2 \overline{z_j}) = \overline{z_j} \Delta \rho_2 + 2\nabla \rho_2 \cdot \nabla \overline{z_j} + \rho_2 \Delta \overline{z_j}, \tag{61}$$

so that

$$\begin{aligned} \int_D (\Delta(\rho_2 \overline{z_j}) - \lambda_0 \rho_2 \overline{z_j}) w_j \mathrm{d}x \\ &= \int_D (\overline{z_j} \Delta \rho_2 + 2\nabla \rho_2 \cdot \nabla \overline{z_j} + \rho_2 \Delta \overline{z_j}, -\lambda_0 \rho_2 \overline{z_j}) w_j \mathrm{d}x \\ &= \int_D (\overline{z_j} \Delta \rho_2 + 2\nabla \rho_2 \cdot \nabla \overline{z_j} + \rho_2 \overline{w_j}) w_j \mathrm{d}x, \end{aligned} \tag{62}$$

using that z_j solves (59). From Lemma 3.2, $z_j \rightharpoonup 0$ in $H^2(D)$, and therefore $z_j \to 0$ strongly $H^1(D)$. Therefore,

$$\frac{1}{\|\Phi'_j\|_x} \int_D (\overline{z_j} \Delta \rho_2 + 2\nabla \rho_2 \cdot \nabla \overline{z_j}) w_j \, \mathrm{d}x \to 0, \tag{63}$$

from which (62) and (60) imply

$$\int_D \rho_2 |w_j|^2 \mathrm{d}x \to 0. \tag{64}$$

From the definition of ρ_2 this implies that $w_j \to 0$ in $L^2(D \setminus R')$. Thus, we have that $w_j \to 0$ in $L^2(D)$.

Finally, we can show $v_j \to 0$ in $H^2(D)$. Take $\Phi'' = (\Delta v_j - \lambda_0 v_j, 0) \in X$. We then have by (32),

$$\frac{1}{\|\Phi''_j\|_X} \int_D |\Delta v_j - \lambda_0 v_j|^2 - q_{\epsilon_j} w_j (\Delta \overline{v_j} - \lambda_0 \overline{v_j}) \, \mathrm{d}x \to 0. \tag{65}$$

Of course,

$$(66)\quad \frac{1}{\|\Phi_j''\|_X}\int_D q_{\epsilon_j} w_j(\Delta\overline{v_j}-\lambda_0\overline{v_j})\,\mathrm{d}x \leq \frac{1}{\|\Delta v_j-\lambda_0 v_j\|_D}\|q_\epsilon\|_{L^\infty(D)}\|w_j\|_{L^2(D)}\|\Delta v_j-\lambda_0 v_j\|_{L^2(D)}\to 0$$

since $w_j \to 0$ in $L^2(D)$ and q_{ϵ_j} is bounded uniformly in ϵ_j. From this, (65) implies

$$(67)\qquad \|\Delta v_j-\lambda_0 v_j\|_D \to 0.$$

Since $v_j \rightharpoonup 0$ in $H^2(D)$, it converges strongly in $L^2(D)$, which implies from the previous line that $\Delta v_j \to 0$ in $L^2(D)$. Therefore, $v_j \to 0$ in $H^2(D)$. This proves that $\|U_j\|_X \to 0$, which is impossible as $\|U_j\|_X = 1$ for all j. □

Corollary 3.6. *Let $\hat{A}^\epsilon_\lambda$ be defined by (14) and the family q_ϵ satisfy Assumption 3.3. Then there exists a $\lambda_0 > 0$ such that the operator $\hat{A}^\epsilon_{\lambda_0}$ is invertible for every $\epsilon \in \mathcal{I}$, and the inverse operator is bounded uniformly with respect to $\epsilon \in \mathcal{I}$.*

Proof. Choose λ_0 as in the statement of Proposition 3.5. Then, the result of Proposition 3.5 is sufficient to apply a generalized version of Lax Milgram (see Theorem 2.22 in [10]) which yields the result. □

We end the section by combining the previous results to show following theorem, which was first shown in [16].

Theorem 3.7. *For a fixed q satisfying Assumption 3.3, the set of transmission eigenvalues is discrete in $\mathbb{C}$ without any finite accumulation point.*

Proof. We have shown that the transmission eigenvalue problem defined by (8) may be written as finding a $k > 0$ such that

$$(68)\qquad A_{-k^2} = \hat{A}_{\lambda_0} + K_{-k^2,\lambda_0}$$

has a non trivial kernel, where $\hat{A}_{\lambda_0}$ is given by (14) and K_{-k^2,λ_0} is given by (16). From Corollary 3.6, we can select λ_0 as so that $\hat{A}_{\lambda_0}$ is invertible. Since $k^2 \to K_{-k^2,\lambda_0}$ is analytic and K is compact, the result follows from the analytic Fredholm theorem. □

Remark 3.8. For a fixed $q \in C^\infty(\overline{D})$ satisfying Assumption 3.3, the existence of an infinite set of transmission eigenvalues as well as completeness of generalized eigenfunctions are proven in [14]. In this case results on the counting function for transmission eigenvalues can be found in [13], [15]. However, techniques from semiclassical analysis used in these papers restrict the existence results to smooth q. For L^∞ contrast q, but under the assumption that q is one sign in D, the existence of an infinite set of real transmission eigenvalues is shown in [3].

4. **Transmission eigenvalue problem in the presence of small volume inhomogeneities.** We now introduce the class of inhomogeneities studied in the remainder of this paper. We present here our convergence analysis and asymptotic formulas for real transmission eigenvalues. This is motivated by the fact that only real transmission eigenvalues can be determined from the scattering data (see e.g. [2]). However our results can be formulated and proven exactly in the same way (with obvious modifications when writing the adjoint operators) for complex

transmission eigenvalues. Hence from now on we set $\tau := -k^2$ and assume that $\tau \in \mathbb{R}$.

For simplicity of exposition, we first consider the case of a single inhomogeneity centered at the origin. Let $q_0 \in C^\infty(D)$ be such that q_0 satisfies

$$q_0(x) > \alpha > 0 \text{ in } R, \tag{69}$$

a neighborhood of the boundary of D as defined in Assumption 3.3. (One could equivalently assume q_0 is negative on R.) Let now B be any open set containing the origin. Consider $\epsilon > 0$, and assume that

$$q_\epsilon = \begin{cases} q_0 & x \in D \setminus (\epsilon B) \\ q_1 & x \in \epsilon B \end{cases} \tag{70}$$

where $q_1 \in \mathbb{R}$. We will assume ϵ is small enough such that $\epsilon B \subset D \setminus \overline{R}$. It is straightforward to check that this family of q_ϵ satisfy the conditions in Assumption 3.3.

