On the asymptotics of a Robin eigenvalue problem

Fioralba Cakoni* Nicolas Chaulet† Houssem Haddar‡

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Abstract

The considered Robin problem can formally be seen as a small perturbation of a Dirichlet problem. However, due to the sign of the impedance value, its associated eigenvalues converge point-wise to $-\infty$ as the perturbation goes to zero. We prove that in this case, Dirichlet eigenpairs are the only accumulation points of the Robin eigenpairs with normalized eigenvectors. We then provide a criteria to select accumulating sequences of eigenvalues and eigenvectors and exhibit their full asymptotic with respect to the small parameter.

1 A Robin eigenvalue problem with negative sign

We are interested in the asymptotic behavior of the eigenvalues λ^δ and eigenfunctions $u^\delta \in H^1(\Omega)$ of the following problem:

$$\Delta u^\delta + \lambda^\delta u^\delta = 0 \qquad \text{in } \Omega \tag{1}$$

$$\partial_\nu u^\delta - \frac{1}{\delta} u^\delta = 0 \qquad \text{on } \Gamma \tag{2}$$

with respect to $\delta > 0$ as it approaches 0, where ν is the outward unit normal vector to Γ which is the C^2-smooth boundary of the bounded connected domain $\Omega \subset \mathbb{R}^d$ for $d \geq 2$.

The eigenvalue problem with Robin boundary condition described by (1)-(2) naturally appear in a number of models related to reaction diffusion problems (see [3]) or scattering theory. For the latter, this eigenvalue problem can be seen as a first approximation to the interior transmission eigenvalue problem associated with the scattering problem by a perfectly conducting body coated with a dielectric layer of width δ (see [1, chapter 8]). It can also be seen as an approximate model to direct scattering problems for perfect conductors coated with metamaterials.

It is well-known that problem (1)-(2) has an infinite sequence of real eigenvalues $\{\lambda_i^\delta\}_{i=1}^\infty$ accumulating at $+\infty$. However, for sufficiently small δ some eigenvalues become negative and their number grows to $+\infty$ as $\delta \to 0$. In fact, for at least C^1 smooth

*Department of Mathematical Sciences, University of Delaware, USA (cakoni@math.udel.edu)
†Department of Mathematics, University College London, UK (n.chaulet@ucl.ac.uk)
‡CMAP, Ecole Polytechnique, Palaiseau, France (houssem.haddar@inria.fr)

boundary Γ, it is known (see for instance [2, 4]) that for every (fixed) $i \geq 1$, $-\delta^2 \lambda_i^\delta \to 1$ as $\delta \to 0$.

In Section 2, we complement this result by indicating that Dirichlet eigenvalues for the $-\Delta$ operator in Ω are the only possible finite accumulation points of λ^δ (extending this way the result obtained in [3] for simple geometries) if the associated H^1 normalized eigenfunctions do not L^2 converge to 0 as δ goes to. We also prove that eigenvectors associated with other accumulation points concentrate at the boundary (in the sense that they converge to zero in any compact set of Ω). Our main result is given in Section 3 which stipulates that some λ^δ does accumulate at Dirichlet eigenvalues providing a full asymptotic development of these sequences as δ goes to zero.

2 Accumulation pairs for Robin eigenpairs

We recall that (1)-(2) are equivalent to the following variational formulation

$$\int_\Omega \nabla u^\delta \nabla v \, dx - \frac{1}{\delta} \int_\Gamma u^\delta v \, ds = \lambda^\delta \int_\Omega u^\delta v \, dx \quad \forall \, v \in H^1(\Omega). \tag{3}$$

Lemma 2.1 *Assume that a sequence $(\lambda^\delta, u^\delta) \in \mathbb{R} \times H^1(\Omega)$ satisfying (3) is such that $\|u^\delta\|_{H^1(\Omega)} = 1$ and $|\lambda^\delta| \leq C$ for some $C > 0$ independent of the δ. Then one can extract a subsequence δ' of δ such that $\lambda^{\delta'} \to \Lambda_0$ and $\|u^{\delta'} - U_0\|_{L^2(\Omega)} \to 0$ as $\delta' \to 0$, where if $U_0 \neq 0$ then (Λ_0, U_0) is some Dirichlet eigenpair for $-\Delta$ in Ω.*

Proof: Since the sequence λ^δ is bounded and u^δ is also bounded in $H^1(\Omega)$, one can extract a subsequence δ' such that $\lambda^{\delta'}$ converges to some $\Lambda_0 \in \mathbb{R}$ and $u^{\delta'}$ converges weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$ to some function $U_0 \in H^1(\Omega)$ as δ' goes to 0. From (3) one deduces that $\int_\Gamma |u^{\delta'}|^2 \, ds \leq C\delta'$ for some $C > 0$ independent of δ', hence $U_0 = 0$ on Γ. Moreover, taking $v \in H_0^1(\Omega)$ in (3) and letting $\delta' \to 0$ proves that U_0 satisfies $\int_\Omega \nabla U_0 \nabla v \, dx = \Lambda_0 \int_\Omega U_0 v \, dx$, which proves, if $U_0 \neq 0$, that (Λ_0, U_0) is a Dirichlet eigenpair. □

Remark 1 We remark that any point on the real axis is a possible accumulation point for $\{\lambda^\delta\}_\delta$. Actually, for a given $i \in \mathbb{N}$ the sequence $\{\lambda_i^\delta\}_\delta$ goes to $-\infty$ continuously. Therefore, for any $\Lambda \in \mathbb{R}$ one can build a sequence $\{\lambda^{\delta_i}\}_{i \in \mathbb{N}}$ such that $\lambda^{\delta_i} = \Lambda$ for any i.

Theorem 2.2 *Consider a sequence $\{\lambda^\delta, u^\delta\}_\delta \in \mathbb{R} \times H^1(\Omega)$ satisfying (3) such that $\|u^\delta\|_{H^1(\Omega)} = 1$ and $\lambda^\delta \leq C < +\infty$ for some constant C independent of δ and let K be a non empty open set compactly included in Ω. Then the sequence $\{\lambda^\delta\}_\delta$ accumulates Dirichlet eigenvalues if and only if there exists $\eta > 0$ and $\delta_0 > 0$ such that for all $\delta \leq \delta_0$,*

$$\|u^\delta\|_{L^2(K)} \geq \eta. \tag{4}$$

Proof: First, let us assume that (4) holds. Then take $\psi \in C_0^\infty(\Omega)$ such that $\psi = 1$ in K and choose $v = \psi^2 u^\delta$ in (3). By developing $\nabla(\psi^2 u^\delta)$ and using Young's inequality we obtain

$$0 \leq \frac{3}{4}\int_\Omega \psi^2 |\nabla u^\delta|^2\,dx \leq \lambda^\delta \int_\Omega \psi^2 u_\delta^2\,dx + 4\int_\Omega |\nabla\psi|^2 u_\delta^2\,dx.$$

This implies $\lambda^\delta \geq -4\|\nabla\psi\|^2_{L^\infty(\Omega)}/\eta$. Then by Lemma 2.1 we obtain that accumulation points are Dirichlet eigenvalues since by (4), any subsequence of u^δ cannot converge to 0 in $L^2(\Omega)$.

Conversely, if $\{\lambda^\delta\}_\delta$ accumulates at Dirichlet eigenvalues, the number of these accumulation point is finite. Then by Lemma 2.1, and since eigenspaces have finite dimensions and $\|u\|_{L^2(K)} > 0$ for all eigenfunctions, accumulation points of $\|u^\delta\|_{L^2(K)}$ are finite discrete positive numbers. This proves (4). □

Lemma 2.1 and Theorem 2.2 prove in particular that accumulating points for Robin eigenpairs $(\lambda^\delta, u^\delta) \in \mathbb{R} \times H^1(\Omega)$ such that $\|u^\delta\|_{H^1(\Omega)} = 1$ are only Dirichlet eigenpairs.

3 Asymptotic of Robin eigenvalues accumulating at Dirichlet eigenvalues

First, it is easy to check that $(\lambda^\delta, u^\delta)$ is a solution of (1)-(2) if and only if $\mu^\delta = \lambda^\delta + \dfrac{\alpha}{\delta^2}$ for some positive constant $\alpha > 0$ and $u^\delta \in H^1(\Omega)$ solve

$$\int_\Omega \nabla u^\delta \nabla v\,dx - \frac{1}{\delta}\int_\Gamma u^\delta v\,ds + \frac{\alpha}{\delta^2}\int_\Omega u^\delta v\,dx = \mu^\delta \int_\Omega u^\delta v\,dx \qquad \text{for all } v \in H^1(\Omega). \quad (5)$$

In the space of $H^1(\Omega)$-functions let us introduce the δ-dependence norm $\|u\|^2_{H^1_\delta(\Omega)} := \|\nabla u\|^2_{L^2(\Omega)} + \frac{1}{\delta^2}\|u\|^2_{L^2(\Omega)}$. We can prove a coercivity result for the variational formulation (5) in $H^1_\delta(\Omega)$ thanks to the following Lemma which is obtained by using the inequality $\|u\|^2_{L^2(\Gamma)} \leq C(\|\nabla u\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)})$.