Assume that λ_0 is not a transmission eigenvalue and is such that the coercivity in Proposition 3.5 holds. We define for $\lambda \geq \lambda_0$,

$$\mathbb{A}_\epsilon := \hat{A}_\lambda^\epsilon = \begin{pmatrix} -q_\epsilon & \Delta - \lambda \\ (\Delta\Delta)^{-1}(\Delta - \lambda) & 0 \end{pmatrix}, \tag{71}$$

and

$$\mathbb{K}_\epsilon(\tau) := K_{\tau,\lambda}^\epsilon = \begin{pmatrix} 0 & \lambda - \tau(1 + q_\epsilon) \\ (\Delta\Delta)^{-1}(\lambda - \tau) & 0 \end{pmatrix}. \tag{72}$$

We further note that $\mathbb{A}_\epsilon$ is self adjoint with respect to the inner product on X, but $\mathbb{K}_\epsilon(\tau)$ is not. Its adjoint is given by

$$\mathbb{K}_\epsilon^* = \begin{pmatrix} 0 & \lambda - \tau \\ (\Delta\Delta)^{-1}((\lambda - \tau) - \tau q_\epsilon) & 0 \end{pmatrix}. \tag{73}$$

In what follows we will use * to denote the adjoint with respect to the inner product on X. Define

$$T_\epsilon(\tau) := -\frac{1}{\tau}\mathbb{A}_\epsilon^{-1}\mathbb{K}_\epsilon(\tau). \tag{74}$$

Then, the transmission eigenvalue problem with scatterer q_ϵ may now be written as finding τ_ϵ such that for $U = (w, v) \in X$,

$$\tau_\epsilon T_\epsilon(\tau_\epsilon)U = U \tag{75}$$

has a nontrivial solution.

5. **Preliminary estimates.**

We now will prove several lemmas about the operators $\mathbb{A}_\epsilon$ and $\mathbb{K}_\epsilon$, which we will later need to apply the perturbation theory.

Lemma 5.1. *Let $\tau \in \mathbb{R}$ and $\mathbb{K}_\epsilon$ be defined by (71). Then we have the operator norm estimate*

$$\|\mathbb{K}_\epsilon(\tau) - \mathbb{K}_0(\tau)\|_{\mathcal{L}(X)} \leq C|\tau|\epsilon^{d/2}.$$

Proof. Let $U := (w, v), \Phi := (\phi, \psi) \in X$. Then,

$$\begin{aligned}((\mathbb{K}_\epsilon(\tau) - \mathbb{K}_0(\tau))U, \Phi)_X =& - \tau \int_D (q_\epsilon - q_0) v\overline{\phi} \,\mathrm{d}x \\ =& - \tau \int_{\epsilon B} (q_1 - q_0) v\overline{\phi} \,\mathrm{d}x \\ \leq& |\tau| \|q_1 - q_0\|_{L^\infty(D)} \|v\|_{L^\infty(D)} \int_D \chi_{\epsilon B} |\phi| \,\mathrm{d}x \\ \leq& C|\tau| \|v\|_{H_0^2(D)} \|\chi_{\epsilon B}\|_{L^2(D)} \|\phi\|_{L^2(D)} \\ \leq& C|\tau|\epsilon^{d/2} \|U\|_X \|\Phi\|_X \end{aligned} \tag{76}$$

where we used Sobolev embedding and the definition of X. □

Lemma 5.2. *Let $\mathbb{A}_\epsilon$ and R be defined by (71) and (69) respectively. Choose $D' \subset D \setminus \overline{R}$ to contain ϵB for all $\epsilon \in \mathcal{I}$. Then, for $U \in X$,*

$$\mathbb{A}_\epsilon U \to \mathbb{A}_0 U \text{ and } \mathbb{A}_\epsilon^{-1} U \to \mathbb{A}_0^{-1} U$$

in the X norm. If in addition we know that the first component of U is in $L^\infty(D')$, we have

$$\|\mathbb{A}_\epsilon U - \mathbb{A}_0 U\|_X \leq C\epsilon^{d/2}.$$

Furthermore, given $F \in X$ and $U_0 = \mathbb{A}_0^{-1} F$, if the first component of U_0 is in $L^\infty(D')$, then

$$\|\mathbb{A}_\epsilon^{-1} F - \mathbb{A}_0^{-1} F\|_X \leq C\epsilon^{d/2}.$$

Proof. We will prove the statement about $\mathbb{A}_\epsilon$ first and use that result to prove the statement about the inverse. Let $U, \Phi \in X$. Then,

$$\begin{aligned}((\mathbb{A}_\epsilon - \mathbb{A}_0)U, \Phi)_X =& \int_D (q_0 - q_\epsilon) w\overline{\phi} \,\mathrm{d}x \\ & \int_{\epsilon B} (q_0 - q_1) w\overline{\phi} \,\mathrm{d}x \\ \leq& \|q_0 - q_1\|_{L^\infty(D)} \|w\|_{L^2(\epsilon B)} \|\phi\|_{L^2(D)} \\ \leq& o(1)\|\Phi\|_X. \end{aligned} \tag{77}$$

This proves the first statement. The second follows from the estimate

$$\begin{aligned}((\mathbb{A}_\epsilon - \mathbb{A}_0)U, \Phi)_X =& \int_D (q_0 - q_\epsilon) w\overline{\phi} \,\mathrm{d}x \\ & \int_{\epsilon B} (q_0 - q_1) w\overline{\phi} \,\mathrm{d}x \\ \leq& \|q_0 - q_1\|_{L^\infty(D)} \|w\|_{L^\infty(D)} \|\chi_{\epsilon B}\|_{L^2(D)} \|\phi\|_{L^2(D)} \\ \leq& C\epsilon^{d/2} \|\Phi\|_X \end{aligned} \tag{78}$$

when $w \in L^\infty(D')$ where $\epsilon B \subset D'$.

Now, we will prove the estimate on the inverse. Let $F \in X$ and define

$$\mathbb{A}_\epsilon U_\epsilon = F \text{ and } \mathbb{A}_0 U_0 = F. \tag{79}$$

Then, for $U_0 := (w_0, v_0)$, we obtain

$$\begin{aligned}(\mathbb{A}_\epsilon(U_\epsilon - U_0), \Phi)_X =& ((\mathbb{A}_0 - \mathbb{A}_\epsilon)U_0, \Phi)_X \\ \leq& \|(\mathbb{A}_0 - \mathbb{A}_\epsilon)U_0\|_X \|\Phi\|_X. \end{aligned} \tag{80}$$

Dividing by $\|\Phi\|_X$ and taking the supremum, we use Proposition 3.5 to obtain

$$\|(\mathbb{A}_\epsilon^{-1} - \mathbb{A}_0^{-1})F\|_X = \|U_\epsilon - U_0\|_X \leq \frac{1}{c}\|(\mathbb{A}_0 - \mathbb{A}_\epsilon)U_0\|_X. \tag{81}$$

Since $\mathbb{A}_\epsilon$ converges strongly to $\mathbb{A}_0$, we have the strong convergence of the inverse from (81). Furthermore, if the first component of U_0 is in $L^\infty(D')$, we have the desired estimate on the inverse from (78) and (81). ☐

We now have an estimate on the composition of the operators.

Lemma 5.3. *Let $\tau \in \mathbb{R}$. Define $\mathbb{A}_\epsilon$ and $\mathbb{K}_\epsilon(\tau)$ by (71). Then,*

$$\|\mathbb{A}_\epsilon^{-1}\mathbb{K}_\epsilon(\tau) - \mathbb{A}_0^{-1}\mathbb{K}_0(\tau)\|_{\mathcal{L}(X)} \leq C\epsilon^{d/2}\max\{|\tau|, 1\}.$$

Proof. Observe that we can write

$$\mathbb{A}_\epsilon^{-1}\mathbb{K}_\epsilon(\tau) - \mathbb{A}_0^{-1}\mathbb{K}_0(\tau) = \mathbb{A}_\epsilon^{-1}(\mathbb{K}_\epsilon(\tau) - \mathbb{K}_0(\tau)) + (\mathbb{A}_\epsilon^{-1} - \mathbb{A}_0^{-1})\mathbb{K}_0(\tau) := I + II. \tag{82}$$

Since $\mathbb{A}_\epsilon^{-1}$ converges in the strong topology by Lemma 5.2, its norm is bounded by the Uniform Boundedness Principle. This norm bound and Lemma 5.1 together imply that I converges in the norm topology, and in particular that

$$\|\mathbb{A}_\epsilon^{-1}(\mathbb{K}_\epsilon(\tau) - \mathbb{K}_0(\tau))\|_{\mathcal{L}(X)} \leq C|\tau|\epsilon^{d/2}. \tag{83}$$

Since $\mathbb{A}_\epsilon^{-1}$ converges and is preceded by a compact operator, II converges in norm as well, but we would like to also estimate the τ dependence. For convenience we prove the estimates for slightly more general operator

$$K(t,f) := \begin{pmatrix} 0 & tf \\ t(\Delta\Delta)^{-1} & 0 \end{pmatrix} \tag{84}$$

for given $f \in C^\infty(D)$ and $t \in \mathbb{R}$. Take $M = (m,n) \in X$. Then, define $M_0 := \mathbb{A}_0^{-1}K(t,f)M$. Notice that the first component of $K(t,f)M$ is $tfn \in H_0^2(D)$. We now compute estimates as in the proof of Lemma 5.2:

$$\begin{aligned}
((\mathbb{A}_\epsilon - \mathbb{A}_0)K(t,f)M, \Phi)_X &= t\int_D (q_0 - q_\epsilon) fn\overline{\phi}\,\mathrm{d}x \\
&= t\int_{\epsilon B}(q_0 - q_1)fn\overline{\phi}\,\mathrm{d}x \\
&\leq |t|\|f\|_{L^\infty(D)}\|q_0 - q_1\|_{L^\infty(D)}\|n\|_{L^\infty(D)}\|\chi_{\epsilon B}\|_{L^2(D)}\|\phi\|_{L^2(D)} \\
&\leq C\epsilon^{d/2}\|n\|_{H_0^2(D)}\|\Phi\|_X \\
&\leq C\epsilon^{d/2}\|M\|_X\|\Phi\|_X
\end{aligned} \tag{85}$$

by the Sobolev Embedding of $H_0^2(D)$ into $C^0(D)$. We hence have shown that

$$\|(\mathbb{A}_\epsilon - \mathbb{A}_0)K(t,f)\|_{\mathcal{L}(X)} \leq C\epsilon^{d/2}. \tag{86}$$

From (81), we obtain

$$\begin{aligned}
\|(\mathbb{A}_\epsilon^{-1} - \mathbb{A}_0^{-1})K(t,f)M\|_X &\leq \frac{1}{c}\|(\mathbb{A}_0 - \mathbb{A}_\epsilon)K(f,t)M\|_X \\
&\leq C\|(\mathbb{A}_\epsilon - \mathbb{A}_0)K(t,f)\|_{\mathcal{L}(X)}\|M\|_X \\
&\leq C\epsilon^{d/2}\|M\|_X,
\end{aligned} \tag{87}$$

implying

$$\|(\mathbb{A}_\epsilon^{-1} - \mathbb{A}_0^{-1})K(t,f)\|_X \leq C\epsilon^{d/2}. \tag{88}$$

Note that $\mathbb{K}_0 = K(\lambda, 1) + \tau K(-1, q_0)$, and so

$$(\mathbb{A}_\epsilon^{-1} - \mathbb{A}_0^{-1})\mathbb{K}_0(\tau) = (\mathbb{A}_\epsilon^{-1} - \mathbb{A}_0^{-1})K(\lambda, 1) + \tau(\mathbb{A}_\epsilon^{-1} - \mathbb{A}_0^{-1})K(-1, q_0). \tag{89}$$

Therefore, by applying (88), we have that II also converges in norm and

$$\|(\mathbb{A}_\epsilon^{-1} - \mathbb{A}_0^{-1})\mathbb{K}_0(\tau)\|_{\mathcal{L}(X)} \leq C(1 + |\tau|)\epsilon^{d/2}. \tag{90}$$

□

5.1. **Convergence on the eigenspace.** Let us define $U = (w, v) \in X$ be a transmission eigenfunction for the background problem (74) when $\epsilon = 0$, corresponding to a transmission eigenvalue τ:

$$\tau T_0(\tau)U = U. \tag{91}$$

We will now define a correction for the difference between $\mathbb{A}_\epsilon^{-1}$ and $\mathbb{A}_0^{-1}$. By standard elliptic regularity, the first component of $\mathbb{A}_0^{-1}U$, w_0, is $H^2_{loc}(D)$. Therefore, it is well defined at the center of the inhomogeneity and we can define $\Psi_\epsilon, C_\epsilon \in X$ by

$$\Psi_\epsilon = \begin{pmatrix} \chi_{\epsilon B} \\ 0 \end{pmatrix} \tag{92}$$

and

$$C_\epsilon = (q_1(0) - q_0(0))w_0(0)\Psi_\epsilon. \tag{93}$$

Lemma 5.4. *Let $\mathbb{A}_\epsilon$ be defined by (71) and C_ϵ by (93). Define $\alpha > 0$ such that $H^2(D) \subset C^{0,\alpha}(D)$. Then*

$$\|\mathbb{A}_\epsilon^{-1}U - \mathbb{A}_0^{-1}U - \mathbb{A}_\epsilon^{-1}C_\epsilon\|_X \leq C\epsilon^{d/2+\alpha},$$

where U is the solution to the background equation (91).