Lemma 3.1 *There exist positive constants α, θ and δ_0 such that for all $\delta \leq \delta_0$*

$$\int_\Omega |\nabla u|^2\,dx - \frac{1}{\delta}\int_\Gamma |u|^2\,ds + \frac{\alpha}{\delta^2}\int_\Omega |u|^2\,dx \geq \theta\left(\int_\Omega |\nabla u|^2\,dx + \frac{1}{\delta^2}\int_\Omega |u|^2\,dx\right)\ \forall u \in H^1(\Omega).$$

Note that from Lemma 3.1 we also have that Problem (5) can be written as a generalized eigenvalue problem $(A^\delta u, v)_{H^1_\delta(\Omega)} = \mu^\delta (B^\delta u, v)_{H^1_\delta(\Omega)}$ where the bounded linear operators $A^\delta : H^1_\delta(\Omega) \to H^1_\delta(\Omega)$ and $B^\delta : H^1_\delta(\Omega) \to H^1_\delta(\Omega)$ are defined by

$$\left(A^\delta u, v\right)_{H^1_\delta(\Omega)} := \int_\Omega \nabla u \nabla v\,dx - \frac{1}{\delta}\int_\Gamma uv\,ds + \frac{\alpha}{\delta^2}\int_\Omega uv\,dx \quad \text{and} \quad \left(B^\delta u, v\right)_{H^1_\delta(\Omega)} := \int_\Omega uv\,dx$$

for all $u, v \in H^1(\Omega)$. The operator $A^\delta : H^1_\delta(\Omega) \to H^1_\delta(\Omega)$ is self-adjoint, coercive with coercivity constant independent of δ from Lemma 3.1 and it satisfies $\|A^\delta\| \leq C$ with a constant $C > 0$ independent of δ, whereas the operator $B^\delta : H^1_\delta(\Omega) \to H^1_\delta(\Omega)$ is self-adjoint and compact. Hence, it is known that there exists a sequence of $\mu_k^\delta > 0$, $k = 0, \dots, +\infty$ accumulating to infinity such that $1/\mu_k^\delta$ are the eigenvalues of the compact self-adjoint operator $(A^\delta)^{-1/2} B^\delta (A^\delta)^{-1/2}$ and the μ_k^δ are eigenvalues of (5).

3.1 Formal asymptotic of the positive eigenvalues

Let us take $(\lambda^\delta, u^\delta)$ a solution to (1)-(2) and introduce the ansatz $U_N^\delta := \sum_{k=0}^N \delta^k u_k$ for u^δ and $\Lambda_N^\delta := \sum_{k=0}^N \delta^k \lambda_k$ for λ^δ. Plugging these two expressions into (1)-(2) enable us to compute all the terms in the expansions explicitly by equating the same powers of δ. To this end, this process first defines λ_0 as being an eigenvalue of $-\Delta$ with Dirichlet boundary conditions in the domain Ω with corresponding eigenvector u_0 normalized such that $\|u_0\|_{L^2(\Omega)} = 1$. Let us assume that λ_0 is simple, otherwise, the definition of the higher order terms in the expansion of λ^δ is much more involved. Next, for some $k > 0$ let us assume that u_p and λ_p for $p < k$ are known. Then, the function $u_k \in H^1(\Omega)$ must be a solution to

$$\Delta u_k + \lambda_0 u_k = -\sum_{p=0}^{k-1} \lambda_{k-p} u_p \text{ in } \Omega, \quad u_k = \partial_\nu u_{k-1} \text{ on } \Gamma \quad \text{and} \quad \int_\Omega u_k u_0 \, dx = 0,$$

where the latter is the compatibility condition that guaranties the existence of u_k. Here, we use the convention that the terms with negative indices are 0. The compatibility condition determines the value of λ_k to $\lambda_k := \int_\Gamma \partial_\nu u_{k-1} \partial_\nu u_0$ and u_k is uniquely defined and for every k there exists $C > 0$ such that $\|u_k\|_{H^2(\Omega)} \le C \|u_0\|_{H^1(\Omega)}$. In addition, for all $v \in H^1(\Omega)$ and $k > 0$, u_k satisfies the following variational equality

$$\int_\Omega \nabla u_k \nabla v dx = \sum_{p=0}^{k} \lambda_{k-p} \int_\Omega u_p v dx + \int_\Gamma \partial_\nu u_k v ds. \tag{6}$$

3.2 A convergence result

For any two functions $u, v \in H^1(\Omega)$, and for $N > 0$, let us denote by

$$E_N^\delta(u, v) := (A^\delta u, v)_{H^1_\delta(\Omega)} - \hat{\mu}_N^\delta (B^\delta u, v)_{H^1_\delta(\Omega)}$$

with $\hat{\mu}_N^\delta := \Lambda_N^\delta + \alpha/\delta^2$. Using equation (6) and the definition of u_0 we obtain after some calculations that for $N \ge 0$,

$$E_N^\delta(U_N^\delta, v) = \delta^N \int_\Gamma \partial_\nu u_N v ds + \sum_{p,k=0, p+k>N}^{N} \delta^{p+k} \lambda_k \int_\Omega u_p v ds.$$