Proof. Define

$$\mathbb{A}_\epsilon U_\epsilon = U \qquad \text{and} \qquad \mathbb{A}_0 U_0 = U. \tag{94}$$

Let $U_0 := (w_0, v_0)$ and note that $w_0 \in H^2_{loc}(D)$. Consider D' to be a C^2 domain satisfying $\epsilon B \subset D' \subset D$ for $\epsilon \in \mathcal{I}$. Sobolev Embedding yields $H^2(D') \subset C^{0,\alpha}(D')$ for some $\alpha > 0$, and therefore, $w_0 \in C^{0,\alpha}(D')$. This allows us to obtain the bound

$$\begin{aligned}(\mathbb{A}_\epsilon(U_\epsilon - U_0 - \mathbb{A}_\epsilon^{-1}C_\epsilon), \Phi)_X &= ((\mathbb{A}_0 - \mathbb{A}_\epsilon)U_0 - C_\epsilon, \Phi)_X \\ &= \int_{\epsilon B} \big((q_1 - q_0)w_0 - (q_1(0) - q_0(0))w_0(0)\big)\overline{\phi}\,\mathrm{d}x \\ &\leq C\epsilon^\alpha \int_{\epsilon B} |\phi|\,\mathrm{d}x \\ &\leq C\epsilon^\alpha \|\chi_{\epsilon B}\|_{L^2(D)}\|\phi\|_{L^2(D)} \\ &\leq C\epsilon^{d/2+\alpha}\|\phi\|_X. \end{aligned} \tag{95}$$

By Proposition 3.5 and the argument at the end of Lemma 5.2, we have

$$\|(\mathbb{A}_\epsilon^{-1} - \mathbb{A}_0^{-1})U - \mathbb{A}_\epsilon^{-1}C_\epsilon\|_X \leq \frac{1}{c}\|(\mathbb{A}_0 - \mathbb{A}_\epsilon)U_0 - C_\epsilon\|_X, \tag{96}$$

which yields the result. □

We will now derive estimates for the strong operator convergence of the adjoint. Recall that our operator

$$T_\epsilon(\tau) := -\frac{1}{\tau}\mathbb{A}_\epsilon^{-1}\mathbb{K}_\epsilon(\tau),$$

where $\mathbb{A}_\epsilon$ is self adjoint, but $\mathbb{K}_\epsilon(\tau)$ is not.

Lemma 5.5. *Let U be the solution of the background equation (91) and $\alpha > 0$ such that $H^2(D) \subset C^{0,\alpha}(D)$. Then,*

$$\|(\mathbb{K}_\epsilon(\tau)^*\mathbb{A}_\epsilon^{-1} - \mathbb{K}_0(\tau)^*\mathbb{A}_0^{-1})U\|_X \leq C(1+|\tau|)O(\epsilon^{d/2+\alpha}).$$

Proof. We begin by adding and subtracting to obtain

$$(\mathbb{K}_\epsilon(\tau)^*\mathbb{A}_\epsilon^{-1} - \mathbb{K}_0(\tau)^*\mathbb{A}_0^{-1})U = (\mathbb{K}_\epsilon(\tau)^* - \mathbb{K}_0(\tau)^*)(\mathbb{A}_\epsilon^{-1} - \mathbb{A}_0^{-1})U + (\mathbb{K}_\epsilon(\tau)^* - \mathbb{K}_0(\tau)^*)\mathbb{A}_0^{-1}U + \mathbb{K}_0(\tau)^*(\mathbb{A}_\epsilon^{-1} - \mathbb{A}_0^{-1})U. \tag{97}$$

The first term on the right hand side above converges with a speed of $O(|\tau|\epsilon^d)$ thanks to Lemmas 5.1 and 5.2 and the regularity of the eigenfunction U. Recall that in the proof of the previous lemma we defined $U_0 := (w_0, v_0)$ by $\mathbb{A}_0 U_0 = U$, and that the first component w_0 is in $H^2_{loc}(D)$. For the second term on the right hand side of (97), we observe that for D' with $\overline{D'} \subset D$ containing ϵB,

$$\begin{aligned}
((\mathbb{K}_\epsilon^*(\tau) - \mathbb{K}_0^*(\tau))U_0, \Phi)_X =& \tau \int_{\epsilon B} (q_1 - q_0) w_0 \overline{\psi} \, \mathrm{d}x \\
\leq& |\tau| \|(q_1 - q_0) w_0\|_{L^\infty(D')} \|\psi\|_{L^\infty(D)} \epsilon^d \\
\leq& C|\tau|\epsilon^d \|\psi\|_{H_0^2(D)} \\
\leq& C|\tau|\epsilon^d \|\Phi\|_X
\end{aligned} \tag{98}$$

using Sobolev Embedding. Finally, for the third term on the right hand side of (97), we have that

$$\mathbb{K}_0(\tau)^*(\mathbb{A}_\epsilon^{-1} - \mathbb{A}_0^{-1})U = \mathbb{K}_0(\tau)^*(\mathbb{A}_\epsilon^{-1}U - \mathbb{A}_0^{-1}U - \mathbb{A}_\epsilon^{-1}C_\epsilon) + \mathbb{K}_0(\tau)^*\mathbb{A}_\epsilon^{-1}C_\epsilon \tag{99}$$

where C_ϵ is defined by (93). The first term in (99) is $O((1+\tau)\epsilon^{d/2+\alpha})$ by Lemma 5.4. We now estimate the second term where we keep in mind the fact that the operator $\mathbb{K}_0$ is smoothing:

$$\begin{aligned}
(\mathbb{K}_0^*(\tau)\mathbb{A}_\epsilon^{-1}C_\epsilon, \Phi)_X =& (C_\epsilon, \mathbb{A}_\epsilon^{-1}\mathbb{K}_0(\tau)\Phi)_X \\
=& (C_\epsilon, (\mathbb{A}_\epsilon^{-1} - \mathbb{A}_0^{-1})\mathbb{K}_0(\tau)\Phi)_X + (C_\epsilon, \mathbb{A}_0^{-1}\mathbb{K}_0(\tau)\Phi)_X.
\end{aligned} \tag{100}$$

Consider the first term on the right hand side of (100). It is obvious that $\|C_\epsilon\|_X \leq C\epsilon^{d/2}$ by the definition, and we recall from (90) that

$$\|(\mathbb{A}_\epsilon^{-1} - \mathbb{A}_0^{-1})\mathbb{K}_0(\tau)\|_{\mathcal{L}(X)} \leq C(1+|\tau|)\epsilon^{d/2}. \tag{101}$$

These results combine to yield

$$\begin{aligned}
(C_\epsilon, (\mathbb{A}_\epsilon^{-1} - \mathbb{A}_0^{-1})\mathbb{K}_0(\tau)\Phi)_X \leq& \|C_\epsilon\|_X \|(\mathbb{A}_\epsilon^{-1} - \mathbb{A}_0^{-1})\mathbb{K}_0(\tau)\|_{\mathcal{L}(X)} \|\Phi\|_X \\
\leq& C\epsilon^{d/2}(1+|\tau|)\epsilon^{d/2}\|\Phi\|_X.
\end{aligned} \tag{102}$$

For the last term on the right hand side of (100), consider the first component of $\mathbb{A}_0^{-1}\mathbb{K}_0\Phi$, and let us denote it by ψ_0. For for some D' compactly contained in D containing ϵB, we use Sobolev embedding on ψ_0:

$$\begin{aligned}
|(C_\epsilon, \mathbb{A}_0^{-1}\mathbb{K}_0(\tau)\Phi)_X| =& \left|(q_1(0) - q_0(0))w_0(0) \int_{\epsilon B} \psi_0 \, \mathrm{d}x\right| \\
\leq& C\|\psi_0\|_{L^\infty(D')}\epsilon^d \\
\leq& C\|\psi_0\|_{H^2(D')}\epsilon^d. \\
\leq& C(1+|\tau|)\|\Phi\|_X \epsilon^d.
\end{aligned} \tag{103}$$

This completes the proof. ∎

5.2. **Inner product estimates.** We will need the following lemmas to derive the asymptotic formula for the eigenvalues.

Lemma 5.6. *Let $d = 2, 3$. For $\mathbb{A}_\epsilon$ be defined by (71) and Ψ_ϵ by*

$$\Psi_\epsilon = \begin{pmatrix} \chi_{\epsilon B} \\ 0 \end{pmatrix} \tag{104}$$

we have that

$$(\Psi_\epsilon, \mathbb{A}_\epsilon^{-1}\Psi_\epsilon)_X = O(\epsilon^{\frac{3}{2}d})$$

holds as $\epsilon \to 0$.

Proof. Let ξ_ϵ denote the first component of $\mathbb{A}_\epsilon^{-1}\Psi_\epsilon$ and R as defined in (69). Then, by definition of $\mathbb{A}_\epsilon^{-1}$, ξ_ϵ is a weak (or distributional) solution to

$$\Delta\xi_\epsilon - \lambda\xi_\epsilon = 0 \qquad \text{in } \ D \tag{105}$$

which implies, in particular, that $\xi_\epsilon \in C^\infty(D)$ [7]. One may use Green's representation formula to show that

$$|\xi_\epsilon(x)| \le C\int_R |\xi_\epsilon(t)|\,\mathrm{d}t \quad \text{for } x \in \epsilon B \subset D \setminus \overline{R}, \tag{106}$$

as in the proof of Lemma 3.4. This implies

$$|\xi_\epsilon(x)| \le C'\|\xi_\epsilon\|_{L^2(D)} \le C'\|\mathbb{A}_{\tau,\epsilon}^{-1}\|_{\mathcal{L}(X)}\|\Psi_\epsilon\|_X := C''\epsilon^{d/2} \tag{107}$$

because the norm of $\mathbb{A}_\epsilon^{-1}$ is bounded due to Lemma 5.2 and the Uniform Boundedness Principle. Therefore, we have the result

$$\left|(\Psi_\epsilon, \mathbb{A}_\epsilon^{-1}\Psi_\epsilon)_X\right| = \left|\int_{\epsilon B} \overline{\xi_\epsilon}\,\mathrm{d}x\right| \le C''\epsilon^{d/2}\int_{\epsilon B}\mathrm{d}x = O(\epsilon^{\frac{3}{2}d}). \tag{108}$$

□

Lemma 5.7. *Let $U = (w, v)$ be the transmission eigenfunction solving (91) and the operators $\mathbb{A}_\epsilon$ and $\mathbb{K}_\epsilon$ be defined by (71). For α such that $H^2(D) \subset C^{0,\alpha}(D)$,*

$$((\mathbb{A}_\epsilon^{-1}\mathbb{K}_\epsilon(\tau) - \mathbb{A}_0^{-1}\mathbb{K}_0(\tau))U, U)_X = \epsilon^d(q_1(0) - q_0(0))\overline{w_0(0)}|B|\,(\tau v(0) - w(0)) + O(\epsilon^{d+\alpha})$$

where w_0 is the first component of $\mathbb{A}_0^{-1}U$.

Proof. First, we observe

$$\begin{aligned}((\mathbb{A}_\epsilon^{-1}\mathbb{K}_\epsilon(\tau) - \mathbb{A}_0^{-1}\mathbb{K}_0(\tau))U, U)_X &= ((\mathbb{A}_\epsilon^{-1} - \mathbb{A}_0^{-1})(\mathbb{K}_\epsilon(\tau) - \mathbb{K}_0(\tau))U, U)_X \\ &\quad + ((\mathbb{A}_\epsilon^{-1} - \mathbb{A}_0^{-1})\mathbb{K}_0(\tau)U, U)_X + (\mathbb{A}_0^{-1}(\mathbb{K}_\epsilon(\tau) - \mathbb{K}_0(\tau))U, U)_X \\ &:= I + II + III.\end{aligned}$$