Since u_0 is uniformly bounded with respect to δ in $H^2(\Omega)$, by using the fact that $\|u\|_{L^2(\Gamma)}^2 \le c\delta \|u\|_{H^1_\delta(\Omega)}^2$ and the bounds on the functions u_k, we obtain that for all $N \ge 0$ it exists $C > 0$ such that for all $\delta > 0$ sufficiently small, $E_N^\delta(U_N^\delta, v) \le C\delta^{N+1/2} \|v\|_{H^1_\delta(\Omega)}$ for all $v \in H^1(\Omega)$. Note that thanks to the normalization $\|u_0\|_{L^2(\Omega)} = 1$ we have that

$\|u_0\|_{H^1_\delta(\Omega)} \geq \delta^{-1}$ and for $N \geq 0$ there exists $C > 0$ such that for all $\delta > 0$ sufficiently small, $\|U_N^\delta\|_{H^1_\delta(\Omega)} \geq C\delta^{-1}$. Hence setting $\hat{U}_N^\delta := U_N^\delta / \|U_N^\delta\|_{H^1_\delta(\Omega)}$ yields

$$\left| E_N^\delta(\hat{U}_N^\delta, v) \right| \leq C\delta^{N+3/2} \|v\|_{H^1_\delta(\Omega)}, \text{ for all } v \in H^1(\Omega). \tag{7}$$

Making use of the Lemma 1.1 in Chapter 3 of [5] we can prove the following theorem.

Theorem 3.2 *Let $N \geq 0$ and Λ_N^δ be as above. There exist $C > 0$ and $\delta_0 > 0$ such that for all $\delta > 0$, $\delta < \delta_0$, there exits an eigenvalue $\lambda^\delta > 0$ of* (1)-(2) *such that $\left|\lambda^\delta - \Lambda_N^\delta\right| \leq C\delta^{N-1/2}$.*

Proof: Let us define $T^\delta := (A^\delta)^{-1/2} B^\delta (A^\delta)^{-1/2}$ as an operator from $H^1_\delta(\Omega)$ into itself. Then (7) becomes $\left\| T^\delta \hat{U}_N^\delta - \hat{U}_N^\delta / \hat{\mu}_N^\delta \right\| \leq C\delta^{N+3/2}/|\hat{\mu}_N^\delta|$. From Lemma 1.1 in Chapter 3 of [5] we obtain that there exists an eigenvalue μ^δ of problem (5) such that $\left|1/\hat{\mu}_N^\delta - 1/\mu^\delta\right| \leq C\delta^{N+3/2}/|\hat{\mu}_N^\delta|$ and as a consequence $|\hat{\mu}_N^\delta - \mu^\delta| \leq C\delta^{N+3/2}|\mu^\delta|$. Therefore it exists $\widetilde{C} > 0$ independent of δ such that $|\mu^\delta| \leq \widetilde{C}|\hat{\mu}_N^\delta| \leq \widetilde{C}(\Lambda_N^\delta + \alpha/\delta^2)$ which yields the desired result. □

This result is not optimal in terms of the power of δ but since for all $N \geq 0$ the error writes $\lambda^\delta - \Lambda_N^\delta = \lambda^\delta - \Lambda_{N+2}^\delta + \delta^{N+1}\lambda_{N+1} + \delta^{N+2}\lambda_{N+2}$ we finally obtain the following result.

Corollary 3.3 *Let $N \geq 0$ and Λ_N^δ be as above. There exist $C > 0$ and $\delta_0 > 0$ such that for all $\delta > 0$, $\delta < \delta_0$, there exits an eigenvalue $\lambda^\delta > 0$ of* (1)-(2) *such that $\left|\lambda^\delta - \Lambda_N^\delta\right| \leq C\delta^{N+1}$.*

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References

[1] N. Chaulet. Modèles d'impédance généralisée en diffraction inverse. PhD Thesis, 2012.

[2] D. Daners, and J. B. Kennedy. On the asymptotic behaviour of the eigenvalues of a Robin problem. Differential integral equation, 27(7/8), 2010.

[3] A. A. Lacey, J. R. Ockendon and J. Sabina. Multidimensional Reaction Diffusion Equations with Nonlinear Boundary Conditions. SIAM J. App. Math., 58(5), 1998.

[4] M. Levitin and L. Parnovski. On the principal eigenvalue of a Robin problem with a large parameter. Math. Nachr., 281(2),2008.

[5] O. A. Oleinik, A. S. Shamaev, and G. A. Yosifian. Mathematical problems in elasticity and homogenization. Elsevier, 1992.