For I, we can use the fact that $\mathbb{A}_0$ is self adjoint and add and subtract a correction term to obtain

$$\begin{aligned} I &= ((\mathbb{K}_\epsilon(\tau) - \mathbb{K}_0(\tau))U, (\mathbb{A}_\epsilon^{-1} - \mathbb{A}_0^{-1})U - \mathbb{A}_\epsilon^{-1}C_\epsilon)_X + ((\mathbb{K}_\epsilon(\tau) - \mathbb{K}_0(\tau))U, \mathbb{A}_\epsilon^{-1}C_\epsilon)_X \\ &\le \|(K_\epsilon(\tau) - K_0(\tau))U\|_X\|(\mathbb{A}_\epsilon^{-1} - \mathbb{A}_0^{-1})U - \mathbb{A}_\epsilon^{-1}C_\epsilon\|_X \\ &\quad + ((\mathbb{K}_\epsilon(\tau) - \mathbb{K}_0(\tau))U, \mathbb{A}_\epsilon^{-1}C_\epsilon)_X \\ &= ((\mathbb{K}_\epsilon(\tau) - \mathbb{K}_0(\tau))U, \mathbb{A}_\epsilon^{-1}C_\epsilon)_X + O((1+\tau)\epsilon^{d+\alpha}) \end{aligned}$$

by combining Lemmas 5.1 and 5.4. Recall that $U := (w, v)$ with $v \in H_0^2(D)$, and let ξ_ϵ denote the first component of $\mathbb{A}_\epsilon^{-1}\Psi_\epsilon$. Using the definition of Ψ_ϵ and C_ϵ in (92) and (93) we find,

$$((\mathbb{K}_\epsilon(\tau) - \mathbb{K}_0(\tau))U, \mathbb{A}_\epsilon^{-1}C_\epsilon)_X = -(q_1(0) - q_0(0))\overline{w(0)}\tau \int_{\epsilon B} (q_1 - q_0)v\overline{\xi_\epsilon}\,\mathrm{d}x. \tag{109}$$

The Hölder continuity of v with exponent α from Sobolev embedding, with α as in Lemma 5.6, gives

$$((\mathbb{K}_\epsilon(\tau)-\mathbb{K}_0(\tau))U, \mathbb{A}_\epsilon^{-1}C_\epsilon)_X = -\tau(q_0(0)-q_1(0))^2 v(0)\overline{w(0)}(\Psi_\epsilon, \mathbb{A}_\epsilon^{-1}\Psi_\epsilon)_X + O(\epsilon^{\frac{3}{2}d+\alpha}).$$

Thus, we conclude that

$$I = O(\epsilon^{d+\alpha}) \tag{110}$$

by Lemma 5.6.

For the second term, we first manipulate it algebraically. Since U satisfies the background equation, the following equality holds:

$$-U = \mathbb{A}_0^{-1}\mathbb{K}_0(\tau)U. \tag{111}$$

Therefore, we compute

$$\begin{aligned} II =&((\mathbb{A}_\epsilon^{-1} - \mathbb{A}_0^{-1})\mathbb{K}_0(\tau)U, U)_X \\ =&(\mathbb{A}_\epsilon^{-1}(\mathbb{A}_0 - \mathbb{A}_\epsilon)\mathbb{A}_0^{-1}\mathbb{K}_0(\tau)U, U)_X \\ =&-((\mathbb{A}_0 - \mathbb{A}_\epsilon)U, \mathbb{A}_\epsilon^{-1}U)_X. \end{aligned} \tag{112}$$

From Lemmas 5.2 and 5.4, we may again use the correction to conclude that

$$II = -((\mathbb{A}_0 - \mathbb{A}_\epsilon)U, \mathbb{A}_0^{-1}U + \mathbb{A}_\epsilon^{-1}C_\epsilon)_X + O\left(\frac{1}{\tau}\epsilon^{d+\alpha}\right). \tag{113}$$

Recall that we denote the first components of $\mathbb{A}_0^{-1}U$ and U by w_0 and w respectively, and that $w_0 \in H^2_{loc}(D)$ and $w \in C^\infty_{loc}(D)$. In fact, w is $C^\infty(D)$ (see Theorem 11.1.1 [7]). In particular, both are in $H^2(D')$ for some $D' \subset D$ containing ϵB. We may also use Hölder continuity as in (5.2) to obtain

$$\begin{aligned} &((\mathbb{A}_0 - \mathbb{A}_\epsilon)U, \mathbb{A}_0^{-1}U)_X \\ &\qquad = \int_{\epsilon B}(q_1 - q_0)w\overline{w_0}\mathrm{d}x = \epsilon^d|B|(q_1(0) - q_0(0))w(0)\overline{w_0(0)} + O(\epsilon^{d+\alpha}). \end{aligned} \tag{114}$$

Let ξ_ϵ denote the first component of $\mathbb{A}^{-1}\Psi_\epsilon$. We may also use the Hölder continuity argument from (5.2) to show

$$\begin{aligned} ((\mathbb{A}_0 - \mathbb{A}_\epsilon)U, \mathbb{A}_\epsilon^{-1}C_\epsilon)_X =&(q_1(0) - q_0(0))\overline{w(0)} \int_{\epsilon B}(q_1 - q_0)w\overline{\xi_\epsilon}\,\mathrm{d}x \\ =&(q_1(0) - q_0(0))^2|w(0)|^2(\Psi_\epsilon, \mathbb{A}_\epsilon^{-1}\Psi_\epsilon)_X + O(\epsilon^{d+\alpha}). \end{aligned} \tag{115}$$

Lemma 5.6 combined with the above yields

$$II = -\epsilon^d(q_1(0) - q_0(0))w(0)\overline{w_0(0)}|B| + O(\epsilon^{d+\alpha}). \tag{116}$$

The estimation of the third term is similar. Again, as v and w_0 are in $H^2_{loc}(D)$, we may use Hölder continuity to conclude that

$$
\begin{aligned}
III =&((\mathbb{K}_\epsilon(\tau)-\mathbb{K}_0(\tau))U, \mathbb{A}_0^{-1}U)_X \\
=&-\tau\int_{\epsilon B}(q_0-q_1)v\overline{w_0}\,\mathrm{d}x \\
=&-\tau\epsilon^d(q_0(0)-q_1(0))v(0)\overline{w_0(0)}|B|+O(\epsilon^{d+\alpha}).
\end{aligned} \tag{117}
$$

We combine all terms to obtain

$$
\begin{aligned}
&((\mathbb{A}_\epsilon^{-1}\mathbb{K}_\epsilon(\tau)-\mathbb{A}_0^{-1}\mathbb{K}_0(\tau))U,U)_X \\
&=-\tau\left(\epsilon^d(q_0(0)-q_1(0))v(0)\overline{w_0(0)}|B|\right)-\left(\epsilon^d(q_1(0)-q_0(0))w(0)\overline{w_0(0)}|B|\right)+O(\epsilon^{d+\alpha}) \\
&=\epsilon^d(q_1(0)-q_0(0))\overline{w_0(0)}|B|\,(\tau v(0)-w(0))+O(\epsilon^{d+\alpha}),
\end{aligned}
$$

which completes the proof. □

6. **Eigenvalue correction formula.**

In this section, we will use the following nonlinear eigenvalue correction result from [11] to obtain an asymptotic formula for a simple transmission eigenvalue.

Theorem 6.1 (Nonlinear Eigenvalue Correction [11])**.** *Let X be a Banach space and $\{T_\epsilon(\lambda): X\to X\}$ a set of compact operator valued functions of λ which are analytic in a region U of the complex plane, such that $T_\epsilon(\lambda)\to T_0(\lambda)$ in norm as $\epsilon\to 0$ uniformly for $\lambda\in U$. Let $\lambda_0\neq 0$, $\lambda_0\in U$ be a simple nonlinear eigenvalue of T_0,*

$$\lambda_0 T_0(\lambda_0)\phi=\phi,$$

define $DT_0(\lambda_0)$ to be the derivative of T_0 with respect to λ evaluated at λ_0, and let ϕ be the normalized eigenfunction and ϕ^* its dual. Then for any ϵ small enough there exists λ_ϵ, a simple nonlinear eigenvalue of T_ϵ, such that if

$$\lambda_0^2\langle DT_0(\lambda_0)\phi,\phi^*\rangle\neq -1$$

we have the following formula

$$
\begin{aligned}
\lambda_\epsilon-\lambda_0=&\lambda_0^2\frac{\langle (T_0(\lambda_0)-T_\epsilon(\lambda_0))\phi,\phi^*\rangle}{1+\lambda_0^2\langle DT_0(\lambda_0)\phi,\phi^*\rangle} \\
&+O\left(\sup_{\lambda\in U}\|(T_\epsilon(\lambda)-T_0(\lambda))\big|_{R(E)}\|\|(T_\epsilon^*(\lambda)-T_0^*(\lambda))\big|_{R(E)^*}\|\right)
\end{aligned}
$$

where $R(E)$ is the space spanned by ϕ and $R(E)^*$ is its dual or the space spanned by ϕ^*.

We now apply this theorem to our operators to obtain the correction formula.

Theorem 6.2. *Let $U=(w,v)\in X$ be the normalized transmission eigenfunction for the background problem (91), with simple eigenvalue τ. Then for any ϵ small enough there exists a simple eigenvalue τ_ϵ of the perturbed problem (74) such that $\tau_\epsilon\to\tau$. Furthermore, for $\|U\|_X=1$ and $\alpha>0$ such that $H^2(D)\subset C^{0,\alpha}(D)$,*

$$\tau_\epsilon-\tau=-\frac{\epsilon^d}{\omega}(q_1(0)-q_0(0))\overline{w_0(0)}|B|\,(\tau v(0)-w(0))+O(\epsilon^{d+\alpha}),$$

when

$$\omega:=\int_D(1+q_0)v\overline{w_0}+w\overline{v_0}\,\mathrm{d}x$$

is nonzero and $(w_0,v_0)\in X$ solves

$$\Delta v_0-\lambda v_0=w+q_0w_0 \qquad \text{in } D$$

$$\Delta w_0 - \lambda w_0 = \Delta\Delta v \qquad \text{in } D$$

for some fixed $\lambda \geq \lambda_0$ *with* λ_0 *as defined in Proposition 3.5.*

Proof. Let T_ϵ and T_0 be defined by (74) and recall $\mathbb{A}_0^{-1}U = (w_0, v_0)$ with $\mathbb{A}_0$ defined by (71). Then, by Lemma 5.3, we have the norm convergence which allows us to apply Theorem 6.1. From Lemmas 5.3 and 5.5, we obtain that the rate of convergence of the tail is $O(\epsilon^{d+\alpha})$. (Note that the norms are restricted to one dimensional subspaces there.) The estimate in the inner product in Lemma 5.7 therefore yields the formula

$$\tau_\epsilon - \tau = \epsilon^d \tau \frac{(q_1(0) - q_0(0))\overline{w_0(0)}|B|\,(\tau v(0) - w(0))}{1 + \tau^2 (DT_0(\tau)U, U)_X} + O(\epsilon^{d+\alpha}), \tag{118}$$

assuming the denominator is nonzero. We now calculate $DT_0(\tau)$. Recall that

$$T_0(\tau) = -\frac{1}{\tau}\mathbb{A}_0^{-1}\mathbb{K}_0$$

with $\mathbb{A}_0$ independent of τ. From the product rule we have that

$$DT_0(\tau) = \frac{1}{\tau^2}\mathbb{A}_0^{-1}\mathbb{K}_0(\tau) - \frac{1}{\tau}\mathbb{A}_0^{-1}D\mathbb{K}_0,$$

and since $\mathbb{K}_0$ is linear in τ , $D\mathbb{K}_0 = \mathbb{C}$ where $\mathbb{C} : X \to X$ is given by

$$\mathbb{C} = -\begin{pmatrix} 0 & -(1+q_0) \\ -(\Delta\Delta)^{-1} & 0 \end{pmatrix}. \tag{119}$$

Hence

$$DT_0(\tau) = \frac{1}{\tau^2}\mathbb{A}_0^{-1}\mathbb{K}_0(\tau) - \frac{1}{\tau}\mathbb{A}_0^{-1}\mathbb{C}.$$

We next calculate that

$$\tau^2 \left(DT_0(\tau)U, U\right)_X = \left((\mathbb{A}_0^{-1}\mathbb{K}_0(\tau) - \tau\mathbb{A}_0^{-1}\mathbb{C})U, U\right)_X = -1 - \tau\left(\mathbb{C}U, \mathbb{A}_0^{-1}U\right)_X \tag{120}$$

since $\mathbb{A}_0^{-1}\mathbb{K}_0(\tau)U = -U$. So, the denominator in (118) becomes

$$1 + \tau^2 \left(DT_0(\tau)U, U\right) = -\tau\left(\mathbb{C}U, \mathbb{A}_0^{-1}U\right),$$

which is precisely $\tau\omega$, from which the result follows. We remark also that the nonzero denominator condition can be written as

$$\left(\begin{pmatrix} 0 & -(1+q_0) \\ -(\Delta\Delta)^{-1} & 0 \end{pmatrix} U, \mathbb{A}_0^{-1}U\right)_X \neq 0. \tag{121}$$

□

We end this section by remarking that the above analysis holds true if the background with contrast $q_0(x)$ is perturbed by many small volume inhomogeneities of arbitrary smooth shape. In particular, for $i = 1, \ldots, m$, we define the bounded open set B_i to be smooth deformations of a ball centered at the origin, so that $z_i + \epsilon B_i$ is a small inhomogeneity centered at z_i. We also assume that ϵ is small enough so that each scaled ball is separated from the others and is inside $D \setminus \overline{R}$, in particular $(z_i + \epsilon\overline{B_i}) \cap (z_j + \epsilon\overline{B_j}) = \emptyset$ for $i \neq j$ and $(z_i + \epsilon\overline{B_i}) \subset D \setminus \overline{R}$, where R is defined by (69). We let W_ϵ be the union of these inhomogeneities, that is

$$W_\epsilon := \bigcup_{i=1}^{m} (z_i + \epsilon B_i)\,,$$

TABLE 1. Parameters for Numerical Example

Domain D	$[-1, 1]$
Background Transmission Eigenvalue $k = \sqrt{-\tau}$	7.12761
Background Coefficient q_0	6.29
Perturbed Coefficient q_1	24
Parameter λ	50.72217

and we define the perturbed contrast q_ϵ:

$$q_\epsilon(x) = \begin{cases} q_i & x \in z_i + \epsilon B_i, \quad i = 1, \dots, m \\ q_0 & x \in D \setminus W_\epsilon \end{cases} \tag{122}$$

where the $q_i \in \mathbb{R}$ are constants. In this case the main result of Theorem (6.2 becomes (see also [4] and [6])

Theorem 6.3. *Let $U = (w, v) \in X$ be the normalized transmission eigenfunction for the background problem (91), with simple eigenvalue τ. Then for any ϵ small enough there exists a simple eigenvalue τ_ϵ of the perturbed problem (74) such that $\tau_\epsilon \to \tau$. Furthermore, for $\|U\|_X = 1$ and $\alpha > 0$ such that $H^2(D) \subset C^{0,\alpha}(D)$,*

$$\tau_\epsilon - \tau = -\frac{\epsilon^d}{\omega} \sum_{i=1}^{m} (q_i(z_i) - q_0(z_i)) \overline{w_0(z_i)} |B_i| \left(\tau v(z_i) - w(z_i)\right) + O(\epsilon^{d+\alpha}),$$

when

$$\omega := \int_D (1 + q_0) v \overline{w_0} + w \overline{v_0} \, \mathrm{d}x$$

is nonzero and $(w_0, v_0) \in X$ solves

$$\begin{aligned} \Delta v_0 - \lambda v_0 &= w + q_0 w_0 && \text{in } D \\ \Delta w_0 - \lambda w_0 &= \Delta\Delta v && \text{in } D \end{aligned}$$

for some fixed $\lambda \geq \lambda_0$ with λ_0 as defined in Proposition 3.5.

7. Numerical example.

We will now attempt to validate our asymptotic formula with a one dimensional numerical experiment. Although the theory here was for dimensions $d = 2, 3$, we expect the same results to hold in dimension one. We choose our scatterer D to be the interval $[-1, 1]$ assume there is a single inhomogeneity centered at the origin. We define q_ϵ on D to be

$$q_\epsilon := \begin{cases} q_1 & x \in (-\epsilon, \epsilon) := \epsilon B \\ q_0 & \text{otherwise} \end{cases} \tag{123}$$

The definition of D and choices for parameters k, q_0 and q_1 are detailed in Table 1. We note that we also chose q_0 so that the background eigenvalue is simple.

Recall that one also needs to choose the parameter λ in the definition of T (which we used to divide the operator into invertible plus compact). The best choice numerically for λ is not obvious, and several different choices were found to yield the same correction accurately.

Figures 1 and 2 show a comparison of the perturbed eigenvalues with the corrected eigenvalues for various values of ϵ, using the formula from Theorem 6.2. An empirical study of the convergence rate $1 + \alpha$ found in Theorem 6.2 yielded

$$\alpha \approx 0.9625, \tag{124}$$

or approximately ϵ^2 convergence, as expected.

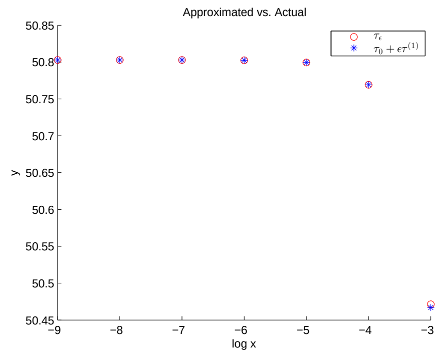

FIGURE 1. Comparison of perturbed eigenvalues and corrected eigenvalues. The red circles are the perturbed transmission eigenvalues (squared) and the blue stars the corrected approximations for various values of ϵ. The x-axis is $\log_{10} \epsilon$.

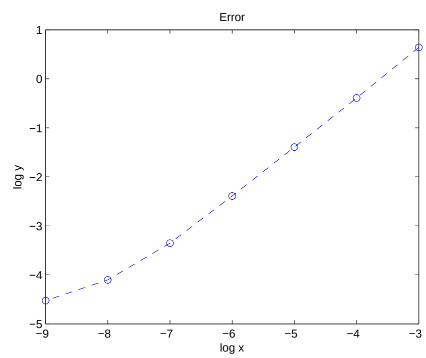

FIGURE 2. Log/log plot of the error $(\tau_\epsilon - (\tau_0 + \epsilon\tau^{(1)}))/\epsilon$.

8. **Conclusion.** In this paper we have derived rigorously a correction formula for the transmission eigenvalues of perturbations of inhomogeneous media without sign restrictions on the contrast (except for a region around the boundary). This was accomplished by using the formulation the nonlinear eigenvalue correction formula [11] which is an extension of Osborn's theorem [12]. We then corroborated the results by conducting a numerical simulation which demonstrated the accuracy of the asymptotic formulas. Note that the formula derived in [6] was based on a different formulation of the transmission eigenvalue problem that requires that q be of one sign. Hence the formula derived here is more general, but has the disadvantage of requiring one to solve an auxiliary partial differential equation. Since the two should of course coincide when q is of one sign, and indeed did in our numerical simulations above, it would be interesting to see if a general formula exists without the need for solving an auxiliary problem. Furthermore, the formula for the correction here requires one to choose the parameter λ which was used to divide the operator into and invertible one plus a compact perturbation. However as mentioned above, in our numerical simulations different λ led to the same value for the correction.

It may also be possible to use the asymptotic formula to reconstruct the location and/or strength of small the inhomogeneities inside the scatterer. A first attempt along these lines is made in [4]. The formulas derived depend on the background medium, the background transmission eigenvalue, the size and contrast of the inhomogeneity, and the location of the center of the inhomogeneity. If knows the size and contrast of the inhomogeneity, perhaps one can use this formula to determine its location. This is the subject of future work.

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